MATH 10023

Core Mathematics III

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Department of Mathematical Sciences
Kent State University

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Contents

Chapter 0

Review

0.1 Basic Factoring Techniques

We begin with a review of basic factoring techniques from your previous algebra classes. The first technique is factoring out the *greatest common factor* or GFC. We find the GCF of all the terms of the polynomial and then factor it out using the distributive law.

Example 1. *Factor out the GCF of each of the following polynomials.*

a. $P(x) = 8x + 4$

b. $P(x) = 9x^3 - 12x^2 + 15x$

c. $P(x, y, z) = 18x^5yz^2 + 63x^3y^3z^3 + 36x^8y^2$

Solution.

a. The GCF of the terms $8x$ and 4 is 4. Thus, we obtain

$$\begin{aligned} P(x) &= 8x + 4 \\ &= 4 \cdot 2x + 4 \cdot 1 \\ &= 4\,(2x + 1)\,. \end{aligned}$$

b. The GCF of the terms $9x^3$, $12x^2$, and $15x$ is $3x$. Thus, we obtain

$$\begin{aligned} P(x) &= 9x^3 - 12x^2 + 15x \\ &= 3x \cdot 3x^2 - 3x \cdot 4x + 3x \cdot 5 \\ &= 3x\left(3x^2 - 4x + 5\right). \end{aligned}$$

c. The GCF of the terms of this polynomial is $9x^3y$. Thus, we obtain

$$\begin{aligned} P(x,y,z) &= 18x^5yz^2 + 63x^3y^3z^3 + 36x^8y^2 \\ &= 9x^3y \cdot 2x^2z^2 + 9x^3y \cdot 7y^2z^3 + 9x^3y \cdot 4x^5y \\ &= 9x^3y\left(2x^2z^2 + 7y^2z^3 + 4x^5y\right). \end{aligned}$$

■

We next review the the technique of factoring the difference of two squares:

$$a^2 - b^2 = (a+b)(a-b)$$

where a and b are real numbers.

Example 2. *Factor each of the following.*

a. $x^2 - 4$ *b.* $9x^2 - 25y^2$

Solution.

a. Since both x^2 and $4 = 2^2$ are perfect squares, we have

$$\begin{aligned} x^2 - 4 &= x^2 - 2^2 \\ &= (x+2)(x-2). \end{aligned}$$

b. Both $9x^2 = (3x)^2$ and $25y^2 = (5y)^2$ are perfect squares, so

$$\begin{aligned} 9x^2 - 25y^2 &= (3x)^2 - (5y)^2 \\ &= (3x+5y)(3x-5y). \end{aligned}$$

Remember that for each term, the *whole term* must first be written as a square. ■

Finally, we recall the technique of factoring trinomials.

Example 3. *Factor* $P(x) = x^2 - x - 12$ *over the integers.*

Solution. We will first list all pairs of integer factors of -12 and their sums.

ab	$a+b$
$-1 \cdot 12$	$-1+12=11$
$1 \cdot -12$	$1-12=-11$
$-2 \cdot 6$	$-2+6=4$
$2 \cdot -6$	$2-6=-4$
$-3 \cdot 4$	$-3+4=1$
$3 \cdot -4$	$3+4=-1 \star$

We are hoping to find a pair of factors of -12 whose sum is -1, since this is the coefficient of x. We see that the last pair in the table satisfies this condition. Therefore,

$$\begin{aligned} P(x) &= x^2 - x - 12 \\ &= (x+3)(x-4). \end{aligned}$$

It's a good idea to check this by multiplying.

	x	-4
x	x^2	$-4x$
3	$3x$	-12

Thus,

$$\begin{aligned} (x+3)(x-4) &= x^2 - 4x + 3x - 12 \\ &= x^2 - x - 12, \end{aligned}$$

which is what we claimed. ■

SECTION 0.1 EXERCISES:
(Answers are found on page 147.)

Factor completely over the integers.

1. $4x^3y - xy^3$
2. $5x^2 + 10xy - 40y^2$
3. $1 - (a+b)^2$
4. $6u^2 + 7uv - 3v^2$
5. $9x^2 - x^6y^2$
6. $2y^4 - 5y^2 - 12$
7. $7a^2 + 17a - 12$
8. $8t^3 + 80t^2 + 150t$
9. $5x^2 - 45$
10. $a^3 - 13a^2 - 90a$

11. $(x-3)^{10}-1$
12. $3x^3y-6x^2y^2+3xy^3$
13. $2z(x-2y)-w(x-2y)$
14. $b^2+20b-800$
15. $12-8x-7x^2$
16. $6x^2+48x+72$
17. $81y^4-16x^4$
18. $(3a+5)a^2+(3a+5)a$
19. $4y^2-29yz-24z^2$
20. $18x^3-8x$
21. $2x^3y-6x^2y^2-8xy^3$
22. $12y^3+44y^2-16y$
23. $3a^4-48a^2$
24. $40x^2+14x-45$
25. x^2y^2-225
26. $81-(a+1)^2$
27. $4x^3-20x^2+25x$
28. $(a-3)^2-(a^2+3)^2$

Chapter 1

Rational Expressions and Functions

1.1 Introduction

Integer analogy

We began our discussion of factored forms in our previous algebra classes with examples involving integers. The set of integers and the set of polynomials share many mathematical properties, so it is useful to consider integer analogs of the polynomial properties and operations that we are discussing. For example, the set of integers is ***closed*** under the operations of addition, subtraction, and multiplication. This means that the sum, difference, or product of any two integers is again an integer. Similarly, as we have seen, the sum, difference, or product of any two polynomials is again a polynomial. Thus, we say that the set of polynomials is ***closed*** under addition, subtraction, and multiplication.

Division is another story. The quotient of two integers might turn out to be an integer, but it might not. For example,

$$6 \div 3 = 2$$

is an integer, while

$$6 \div 4 = \frac{6}{4} = \frac{3}{2}$$

is not an integer. The set of ***rational numbers***[*] is defined as the set of numbers that can be written as quotients of two integers. This set includes all

[*] The term *rational* in this context comes from the word *ratio*, meaning *quotient*, and has nothing to do with how "reasonable" these numbers might be!

integers since any integer can be written as itself divided by one. Unlike the set of integers, the set of rational numbers is closed under all four basic operations: addition, subtraction, multiplication, and division.

In a similar way, the quotient of two polynomials might be a polynomial, but it might not. Consider the following examples involving monomials. We see that

$$x^2yz^4 \div xz^2 = \frac{x^2yz^4}{xz^2} = xyz^2$$

is a polynomial, while

$$x^2yz^4 \div x^3z^2 = \frac{x^2yz^4}{x^3z^2} = \frac{yz^2}{x}$$

is not a polynomial. Here again we will define a new set of mathematical objects to include all polynomials and all possible quotients of polynomials. We will call this the set of ***rational expressions***. Unlike the set of polynomials, the set of rational expressions is closed under all four basic operations: addition, subtraction, multiplication, and division. In this chapter we will study the set of rational expressions and the rational functions defined by them.

We will first look at examples of expressions that are rational expressions and others that are not.

Example 1. *Determine which of the following are rational expressions.*

a. $\dfrac{4x^3 - 6x^2}{5x^2 + 1}$ *b.* $\dfrac{\sqrt{x^2 + 3}}{x - 2}$ *c.* $8t^{10} - 15t^5 + 9$

Solution.

a. The expression $\frac{4x^3-6x^2}{5x^2+1}$ is a quotient of two polynomials. Therefore it is a rational expression.

b. In the expression $\frac{\sqrt{x^2+3}}{x-2}$ the variable appears under a radical (square root). Therefore, this is not a rational expression.

c. The expression $8t^{10} - 15t^5 + 7t^3 + 9$ is a polynomial. Since the set of rational expressions includes the set of polynomials, this is a rational expression. ■

Practice 1. *Determine which of the following are rational expressions.* *(Answers on page 10.)*

a. $\dfrac{3\sqrt{x}-1}{x^2+2}$ *b.* $x^2+2xy+y^2$ *c.* $\dfrac{1}{x}$

Division involving zero

We have seen previously that zero has some special properties. In previous courses, we exploited the Zero Product Property to solve polynomial equations by factoring. We have learned that 0 is the additive identity element of the set of real numbers. A consequence of this is the ***Multiplication Property of Zero***:

If a is any real number, then $a \cdot 0 = 0$.

Remember that multiplication and division are closely connected. We can regard a division problem as a "missing factor" multiplication problem. For example, solving

$$\frac{10}{5} = ?$$

is equivalent to solving

$$5 \times ? = 10.$$

That is,

$$\frac{10}{5} = 2$$

precisely because

$$5 \times 2 = 10.$$

What happens if we attempt to divide 10 by 0? We know that the division problem

$$\frac{10}{0} = ?$$

is equivalent to the multiplication problem

$$0 \times ? = 10.$$

However, the multiplication problem has no solution since 0 times any real number equals 0 by the Multiplication Property of Zero. Thus, we say that $\frac{10}{0}$ is ***undefined***.

There was nothing special about 10 here—we could replace 10 with any nonzero real number and draw the same conclusion. But what if we divide 0 by 0? Now the division problem

$$\frac{0}{0} = ?$$

is equivalent to the multiplication problem

$$0 \times ? = 0.$$

In this case, *every* real number is a solution to the multiplication problem. For this reason, we say that $\frac{0}{0}$ is ***indeterminate***, and so it is also undefined. In summary:

Division by zero is undefined.

Practice 2. *Convert each of the following division problems into an equivalent multiplication problem, then solve or else explain why no solution exists. (Answers on page 10.)*

a. $\frac{24}{4} = ?$ *b.* $\frac{24}{0} = ?$ *c.* $\frac{0}{4} = ?$ *d.* $\frac{-6}{0} = ?$

Rational functions

We now turn our attention to functions whose rules are given by rational expressions. These functions are called ***rational functions***. We wish to evaluate rational functions for various values of the variable and to determine their domains. Since the domain of any polynomial function is the set of all real numbers, the domain of a rational expression will consist of all real numbers that do not make the denominator zero.

Example 2. *Let* $r(x) = \dfrac{2x-5}{x^2-1}$ *Evaluate each of the following, or indicate "undefined."*

a. $r(3)$ *b.* $r(-10)$ *c.* $r(1)$

Solution. We substitute the number indicated for the variable, provided this will not produce 0 in the denominator (since division by zero is undefined).

a. Substituting 3 for x will not produce 0 in the denominator. Thus,

$$\begin{aligned} r(3) &= \frac{2(3)-5}{(3)^2-1} \\ &= \frac{6-5}{9-1} \\ &= \frac{1}{8}. \end{aligned}$$

b. Substituting -10 for x will not produce 0 in the denominator. Thus,

$$\begin{aligned} r(-10) &= \frac{2(-10)-5}{(-10)^2-1} \\ &= \frac{-20-5}{100-1} \\ &= \frac{-25}{99} \\ &= -\frac{25}{99}. \end{aligned}$$

c. Substituting 1 for x in the denominator, we obtain

$$\begin{aligned} (1)^2-1 &= 1-1 \\ &= 0. \end{aligned}$$

Thus, $r(1)$ is undefined. ■

Practice 3. *Let* $r(x) = \dfrac{x^2+4x-1}{x-6}$ *Evaluate each of the following, or indicate "undefined."* *(Answers on the next page.)*

a. $r(-2)$ *b.* $r(3)$ *c.* $r(6)$

Example 3. *Let* $r(x) = \dfrac{x^2+9}{x^2-9}$. *Find all values of* x *for which* $r(x)$ *is undefined. Then express the domain of* r *in interval notation.*

Solution. Since a rational expression is undefined for any value(s) of x which make the denominator zero, we set the denominator equal to zero and solve for x.

$$\begin{aligned} x^2-9 &= 0 \\ (x-3)(x+3) &= 0 \end{aligned}$$

By the Zero Product Property,

$$\begin{aligned} x-3&=0 \\ x&=3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x+3&=0 \\ x&=-3 \end{aligned}$$

Now ± 3 are the only real numbers for which this rational expression is *undefined*, so the domain of r consists of all real numbers *except* ± 3. In interval notation,

$$\operatorname{dom}(r) = (-\infty, -3) \cup (-3, 3) \cup (3, \infty). \qquad \blacksquare$$

Practice 4. *Let* $r(t) = \dfrac{6t-5}{t^3-t^2-2t}$. *Find all values of* t *for which* $r(t)$ *is undefined. Then express the domain of* r *in interval notation.* *(Answers below.)*

ANSWERS TO SECTION 1.1 PRACTICE PROBLEMS

1. (a) not a rational expression
   (b) rational expression
   (c) rational expression

2. (a) $4 \times ? = 24$; $? = 6$
   (b) $0 \times ? = 24$; no solution
   (c) $4 \times ? = 0$; $? = 0$
   (d) $0 \times ? = -6$; no solution

3. (a) $r(-2) = \frac{5}{8}$
   (b) $r(3) = -\frac{20}{3}$
   (c) $r(6)$ is undefined

4. r is undefined for $t = -1, 0, 2$
   $\operatorname{dom}(r) = (-\infty, -1) \cup (-1, 0) \cup (0, 2) \cup (2, \infty)$

SECTION 1.1 EXERCISES:

(Answers are found on page 147.)

Determine which of the following are rational expressions.

1. $\dfrac{t^3-6}{5+t^2}$
2. $\dfrac{y^{10}+7y^5}{5y^3}$
3. $\sqrt{x^2-8x+1}$
4. $\sqrt[3]{t}+t-2$
5. $\sqrt{2}\,x^3+x^2-0.81x-\dfrac{1}{2}$
6. $\dfrac{2}{x^3}-\dfrac{x}{4}$
7. $\dfrac{1-x^{-2}}{1+x^2}$
8. $2-\dfrac{3x-1}{4x+1}$
9. $\dfrac{2x^2(x^3-4x)}{4x^3-\sqrt{7}}$
10. $\dfrac{3|x|}{2x^2-7x+5}$

Convert each of the following division problems into an equivalent multiplication problem, then solve or else explain why no solution exists.

11. $\frac{105}{5} = ?$
12. $\frac{66}{11} = ?$
13. $\frac{81}{0} = ?$
14. $\frac{-45}{0} = ?$
15. $\frac{0}{81} = ?$
16. $\frac{0}{-45} = ?$
17. $\frac{0}{0} = ?$
18. $\frac{0}{\pi} = ?$
19. $\frac{-81}{-3} = ?$
20. $\frac{\sqrt{5}}{0} = ?$

Evaluate each of the following, or indicate "undefined."

21. For $r(y) = \frac{y^2 - 1}{2y}$, evaluate $r(-1)$, $r(0)$, and $r(2)$.
22. For $r(x) = \frac{x}{x^2 - 4}$, evaluate $r(-2)$, $r(0)$, and $r(1)$.
23. For $r(x) = \frac{x+5}{x-5}$, evaluate $r(-5)$, $r(0)$, and $r(5)$.
24. For $r(t) = \frac{t^3 + t - 1}{t^2 + 1}$, evaluate $r(-1)$, $r(0)$, and $r(1)$.
25. For $r(t) = \frac{5t}{16 - t^2}$, evaluate $r(-4)$, $r(0)$, and $r(2)$.
26. For $r(x) = \frac{x+1}{x^2 - 9}$, evaluate $r(-4)$, $r(0)$, and $r(3)$.
27. For $r(x) = \frac{x}{3x - 5}$, evaluate $r(-2)$, $r(0)$, and $r(3)$.
28. For $r(y) = \frac{y+4}{2y^2 - 4y}$, evaluate $r(-4)$, $r(0)$, and $r(2)$.
29. For $r(t) = \frac{6t}{t^2 - 3t + 2}$, evaluate $r(-1)$, $r(0)$, and $r(1)$.
30. For $r(s) = \frac{s^4 - 4s^2 + 1}{s^3}$, evaluate $r(-2)$, $r(0)$, and $r(1)$.

Find all values of the variable for which the given rational function is undefined. Then express the domain in interval notation.

31. $r(x) = \dfrac{x-1}{x^2+8}$

32. $r(t) = \dfrac{t^2-t}{t-7}$

33. $r(x) = \dfrac{5x-3}{x^3+2x^2}$

34. $r(x) = \dfrac{4}{x^2+14x+49}$

35. $r(t) = \dfrac{6t}{t^2-3t-10}$

36. $r(x) = \dfrac{x^3-8}{x}$

37. $r(y) = \dfrac{3y+1}{5y^2-4y}$

38. $r(t) = \dfrac{2t-7}{3t^2+11t-4}$

39. $r(x) = \dfrac{3x-2}{2x^2+4}$

40. $r(x) = \dfrac{5x^3-2x^2+1}{3x^2-27}$

1.2 Zeros and Vertical Asymptotes

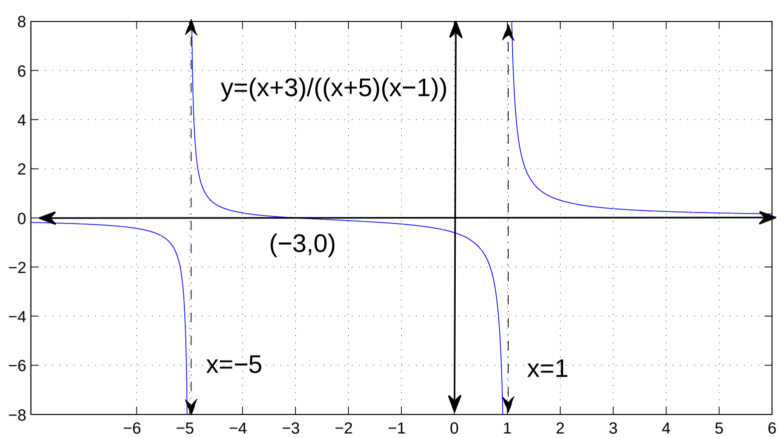
In your previous algebra courses, you studied the relationship between the linear factors, zeros, and x-intercepts of a polynomial. In particular, the following conditions are all equivalent:

- $(x - c)$ is a factor of the polynomial $P(x)$;
- $P(c) = 0$ (that is, c is a ***zero*** or ***root*** of $P(x)$);
- the point $(c, 0)$ is an x-intercept of the graph $y = P(x)$.

Since rational functions are ratios of polynomials, it's reasonable to have a similar situation for rational functions.

Example 1. *Let* $r(x) = \dfrac{x+3}{(x+5)(x-1)}$. *Examine the graph* $y = r(x)$ *and locate the x-intercepts (if any) and any other interesting behavior.*

Solution. Here is the graph of $r(x)$:



We observe that the x-intercept is the point $(-3, 0)$. This is not unexpected, since $x + 3$ is a factor of the numerator of $r(x)$. We also notice some interesting behavior near $x = -5$ and $x = 1$. The graph of $r(x)$ gets arbitrarily close to the vertical lines $x = -5$ and $x = 1$. We call these lines ***vertical asymptotes*** of $r(x)$. Look back at the formula for the function. Notice that $(x + 5)$ and $(x - 1)$ are factors of the denominator of $r(x)$. ■

We will assume that our rational function $r(x)$ is in *lowest terms*. If so, then the following conditions are all equivalent:

- $(x - c)$ is a factor of the *numerator* of $r(x)$;
- $r(c) = 0$ (that is, c is a ***zero*** of $r(x)$);
- the point $(c, 0)$ is an x-intercept of the graph $y = r(x)$.

This takes care of the factors of the numerator, but what about the factors of the denominator? We have seen in Section 1.1 that when the denominator is equal to zero, the rational function is undefined. As in the previous example, if the function is in lowest terms, this will correspond to a vertical asymptote on the graph $y = r(x)$. We have the following equivalent statements for a rational function in lowest terms:

- $(x - c)$ is a factor of the *denominator* of $r(x)$;
- $r(c)$ is undefined (that is, c is not in the domain of $r(x)$);
- the line $x = c$ is an vertical asymptote for the graph $y = r(x)$.

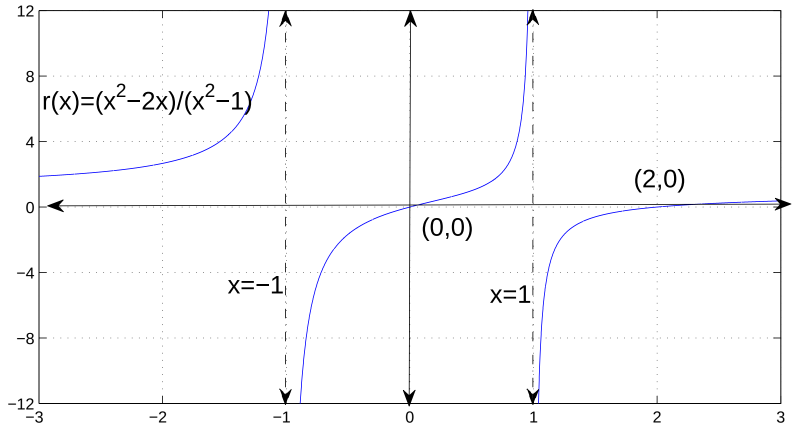
Example 2. *Find the x-intercepts and vertical asymptotes of the graph of the rational function $r(x) = \dfrac{x^2 - 2x}{x^2 - 1}$.*

Solution. We first factor the numerator and denominator completely over the real numbers.

$$\begin{aligned} r(x) &= \frac{x^2 - 2x}{x^2 - 1} \\ &= \frac{x(x-2)}{(x+1)(x-1)}. \end{aligned}$$

Notice that $r(x)$ is written in lowest terms. Then each linear factor of the numerator corresponds to an x-intercept, so the x-intercepts are $(0, 0)$ and $(2, 0)$. Each linear factor of the denominator corresponds to vertical asymptote. Thus, the vertical asymptotes are the lines $x = -1$ and $x = 1$.

We can confirm this by looking at the graph $y = r(x)$.



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SECTION 1.2 EXERCISES:
(Answers are found on page 148.)

Find the x-intercepts and vertical asymptotes of the graphs of each of the following rational functions. Express each intercept as an ordered pair and each asymptote as the equation of the line.

1. $r(x) = \dfrac{x-8}{x+3}$
2. $r(x) = \dfrac{x+4}{x+2}$
3. $r(x) = \dfrac{2x+10}{x}$
4. $r(x) = \dfrac{5x-15}{x-4}$
5. $r(x) = \dfrac{x^2-9}{2x-1}$
6. $r(x) = \dfrac{x^2-16}{3x+5}$
7. $r(x) = \dfrac{x^2+1}{x-6}$
8. $r(x) = \dfrac{x^2+7}{x}$
9. $r(x) = \dfrac{5-x}{x^2+2x+1}$
10. $r(x) = \dfrac{12-2x}{x^2+4x+4}$
11. $r(x) = \dfrac{x^2-x-6}{x^2+8}$
12. $r(x) = \dfrac{x^2-7x+12}{x^2+16}$
13. $r(x) = \dfrac{5x^3-20x}{x+3}$
14. $r(x) = \dfrac{2x^3-18x}{x^2+2}$
15. $r(x) = \dfrac{x^2-25}{36-x^2}$
16. $r(x) = \dfrac{100-x^2}{121-x^2}$

1.3 Simplification of Rational Expressions

Recall that the fractions $\frac{3}{6}$, $\frac{-50}{-100}$, and $\frac{1}{2}$ are all ***equivalent***—they all represent the same rational number. However, we usually prefer the form $\frac{1}{2}$ because it is in ***lowest terms***. That is, the only common factors of the numerator and denominator are ± 1.

Similarly, the rational expressions $\frac{2x}{2y}$ and $\frac{x}{y}$ are ***equivalent***. They represent the same real number for all *permissible* values of the variables x and y. Here again, we usually prefer $\frac{x}{y}$ since this is in lowest terms. We will write rational expressions in lowest terms in the same way that we write number fractions in lowest terms—by factoring the numerator and denominator and then dividing out the greatest common factor of the numerator and denominator. There is one slight twist: the domain of the original rational expression and the domain of the simplified expression might not be the same.

Example 1. *Simplify by writing in lowest terms.* $\dfrac{18x^3y^2}{30xy^5}$

Solution. The numerator and denominator are both in factored form, so we pull out the greatest common factor and simplify.

$$\begin{aligned}\frac{18x^3y^2}{30xy^5} &= \frac{6xy^2 \cdot 3x^2}{6xy^2 \cdot 5y^3} \\ &= \frac{6xy^2}{6xy^2} \cdot \frac{3x^2}{5y^3} \\ &= 1 \cdot \frac{3x^2}{5y^3} \\ &= \frac{3x^2}{5y^3}\end{aligned}$$

where $x, y \neq 0$ since these would make the denominator of the original expression zero. ■

Practice 1. *Simplify by writing in lowest terms.* $\dfrac{49a^5bc^2}{56a^3b^5c}$

(Answers on page 22.)

Example 2. *Simplify by writing in lowest terms.* $\dfrac{7y-7}{y^5-y^4}$

Solution. Here we must first factor the numerator and denominator fully.

$$\begin{aligned}\frac{7y-7}{y^5-y^4} &= \frac{7(y-1)}{y^4(y-1)} \\ &= \frac{7}{y^4}\cdot\frac{(y-1)}{(y-1)} \\ &= \frac{7}{y^4}\cdot 1 \\ &= \frac{7}{y^4}\end{aligned}$$

where $y \neq 0, 1$ since these would make the denominator of the original expression zero. ■

Practice 2. *Simplify by writing in lowest terms.* $\dfrac{2t^3+3t^2}{-10t-15}$

(Answers on page 22.)

Example 3. *Simplify by writing in lowest terms.* $\dfrac{t^2-25}{t^2+2t-15}$

Solution. Again, we begin by factoring the numerator and denominator fully using techniques from Chapter **??**.

$$\begin{aligned}\frac{t^2-25}{t^2+2t-15} &= \frac{(t-5)(t+5)}{(t+5)(t-3)} \\ &= \frac{(t+5)}{(t+5)}\cdot\frac{(t-5)}{(t-3)} \\ &= 1\cdot\frac{(t-5)}{(t-3)} \\ &= \frac{t-5}{t-3}\end{aligned}$$

where $t \neq -5, 3$ since these would make the denominator of the original expression zero. ■

Practice 3. *Simplify by writing in lowest terms.* $\dfrac{2x^2+5x-3}{x^2+6x+9}$

(Answers on page 22.)

Example 4. *Simplify by writing in lowest terms.*

a. $\dfrac{x-2}{2-x}$ *b.* $\dfrac{15t^2-5t-20}{16-9t^2}$

Solution.

a. The numerator and denominator of this expression are additive inverses of one another. We can factor -1 out of the denominator and then simplify.

$$\begin{aligned}\frac{x-2}{2-x} &= \frac{x-2}{-1(-2+x)}\\ &= \frac{x-2}{-1(x-2)}\\ &= -1\cdot\frac{x-2}{x-2}\\ &= -1\cdot 1\\ &= -1\end{aligned}$$

where $x \neq 2$. Note that we could have factored -1 out of the numerator instead.

b. We begin by factoring the numerator and denominator fully. We will watch for factors that are additive inverses of one another.

$$\begin{aligned}\frac{15t^2-5t-20}{16-9t^2} &= \frac{5(3t^2-t-4)}{(4-3t)(4+3t)}\\ &= \frac{5(3t-4)(t+1)}{(4-3t)(4+3t)}\\ &= \frac{5(3t-4)(t+1)}{(-1)(3t-4)(3t+4)}\\ &= \frac{3t-4}{3t-4}\cdot\frac{5(t+1)}{(-1)(3t+4)}\\ &= 1\cdot\frac{5(t+1)}{(-1)(3t+4)}\\ &= -\frac{5(t+1)}{3t+4}\end{aligned}$$

where $x \neq \pm\frac{4}{3}$. ■

Practice 4. *Simplify by writing in lowest terms.* *(Answers on page 22.)*

a. $\dfrac{a-b}{b-a}$

b. $\dfrac{49-4y^2}{2y^2+3y-35}$

Compound rational expressions

We next consider ***compound*** (or ***complex***) rational expressions which are rational expressions whose numerators and/or denominators contain fractions. We wish to write these as simple rational expressions in lowest terms. One way to do this is to clear the denominators of the minor fractions by multiplying the numerator and denominator of the main fraction by the least common denominator (LCD) of all of the minor fractions.

Example 5. *Simplify.* $\dfrac{1-\dfrac{2}{3}}{\dfrac{1}{2}+5}$

Solution. The minor fractions in this compound rational expression are $\frac{2}{3}$ and $\frac{1}{2}$. Their least common denominator is $3 \cdot 2 = 6$. Therefore, we multiply the main fraction by $\frac{6}{6}$. Since $\frac{6}{6} = 1$, this does not change the value of the main expression.

$$\begin{aligned}
\frac{1-\dfrac{2}{3}}{\dfrac{1}{2}+5} &= 1 \cdot \frac{1-\dfrac{2}{3}}{\dfrac{1}{2}+5} \\
&= \frac{6}{6} \cdot \frac{1-\dfrac{2}{3}}{\dfrac{1}{2}+5} \\
&= \frac{6 \cdot \left(1-\dfrac{2}{3}\right)}{6 \cdot \left(\dfrac{1}{2}+5\right)} \\
&= \frac{6 \cdot 1 - \dfrac{6 \cdot 2}{3}}{\dfrac{6 \cdot 1}{2} + 6 \cdot 5}
\end{aligned}$$

$$= \frac{6-4}{3+30}$$
$$= \frac{2}{33}.$$

■

Practice 5. *Simplify.* $\dfrac{\dfrac{3}{5}+1}{2-\dfrac{1}{3}}$

(Answers on page 22.)

Example 6. *Simplify.*

a. $\dfrac{\dfrac{1}{x}+\dfrac{1}{5}}{\dfrac{1}{x}-\dfrac{1}{5}}$ *b.* $\dfrac{1+\dfrac{x}{2y}}{\dfrac{x-1}{y}}$

Solution.

a. The minor fractions are $\frac{1}{x}$ and $\frac{1}{5}$ and their LCD is $5x$.

$$\frac{\dfrac{1}{x}+\dfrac{1}{5}}{\dfrac{1}{x}-\dfrac{1}{5}} = \frac{5x}{5x}\cdot\frac{\dfrac{1}{x}+\dfrac{1}{5}}{\dfrac{1}{x}-\dfrac{1}{5}}$$
$$= \frac{5x\cdot\left(\dfrac{1}{x}+\dfrac{1}{5}\right)}{5x\cdot\left(\dfrac{1}{x}-\dfrac{1}{5}\right)}$$
$$= \frac{\dfrac{5x\cdot 1}{x}+\dfrac{5x\cdot 1}{5}}{\dfrac{5x\cdot 1}{x}-\dfrac{5x\cdot 1}{5}}$$
$$= \frac{\dfrac{x}{x}\cdot 5+\dfrac{5}{5}\cdot x}{\dfrac{x}{x}\cdot 5-\dfrac{5}{5}\cdot x}$$
$$= \frac{5+x}{5-x}$$

where $x \neq 0, 5$ since these would make denominators in the original expression zero.

b. The minor fractions are $\frac{x}{2y}$ and $\frac{x-1}{y}$ and their LCD is $2y$.

$$\begin{aligned}
\frac{1+\dfrac{x}{2y}}{\dfrac{x-1}{y}} &= \frac{2y}{2y} \cdot \frac{1+\dfrac{x}{2y}}{\dfrac{x-1}{y}} \\
&= \frac{2y \cdot \left(1+\dfrac{x}{2y}\right)}{2y \cdot \left(\dfrac{x-1}{y}\right)} \\
&= \frac{2y \cdot 1 + \dfrac{2y \cdot x}{2y}}{\dfrac{2y \cdot (x-1)}{y}} \\
&= \frac{2y + \dfrac{2y}{2y} \cdot x}{\dfrac{y}{y} \cdot 2 \cdot (x-1)} \\
&= \frac{2y+x}{2(x-1)}
\end{aligned}$$

where $y \neq 0$ and $x \neq 1$. Note that 2 is a factor of the denominator, but not of the entire numerator, so no further simplification is possible. ■

Practice 6. *Simplify.* *(Answers on the following page.)*

a. $\dfrac{\dfrac{3}{y}+\dfrac{1}{4}}{\dfrac{1}{y}+\dfrac{1}{2}}$

b. $\dfrac{\dfrac{t+2}{t}}{3-\dfrac{1}{5t}}$

ANSWERS TO SECTION 1.3 PRACTICE PROBLEMS

1. $\frac{7a^2c}{8b^4}$
2. $-\frac{t^2}{5}$ $(= -\frac{1}{5}t^2)$
3. $\frac{2x-1}{x+3}$
4. (a) -1
   (b) $-\frac{2y+7}{y+5}$
5. $\frac{24}{25}$
6. (a) $\frac{y+12}{2(y+2)}$
   (b) $\frac{5(t+2)}{15t-1}$

SECTION 1.3 EXERCISES:

(Answers are found on page 148.)

Simplify by writing in lowest terms.

1. $\dfrac{32x^3y^2}{-8xy^5}$
2. $\dfrac{2a^6b^5}{10a^6b^4}$
3. $\dfrac{3(x-12)(x+8)}{6(x-12)}$
4. $\dfrac{x(2x+1)}{(2x+1)(x-3)}$
5. $\dfrac{3t-1}{1-3t}$
6. $\dfrac{5x-10}{2-x}$
7. $\dfrac{y-5}{y^2-10y+25}$
8. $\dfrac{x^2+12x+36}{x+6}$
9. $\dfrac{t^2-10t}{t^3-17t^2+70t}$
10. $\dfrac{5x^2+30x}{x+6}$
11. $\dfrac{x^2-81}{x^2+18x+81}$
12. $\dfrac{a^2-20a+100}{a^2-100}$
13. $\dfrac{2m^2+7m-4}{2m-1}$
14. $\dfrac{25y^2-1}{5y^2+4y-1}$
15. $\dfrac{4-x^2}{x^2-2x}$
16. $\dfrac{x^2-10x-11}{121-x^2}$
17. $\dfrac{3y^2+13y+4}{3y^2+7y+2}$
18. $\dfrac{2t^2+9t-5}{4t^2-4t+1}$
19. $\dfrac{x\left(x^2+4\right)-4\left(x^2+4\right)}{x^4-16}$
20. $\dfrac{7(3x+2)+x(3x+2)}{3x^2+5x+2}$
21. $\dfrac{5a-3b}{6b-10a}$
22. $\dfrac{x^3}{x^2+2x}$

23. $\dfrac{x^2+5x-6}{x^2+6x}$

24. $\dfrac{t^3-4t}{12-3t^2}$

25. $\dfrac{x^3-x}{x^3-2x^2+x}$

26. $\dfrac{4a^2-4ab}{3b^2-3ab}$

27. $\dfrac{5y^2-20}{10y^2+10y-60}$

28. $\dfrac{6a^2b^6-8a^4b^2}{2a^2b^2}$

29. $\dfrac{4a^2+12a+9}{4a^2-9}$

30. $\dfrac{x^2+6x+5}{x^2-x-2}$

Simplify each compound rational expression.

31. $\dfrac{1+\frac{3}{5}}{\frac{1}{3}-1}$

32. $\dfrac{\frac{1}{4}-2}{\frac{3}{4}+1}$

33. $\dfrac{\frac{2}{3}-\frac{2}{5}}{\frac{5}{4}-\frac{4}{5}}$

34. $\dfrac{\frac{1}{h}-\frac{1}{5}}{h}$

35. $\dfrac{\frac{1}{x}-\frac{1}{3}}{\frac{1}{x}+\frac{1}{3}}$

36. $\dfrac{\frac{1}{y}+\frac{2}{5}}{\frac{1}{y}-\frac{2}{5}}$

37. $\dfrac{1+\frac{1}{a}}{\frac{a+1}{a}}$

38. $\dfrac{\frac{t+1}{t}}{\frac{2}{t}-3}$

39. $\dfrac{3-\frac{1}{x}}{9-\frac{1}{x^2}}$

40. $\dfrac{\frac{a}{b}-1}{\frac{a^2}{b^2}-1}$

41. $\dfrac{\frac{1}{x}-\frac{1}{7}}{x-7}$

42. $\dfrac{\frac{2}{y}-\frac{2}{3}}{y-3}$

43. $\dfrac{\frac{1}{x}-\frac{1}{4}}{x-4}$

44. $\dfrac{\frac{1}{x^2}-\frac{1}{9}}{\frac{1}{x}-\frac{1}{3}}$

45. $\dfrac{\dfrac{1}{w^2} - \dfrac{1}{4}}{\dfrac{1}{w} - \dfrac{1}{2}}$

46. $\dfrac{\dfrac{1}{x^2} - \dfrac{1}{y^2}}{\dfrac{1}{x} - \dfrac{1}{y}}$

47. $\dfrac{\dfrac{4}{x^2} - \dfrac{1}{y^2}}{\dfrac{2}{x} + \dfrac{1}{y}}$

48. $\dfrac{\dfrac{1}{y^2} - \dfrac{1}{49}}{\dfrac{1}{y} + \dfrac{1}{7}}$

49. $\dfrac{1 - \dfrac{2}{x}}{\dfrac{4}{x} - \dfrac{3}{2}}$

50. $\dfrac{2 - \dfrac{2}{x}}{\dfrac{1}{x^2} - 1}$

1.4 Multiplication and Division of Rational Expressions

Since rational expressions represent numbers, the rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing number fractions.

Multiplication of rational expressions

If $\frac{P}{Q}$ and $\frac{R}{S}$ are rational expressions, then

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}.$$

Example 1. *Find the product and simplify.* $-\frac{15x^2}{2y^4} \cdot \frac{14y^3}{25x^5}$

Solution.

$$\begin{aligned} -\frac{15x^2}{2y^4} \cdot \frac{14y^3}{25x^5} &= -\frac{15x^2 \cdot 14y^3}{2y^4 \cdot 25x^5} \\ &= -\frac{5 \cdot 2x^2y^3 \cdot 3 \cdot 7}{5 \cdot 2x^2y^3 \cdot 5x^3y} \\ &= -\frac{10x^2y^3}{10x^2y^3} \cdot \frac{21}{5x^3y} \\ &= -1 \cdot \frac{21}{5x^3y} \\ &= -\frac{21}{5x^3y} \end{aligned}$$

■

Practice 1. *Find the product and simplify.* $\frac{66a^3b^2}{c^6} \cdot \frac{b^5c^3}{121a}$

(Answers on page 31.)

We usually want to express our answers in lowest terms. Therefore, we write the numerators and denominators in factored form and simplify whenever possible.

Example 2. *Find the product and simplify.* $\frac{2y^2+y}{3} \cdot \frac{6y}{4y^2-1}$

Solution.

$$\begin{aligned}
\frac{2y^2+y}{3}\cdot\frac{6y}{4y^2-1} &= \frac{\left(2y^2+y\right)\cdot 6y}{3\cdot\left(4y^2-1\right)} \\
&= \frac{y\,(2y+1)\cdot 3\cdot 2y}{3\cdot(2y-1)\,(2y+1)} \\
&= \frac{3\,(2y+1)\cdot 2y^2}{3\,(2y+1)\cdot(2y-1)} \\
&= \frac{3\,(2y+1)}{3\,(2y+1)}\cdot\frac{2y^2}{2y-1} \\
&= 1\cdot\frac{2y^2}{2y-1} \\
&= \frac{2y^2}{2y-1}
\end{aligned}$$

■

Practice 2. *Find the product and simplify.* $\dfrac{5x}{x^2-9}\cdot\dfrac{2x-6}{x}$

(Answers on page 31.)

Example 3. *Find the product and simplify.* $\dfrac{t^2+t-2}{3t^2-14t-5}\cdot\dfrac{6t^2+2t}{t^2+4t+4}$

Solution.

$$\begin{aligned}
\frac{t^2+t-2}{3t^2-14t-5}\cdot\frac{6t^2+2t}{t^2+4t+4} &= \frac{(t+2)(t-1)}{(3t+1)(t-5)}\cdot\frac{2t(3t+1)}{(t+2)^2} \\
&= \frac{(t+2)(t-1)\cdot 2t(3t+1)}{(3t+1)(t-5)\cdot(t+2)^2} \\
&= \frac{(t+2)(3t+1)\cdot 2t(t-1)}{(t+2)(3t+1)\cdot(t-5)(t+2)} \\
&= \frac{(t+2)(3t+1)}{(t+2)(3t+1)}\cdot\frac{2t(t-1)}{(t-5)(t+2)} \\
&= \frac{2t(t-1)}{(t-5)(t+2)}
\end{aligned}$$

■

Practice 3. *Find the product and simplify.* $\dfrac{(y+5)^2}{6y^2-24} \cdot \dfrac{3y+6}{y^2+6y+5}$

(Answers on page 31.)

Multiplicative inverses

Recall from your previous algebra courses that the ***multiplicative inverse*** (or ***reciprocal***) of a nonzero number a is the number whose product with a is 1. The multiplicative inverse of a can be written $\frac{1}{a}$ or a^{-1} since

$$a \cdot \frac{1}{a} = \frac{a}{a} = 1 \qquad \text{and} \qquad a \cdot a^{-1} = a^{1+(-1)} = a^0 = 1.$$

For example, the multiplicative inverse of 5 is $\frac{1}{5}$ $(= 5^{-1})$ since

$$5 \cdot \frac{1}{5} = \frac{5}{5} = 1 \qquad \text{and} \qquad 5 \cdot 5^{-1} = 5^{1+(-1)} = 5^0 = 1.$$

Suppose $a \neq 0$ and $b \neq 0$. Then the multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$ since

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = 1.$$

Note that 0 does not have a multiplicative inverse because the product of 0 with any number is 0 by the Multiplication Property of Zero.

Example 4. *Find the multiplicative inverse of each of the following.*

a. $\dfrac{1}{3}$ *c.* $\dfrac{2}{5}$ *e.* $\dfrac{8a}{7b}$

b. $\dfrac{1}{x}$ *d.* y^2 *f.* $\dfrac{x-1}{2y+5}$

Solution.

a. The multiplicative inverse of $\frac{1}{3}$ is 3 since

$$\frac{1}{3} \cdot 3 = \frac{1}{3} \cdot \frac{3}{1} = \frac{3}{3} = 1.$$

b. The multiplicative inverse of $\frac{1}{x}$ is x, since

$$\frac{1}{x} \cdot x = \frac{1}{x} \cdot \frac{x}{1} = \frac{x}{x} = 1.$$

c. The multiplicative inverse of $\frac{2}{5}$ is $\frac{5}{2}$ since

$$\frac{2}{5} \cdot \frac{5}{2} = \frac{10}{10} = 1.$$

d. The multiplicative inverse of y^2 is $\frac{1}{y^2}$, provided $y \neq 0$, since

$$y^2 \cdot \frac{1}{y^2} = \frac{y^2}{1} \cdot \frac{1}{y^2} = \frac{y^2}{y^2} = 1.$$

e. The multiplicative inverse of $\frac{8a}{7b}$ is $\frac{7b}{8a}$, provided $a \neq 0$, since

$$\frac{8a}{7b} \cdot \frac{7b}{8a} = \frac{56ab}{56ab} = 1.$$

f. The multiplicative inverse of $\frac{x-1}{2y+5}$ is $\frac{2y+5}{x-1}$, provided $x \neq 1$, since

$$\frac{x-1}{2y+5} \cdot \frac{2y+5}{x-1} = \frac{(x-1)(2y+5)}{(2y+5)(x-1)} = 1.$$

■

Practice 4. *Find the multiplicative inverse of each of the following.* *(Answers on page 31.)*

a. $\frac{1}{10}$

b. $\frac{1}{a}$

c. $\frac{11}{12}$

d. x^3

e. $\frac{5x}{11y}$

f. $\frac{a+b}{a-b}$

Division of rational expressions

Dividing by a nonzero number a is equivalent to multiplying by its multiplicative inverse $\frac{1}{a}$. This is true for whole numbers, as in

$$10 \div 2 = 10 \cdot \frac{1}{2} = \frac{10}{2} = 5,$$

as well as for fractions, as in

$$\frac{2}{3} \div \frac{1}{6} = \frac{2}{3} \cdot \frac{6}{1} = \frac{2 \cdot 6}{3} = \frac{2 \cdot 2 \cdot 3}{3} = 4.$$

Thus, it is true for rational expressions, too.

Let $\frac{P}{Q}$ and $\frac{R}{S}$ be rational expressions with $\frac{R}{S} \neq 0$. Then

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}.$$

Example 5. *Find the quotient and simplify.*

a. $\frac{5}{6} \div \frac{x}{10}$ *b.* $\frac{6}{y^2} \div \frac{3}{y}$ *c.* $\frac{11x^3}{6y} \div \frac{22x^2}{3y^4}$

Solution.

a.

$$\begin{aligned}\frac{5}{6} \div \frac{x}{10} &= \frac{5}{6} \cdot \frac{10}{x} \\ &= \frac{5 \cdot 10}{6 \cdot x} \\ &= \frac{5 \cdot 5 \cdot 2}{2 \cdot 3 \cdot x} \\ &= \frac{2}{2} \cdot \frac{25}{3x} \\ &= \frac{25}{3x}\end{aligned}$$

b.

$$\begin{aligned}\frac{6}{y^2} \div \frac{3}{y} &= \frac{6}{y^2} \cdot \frac{y}{3} \\ &= \frac{6 \cdot y}{y^2 \cdot 3} \\ &= \frac{2 \cdot 3 \cdot y}{3 \cdot y \cdot y} \\ &= \frac{3y}{3y} \cdot \frac{2}{y} \\ &= \frac{2}{y}\end{aligned}$$

c.

$$\begin{aligned}
\frac{11x^3}{6y} \div \frac{22x^2}{3y^4} &= \frac{11x^3}{6y} \cdot \frac{3y^4}{22x^2} \\
&= \frac{11x^3 \cdot 3y^4}{2 \cdot 3 \cdot y \cdot 2 \cdot 11 \cdot x^2} \\
&= \frac{33x^2y \cdot xy^3}{33x^2y \cdot 4} \\
&= \frac{33x^2y}{33x^2y} \cdot \frac{xy^3}{4} \\
&= \frac{xy^3}{4}
\end{aligned}$$

■

Practice 5. *Find the quotient and simplify.* *(Answers on the facing page.)*

a. $\frac{a}{2} \div \frac{b}{4}$ *b.* $\frac{15}{t} \div \frac{20}{t^3}$ *c.* $\frac{30y^2}{49x} \div \frac{18y}{7x^3}$

Example 6. *Find the quotient and simplify.* $\frac{a^2-49}{8} \div \frac{4a+28}{5}$

Solution.

$$\begin{aligned}
\frac{a^2-49}{8} \div \frac{4a+28}{5} &= \frac{a^2-49}{8} \cdot \frac{5}{4a+28} \\
&= \frac{(a-7)(a+7) \cdot 5}{8 \cdot 4 \cdot (a+7)} \\
&= \frac{(a+7) \cdot 5(a-7)}{(a+7) \cdot 32} \\
&= \frac{a+7}{a+7} \cdot \frac{5(a-7)}{32} \\
&= \frac{5(a-7)}{32} \\
&= \frac{5}{32}(a-7)
\end{aligned}$$

■

Practice 6. *Find the quotient and simplify.* $\dfrac{6t+2}{25} \div \dfrac{9t^2-1}{10}$
(Answers below.)

Example 7. *Find the quotient and simplify.* $\dfrac{2x^2+3x-5}{x^2+5x+6} \div \dfrac{x^2-1}{x^2+3x}$

Solution.

$$\begin{aligned}
\frac{2x^2+3x-5}{x^2+5x+6} \div \frac{x^2-1}{x^2+3x} &= \frac{2x^2+3x-5}{x^2+5x+6} \cdot \frac{x^2+3x}{x^2-1} \\
&= \frac{(2x+5)(x-1)\cdot x(x+3)}{(x+2)(x+3)\cdot(x-1)(x+1)} \\
&= \frac{(x+3)(x-1)\cdot x(2x+5)}{(x+3)(x-1)\cdot(x+2)(x+1)} \\
&= \frac{(x+3)(x-1)}{(x+3)(x-1)} \cdot \frac{x(2x+5)}{(x+2)(x+1)} \\
&= \frac{x(2x+5)}{(x+2)(x+1)}
\end{aligned}$$

■

Practice 7. *Find the quotient and simplify.* $\dfrac{5x^2+6x+1}{x^2-9} \div \dfrac{5x^3+x^2}{x^2+4x-21}$
(Answers below.)

ANSWERS TO SECTION 1.4 PRACTICE PROBLEMS

1. $\frac{6a^2b^7}{11c^3}$
2. $\frac{10}{x+3}$
3. $\frac{y+5}{2(y-2)(y+1)}$
4. (a) 10
   (b) a
   (c) $\frac{12}{11}$
   (d) $\frac{1}{x^3}$
   (e) $\frac{11y}{5x}$
   (f) $\frac{a-b}{a+b}$
5. (a) $\frac{2a}{b}$
   (b) $\frac{3t^2}{4}$
   (c) $\frac{5x^2y}{21}$
6. $\frac{4}{5(3t-1)}$
7. $\frac{(x+1)(x+7)}{x^2(x+3)}$

SECTION 1.4 EXERCISES:

(Answers are found on page 149.)

Find the product and simplify.

1. $-\frac{5}{8} \cdot \frac{16}{25}$

2. $\frac{42}{105} \cdot \frac{100}{9}$

3. $\frac{49}{40} \cdot \frac{45}{196}$

4. $9\frac{4}{5} \cdot 2\frac{4}{7}$

5. $\frac{4x}{y^2} \cdot \frac{5y}{2x^2}$

6. $\frac{169ab^3}{5a^2b} \cdot \frac{a^3b^2}{130ab^5}$

7. $\frac{t^2+t}{10} \cdot \frac{20}{t+1}$

8. $\frac{x^3-x^2}{3x} \cdot \frac{45}{5x-5}$

9. $\frac{x}{x+y} \cdot \frac{3x+3y}{5x^2+5x}$

10. $\frac{2y}{3y-18} \cdot \frac{y^2-12y+36}{4y^2}$

11. $\frac{m^2+6m+9}{m^2-25} \cdot \frac{m-5}{9-m^2}$

12. $\frac{y^2-1}{y^2+1} \cdot \frac{-2y^2-2}{3-3y}$

13. $\frac{a^4-b^4}{3} \cdot \frac{6}{a^2+b^2}$

14. $\frac{x^4+6x^2+9}{2x} \cdot \frac{4x^2}{x^2+3}$

15. $\frac{4-2x}{8} \cdot \frac{x+2}{x^2-4}$

16. $\frac{t^2-4}{t^2+4} \cdot \frac{2t^2+8}{t^3-4t}$

17. $\frac{5t-25}{10} \cdot \frac{20}{30-6t}$

18. $\frac{x+1}{x^2-1} \cdot \frac{2-x-x^2}{5x}$

19. $\frac{12a-16}{4a-12} \cdot \frac{6a-18}{9a^2+6a-24}$

20. $\frac{x+y}{x-y} \cdot \frac{x^2-2xy+y^2}{x^2-y^2}$

21. $\frac{x-1}{6} \cdot \frac{2x^2+2}{x^2-1}$

22. $\frac{y^2-y-6}{y^2-3y} \cdot \frac{y^3+y^2}{y+2}$

23. $\frac{6a^2-7a-3}{a^2-2a+1} \cdot \frac{a-1}{6a^2+2a}$

24. $\frac{x^2-9}{x^3+4x^2+4x} \cdot \frac{2x^2+4x}{x^2+2x-15}$

Find the multiplicative inverse.

25. $\frac{1}{5}$
26. $\frac{1}{17}$
27. -23
28. 100
29. $\frac{4}{9}$
30. $-\frac{11}{3}$
31. $\frac{0}{4}$
32. $\frac{0}{1}$
33. $\frac{1}{x^6}$
34. $\frac{1}{t^2}$
35. $\frac{2a}{3b}$
36. $\frac{5y}{z}$
37. $\frac{x-1}{x+1}$
38. $\frac{3x^2}{x^2+5}$
39. $\frac{y^2-y+9}{5y+7}$
40. $\frac{x+y}{xy}$

Find the quotient and simplify.

41. $\frac{16}{25} \div \frac{64}{15}$
42. $\frac{42}{5} \div \frac{15}{2}$
43. $16\frac{1}{4} \div 5$
44. $\frac{-5}{32} \div \frac{-1}{6}$
45. $\frac{12}{x^3} \div \frac{15}{x^4}$
46. $\frac{10}{t^8} \div \frac{35}{t^3}$
47. $5a^2b \div \frac{25ab^2}{6}$
48. $x^5y^4 \div \frac{x^2y^{10}}{7}$
49. $\frac{x^2}{15} \div \frac{x}{3}$
50. $\frac{a^3}{b^3} \div \frac{a}{b}$
51. $\frac{2t^7}{5} \div 4t^2$
52. $\frac{6}{x^5} \div 2x$
53. $\frac{x^2-144}{x} \div \frac{3x+36}{x}$
54. $\frac{t^2+2t+1}{50} \div \frac{t^2-1}{10}$
55. $\frac{5y^2}{9-y^2} \div \frac{y}{y-3}$
56. $\frac{x^2-y^2}{x^2+2xy+y^2} \div \frac{x-y}{x+y}$
57. $\frac{5}{8t+16} \div \frac{7}{12t+24}$
58. $\frac{2y-8}{6} \div \frac{12-3y}{2}$
59. $\frac{a}{b} \div \frac{a^2-ab}{ab+b^2}$
60. $\frac{1-x}{2+x} \div \frac{x^2-x}{x^2+2x}$

61. $\dfrac{x-2}{3x+3} \div \dfrac{x^2+2x}{x+2}$

62. $\dfrac{(x+1)^2}{x^2-6x+9} \div \dfrac{3x+3}{x-3}$

63. $\dfrac{y^2-9}{y^2-y-20} \div \dfrac{4y+12}{2y-10}$

64. $\dfrac{2x^2+4x}{x^2-4x-12} \div \dfrac{x^3-4x^2+4x}{x^3-6x^2}$

65. $\dfrac{3x^2-2x-8}{3x^2+14x+8} \div \dfrac{3x+4}{3x+2}$

66. $\dfrac{z^2+3z+2}{z^2+4z+3} \div \dfrac{z^2+z-2}{z^2+3z-4}$

1.5 Addition and Subtraction of Rational Expressions

Fractions to be added or subtracted must first be written with the same denominator. The same is true for rational expressions. We will start by adding and subtracting rational expressions with like denominators.

Adding and subtracting rational expressions with like denominators

If $\frac{P}{Q}$ and $\frac{R}{Q}$ are rational expressions, then

$$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$$

and

$$\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}.$$

Example 1. *Perform the indicated operations and simplify.*

a. $\frac{3x}{5} + \frac{x}{5}$ *b.* $\frac{4a}{b} - \frac{7}{b} + \frac{6a}{b}$ *c.* $\frac{5t+7}{3t} - \frac{2t+1}{3t}$

Solution.

a. We note that both fractions have the denominator 5.

$$\begin{aligned}\frac{3x}{5} + \frac{x}{5} &= \frac{3x+x}{5} \\ &= \frac{4x}{5}.\end{aligned}$$

b. All fractions have the denominator b.

$$\begin{aligned}\frac{4a}{b} - \frac{7}{b} + \frac{6a}{b} &= \frac{4a-7+6a}{b} \\ &= \frac{10a-7}{b}.\end{aligned}$$

c. Both fractions have the denominator $3t$.

$$\begin{aligned}
\frac{5t+7}{3t} - \frac{2t+1}{3t} &= \frac{(5t+7)-(2t+1)}{3t} \\
&= \frac{5t+7-2t-1}{3t} \\
&= \frac{3t+6}{3t} \\
&= \frac{3(t+2)}{3t} \\
&= \frac{3}{3} \cdot \frac{t+2}{t} \\
&= \frac{t+2}{t}.
\end{aligned}$$

Be careful when subtracting a fraction whose numerator has more than one term. As in this example, we must change the sign of *every* term in the numerator of the fraction being subtracted. ■

Practice 1. *Perform the indicated operations and simplify.* *(Answers on page 43.)*

a. $\dfrac{11}{t^2} + \dfrac{9}{t^2}$ *b.* $\dfrac{x}{2y} - \dfrac{5x}{2y} - \dfrac{3x}{2y} + \dfrac{x}{2y}$ *c.* $\dfrac{5n+4}{n} - \dfrac{4-n}{n}$

Example 2. *Perform the indicated operations and simplify.*

a. $\dfrac{6}{x-6} - \dfrac{x}{x-6}$

b. $\dfrac{3a-1}{a^2+2a} + \dfrac{a+1}{a^2+2a}$

c. $\dfrac{(x+1)^2}{x^2(x-10)^3} - \dfrac{12x+1}{x^2(x-10)^3}$

Solution.

a. Both fractions have the denominator $x - 6$.

$$\begin{aligned}\frac{6}{x-6} - \frac{x}{x-6} &= \frac{6-x}{x-6} \\ &= \frac{-1(x-6)}{x-6} \\ &= -1 \cdot \frac{x-6}{x-6} \\ &= -1.\end{aligned}$$

b. Both fractions have the denominator $a^2 + 2a$.

$$\begin{aligned}\frac{3a-1}{a^2+2a} + \frac{a+1}{a^2+2a} &= \frac{(3a-1)+(a+1)}{a^2+2a} \\ &= \frac{3a-1+a+1}{a^2+2a} \\ &= \frac{4a}{a(a+2)} \\ &= \frac{a}{a} \cdot \frac{4}{a+2} \\ &= \frac{4}{a+2}.\end{aligned}$$

c. Both fractions have the denominator $x^2(x-10)^3$.

$$\begin{aligned}
\frac{(x+1)^2}{x^2(x-10)^3} - \frac{12x+1}{x^2(x-10)^3} &= \frac{(x^2+2x+1)-(12x+1)}{x^2(x-10)^3} \\
&= \frac{x^2+2x+1-12x-1}{x^2(x-10)^3} \\
&= \frac{x^2-10x}{x^2(x-10)^3} \\
&= \frac{x(x-10)}{x^2(x-10)^3} \\
&= \frac{x(x-10)}{x(x-10)} \cdot \frac{1}{x(x-10)^2} \\
&= \frac{1}{x(x-10)^2}.
\end{aligned}$$

Note that it was necessary to expand the expressions in the numerator in order to subtract them. However, the denominator was left in factored form. ■

Practice 2. *Perform the indicated operations and simplify.* *(Answers on page 43.)*

a. $\dfrac{4}{y+3} + \dfrac{y-1}{y+3}$ *b.* $\dfrac{(x-5)^2}{x^3-15x^2} - \dfrac{3x+55}{x^3-15x^2}$

Adding and subtracting rational expressions with unlike denominators

When we wish to add or subtract fractions with unlike denominators, we must first rewrite them with the same denominator. To keep things as simple as possible, it is best to use the least common denominator. Recall that the ***least common denominator*** of a collection of simplified number fractions is the smallest positive number that is a multiple of all of the denominators of the fractions. Similarly, a ***least common denominator*** (***LCD***) of a collection of simplified rational expressions is a polynomial of lowest degree and with smallest coefficients that is a multiple of all of the denominators of the rational expressions. We used least common denominators in Section 1.3 to simplify compound rational expressions. Let us practice finding LCDs of collections of rational expressions with somewhat more complicated denominators than we considered there.

Example 3. *Find the least common denominator of each pair. Then rewrite each rational expression as an equivalent expression with the common denominator.*

a. $\dfrac{x}{x-5}, \dfrac{3}{x+5}$ *b.* $\dfrac{3y-1}{y^2-4}, \dfrac{y}{y^2-y-2}$

Solution.

a. The greatest common factor of these denominators is 1, so the LCD of the fractions is the product of the denominators: $(x-5)(x+5)$. Rewriting each with this common denominator, we obtain,

$$\frac{x}{x-5} = \frac{x}{x-5} \cdot \frac{x+5}{x+5} = \frac{x(x+5)}{(x-5)(x+5)}$$

and

$$\frac{3}{x+5} = \frac{3}{x+5} \cdot \frac{x-5}{x-5} = \frac{3(x-5)}{(x-5)(x+5)}.$$

b. We must first factor each denominator fully.

$$\frac{3y-1}{y^2-4} = \frac{3y-1}{(y-2)(y+2)} \quad \text{and} \quad \frac{y}{y^2-y-2} = \frac{y}{(y-2)(y+1)}.$$

The LCD of these fractions is $(y-2)(y+2)(y+1)$. Rewriting each with this common denominator, we obtain,

$$\frac{3y-1}{(y-2)(y+2)} = \frac{3y-1}{(y-2)(y+2)} \cdot \frac{y+1}{y+1} = \frac{(3y-1)(y+1)}{(y-2)(y+2)(y+1)}$$

and

$$\frac{y}{(y-2)(y+1)} = \frac{y}{(y-2)(y+1)} \cdot \frac{y+2}{y+2} = \frac{y(y+2)}{(y-2)(y+2)(y+1)}.$$

■

Practice 3. *Find the least common denominator of each pair and then rewrite each rational expression as an equivalent expression with the common denominator. (Answers on page 43.)*

a. $\dfrac{y}{2y+1}, \dfrac{5}{y-3}$

b. $\dfrac{m^2}{m^2+14m+49}, \dfrac{5m+1}{m^2-m-56}$

Now we are ready to put these skills together to add and subtract fractions with unlike denominators.

Example 4. *Perform the indicated operations and simplify.*

a. $\dfrac{2}{5} - \dfrac{1}{x} + \dfrac{3}{x^2}$

b. $\dfrac{4}{x-1} + \dfrac{1}{x+2}$

Solution.

a. The LCD is $5x^2$.

$$\begin{aligned}\frac{2}{5} - \frac{1}{x} + \frac{3}{x^2} &= \frac{2}{5} \cdot \frac{x^2}{x^2} - \frac{1}{x} \cdot \frac{5x}{5x} + \frac{3}{x^2} \cdot \frac{5}{5} \\ &= \frac{2x^2}{5x^2} - \frac{5x}{5x^2} + \frac{15}{5x^2} \\ &= \frac{2x^2 - 5x + 15}{5x^2}.\end{aligned}$$

b. The LCD is $(x-1)(x+2)$.

$$\begin{aligned}\frac{4}{x-1} + \frac{1}{x+2} &= \frac{4}{x-1} \cdot \frac{x+2}{x+2} + \frac{1}{x+2} \cdot \frac{x-1}{x-1} \\ &= \frac{4(x+2)}{(x-1)(x+2)} + \frac{x-1}{(x-1)(x+2)} \\ &= \frac{4x+8}{(x-1)(x+2)} + \frac{x-1}{(x-1)(x+2)} \\ &= \frac{(4x+8)+(x-1)}{(x-1)(x+2)} \\ &= \frac{4x+8+x-1}{(x-1)(x+2)}\end{aligned}$$

$$= \frac{5x+7}{(x-1)(x+2)}$$

■

Practice 4. *Perform the indicated operations and simplify.* *(Answers on page 43.)*

a. $\dfrac{3}{y^3} - \dfrac{1}{2y} + \dfrac{3}{8}$ *b.* $\dfrac{2}{x-4} - \dfrac{2}{x+5}$

Example 5. *Perform the indicated operations and simplify.*

a. $\dfrac{1}{x-3} - \dfrac{1}{x^2} + \dfrac{5}{x^2-3x}$ *b.* $\dfrac{-16t}{36-t^2} + \dfrac{2-15t}{t^2-t-30}$

Solution.

a. We will start by factoring the denominators.

$$\frac{1}{x-3} - \frac{1}{x^2} + \frac{5}{x^2-3x} = \frac{1}{x-3} - \frac{1}{x^2} + \frac{5}{x(x-3)}$$

so the LCD is $x^2(x-3)$;

$$= \frac{1}{x-3} \cdot \frac{x^2}{x^2} - \frac{1}{x^2} \cdot \frac{x-3}{x-3} + \frac{5}{x(x-3)} \cdot \frac{x}{x}$$

$$= \frac{x^2}{x^2(x-3)} - \frac{x-3}{x^2(x-3)} + \frac{5x}{x^2(x-3)}$$

$$= \frac{x^2-(x-3)+5x}{x^2(x-3)}$$

$$= \frac{x^2-x+3+5x}{x^2(x-3)}$$

$$= \frac{x^2+4x+3}{x^2(x-3)}$$

$$= \frac{(x+1)(x+3)}{x^2(x-3)}$$

b. Again, we factor the denominators first.

$$\begin{aligned}
\frac{-16t}{36-t^2}+\frac{2-15t}{t^2-t-30} &= \frac{-16t}{(6-t)(6+t)}+\frac{2-15t}{(t-6)(t+5)} \\
&= \frac{-16t}{(-1)(t-6)(t+6)}+\frac{2-15t}{(t-6)(t+5)} \\
&= \frac{(-1)(-16t)}{(t-6)(t+6)}+\frac{2-15t}{(t-6)(t+5)}
\end{aligned}$$

so the LCD is $(t-6)(t+6)(t+5)$;

$$\begin{aligned}
&= \frac{16t}{(t-6)(t+6)}\cdot\frac{t+5}{t+5}+\frac{2-15t}{(t-6)(t+5)}\cdot\frac{t+6}{t+6} \\
&= \frac{16t(t+5)}{(t-6)(t+6)(t+5)}+\frac{(2-15t)(t+6)}{(t-6)(t+6)(t+5)} \\
&= \frac{16t^2+80t}{(t-6)(t+6)(t+5)}+\frac{-15t^2-88t+12}{(t-6)(t+6)(t+5)} \\
&= \frac{(16t^2+80t)+(-15t^2-88t+12)}{(t-6)(t+6)(t+5)} \\
&= \frac{16t^2+80t-15t^2-88t+12}{(t-6)(t+6)(t+5)} \\
&= \frac{t^2-8t+12}{(t-6)(t+6)(t+5)} \\
&= \frac{(t-6)(t-2)}{(t-6)(t+6)(t+5)} \\
&= \frac{(t-6)}{(t-6)}\cdot\frac{t-2}{(t+6)(t+5)}
\end{aligned}$$

$$= \frac{t-2}{(t+6)(t+5)}.$$

■

Practice 5. *Perform the indicated operations and simplify.* *(Answers below.)*
$\dfrac{x+1}{x^2+x-2} + \dfrac{3}{x^2-1}$

ANSWERS TO SECTION 1.5 PRACTICE PROBLEMS

1. (a) $\frac{20}{t^2}$
   (b) $-\frac{3x}{y}$
   (c) 6
2. (a) 1
   (b) $\frac{x+2}{x^2}$
3. (a) the LCD is $(2y+1)(y-3)$
   $\frac{y(y-3)}{(2y+1)(y-3)}, \frac{5(2y+1)}{(2y+1)(y-3)}$
   (b) the LCD is $(m+7)^2(m-8)$
   $\frac{m^2(m-8)}{(m+7)^2(m-8)}, \frac{(5m+1)(m+7)}{(m+7)^2(m-8)}$
4. (a) $\frac{3y^3-4y^2+24}{8y^3}$
   (b) $\frac{18}{(x-4)(x+5)}$
5. $\frac{x^2+5x+7}{(x+2)(x-1)(x+1)}$

SECTION 1.5 EXERCISES:

(Answers are found on page 150.)

Perform the indicated operations and simplify.

1. $\dfrac{5}{x^3} - \dfrac{4}{x^3}$
2. $\dfrac{a}{b} + \dfrac{3a}{b}$
3. $\dfrac{2}{5t^2} - \dfrac{1}{5t^2} + \dfrac{4}{5t^2}$
4. $\dfrac{2}{3x} - \dfrac{7}{3x} - \dfrac{1}{3x}$
5. $\dfrac{y+1}{y} - \dfrac{2y+1}{y}$
6. $\dfrac{2-x}{x} - \dfrac{1-x}{x}$
7. $\dfrac{2y}{5-y} - \dfrac{10}{5-y}$
8. $\dfrac{1}{2m-1} - \dfrac{2m}{2m-1}$
9. $\dfrac{2t-5}{t^2-3t} + \dfrac{2-t}{t^2-3t}$
10. $\dfrac{6x+1}{x^3+x^2} - \dfrac{x-6}{x^3+x^2}$
11. $\dfrac{(2x+1)^2}{x^3(7x+1)^2} + \dfrac{3x-4x^2}{x^3(7x+1)^2}$
12. $\dfrac{y^2+2y+1}{y^2(y+2)^4} + \dfrac{3y+5}{y^2(y+2)^4}$

13. $\dfrac{3}{5-y}+\dfrac{7}{5-y}-\dfrac{2y}{5-y}$

14. $\dfrac{2x^2}{(x-3)^2}-\dfrac{12x-18}{(x-3)^2}$

15. $\dfrac{4x}{x^2-x-2}+\dfrac{4}{x^2-x-2}$

16. $\dfrac{x^2}{3x^2+6}-\dfrac{1+x^2}{3x^2+6}-\dfrac{5}{3x^2+6}$

17. $\dfrac{3}{5a^6}+\dfrac{4}{5a^6}+\dfrac{3}{5a^6}$

18. $\dfrac{4b}{a^2-a}+\dfrac{-6b}{a^2-a}+\dfrac{2b}{a^2-a}$

19. $\dfrac{5x}{x^2-9}-\dfrac{2x-1}{x^2-9}+\dfrac{8}{x^2-9}$

20. $\dfrac{-8}{3t+7}+\dfrac{t+3}{3t+7}-\dfrac{5t}{3t+7}$

21. $\dfrac{4x+3}{4x^2-8x-5}-\dfrac{x+6}{4x^2-8x-5}-\dfrac{x+2}{4x^2-8x-5}$

22. $\dfrac{7x^2}{x^2-2xy+y^2}-\dfrac{2y^2}{x^2-2xy+y^2}+\dfrac{y^2-6x^2}{x^2-2xy+y^2}$

Find the least common denominator of each pair or triple. Then rewrite each rational expression as an equivalent expression with the common denominator.

23. $\dfrac{5}{x+1}, \dfrac{x}{x-2}$

24. $\dfrac{1}{2y-1}, \dfrac{y}{2y+1}$

25. $\dfrac{x+10}{x}, \dfrac{1}{6-x}$

26. $\dfrac{t-1}{t+8}, \dfrac{t+3}{2t}$

27. $\dfrac{m+2}{25m^2-4}, \dfrac{7}{5m^2+3m-2}$

28. $\dfrac{2x}{x^2-9x-10}, \dfrac{5}{x^2+2x+1}$

29. $\dfrac{1}{5y}, \dfrac{1}{5(y+1)}$

30. $\dfrac{x+2}{6x}, \dfrac{x}{6(x-2)}$

31. $\dfrac{5}{4y^2}, \dfrac{11}{12y^5}, \dfrac{-3}{8y^3}$

32. $\dfrac{3x}{x-1}, \dfrac{x}{x^2-1}, \dfrac{2}{3x+3}$

33. $\dfrac{7}{9a}, \dfrac{2a}{3a+6}, \dfrac{a+1}{a+2}$

34. $\dfrac{3}{4-2t}, \dfrac{-t}{t-2}, \dfrac{t+1}{t^2-4}$

Perform the indicated operations and simplify.

35. $\dfrac{2}{x^2}+\dfrac{1}{3x}+\dfrac{1}{9}$

36. $\dfrac{1}{6}-\dfrac{1}{2x}+\dfrac{1}{x^2}$

37. $\dfrac{3}{y-1}-\dfrac{1}{2y+1}$

38. $\dfrac{t}{3t+2}+\dfrac{4}{t+5}$

39. $x+2-\dfrac{4x-1}{2x}$

40. $y-1+\dfrac{3y+2}{y}$

41. $\dfrac{2t-3}{t^2-25}-\dfrac{6}{5t-25}$

42. $\dfrac{1}{3x+6}+\dfrac{x+5}{x^2-4}$

43. $\dfrac{2}{x+4}-\dfrac{1}{x^2+4x}+\dfrac{1}{x^2}$

44. $\dfrac{1}{y^2}+\dfrac{1}{y-1}-\dfrac{1}{y^2-y}$

45. $\dfrac{2}{x^2-4}-\dfrac{x+1}{x^2-5x+6}$

46. $\dfrac{x+3}{x^2+3x-10}+\dfrac{4}{x^2-x-2}$

47. $\dfrac{4a^2}{a^2+a}+\dfrac{2a}{a+1}$

48. $\dfrac{6x+6}{2x^2-x-1}+\dfrac{2}{2x+1}$

49. $\dfrac{2}{n^2+4n+4}-\dfrac{1}{n+2}$

50. $\dfrac{x+2}{x+1}-6$

51. $1-\dfrac{(a-b)^2}{(a+b)^2}$

52. $\dfrac{t}{t-2}+\dfrac{4+2t}{t^2-4}$

53. $\dfrac{1-3y}{3-2y}-\dfrac{3y+3}{2y^2-y-3}$

54. $6-\dfrac{8}{x}-x$

55. $\dfrac{1}{21x^2}+\dfrac{1}{28x}-\dfrac{1}{6x^3}$

56. $\dfrac{6}{x-2}-\dfrac{16}{x^2-4}-\dfrac{4}{x+2}$

57. $\dfrac{y-5}{y+5}-\dfrac{y+3}{5-y}-\dfrac{2y^2+30}{y^2-25}$

58. $\dfrac{1}{2}-\dfrac{t-3}{t+3}+\dfrac{t^2+27}{2t^2+6t}$

59. $\dfrac{y}{y-1}+\dfrac{3y-6}{y^2+y-2}-\dfrac{2}{y+2}$

60. $\dfrac{2z}{z^2-3z+2}-\dfrac{z}{z-2}+\dfrac{2z}{z-1}$

1.6 Rational Equations

In this section we will solve equations involving rational expressions. We will use many skills that we have developed over the past several courses including combining and simplifying rational expressions and solving linear and quadratic equations.

There are many approaches to solving rational equations. Here we will discuss two methods and the advantages and disadvantages of each.

Method 1

This method is similar to the method that we used to solve quadratic and higher degree equations in Section **??**. We rewrite the equation so that one side is zero and the other side is a single rational expression in lowest terms.

Example 1. *Solve for the variable.* $\dfrac{x^2}{x-5} = \dfrac{25}{x-5}$

Solution. Once we have both expressions on the same side of the equation, combining them is a simple matter, since they have the same denominator.

$$\begin{aligned}
\frac{x^2}{x-5} &= \frac{25}{x-5} \\
\frac{x^2}{x-5} - \frac{25}{x-5} &= \frac{25}{x-5} - \frac{25}{x-5} \\
\frac{x^2-25}{x-5} &= 0 \\
\frac{(x-5)(x+5)}{x-5} &= 0 \\
\frac{x+5}{1} &= 0 \\
x+5 &= 0 \\
x &= -5
\end{aligned}$$

We will check this solution by substituting -5 for x in the left-hand side

(LHS) and in the right-hand side (RHS) of the original equation.

$$\begin{aligned} \text{LHS} &= \frac{(-5)^2}{(-5)-5} \\ &= \frac{25}{-10} \\ &= -\frac{5}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \text{RHS} &= \frac{25}{(-5)-5} \\ &= \frac{25}{-10} \\ &= -\frac{5}{2}. \end{aligned}$$

Since the left-hand side and the right-hand side are equal, $x = -5$ is the solution to the original rational equation. ■

Practice 1. *Solve for the variable using Method 1. Check your solutions.* *(Answer on page 56.)*

$$\frac{y^2+3}{y-4} = \frac{19}{y-4}.$$

Example 2. *Solve for the variable. Check your solutions.*

a. $\frac{1}{2} + \frac{1}{t} = \frac{1}{4} + \frac{3}{t}$ *b.* $\frac{3}{x+1} = \frac{x}{x-1} - 1$

Solution.

a. First we must rewrite the equation so that one side is zero. Then to combine the fractions on the other side, we must find a common denominator. We see that the LCD of all of the fractions is $4t$.

$$\begin{aligned} \frac{1}{2} + \frac{1}{t} &= \frac{1}{4} + \frac{3}{t} \\ \frac{1}{2} + \frac{1}{t} - \frac{1}{4} - \frac{3}{t} &= \frac{1}{4} + \frac{3}{t} - \frac{1}{4} - \frac{3}{t} \\ \frac{1}{2} + \frac{1}{t} - \frac{1}{4} - \frac{3}{t} &= 0 \\ \frac{1}{2} \cdot \frac{2t}{2t} + \frac{1}{t} \cdot \frac{4}{4} - \frac{1}{4} \cdot \frac{t}{t} - \frac{3}{t} \cdot \frac{4}{4} &= 0 \\ \frac{2t}{4t} + \frac{4}{4t} - \frac{t}{4t} - \frac{12}{4t} &= 0 \\ \frac{2t+4-t-12}{4t} &= 0 \\ \frac{t-8}{4t} &= 0 \end{aligned}$$

A fraction in lowest terms equals 0 if and only if the numerator equals 0, so we obtain

$$\begin{aligned} t - 8 &= 0 \\ t &= 8. \end{aligned}$$

We check 8 in the original equation.

$$\begin{aligned} \text{LHS} &= \frac{1}{2} + \frac{1}{8} \\ &= \frac{4}{8} + \frac{1}{8} \\ &= \frac{5}{8} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \text{RHS} &= \frac{1}{4} + \frac{3}{8} \\ &= \frac{2}{8} + \frac{3}{8} \\ &= \frac{5}{8} \end{aligned}$$

Since LHS = RHS, 8 is the solution of the original equation.

b. Here the LCD of all of the fractions is $(x+1)(x-1)$.

$$\begin{aligned}
\frac{3}{x+1} &= \frac{x}{x-1} - 1\\
\frac{3}{x+1} - \frac{x}{x-1} + 1 &= \frac{x}{x-1} - 1 - \frac{x}{x-1} + 1\\
\frac{3}{x+1} - \frac{x}{x-1} + 1 &= 0\\
\frac{3}{x+1}\cdot\frac{x-1}{x-1} - \frac{x}{x-1}\cdot\frac{x+1}{x+1} + 1\cdot\frac{(x+1)(x-1)}{(x+1)(x-1)} &= 0\\
\frac{3(x-1)}{(x+1)(x-1)} - \frac{x(x+1)}{(x-1)(x+1)} + \frac{(x+1)(x-1)}{(x+1)(x-1)} &= 0\\
\frac{3x-3}{(x+1)(x-1)} - \frac{x^2+x}{(x-1)(x+1)} + \frac{(x^2-1)}{(x+1)(x-1)} &= 0\\
\frac{(3x-3)-(x^2+x)+(x^2-1)}{(x+1)(x-1)} &= 0\\
\frac{3x-3-x^2-x+x^2-1}{(x+1)(x-1)} &= 0\\
\frac{2x-4}{(x+1)(x-1)} &= 0\\
\frac{2(x-2)}{(x+1)(x-1)} &= 0
\end{aligned}$$

Since the expression is in lowest terms, this is equivalent to

$$\begin{aligned}
2(x-2) &= 0\\
x-2 &= 0\\
x &= 2.
\end{aligned}$$

We check our solution in the original equation.

$$\begin{aligned} \text{LHS} &= \frac{3}{2+1} \\ &= \frac{3}{3} \\ &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \text{RHS} &= \frac{2}{2-1} - 1 \\ &= \frac{2}{1} - 1 \\ &= 2 - 1 \\ &= 1. \end{aligned}$$

Since LHS = RHS, 2 is the solution of the original equation. ■

Practice 2. *Solve for the variable using Method 1. Check your solutions.* *(Answer on page 56.)*

$$\frac{5}{t+2} - \frac{2}{t} = \frac{2t+3}{t^2+2t}$$

Example 3. *Solve for the variable. Check your solutions.*

a. $\dfrac{3}{x+5} + \dfrac{1}{x-2} = \dfrac{4x-1}{x^2+3x-10}$ *b.* $\dfrac{x+1}{x-3} - \dfrac{1}{x} = \dfrac{1}{x^2-3x}$

Solution.

a. The LCD of all of the fractions in the equation is $(x+5)(x-2) = x^2+3x-10$.

$$\begin{aligned} \frac{3}{x+5} + \frac{1}{x-2} &= \frac{4x-1}{x^2+3x-10} \\ \frac{3}{x+5} + \frac{1}{x-2} - \frac{4x-1}{(x+5)(x-2)} &= 0 \\ \frac{3}{x+5} \cdot \frac{x-2}{x-2} + \frac{1}{x-2} \cdot \frac{x+5}{x+5} - \frac{4x-1}{(x+5)(x-2)} &= 0 \\ \frac{3(x-2) + (x+5) - (4x-1)}{(x+5)(x-2)} &= 0 \\ \frac{3x-6+x+5-4x+1}{(x+5)(x-2)} &= 0 \\ \frac{0}{(x+5)(x-2)} &= 0. \end{aligned}$$

This equation is true for all permissible values of the variable. Looking back at the original equation, we see that the domain is the set of all real numbers except -5 and 2. Thus, the solution set is the set of all real numbers except -5 and 2. In interval notation, the solution set is $(-\infty, -5) \cup (-5, 2) \cup (2, \infty)$.

b. The LCD is $x(x-3) = x^2 - 3x$.

$$\begin{aligned}
\frac{x+1}{x-3} - \frac{1}{x} &= \frac{1}{x^2-3x} \\
\frac{x+1}{x-3} - \frac{1}{x} - \frac{1}{x(x-3)} &= 0 \\
\frac{x+1}{x-3} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x-3}{x-3} - \frac{1}{x(x-3)} &= 0 \\
\frac{x^2+x}{x(x-3)} - \frac{x-3}{x(x-3)} - \frac{1}{x(x-3)} &= 0 \\
\frac{(x^2+x)-(x-3)-1}{x(x-3)} &= 0 \\
\frac{x^2+x-x+3-1}{x(x-3)} &= 0 \\
\frac{x^2+2}{x(x-3)} &= 0
\end{aligned}$$

Since the fraction is in lowest terms, this is equivalent to

$$\begin{aligned}
x^2 + 2 &= 0 \\
x^2 &= -2.
\end{aligned}$$

However, the square of a real number cannot be negative. Therefore, this equation has no real solutions. ■

Practice 3. *Solve for the variable using Method 1.* *(Answers on page 56.)*

a. $\dfrac{y}{y-1} = \dfrac{1}{y+1} + \dfrac{1}{2}$ *b.* $\dfrac{13a-8}{3a^2-2a} - \dfrac{4}{a} = \dfrac{1}{3a-2}$

The primary disadvantage of Method 1 is that it entails a great deal of writing. Our next method is more concise.

Method 2

This method is similar to one of the methods that we used to solve linear equations with fractional coefficients previously. We eliminate the denominators by multiplying both sides of the equation by the LCD of all of the rational expressions involved. Unfortunately, the resulting polynomial equation might not be equivalent to the original rational equation; it might have more solutions. Therefore, we must be very careful to check each proposed solution in the original rational equation. We will revisit each of the examples we solved by Method 1, this time using Method 2.

Example 4. *Solve for the variable.* $\dfrac{x^2}{x-5} = \dfrac{25}{x-5}$

Solution. The LCD is $x-5$.

$$\begin{aligned}
\frac{x^2}{x-5} &= \frac{25}{x-5} \\
(x-5)\cdot\frac{x^2}{x-5} &= (x-5)\cdot\frac{25}{x-5} \\
x^2 &= 25 \\
\sqrt{x^2} &= \sqrt{25} \\
|x| &= 5 \\
x &= \pm 5
\end{aligned}$$

Thus, we would appear to have two solutions. However, note that 5 is not in the domain of the original rational expressions (since it would make the denominators 0). Therefore, the only solution to the equation is $x = -5$. We should check this solution by substituting -5 for x in the left-hand side (LHS) and in the right-hand side (RHS) of the original equation. However, we already performed this check when we solved this equation using Method 1, so we will dispense with it here. ■

The number 5 in the previous example is what is known as an ***extraneous solution*** to the equation. That is, it is a solution to the polynomial equation obtained when the denominators are cleared, but not a solution to the original rational equation. This is because 5 was not in the domain of the original rational expressions, making their denominator zero. We must always check for extraneous solutions when solving rational equations by this method.

Practice 4. *Solve for the variable using Method 2. Check for extraneous solutions.* *(Answer on* *page 56.)*

$$\frac{y^2+3}{y-4} = \frac{19}{y-4}.$$

Example 5. *Solve for the variable. Check your solutions.*

a. $\frac{1}{2}+\frac{1}{t} = \frac{1}{4}+\frac{3}{t}$ *b.* $\frac{3}{x+1} = \frac{x}{x-1} - 1$

Solution.

a. The LCD of all of the fractions is $4t$.

$$\begin{aligned}
\frac{1}{2}+\frac{1}{t} &= \frac{1}{4}+\frac{3}{t} \\
4t\cdot\left(\frac{1}{2}+\frac{1}{t}\right) &= 4t\cdot\left(\frac{1}{4}+\frac{3}{t}\right) \\
4t\cdot\frac{1}{2}+4t\cdot\frac{1}{t} &= 4t\cdot\frac{1}{4}+4t\cdot\frac{3}{t} \\
2t+4 &= t+12 \\
2t-t &= 12-4 \\
t &= 8.
\end{aligned}$$

We would check 8 in the original equation if we hadn't done it when we solved the equation by Method 1.

b. Here the LCD of all of the fractions is $(x+1)(x-1)$.

$$\begin{aligned}
\frac{3}{x+1} &= \frac{x}{x-1} - 1 \\
(x+1)(x-1)\cdot\frac{3}{x+1} &= (x+1)(x-1)\cdot\left(\frac{x}{x-1} - 1\right) \\
(x+1)(x-1)\cdot\frac{3}{x+1} &= (x+1)(x-1)\cdot\frac{x}{x-1} - (x+1)(x-1)\cdot 1 \\
(x-1)\cdot 3 &= (x+1)\cdot x - (x+1)(x-1) \\
3x-3 &= x^2+x-(x^2-1) \\
3x-3 &= x^2+x-x^2+1 \\
3x-3 &= x+1 \\
3x-x &= 1+3 \\
2x &= 4 \\
x &= 2.
\end{aligned}$$

Again, we would check our solution in the original equation if we hadn't done so before. ■

Practice 5. *Solve for the variable using Method 2. Check your solutions.* *(Answer on page 56.)*

$$\frac{5}{t+2} - \frac{2}{t} = \frac{2t+3}{t^2+2t}$$

Example 6. *Solve for the variable. Check your solutions.*

a. $\dfrac{3}{x+5} + \dfrac{1}{x-2} = \dfrac{4x-1}{x^2+3x-10}$ *b.* $\dfrac{x+1}{x-3} - \dfrac{1}{x} = \dfrac{1}{x^2-3x}$

Solution.

a. The LCD of all of the fractions in the equation is $(x+5)(x-2) =$

$x^2 + 3x - 10$.

$$\frac{3}{x+5} + \frac{1}{x-2} = \frac{4x-1}{x^2+3x-10}$$

$$(x+5)(x-2) \cdot \left(\frac{3}{x+5} + \frac{1}{x-2} \right) = (x+5)(x-2) \cdot \frac{4x-1}{(x+5)(x-2)}$$

$$(x+5)(x-2) \cdot \frac{3}{x+5} + (x+5)(x-2) \cdot \frac{1}{x-2} = (x+5)(x-2) \cdot \frac{4x-1}{(x+5)(x-2)}$$

$$\begin{aligned} (x-2) \cdot 3 + (x+5) \cdot 1 &= 4x - 1 \\ 3x - 6 + x + 5 &= 4x - 1 \\ 4x - 1 &= 4x - 1. \end{aligned}$$

This is an ***identity***, so the solution set is the set of all real numbers in the domain of the original rational equation. Since the original rational equation is undefined for $x = -5$ and for $x = 2$, the solution set is $(-\infty, -5) \cup (-5, 2) \cup (2, \infty)$.

b. The LCD is $x(x-3) = x^2 - 3x$.

$$\frac{x+1}{x-3} - \frac{1}{x} = \frac{1}{x^2 - 3x}$$

$$x(x-3) \cdot \left(\frac{x+1}{x-3} - \frac{1}{x} \right) = x(x-3) \cdot \frac{1}{x^2 - 3x}$$

$$x(x-3) \cdot \frac{x+1}{x-3} - x(x-3) \cdot \frac{1}{x} = x(x-3) \cdot \frac{1}{x(x-3)}$$

$$\begin{aligned} x \cdot (x+1) - (x-3) &= 1 \\ x^2 + x - x + 3 &= 1 \\ x^2 &= 1 - 3 \\ x^2 &= -2. \end{aligned}$$

We see again that this equation has no real solutions (since the square of a real number cannot be negative). ■

Practice 6. *Solve for the variable using Method 2. (Answers below.)*

a. $\dfrac{y}{y-1} = \dfrac{1}{y+1} + \dfrac{1}{2}$ *b.* $\dfrac{13a-8}{3a^2-2a} - \dfrac{4}{a} = \dfrac{1}{3a-2}$

Comparing the solutions using Method 2 to the solutions using Method 1, we see that Method 2 yields shorter solutions. However, we must always be careful with Method 2 to check for extraneous solutions.

ANSWERS TO SECTION 1.6 PRACTICE PROBLEMS

1. $y = -4$
2. $t = 7$
3. (a) no real solutions
   (b) $(-\infty, 0) \cup \left(0, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$
4. $y = -4$
5. $t = 7$
6. (a) no real solutions
   (b) $(-\infty, 0) \cup \left(0, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$
7. $(-\infty, 0] \cup (8, \infty)$
8. $(-\infty, -8] \cup (8, \infty)$

SECTION 1.6 EXERCISES:

(Answers are found on page 151.)

Solve for the variable using Method 1. Be sure to check your solution(s) in the original equation.

1. $\dfrac{2t^2+1}{t+12} = \dfrac{t^2+145}{t+12}$
2. $\dfrac{x^2+5x-1}{x-3} = \dfrac{5x+8}{x-3}$
3. $\dfrac{1}{4x} + \dfrac{3}{5x} = \dfrac{51}{20}$
4. $\dfrac{1}{2t} + \dfrac{2}{3t} = \dfrac{1}{6}$
5. $\dfrac{2x+5}{x+1} - \dfrac{3x}{x^2+x} = 2$
6. $\dfrac{4}{x-2} - \dfrac{6}{2x-4} = \dfrac{4}{x^2-4} + \dfrac{1}{x+2}$
7. $\dfrac{3x}{x+7} = \dfrac{6}{5}$
8. $\dfrac{5-2x}{x+1} = \dfrac{3x}{x^2+x}$
9. $\dfrac{6}{x^2+2x} + \dfrac{3}{x+2} = \dfrac{3-x}{x}$
10. $\dfrac{2}{x+6} - \dfrac{2}{x-6} = \dfrac{x-18}{x^2-36}$

Solve for the variable using Method 2. Be sure to check your solution(s) in the original equation.

11. $\frac{2}{x} - \frac{1}{3} = \frac{1}{x} + \frac{5}{6}$
12. $\frac{1}{5y} + \frac{1}{3} = \frac{2}{3y} - \frac{1}{15}$
13. $\frac{4}{x-3} - \frac{1}{x+1} = \frac{3}{x+1}$
14. $\frac{13}{4x-2} = \frac{3x^2-4x-15}{2x^2-7x+3}$
15. $\frac{3}{y-5} = \frac{5}{y-3}$
16. $\frac{2}{x-1} = \frac{1}{x-2}$
17. $\frac{x^2}{x^2+1} - 1 = \frac{-2}{2x^2+2}$
18. $\frac{2-x}{x+1} + \frac{x+8}{x-2} = \frac{13x+4}{x^2-x-2}$
19. $\frac{x-2}{x+3} - \frac{x-1}{x+4} = \frac{x}{x+3} + \frac{x-1}{x^2+7x+12}$
20. $\frac{3}{2x^2-3x-2} - \frac{x+2}{2x+1} = \frac{2x}{10-5x}$

Solve for the variable using any method approved by your instructor. Be sure to check your solution(s) in the original equation.

21. $\frac{2x}{x-4} = \frac{2}{1-x}$
22. $\frac{x+5}{x+3} = \frac{x+18}{8-2x}$
23. $\frac{3}{y-4} + \frac{5}{y+4} = \frac{16}{y^2-16}$
24. $\frac{5}{x^2-9} + \frac{2}{x-3} = \frac{3}{x+3}$
25. $\frac{t}{t-6} = \frac{1}{t-6} - \frac{2}{t}$
26. $\frac{2}{x} - \frac{x}{3} = \frac{5}{3x}$
27. $\frac{z+1}{z-1} - \frac{4}{z} = \frac{2}{z^2-z}$
28. $\frac{20}{x^2-25} - \frac{2}{x-5} = \frac{4}{x+5}$
29. $\frac{2}{2y+3} + \frac{6y-5}{2y^2-y-6} = \frac{1}{y-2}$
30. $\frac{x^2-10}{x^2-x-20} - 1 = \frac{5}{x-5}$
31. $\frac{2}{x-3} + 3 = \frac{2}{x+3}$
32. $\frac{a}{a-1} - \frac{3}{4} = \frac{3}{4a} - \frac{1}{a-1}$
33. $\frac{12}{m^2-1} = \frac{6}{m+1} + 2$
34. $\frac{4}{x^2-4} - 1 = \frac{1}{x-2}$
35. $\frac{t}{t+3} = \frac{1}{t-3} + \frac{t^2-4t-3}{t^2-9}$
36. $\frac{t^2}{t^2+5t-14} = \frac{1}{t-2} + \frac{t}{t+7}$
37. $\frac{-y}{2y^2+5y-3} = \frac{y}{2y-1} - \frac{1}{y+3}$
38. $\frac{5}{3x+2} + \frac{2x}{x+1} = \frac{2}{3x^2+5x+2}$

39. $\dfrac{x^2 - 4x}{x^2 - 6x + 5} = \dfrac{x}{x - 5} - \dfrac{3}{x - 1}$

40. $\dfrac{x^2 + 10x + 1}{5x^2 + x} = \dfrac{2}{x} + \dfrac{x}{5x + 1}$

1.7 Applications of Rational Equations

In this section we solve application problems using rational equations.

Example 1. *Four times the reciprocal of a number equals one-ninth the number. Find the number.*

Solution. Let x be the number.

"Four times the reciprocal of a number" can be written $4 \cdot \frac{1}{x}$, while "one-ninth the number" is $\frac{1}{9} \cdot x$. We set these equal to one another and solve for x.

$$\begin{aligned} 4 \cdot \frac{1}{x} &= \frac{1}{9} \cdot x \\ \frac{4}{x} &= \frac{x}{9} \\ \frac{4}{x} \cdot \frac{9}{9} &= \frac{x}{9} \cdot \frac{x}{x} \\ \frac{36}{9x} &= \frac{x^2}{9x} \end{aligned}$$

Since the denominators are equal, this implies

$$\begin{aligned} 36 &= x^2 \\ \pm 6 &= x. \end{aligned}$$

Therefore, the number is 6 or -6. ■

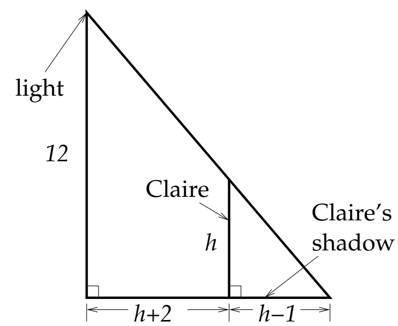
Practice 1. *A number added to twice the reciprocal of the number equals -3. Find the number.* (*Answers on page 70.*)

Example 2. *Claire is standing near a twelve-foot-tall street light. Her distance from the base of the light post is two feet more than her height. The length of her shadow on the ground is one foot less than her height. How tall is Claire?*

Solution. Let h be Claire's height, in feet.

Then her distance from the base of the post is $h + 2$ feet and the length of her shadow is $h - 1$ feet. This situation can be modeled geometrically by two right triangles, as shown in the diagram. These triangles are *similar* since their angles are equal. Hence, their sides are proportional. In particular, the ratio of the height to the base of the small triangle is equal to the ratio of the height to the base of the large triangle. The base of the large triangle is

$$(h+2)+(h-1)=h+2+h-1=2h+1.$$



Thus, by similar triangles,

$$\begin{aligned}\frac{h}{h-1} &= \frac{12}{2h+1}\\ \frac{h}{h-1}\cdot\frac{2h+1}{2h+1} &= \frac{12}{2h+1}\cdot\frac{h-1}{h-1}\\ \frac{h(2h+1)}{(h-1)(2h+1)} &= \frac{12(h-1)}{(h-1)(2h+1)}\end{aligned}$$

Since the denominators are the same, this implies

$$\begin{aligned}h(2h+1) &= 12(h-1)\\ 2h^2+h &= 12h-12\\ 2h^2+h-12h+12 &= 12h-12-12h+12\\ 2h^2-11h+12 &= 0\\ (2h-3)(h-4) &= 0\end{aligned}$$

By the Zero Product Property,

$$\begin{aligned}2h-3 &= 0\\ 2h &= 3\\ \frac{2h}{2} &= \frac{3}{2}\\ h &= \frac{3}{2}\end{aligned} \qquad \text{or} \qquad \begin{aligned}h-4 &= 0\\ h &= 4\end{aligned}$$

Both proposed solutions are in the domain of the original expressions and the reader can check that both are indeed solutions to the original problem. So Claire is either 4 feet or $\frac{3}{2}$ feet (18 inches) tall. ■

Practice 2. *Michael is standing near a fifteen-foot-tall street light. His distance from the base of the light post is double his height. The length of his shadow on the ground is two feet more than his height. How tall is Michael?* *(Answers on page 70.)*

Rates

One of the most important ideas in mathematics and the sciences is the concept of *rate*. We may define ***rate*** to be a ratio of two measurements. For example, if Ken makes \$105 for working 20 hours, he is paid at a *rate* of

$$\frac{\$105}{20 \text{ hours}}.$$

If we write a rate with 1 in the denominator, we get the equivalent ***unit rate***. Unit rates are often easier to conceptualize than other rates. Ken's pay rate is converted to the equivalent unit rate as follows.

$$\frac{\$105}{20 \text{ hours}} = \frac{\$\ 20 \cdot 5.25}{20 \text{ hours}} = \frac{\$5.25}{1 \text{ hour}} = \$5.25 \text{ per hour.}$$

Note that the division sign in a rate is often read "per."

Example 3. *Convert each of the following to the equivalent unit rate (unless otherwise specified).*

a. *The pop cost 84 cents for 12 ounces.*

b. *The nurse counted 16 heartbeats in 1/4 minute.*

c. *There is a rise of 5 units for a run of 2 units.*

d. *There are 180 minutes in 3 hours.*

e. *The account pays \$0.06 in interest for each \$1.00 invested. (Convert to a percentage.)*

Solution.

a. $\frac{84¢}{12 \text{ oz}} = \frac{7 \cdot 12\ ¢}{12 \text{ oz}} = \frac{7¢}{1 \text{ oz}} = 7¢ \text{ per oz.}$

b. $\dfrac{16 \text{ beats}}{1/4 \text{ min}} = \dfrac{16 \cdot 4 \text{ beats}}{1 \text{ min}} = 64$ beats per min.

c. $\dfrac{5 \text{ units}}{2 \text{ units}} = \dfrac{\frac{5}{2} \text{ units}}{1 \text{ unit}} = \dfrac{5}{2}.$

This is an example of the *slope* of a line, as we studied in a previous course. In this case, the units of the measurement in the numerator are the same as the units of measurement in the denominator, so the rate itself has no units.

d. $\dfrac{180 \text{ minutes}}{3 \text{ hours}} = \dfrac{3 \cdot 60 \text{ minutes}}{3 \text{ hours}} = \dfrac{60 \text{ minutes}}{1 \text{ hour}} = 60$ minutes per hour.

This is an example of a *conversion factor* (also studied in a previous course) which converts from one unit of measurement to another for the same quantity.

e. This is already given as a unit rate. However, it is customary to express *interest rates* as percentages rather than as unit rates. The term "percent" means "per 100," so a percentage is really a rate with a denominator of 100 (where the "cent" is the 100). Here again, the units of the numerator are the same as the units of the denominator, so the rate has no units of measurement.

$$\frac{\$0.06}{\$1.00} = \frac{0.06 \cdot 100}{1.00 \cdot 100} = \frac{6}{100} = 6\%.$$ ■

Practice 3. *Convert each of the following to the equivalent unit rate (unless otherwise specified).* *(Answers on page 70.)*

a. *There are 12 girl scouts sleeping in 3 tents.*

b. *The car went 210 miles on 10 gallons.*

c. *The sprinkler used 300 gallons in 15 minutes.*

d. *There are 2640 feet in 1/2 mile.*

e. *The student earned 450 points out of 500 points. (Convert to a percentage.)*

Applications where rates are additive

Example 4. *Carol can grade her class's exams in five hours. It takes her assistant seven hours. How long will it take them to grade the exams working together?*

Solution. Let us first think carefully about the situation and find a rough estimate of the answer. If both graders worked at Carol's rate, together they would complete the task in half Carol's time, or in two and a half hours. On the other hand, if both worked at the assistant's rate, together they could grade the exams in three and a half hours. Therefore, we should expect it to take between two and a half and three and a half hours. We model this situation with a rational equation to find a more precise answer to the question.

We are given the amount of *time* that different people can complete the same task. Let us find the *rate* at which each person works, writing each as a unit rate. Since Carol completes the task in five hours, she works at the rate of

$$\frac{1 \text{ task}}{5 \text{ hours}} = \frac{1}{5} \text{ task per hour.}$$

Carol's assistant completes the task in seven hours, so he works at the rate of

$$\frac{1 \text{ task}}{7 \text{ hours}} = \frac{1}{7} \text{ task per hour.}$$

Let t denote the amount of time, in hours, that it takes for Carol and her assistant to grade the exams together. Then the rate that they work together is

$$\frac{1 \text{ task}}{t \text{ hours}} = \frac{1}{t} \text{ task per hour.}$$

Assuming that the people don't slow each other down when they work together (which is entirely possible!), the rate that they work together will be the *sum* of their individual rates. Thus we obtain

$$\begin{aligned}
\frac{1}{5} + \frac{1}{7} &= \frac{1}{t} \\
\frac{1}{5} \cdot \frac{7t}{7t} + \frac{1}{7} \cdot \frac{5t}{5t} &= \frac{1}{t} \cdot \frac{35}{35} \\
\frac{7t}{35t} + \frac{5t}{35t} &= \frac{35}{35t} \\
\frac{7t + 5t}{35t} &= \frac{35}{35t} \\
\frac{12t}{35t} &= \frac{35}{35t}
\end{aligned}$$

Since both fractions have the same denominator, this implies

$$\begin{aligned} 12t &= 35 \\ t &= \frac{35}{12} \\ t &= 2\,\frac{11}{12} \text{ hours.} \end{aligned}$$

Converting the fractional part of the hour to minutes, we have:

$$\frac{11}{12}\text{ hour} = \frac{11}{12}\text{ hour} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = 11 \cdot 5 \text{ minutes} = 55 \text{ minutes.}$$

Thus, working together, Carol and her assistant can grade the exams in 2 hours 55 minutes. Note that this falls in the range that we expected from our rough analysis of the problem. ■

Practice 4. *Meredith can weed the garden in two hours. It takes Tim four hours. How long will it take them to weed the garden working together?* *(Answers on page 70.)*

Example 5. *A tub with a leak fills in twenty minutes. The water will all leak out of the full tub in one hour. How long would it take to fill the tub if the leak is plugged?*

Solution. Let t be the amount of time, in minutes, for the spigot to fill the tub if the leak is plugged. If the leak is plugged, the tub will fill *faster*, taking *less than* twenty minutes to fill. Thus, we expect t to be less than twenty.

Once again, we find the corresponding *rates*. The rate at which the tub is leaking is

$$\frac{1 \text{ tub}}{1 \text{ hour}} = \frac{1 \text{ tub}}{60 \text{ minutes}} = \frac{1}{60} \text{ tub per minute.}$$

The rate at which the tub can be filled while leaking is

$$\frac{1 \text{ tub}}{20 \text{ minutes}} = \frac{1}{20} \text{ tub per minute.}$$

The rate at which the spigot fills the tub after the leak is plugged is

$$\frac{1 \text{ tub}}{t \text{ minutes}} = \frac{1}{t} \text{ tub per minute.}$$

Here the spigot and the leak are working *against* each other, so their rates should be *subtracted* to get the rate at which the tub fills while leaking.

$$\begin{aligned}
\frac{1}{t} - \frac{1}{60} &= \frac{1}{20} \\
\frac{1}{t} \cdot \frac{60}{60} - \frac{1}{60} \cdot \frac{t}{t} &= \frac{1}{20} \cdot \frac{3t}{3t} \\
\frac{60}{60t} - \frac{t}{60t} &= \frac{3t}{60t} \\
\frac{60 - t}{60t} &= \frac{3t}{60t}
\end{aligned}$$

Since the denominators are equal, this implies

$$\begin{aligned}
60 - t &= 3t \\
60 - t + t &= 3t + t \\
60 &= 4t \\
\frac{60}{4} &= \frac{4t}{4} \\
15 &= t.
\end{aligned}$$

Leaving it to the reader to check the answer in the original equation, we conclude that the spigot would fill the tub in fifteen minutes if the leak were plugged. Note that this is less than twenty minutes, as expected. ■

Practice 5. *A tub with a leak fills in fifty minutes. If the leak were plugged, it would fill in thirty minutes. How long would it take the water to leak out of the full tub?* (*Answers on page 70.*)

Example 6. *It took Lynne ten minutes to walk from the ticket counter to her gate at the airport. If she walked on the moving walkway, it would take her six minutes. How long would it take her if she stood on the moving walkway?*

Solution. This is a "distance/rate/time" problem. However, we are not given the distance in conventional units such as miles or feet. We use the length of Lynne's walk to the gate as our unit, and find expressions for the rates involved just as we did in the previous rate problems. Let t be the time, in minutes, it would take Lynne to get to her gate if she stood on the moving walkway. The the walkway is moving at a rate of

$$\frac{1 \text{ distance to the gate}}{t \text{ minutes}} = \frac{1}{t} \text{ distance to gate per minute.}$$

Lynne can walk at a rate of

$$\frac{1 \text{ distance to the gate}}{10 \text{ minutes}} = \frac{1}{10} \text{ distance to gate per minute.}$$

On the moving walkway, Lynne travels at a rate of

$$\frac{1 \text{ distance to the gate}}{6 \text{ minutes}} = \frac{1}{6} \text{ distance to gate per minute.}$$

Since Lynne and the walkway are moving in the same direction, their rates can be *added*.

$$\begin{aligned} \frac{1}{t} + \frac{1}{10} &= \frac{1}{6} \\ \frac{1}{t} \cdot \frac{30}{30} + \frac{1}{10} \cdot \frac{3t}{3t} &= \frac{1}{6} \cdot \frac{5t}{5t} \\ \frac{30}{30t} + \frac{3t}{30t} &= \frac{5t}{30t} \\ \frac{30 + 3t}{30t} &= \frac{5t}{30t} \end{aligned}$$

Since the denominators are equal, this implies

$$\begin{aligned} 30 + 3t &= 5t \\ 30 + 3t - 3t &= 5t - 3t \\ 30 &= 2t \\ 15 &= t. \end{aligned}$$

Lynne would take fifteen minutes to get to her gate if she stood on the moving walkway. ■

Practice 6. *Ginny can walk from the ticket counter to his gate at the airport in thirty-six minutes. If she stands on the moving walkway, it takes eighteen minutes. How long would it take her if she walked on the moving walkway?* *(Answers on page 70.)*

Applications where rates are not additive

Example 7. *A car travels thirty miles in the time it takes a bicycle to go ten. The car's speed is forty miles per hour faster than the bike's. Find the speed of each.*

Solution. In this problem, we are given the relationship between the rates of the two vehicles. Let r be the rate (speed) of the bicycle, in miles per hour. Then the rate of the car is $r + 40$ miles per hour. We can organize the given information in a table.

	distance	***rate***	***time***
bike	10 miles	$r\ \dfrac{\text{miles}}{\text{hour}}$	
car	30 miles	$(r+40)\ \dfrac{\text{miles}}{\text{hour}}$	

We know that rate is defined to be the distance traveled over the time elapsed. In symbols,

$$r = \frac{d}{t}.$$

We can solve this for time, t.

$$\begin{aligned} r &= \frac{d}{t} \\ r \cdot t &= \frac{d}{t} \cdot t \\ r \cdot t &= d \\ \frac{r \cdot t}{r} &= \frac{d}{r} \\ t &= \frac{d}{r}. \end{aligned}$$

We fill in the last column of our table using the formula for time that we just derived.

	distance	***rate***	***time***
bike	10 miles	$r\ \dfrac{\text{miles}}{\text{hour}}$	$\dfrac{10}{r}$ hours
car	30 miles	$(r+40)\ \dfrac{\text{miles}}{\text{hour}}$	$\dfrac{30}{r+40}$ hours

Recall that the time it takes the bike to go ten miles *equals* the time it takes the car to go thirty miles. This gives us an equation to solve.

$$\begin{aligned}
\frac{10}{r} &= \frac{30}{r+40} \\
\frac{10}{r} \cdot \frac{r+40}{r+40} &= \frac{30}{r+40} \cdot \frac{r}{r} \\
\frac{10(r+40)}{r(r+40)} &= \frac{30r}{r(r+40)}
\end{aligned}$$

Since the denominators are equal, this implies

$$\begin{aligned}
10(r+40) &= 30r \\
10r + 400 &= 30r \\
10r + 400 - 10r &= 30r - 10r \\
400 &= 20r \\
\frac{400}{20} &= \frac{20r}{20} \\
20 &= r
\end{aligned}$$

and so

$$r + 40 = 60.$$

Hence, the bicycle is traveling 20 miles per hour and the car is traveling 60 miles per hour. ■

Practice 7. *A car travels fifty miles in the time it takes a bicycle to go twenty. The car's speed is fifteen miles per hour faster than the bike's. Find the speed of each.* *(Answers on page 70.)*

Example 8. *Nuts are twice as expensive per pound as raisins. To make seven pounds of trail mix, \$15 worth of nuts is combined with \$10 worth of raisins. What is the price per pound of each?*

Solution. This is a rate problem where the rate is the quotient of cost and weight (measured in dollars per pound). In symbols,

$$r = \frac{c}{w}.$$

Let r be the rate (cost per pound) of the raisins. Then the cost per pound of the nuts is $2r$. We organize our data in a table.

	rate	***total cost***	***weight***
raisins	$r\ \frac{\$}{\text{pound}}$	\$10	
nuts	$2r\ \frac{\$}{\text{pound}}$	\$15	

To find an expression for the weight of each, we solve $r = \frac{c}{w}$ for w.

$$\begin{aligned} r &= \frac{c}{w} \\ r \cdot w &= \frac{c}{w} \cdot w \\ r \cdot w &= c \\ \frac{r \cdot w}{r} &= \frac{c}{r} \\ w &= \frac{c}{r}. \end{aligned}$$

We use this formula to complete the table.

	rate	***total cost***	***weight***
raisins	$r\ \frac{\$}{\text{pound}}$	\$10	$\frac{10}{r}$ pounds
nuts	$2r\ \frac{\$}{\text{pound}}$	\$15	$\frac{15}{2r}$ pounds

We use the fact that the total weight of the trail mix is seven pounds to write an equation.

$$\begin{aligned} \frac{10}{r} + \frac{15}{2r} &= 7 \\ \frac{10}{r} \cdot \frac{2}{2} + \frac{15}{2r} &= 7 \cdot \frac{2r}{2r} \\ \frac{20}{2r} + \frac{15}{2r} &= \frac{14r}{2r} \\ \frac{20+15}{2r} &= \frac{14r}{2r} \\ \frac{35}{2r} &= \frac{14r}{2r} \end{aligned}$$

Since the denominators are equal, this implies

$$\begin{aligned} 35 &= 14r \\ \frac{35}{14} &= \frac{14r}{14} \\ \frac{5}{2} &= r \\ r &= \$2.50 \text{ per pound} \end{aligned}$$

and so

$$2r = \$5.00 \text{ per pound.}$$

Thus, raisins cost $2.50 per pound and nuts cost $5 per pound. ■

Practice 8. *Nuts are three times as expensive per pound as raisins. To make four pounds of trail mix, $9 worth of nuts is combined with $5 worth of raisins. What is the price per pound of each?* (*Answers below.*)

ANSWERS TO SECTION 1.7 PRACTICE PROBLEMS

1. The number is -2 or -1.
2. Michael is 6 feet tall. ($-\frac{5}{3}$ is not in the domain.)
3. (a) 4 girls per tent
   (b) 21 miles per gallon
   (c) 20 gallons per minute
   (d) 5280 feet per mile
   (e) 90%
4. It would take them 1 hour 20 minutes working together.
5. The water would leak out in 75 minutes.
6. It would take her 12 minutes.
7. The bicycle is traveling 10 miles per hour and the car is traveling 25 miles per hour.
8. Raisins cost $2 per pound and nuts cost $6 per pound.

SECTION 1.7 EXERCISES:

(Answers are found on page 151.)

Solve each problem. Be sure to:

- *Introduce your variable with a "Let" statement.*
- *Set up and solve an equation.*
- *State your answer in a complete sentence in the context of the original problem.*

1. Twice the reciprocal of a number equals one-eighth the number. Find the number.

2. Five times the reciprocal of a number equals one-fifth the number. Find the number.

3. Twice a number plus its reciprocal equals 3. Find the number.

4. Three times a number minus its reciprocal equals -2. Find the number.

5. Six times the reciprocal of a number is four times the number, plus two. Find the number.

6. The sum of the reciprocals of 3 and a number is -2. Find the number.

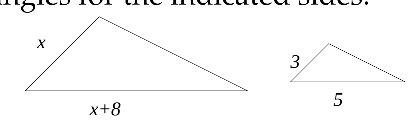
7. Ten divided by the difference of a number and one equals the quotient of the number and two. Find the number.

8. A number minus twice its reciprocal is the same as the reciprocal of four times the number. Find the number.

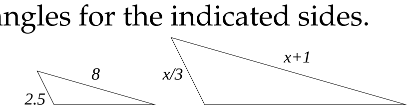
9. The ratio of six and a number squared, reduced by one, is equal to the reciprocal of the number. Find the number.

10. The product of the reciprocal of a number and the sum of the number plus four is the same as the ratio of the number and the sum of the number plus four. Find the number.

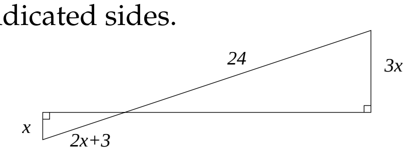
11. Solve the similar triangles for the indicated sides.



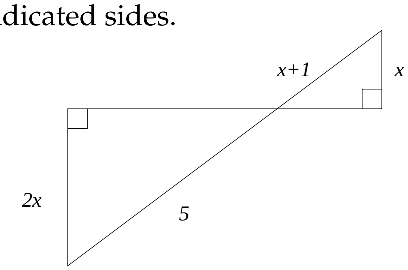
12. Solve the similar triangles for the indicated sides.



13. Solve for the three indicated sides.



14. Solve for the three indicated sides.



15. Thelma is standing near a seven-foot-tall street lamp. Her distance from the base of the lamp post is one foot less than her height. The length of her shadow on the ground is five feet more than her height. How tall is Thelma?

16. Billy is standing near a nine-foot-tall street lamp. His shadow is one foot longer than his height. His distance to the base of the lamp post is twice the length of his shadow. How tall is Billy?

17. A 20-foot tree casts a shadow that is four times as long as the shadow of the man. The man's shadow is 2 feet more than his height. What is his height?

18. A fence casts a shadow that is 3 feet more than the height of the fence. The shadow of a nearby 20-foot tree is six feet less than four times the fence's shadow. How high is the fence?

Convert each of the following to the equivalent unit rate.

19. Linda earns $225 for working 30 hours.

20. It takes 40 gallons of sap to make 8 pints of maple syrup.

21. A total of 1170 boxes of cookies were sold by the 9 girl scouts in the troop.

22. There are 120 boxes of cookies in 10 cases.

23. There are 300 feet in 100 yards.

24. There are 10 cm in 100 mm.

25. There is a rise of 7 units for a run of 28 units.

26. There is a rise of 10 units for a run of 2 units.

Convert each of the following to the equivalent percentage.

27. The account pays $7 in interest for each $200 invested.
28. The credit card company charges $140 in interest for each $1000 charged on the card.
29. Brock earned 47 out of 50 points on the exam.
30. Kendra earned 19 out of 20 points on the quiz.

Solve each problem. Be sure to:

- *Introduce your variable with a "Let" statement.*
- *Set up and solve an equation.*
- *State your answer in a complete sentence in the context of the original problem.*

31. Dan can shovel the driveway in 90 minutes. It takes Jim 60 minutes. How long will it take them to shovel the driveway working together?
32. Lisa can paint the dining room in 4 hours. It takes Clive 6 hours. How long will it take them to paint the dining room working together?
33. Two student workers pick up trash after a home game. Jim alone can clear the field in two hours. With Todd helping, the two can finish the job in 1 hour 10 minutes. How long would it take Todd alone?
34. Maggie can mow the school lawn in 4 hours. Seth can mow the same lawn in 3 hours. How long would it take them to mow the school lawn if they work together?
35. It takes Craig twice as long to clean out the garage as his father. Working together, father and son can clean the garage in 3 hours. How long would it take Craig alone?
36. Bob can wash the truck in 45 minutes. With Janet working alongside, the two can wash the car in 30 minutes. How long would it take Janet to wash the car by herself?
37. A train travels fifty miles in the time it takes a truck to go thirty. The train's speed is forty miles per hour faster than the truck's. Find the speed of each.

38. A tub with a leak fills in fifty minutes. If the leak were plugged, it would fill in thirty minutes. How long would it take the water to leak out of the full tub?

39. A tub takes three times as long to fill now that it has sprung a leak than it did before. All of the water will leak out of the full tub in 1 hour. How long does it take to fill the tub with and without the leak?

40. Bryan took a 4-mile walk. The second half of his walk he averaged 1 mile per hour less than the first half, and it took him $\frac{1}{3}$ hour longer. What rate did he average the first half of the walk?

41. A fisherman puts a small trolling motor on his boat. He can travel downstream 27 miles in the same time he can travel upstream 9 miles. The speed of the current is 6 miles per hour. How fast is the boat in still water?

42. A train travels 315 miles across the plains in the same time it travels 175 miles in the mountains. If the rate of the train is 40 miles per hour slower in the mountains, find the rate in both the plains and the mountains.

43. A boat travels 15 miles with a 5 mile-per-hour current in the same time it can travel 10 miles against the 5 mile-per-hour current. Find the speed of the boat in still water.

44. A motorized raft can travel 6 miles per hour in still water. If the raft can travel 12 miles downstream in the same time it can travel 6 miles upstream, what is the speed of the current?

45. A plane flies 460 miles with a tail wind of 30 miles per hour. Flying against the wind, it flies 340 miles in the same amount of time. How fast would the plane fly in still air?

46. Ron can walk from the ticket counter to his gate at the airport in three minutes. If he walks on the moving walkway, it takes two minutes. How long would it take him if he stood on the moving walkway?

47. Cashews cost $8 more per pound than peanuts. To make one and one-half pounds of nut mix, $5 worth of cashews is combined with $2 worth of peanuts. What is the price per pound of each?

48. A 20-pound bag of grass seed mixture consists of \$18 worth of annual rye grass seed and \$20 worth of perennial bluegrass seed. What is the price per pound of each type of seed if bluegrass seed costs one dollar more per pound than the rye grass seed?

Chapter 2

Intermediate Factoring Techniques

2.1 Introduction: The *why* behind the *how*

So why do we factor in the first place? Why should we care? We factor polynomials because the factored form

- is often easier to work with than the original;
- helps us solve problems;
- helps us analyze the graph of a function.

Scenario 1.

We know from physics that the height of an object thrown or launched from ground level at any given time, t, is given by the formula

$$h(t) = -16t^2 + v_0 t$$

where v_0 is the initial velocity.

Suppose the initial velocity of a model rocket launched from the ground is 80 feet per second. The formula then becomes

$$h(t) = -16t^2 + 80t.$$

If we wanted to figure out when the rocket hits the ground again, what is the final height? Write the equation we would need to solve.

The factored form of this equation makes it quite easy to solve.

$$0 = -16t(t - 5)$$

What values of t solve the equation? What does the solution mean in the context of the problem?

Scenario 2.

Suppose an aluminum pop can maker wants to compare the total amount of aluminum needed for cans of different sizes. The formula for surface area of a cylinder is

$$SA = 2\pi r^2 + 2\pi rh,$$

where h is the height and r is the base radius. If we factor the right side of this equation like so:

$$SA = 2\pi r(r + h),$$

it becomes easier to calculate and estimate with.

Scenario 3.

If we want to analyze the graph of

$$y = x^2 - x - 6,$$

we might find it easier to hand graph or recognize its given graph if the expression is in factored form:

$$y = (x - 3)(x + 2).$$

Graph this equation with your grapher.

How are the factors related to the graph? Why is this the case? Will this relationship always hold?

Do the same for the following factored forms:

a. $y = (x - 2)(x + 4)$

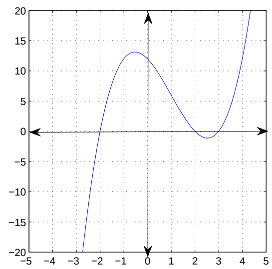
b. $y = (x + 5)(x - 1)$

c. $y = (x + 6)(x + 2)$

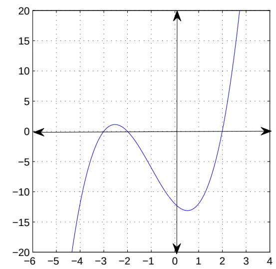
How are these equations related to their graphs? *Why* does this pattern hold?

Practice 1. *(Answers on the following page.) Which of the following graphs could be the graph of the equation* $y = (x+3)(x+2)(x-2)$*?*

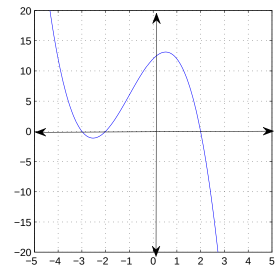
a.



b.



c.



Practice 2. *(Answers on the next page.) Write a quadratic equation in factored form* $y = (\quad)(\quad)$ *that could have the following numeric representation.*

x	y
-3	12
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5
5	12
6	21

Scenario 4.

With algebraic functions, factoring allows us to simplify expressions and thus make them easier to work with. For example,

$$\frac{x^2 - 5x + 4}{x - 1} = \frac{(x-1)(x-4)}{x-1} = x - 4,$$

which is short and sweet, though we note a difference in these two expressions when $x = 1$. How are these expressions different when $x = 1$? The expression on the left is not defined at $x = 1$, but the expression on the right is.

In this chapter, we will discuss three main factoring techniques:

- factoring by grouping;
- factoring a difference of squares and sums and differences of cubes;
- factoring quadratic type expressions.

ANSWERS TO SECTION 2.1 PRACTICE PROBLEMS

1. b
2. $y = (x+1)(x-3)$

2.2 Factoring by Grouping

For polynomials that have more than three terms, a common technique is factoring by grouping. We group terms together and look for common factors in each group.

Example 1. *Factor* $12ab + 18a + 10b + 15.$

Solution. Group the first two terms and the last two terms together:

$$12ab + 18a + 10b + 15 = \underbrace{12ab + 18a} + \underbrace{10b + 15}$$

Factor the greatest common factor from each group:

$$= 6a(2b + 3) + 5(2b + 3)$$

Factor out the binomial common factor:

$$= (2b + 3)(6a + 5).$$

Note that $6a(2b + 3) + 5(2b + 3)$ is *not* the factored form. Why not?

Check. We can check our work by substituting values in to the original expression and in to our factored form. If a = 1 and b =-5 , the original expression is

$$12(1)(-5) + 18(1) + 10(-5) + 15 = -60 + 18 - 50 + 15 = -77.$$

Find the value of the factored form if $a = 1$ and $b = -5$.

Make a table for several different values of a and b. What did you notice? ■

Example 2. *Factor* $2x^2 + 3y + 6x + xy.$

Solution. At first glance, it looks like this expression cannot be factored because nothing can be factored out of the first two terms. If we rearrange the terms, however, factoring becomes possible.

$$\begin{aligned} 2x^2 + 3y + 6x + xy &= \underbrace{2x^2 + xy} + \underbrace{6x + 3y} \\ &= x(2x + y) + 3(2x + y) \\ &= (2x + y)(x + 3). \end{aligned}$$

Check. We can check our work by substituting values in to the original expression and in to our factored form. If $x = 1$ and $y = -5$, the original expression is

$$2(1)^2 + 3(-5) + 6(1) + (1)(-5) = 2 + (-15) + 6 + (-5) = -12.$$

Evaluate the factored expression at $x = 1$ and $y = -5$. Make a table of several such values. What did you notice?

This example shows that rearranging the terms may be necessary before factoring. ∎

Example 3. *Factor* $x^3 + 3x^2 - 4x - 12.$

Solution. We group the first two terms and the last two terms together, noting that a plus sign replaces the minus, but the last two terms are negative.

$$\begin{aligned} x^3 + 3x^2 - 4x - 12 &= \underbrace{x^3 + 3x^2} + \underbrace{-4x - 12} \\ &= x^2(x+3) - 4(x+3) \\ &= (x+3)(x^2-4) \end{aligned}$$

Note that we can still factor further:

$$= (x+3)(x+2)(x-2).$$

What would be the x-intercepts of the graph of

$$y = (x+3)(x+2)(x-2)?$$

Graph this function on your grapher and make a note of the x-intercepts.

Now graph the original expression on the same set of axes. (Let $Y_2 = x^3 + 3x^2 - 4x - 12$.) What do you notice about the two graphs?

Check. As before, we can observe what happens when we substitute values for x into each expression. For example, if $x = 1$, the original expression is

$$(1)^3 + s(1)^2 - 4(1) - 12 = -12.$$

Substitute $x = 1$ into the factored form. What is the result?

Put the original expression into Y1 and the factored expression into Y2 on your graphing calculator. Go to TBLSET and let TblStart= 0 and TBL = 1. Go to 2nd GRAPH and observe the values in the table. How do the values of the original form compare to those in the factored form? Will this always happen? Why? ∎

Example 4. *Factor* $4x^3 + 2x^2 + 4x + 2.$

Solution. A good first step is to check if there is a GCF to factor out.

$$\begin{aligned} 4x^3 + 2x^2 + 4x + 2 &= 2(2x^3 + x^2 + 2x + 1) \\ &= 2(\underbrace{2x^3 + x^2} + \underbrace{2x + 1}) \\ &= 2[x^2(2x+1) + 1(2x+1)] \\ &= 2[(2x+1)(x^2+1)] \\ &= 2(2x+1)(x^2+1). \end{aligned}$$

Remember to *always* factor out the GCF first. ■

Practice 1. *(Answers on the following page.) Factor each of the following.*

a. $5x^2 + 30x + 4x + 24$

b. $15ac + 20ad + 4bd + 3bc$

c. $10m^2 - 12n + 15m - 8mn$

Example 5. *Factor* $x^2 + 10x + 25 - y^2$.

Solution. This time, we group the first three terms together because they are readily factorable.

$$\begin{aligned} x^2 + 10x + 25 - y^2 &= \underbrace{x^2 + 10x + 25} - \underbrace{y^2} \\ &= (x+5)^2 - y^2 \end{aligned}$$

Note that now we have a difference of two perfect squares:

$$\begin{aligned} &= [(x+5) - y][(x+5) + y] \\ &= (x + 5 - y)(x + 5 + y). \end{aligned}$$

This shows that sometimes we group three terms instead of two. ■

ANSWERS TO SECTION 2.2 PRACTICE PROBLEMS

1. (a) $(5x+4)(x+6)$ (b) $(5a+b)(3c+4d)$ (c) $(5m-4n)(2m+3)$

SECTION 2.2 EXERCISES:

(Answers are found on page 152.)

Factor each of the following.

1. $x^3+3x^2+4x+12$
2. $4x^2-12x+5x-15$
3. $2x^4-2x^3-5x+5$
4. $7x^3+x^2-42x-6$
5. $4xy^2-3x+3y-4x^2y$
6. $12ac+6ad+4bd+8bc$
7. $6x^4-6x^3-3x^2+3x$
8. $x^4-7x^3+6x^2-42x$
9. $x^2y-yz+xy-xyz$
10. $12x^3z-3x^2y^3z-4yxz+y^4z$
11. x^3+x^2+x+1
12. $24m^3-4m^2-6m+1$
13. $2a^3b-2a^2b^2+8a^2b-8ab^2$
14. $8ac+20ad-25bd-10bc$
15. $-2x^3+2x^2+3x-3$
16. $12x^3+2x^2-60x-10$
17. $2x^3y-2x^2z+5xy-5z$
18. $10x^3-2x^2y^3-5yx+y^4$
19. $21x^5-9x^3-7x^2+3$
20. $30m^4-5m^3-6m^2+m$

2.3 Special Binomial Forms: Sums and Differences of Squares and Cubes

Difference of two squares

Recall the technique of factoring the difference of two squares.

$$a^2 - b^2 = (a+b)(a-b)$$

where a and b are real numbers.

Example 1. *Factor $a^4 - 100$.*

Solution. The term a^4 is a perfect square since $(a^2)^2 = a^4$.

$$\begin{aligned} a^4 - 100 &= (a^2)^2 - (10)^2 \\ &= (a^2+10)(a^2-10). \end{aligned}$$

If we are factoring over the integers, then we are done. Recall that "factoring over the integers" means that only integers are allowed as coefficients in the factors. ■

Example 2. *Factor $(x-5)^2 - 81$.*

Solution. The expression $(x-5)^2$ is a perfect square, so this follows our pattern.

$$\begin{aligned} (x-5)^2 - 81 &= (x-5)^2 - (9)^2 \\ &= [(x-5)+9][(x-5)-9] \\ &= (x-5+9)(x-5-9) \\ &= (x+4)(x-14). \end{aligned}$$

Note that each factor was simplified by combining like terms. ■

Practice 1. *(Answers on page 88.) Factor each of the following over the integers.*

a. $(m+5)^2 - 49$
b. $(y-3)^2 - 121$
c. $(x-y)^2 - z^2$
d. $x^{10} - 81$
e. $x^{22} - y^8$

Let's take a close look at the binomial $a^2 - 10$ from the answer in Example 1. Is there any way we can consider it to be a difference of two squares? What is the square root of 10? Try writing $a^2 - 10$ as a difference of two squares using *irrational* numbers. We call this technique "factoring over the real numbers."

Example 3. *Factor each of the following over the real numbers.*

a. $a^2 - 10$ *b.* $x^2 - 5$ *c.* $2y^2 - 3$

Solution.

a. Since $10 = \left(\sqrt{10}\right)^2$, we can factor

$$\begin{aligned} a^2 - 10 &= a^2 - \left(\sqrt{10}\right)^2 \\ &= \left(a + \sqrt{10}\right)\left(a - \sqrt{10}\right). \end{aligned}$$

b. Since $5 = \left(\sqrt{5}\right)^2$, we have

$$\begin{aligned} x^2 - 5 &= x^2 - \left(\sqrt{5}\right)^2 \\ &= \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right). \end{aligned}$$

c. Note here that $2y^2 = \left(\sqrt{2}\,y\right)^2$ and $3 = \left(\sqrt{3}\right)^2$.

$$\begin{aligned} 2y^2 - 3 &= \left(\sqrt{2}\,y\right)^2 - \left(\sqrt{3}\right)^2 \\ &= \left(\sqrt{2}\,y + \sqrt{3}\right)\left(\sqrt{2}\,y - \sqrt{3}\right). \end{aligned}$$

■

Practice 2. *(Answers on page 88.) Factor each of the following over the real numbers.*

a. $b^2 - 15$ b. $3m^2 - 7n^2$ c. $x^2 - 99$

Sum or difference of two cubes

Sometimes we need to factor expressions that are sums or differences of cubes, like the following:

$$x^3 - 8 \qquad x^6 + 27 \qquad 27m^3 - 64n^3$$

Note that we will consider both *sums and differences* of two cubes. In the case of squares, we considered only differences, since the sum of two squares is prime over the real numbers.

One factor is the sum or difference of the cube root of each original term. For example, one factor of $x^3 - 8$ is $x - 2$;

one factor of $x^6 + 27$ is $x^2 + 3$;

one factor of $27m^3 - 64n^3$ is $3m - 4n$.

Note that each of these binomial factors has the *same sign of operation* as the original binomial.

To find the other factor, we could use long division, or think it through until we notice the following pattern.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Note that the second factor is a trinomial and that the first sign of operation is *opposite* that in the original binomial. Note also that the last sign is *always positive*. (Some students use the acronym SOAP for Same–Opposite–Always Positive to help them remember the signs.)

Example 4. *Factor $x^3 + 27$.*

Solution. To get the first (binomial) factor, we take the cube root of each term and use the *Same* sign of operation: $(x + 3)$.

For the trinomial factor, we square the first term in the first factor (to get the cube back); multiply the two terms and use the *Opposite* sign of operation; the *Add* the square of the second term: $(x^2 - 3x + 9)$.

Thus, the factorization is

$$x^3 + 27 = (x + 3)(x^2 - 3x + 9).$$

Check. We multiply using the distributive law and then combine like terms.

$$\begin{aligned}(x+3)(x^2 - 3x + 9) &= x(x^2 - 3x + 9) + 3(x^2 - 3x + 9)\\ &= x^3 - 3x^2 + 9x + 3x^2 - 9x + 27\\ &= x^3 + 27\end{aligned}$$

which is what we started with.

It is always good practice to check our answers by multiplying. ■

Example 5. *Factor $27m^3 - 64n^3$.*

Solution. We note that this is the difference of two cubes.

$$\begin{aligned} 27m^3 - 64n^3 &= (3m)^3 - (4n)^3 \\ &= (3m - 4n)\left((3m)^2 + (3m)(4n) + (4n)^2\right) \\ &= (3m - 4n)\left(9m^2 + 12mn + 16n^2\right). \end{aligned}$$

Now check the answer by multiplying. ■

ANSWERS TO SECTION 2.3 PRACTICE PROBLEMS

1. (a) $(m-2)(m+12)$
   (b) $(y-14)(y+8)$
   (c) $(x-y-z)(x-y+z)$
   (d) $\left(x^5-9\right)\left(x^5+9\right)$
   (e) $\left(x^{11}-y^4\right)\left(x^{11}+y^4\right)$
2. (a) $\left(b-\sqrt{15}\right)\left(b+\sqrt{15}\right)$
   (b) $\left(\sqrt{3}\,m-\sqrt{7}\,n\right)\left(\sqrt{3}\,m+\sqrt{7}\,n\right)$
   (c) $\left(x-3\sqrt{11}\right)\left(x+3\sqrt{11}\right)$

SECTION 2.3 EXERCISES:

(Answers are found on page 152.)

Factor each of the following completely over the real numbers.

1. $x^2 - 144$
2. $a^2 - 81b^4$
3. $m^4 - 36$
4. $4x^2 - 9y^2$
5. $9a^2 - 64$
6. $25m^2 - 49y^2$
7. $4 - 9y^2$
8. $169 - x^2$
9. $25 - 121a^2$
10. $49 - m^2$
11. $81 - b^2$
12. $1 - 4x^2$
13. $x^3y - xy^3$
14. $8x - 18x^3$
15. $3y^4 - 27y^2$
16. $32m^2 - 2n^2$
17. $(x-1)^2 - 4$
18. $(x+7)^2 - 9$
19. $(x+3)^2 - y^2$
20. $(4a-5)^2 - b^2$
21. $m^2(3n+1)^2 - 4m^2$
22. $a(b+8)^2 - 121a^3$
23. $a^2 - 13$

24. $x^2 - 14$
25. $b^2 - 21$
26. $a^2 - 2$
27. $b^2 - 33$
28. $x^4 - 6$
29. $2m^2 - 3$
30. $25x^2 - 19$
31. $7x^2 - 11$
32. $5a^2 - 7$
33. $16b^2 - 5$
34. $21x^2 - 4$
35. $1 - 100x^2$
36. $25ab^2 - a^3$
37. $x^2 - 196$
38. $9m^2 - 5$
39. $72x^2 - 50$
40. $16a^4 - 1$
41. $x^2 - 7$
42. $a^4b^2 - a^2b^4$
43. $625x^4 - 81y^4$
44. $m^2 - 3$
45. $(x+3)^2 - 36y^2$
46. $3a^2 - 5$
47. $y^3(2x-3)^2 - 169y$

Factor each of the following completely over the real numbers by first using the grouping technique from Example 5 of Section 2.2 and then factoring the resulting difference of two squares.

48. $x^2 + 4x + 4 - y^2$
49. $a^2 - 2a + 1 - b^2$
50. $m^2 - 6m + 9 - n^2$
51. $p^2 + 10p + 25 - q^2$
52. $4x^2 - 4x + 1 - y^2$
53. $x^2 + 4x + 4 - y^2$

Factor each of the following.

54. $x^3 + 1$
55. $x^3 + 8$
56. $x^3 + y^3$
57. $8x^3 + 27$
58. $54x^3y + 16y^4$
59. $125x^3y^3 + 1$
60. $216x + x^4$
61. $x^6 + 64y^3$
62. $432 - 16x^3$
63. $x^3y^3z^3 + 343$
64. $x^{12} + 1$
65. $x^3 - 1$

66. $x^3 - 8$
67. $8 - x^3$
68. $54x^3 - 2$
69. $x^3y^3 - 343$
70. $16x^4y - 16xy$
71. $x^6 - 64y^3$
72. $216x^3 - 125$
73. $125 - 216x^3$
74. $x^9 - y^9$

2.4 Factoring Trinomials Reducible to Quadratic Form

A complicated-looking polynomial may just be a "quadratic in disguise."

Example 1. *Factor* $(x-1)^2+7(x-1)+12$.

Solution. Notice that the binomial $(x-1)$ is in two of the terms in this polynomial and is squared in one of them. If we substitute u for $(x-1)$, the result is much easier to work with:

$$\begin{array}{ll} \text{Original polynomial:} & (x-1)^2 + 7(x-1) + 12 \\ \text{Substitute } u \text{ for } (x-1): & \quad u^2 + \quad 7u \quad\quad + 12 \end{array}$$

Now, we'll factor:

$$u^2+7u+12=(u+3)(u+4).$$

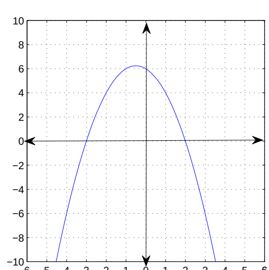
We're not finished, however, since the original polynomial was in $(x-1)$, not u. Now we need to reverse the substitution by replacing the u with $(x-1)$:

$$\begin{aligned}(u+3)(u+4) &= [(x-1)+3][(x-1)+4] \\ &= (x+2)(x+3).\end{aligned}$$

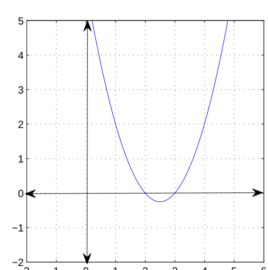
So $(x+2)(x+3)$ is the factored form. ■

Which of the following could be the graphs of the function, g, given by $g(x)=(x+2)(x-3)$?

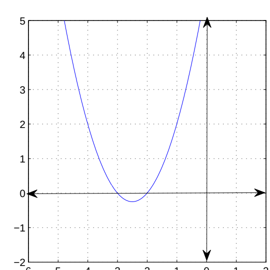
a.



b.



c.



Alternate Solution. Some people might take this same problem, notice that the left side is a "quadratic in disguise" and factor it directly, without using substitution.

$$\begin{aligned}(x-1)^2+7(x-1)+12 &= [(x-1)+3][(x-1)+4]\\ &= (x+2)(x+3).\end{aligned}$$

■

Example 2. *Factor* $2(x-5)^2+3(x-5)-5$.

Solution. Again, substitution will make this polynomial easier to factor. Since the binomial $(x-5)$ appears twice— once as a linear term and once squared— we'll substitute in u for it.

$$\begin{aligned}2(x-5)^2+3(x-5)-5 &= 2u^2+3u-5\\ &= (2u+5)(u-1)\\ &= [2(x-5)+5][(x-5)-1]\\ &= (2x-10+5)(x-6)\\ &= (2x-5)(x-6).\end{aligned}$$

As in the previous example, you may prefer to factor it directly without using substitution.

$$\begin{aligned}2(x-5)^2+3(x-5)-5 &= [2(x-5)+5][(x-5)-1]\\ &= (2x-10+5)(x-6)\\ &= (2x-5)(x-6).\end{aligned}$$

Don't forget to check your answers by multiplying. ■

Practice 1. *(Answers on the next page.) Factor each of the following.*

a. $(x+4)^2+3(x+4)-18$

b. $(x+2)^2-2(x+2)-15$

c. $3(x-1)^2+5(x-1)-2$

Example 3. *Factor* $3x^{2/3}-5x^{1/3}-2$.

Solution. Look at the power of the variable in the middle term: $x^{1/3}$. If we square it, we get the variable in the first term: $\left(x^{1/3}\right)^2=x^{2/3}$. This is how we know that this is reducible to a quadratic, or in quadratic form, a "quadratic in disguise."

To factor, we use substitution, letting $u = x^{1/3}$.

$$\begin{aligned} 3x^{2/3} - 5x^{1/3} - 2 &= 3\left(x^{1/3}\right)^2 - 5x^{1/3} - 2 \\ &= 3u^2 - 5u - 2 \\ &= (3u+1)(u-2) \\ &= \left(3x^{1/3}+1\right)\left(x^{1/3}-2\right). \end{aligned}$$

Don't forget to substitute $x^{1/3}$ back in for u after factoring.

Again, you may choose to factor directly:

$$\begin{aligned} 3x^{2/3} - 5x^{1/3} - 2 &= 3\left(x^{1/3}\right)^2 - 5x^{1/3} - 2 \\ &= \left(3x^{1/3}+1\right)\left(x^{1/3}-2\right). \end{aligned}$$

Use whichever method seems most natural to you. ■

Practice 2. *(Answers below.) Factor each of the following.*

a. $x^{2/3} + 3x^{1/3} - 18$

b. $m^{1/2} - 2m^{1/4} - 15$

c. $3p^{2/5} + 5p^{1/5} - 2$

ANSWERS TO SECTION 2.4 PRACTICE PROBLEMS

1. (a) $(x+10)(x+1)$
   (b) $(x-3)(x+5)$
   (c) $(3x-4)(x+1)$
2. (a) $\left(x^{1/3}+6\right)\left(x^{1/3}-3\right)$
   (b) $\left(m^{1/4}-5\right)\left(m^{1/4}+3\right)$
   (c) $\left(3p^{1/5}-1\right)\left(p^{1/5}+2\right)$

SECTION 2.4 EXERCISES:
(Answers are found on page 153.)

1. $x^4 + 6x^2 + 5$
2. $x^6 - 5x^3 + 6$
3. $2x^8 + 5x^4 - 3$
4. $12x^5 + 7x^3 - 12x$
5. $x^{10} - x^5 - 2$
6. $(x+1)^2 + 6(x+1) + 5$

7. $4(x-2)^2+(x-2)-18$
8. $6(2x-1)^2-7(2x-1)-24$
9. $8(x+3)^3-10(x+3)^2-12(x+3)$
10. $(x+2)^4-2(x+2)^2-3$
11. $2x^{1/2}-9x^{1/4}-18$
12. $10x^{2/3}-43x^{1/3}-9$
13. $12x^{4/5}+6x^{2/5}-36$
14. $24x-58x^{1/2}-42$
15. $2x^5+3x^{5/2}-5$
16. $2(x+1)^{1/2}-9(x+1)^{1/4}-18$
17. $(2x+1)^{2/3}-2(2x+1)^{1/3}-8$
18. $2(x-1)^{3/2}+7(x-1)^{3/4}-4$
19. $(x+1)^{3/2}+2(x+1)^{1/2}-24(x+1)^{-1/2}$
20. $(x+2)^{1/2}-8(x+2)^{1/4}-9$

Chapter 3

Quadratic Expressions and Functions

3.1 Introduction: The nature of quadratic functions

In your previous algebra courses, you have studied linear functions, written equations for straight lines, found slope and intercepts, and hopefully modeled a few real world scenarios using linear functions. If you are rusty on any of these skills or vocabulary, you might want to review a bit.

What you need to remember about a linear function for now is that it represents, or models, a constant rate of change. For example, if you earn an annual salary of $30,000 and are promised a $500 raise each year, your salary will be increasing at the same amount every single year.

A table of inputs (years) and outputs (salary) might look like this:

Year	Calculation	Salary
0	$30000 + 500(0)$	30000
1	$30000 + 500(1)$	30500
2	$30000 + 500(2)$	31000
3	$30000 + 500(3)$	31500
4	$30000 + 500(4)$	32000
5	.	32500
6	.	33000
7	.	33500
8		34000
9		34500
10		35000
.	.	.
t	$30000 + 500(t)$	.

Note that each successive value in the salary column is 500 more than the previous one. We could figure out the salary for any given year by adding 500× number of years to $30,000. In other words,

$$\text{Salary} = 30,000 + 500 \times \text{ number of years}$$

which follows the standard form of a straight line,

$$y = b + m \cdot x$$

where y is the output, b is the initial value, , is the rate of change or slope, and x is the input.

We are now going to investigate what happens if we take two linear functions and multiply them together. We will do this graphically, then symbolically. You will need to know how to write an equation for a line and know the meaning of *x-intercept* and *y-intercept*.

Please print out and complete Activity 1.

We are ready to investigate the rate of change of these new functions. You probably noticed that the result of multiplying two linear functions together is a function with a degree of two. A polynomial function with a degree of two is called a ***quadratic function***.

Please print out and complete Activity 2.

3.2 More About Quadratic Functions

Real-world functions, Dependent and Independent Variables

Scenario 1. *Suppose that you run a music store and the revenue, in thousands of dollars, from the sale of n thousand CDs is given by*

$$Revenue = 2n^2 + n.$$

If the cost, in thousands of dollars, of producing n thousand CDs is given by

$$Cost = n^2 - 2n + 10,$$

how many CDs do you need to sell in order to break even?

Scenario 2. *Suppose to celebrate the grand opening of your music store, you hire someone to set off a fireworks display from the 80-foot high rooftop of the store with an initial velocity of 64 feet per second. The height of the display t seconds after having been launched is given by*

$$Height = -16t^2 + 64t + 80.$$

When will the cardboard shell from the fireworks reach the ground?

Practice 1. *(Answers on page 103.) In both of the above situations, two quantities are changing. What are these "variables" for each scenario?*

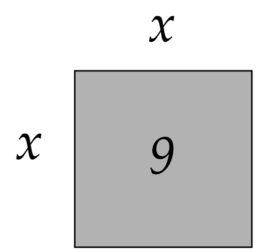
Usually when we have two variables like this, we can think of one as being dependent upon the other. We call this variable the ***dependent*** or ***output*** variable. The other is the ***independent*** or ***input*** variable.

Practice 2. *(Answers on page 103.)*

a. Identify the dependent and the independent variable in each of the above scenarios.

b. Think of three other scenarios that have inputs and outputs. Identify the dependent and the independent variable in each.

The variable in the equations in two scenarios about the music store has a highest power of 2. We call this type of equation a ***quadratic***. The word *quadratic* comes from the Latin word for *square*. As a matter of fact, we might think of solving a quadratic equation as finding the side of a square with a given area. For example, to solve $x^2 = 9$, we might ask, "What is the side of the square whose area is 9?"



Let's take a closer look at the Scenario 1 in the prep assignment. Notice how we wrote the expressions for revenue and cost:

$$Revenue = 2n^2 + n \qquad \text{and} \qquad Cost = n^2 - 2n + 10.$$

We write expression this way to show that the revenue (or cost) *depends upon* the number, n, of CDs sold.

Complete the table below, for the revenue and cost for different numbers of CDs made.

Number of CDs made (n)	Revenue	Cost
0		
1		
2		
3		
4		
5		
10		
50		
100		
1000		
10000		

The table above and equations are representations of the mathematical notion of function. We can think of a function as a process that takes the input (in this case the number of CDs) and transforms it into a unique output. (In this case, the function is revenue in dollars; the cost would represent a different function). We could write the revenue function this way:

$$Revenue(number\ of\ CDs) = 2 \cdot (number\ of\ CDs)^2 + number\ of\ CDs$$

where $Revenue$ is the name of the function and $number\ of\ CDs$ is the input. To shorten our writing a bit, we use the variable n to represent the number of CDs sold. Notice how much more efficient the equation becomes:

$$Revenue(n) = 2n^2 + n.$$

We'll call this the Revenue function.

Similarly we could call the equation for cost the cost function and write it the long way:

$$Cost(number\ of\ CDs) = (number\ of\ CDs)^2 - 2 \cdot number\ of\ CDs + 10$$

or the short way:

$$Cost(n) = n^2 - 2n + 10.$$

So, $Cost(20) = 20^2 - 2(20) + 10 = 370$. In words: the cost of making 20 thousand CDs is \$370 thousand dollars. We can write this information in couple shorthand ways:

$$C(20) = 370 \qquad \text{or} \qquad (20, 370).$$

They mean the same thing. Note that in the above scenario, we are assuming that the number of CDs made is the same as the number of CDs sold.

Suppose we wanted to answer the question, "How many CDs do we need to sell in order to break even?" At the break-even point, the costs are the same as the revenue, so we are neither making nor losing money. We therefore set the outputs of the two functions equal to each other:

$$Revenue(n) = Cost(n)$$
$$2n^2 + n = n^2 - 2n + 10$$

In doing so, we obtain an equation that we want to solve.

In the Scenario 2, we might set the height function equal to 0 and then solve for the variable t to find the time when the firework shells reach the ground:

$$Height(t) = -16t^2 + 63t + 80 = 0,$$

where *Height* is the name of the function and t is the input.

We can thus see a relationship between *functions* and *equations*. A function is a process that takes an input and gives an output. It can often be written as a formula showing a relationship between the independent variable (input) and the dependent variable (the output). When we either set two function outputs equal to each other, or set one function output equal to 0, we obtain an equation to solve.

We'll spend quite a bit of time later in this chapter reviewing and learning the procedures for solving such quadratic equations.

Practice 3. *(Answers on page 103.) Identitfy the input and output values in each of the following scenarios. Then represent the scenario with function notation.*

a. *The owner of the music store is debating how much floor space to allow for the classical music section. She's very artistic and wants to make it in the shape of the golden rectangle. (The length, l, needs to be 1.6 times the width, w.) The area, then, is $Area = 1.6w^2$.*

b. *She wants the hip hop section to be a rectangle with length five feet more than the width ($length = 5 + w$). The area, then, is $Area = w(w + 5)$.*

c. *The manufacturer of the CDs charges for shipping based partly on the cost of gas and he wants to know how fast his trucks should travel in order to achieve fuel efficiency. The gas mileage, M for their trucks depends upon the speed, s of travel and is given by $Mileage = -\frac{1}{28}s^2 + 3s - 31$, as long as the speed does not exceed 70 mph.*

d. *The effectiveness of advertising on television depends upon how many times a viewer watches it. The advertising agency you hired determined that the effectiveness, E, (measured on scale from 0 to 10) is given by*

$$Effectiveness = \frac{2}{3}n - \frac{1}{90}n^2.$$

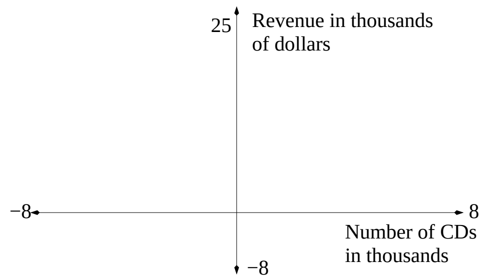
Practice 4. *(Answers on page 103.) Create a table of values for each of the scenarios in Practice 3. Then choose three pairs from the table and interpret in a full sentence.*

Practice 5. *(Answers on page 103.) Extend each of the scenarios in Practice 3 to one which requires solving a quadratic equation. Then write the equation. (You need not solve it.)*

Graphs of Quadratics

Another way of representing functions is a graph. We could put the independent variable on the horizontal axis (usually referred to as the x-axis) and the dependent variable on the vertical axis (usually referred to as the y-axis)

Use the table of values for the Revenue function above and plot the points on graph paper with axis labeled as follows.



Now on the same set of axes, plot values for the cost function.

The graph of a quadratic function $f(x) = ax^2 + bx + c$ is a ***parabola***.

The only functions whose graphs are called *parabolas* are quadratic functions. The graphs of other functions such as $y = x^4$ or $y = x^6$ are not parabolas, even though they are similar to parabolas.

Complete the following table of values for the revenue function.

Number of CDs (n)	Revenue
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

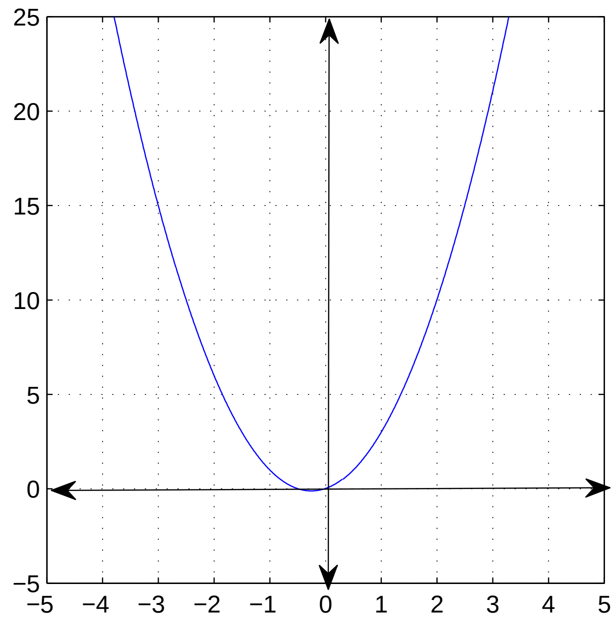
Can you substitute *any number* for the input?

Can the output be *any real number*?

Are the values of the revenue getting larger (increasing) as the number of CDs (the input) increases or are they getting smaller (decreasing)?

Does there appear to be a *maximum value* for the revenue? A *minimum value*?

Below is the graph of the revenue function.



Where on the graph is the max or minimum value of the revenue? Where is the revenue equal to 0?

Suppose we wanted to know when the revenue was negative, i.e., the business is going in the hole. We need to find the exact point where the revenue changes from positive to negative. In other words, we need to find where the revenue is zero. We could look at the table of values, but finding the exact place at which it is zero is difficult. We could set the function equal to zero and solve:

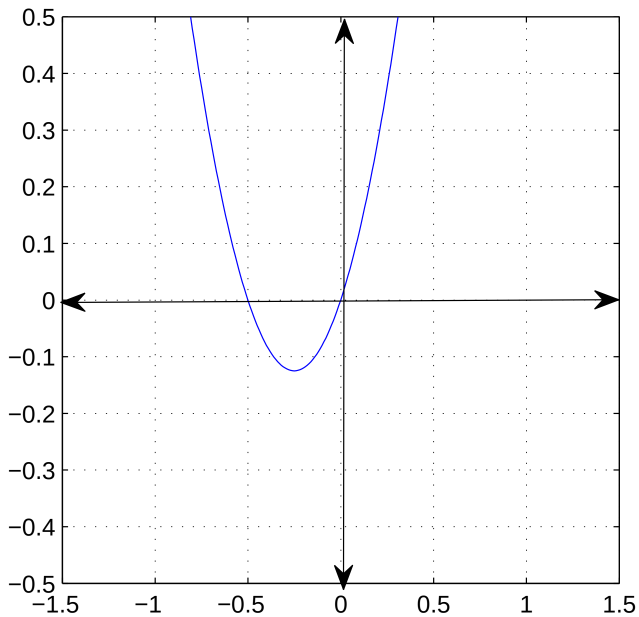
$$\begin{aligned} & 2n^2 + n = 0 \\ \text{Factor:} \qquad & n(2n+1) = 0 \end{aligned}$$

Set each factor equal to 0:

$$\begin{aligned} n = 0 \quad &\text{or} \quad 2n + 1 = 0 \\ n = 0 \quad &\text{or} \quad 2n = -1 \\ n = 0 \quad &\text{or} \quad n = -1/2 \end{aligned}$$

Thus we have two values of n for which the Revenue is zero: $n = 0$ or $n = 1/2$. This means that when we sell 0 CDs, we have 0 revenue, which makes sense. If we sell $-1/2$ CDs, we have 0 revenue. This value for n does not make sense in the context of the problem, so we can ignore it.

If we enlarged the part of the graph between -1.5 and 1.5 the graph looks like this:



Notice that the revenue is negative for n in the interval $(-1/2, 0)$ (that is, for $-1/2 < n < 0$). Again, the negative values of n don't make sense in this context, so we can ignore them for now.

Reading Graphs of Functions

Worked examples can be found at
www.math.kent.edu/ebooks/FUNMATHV/more2_1b.htm.

ANSWERS TO SECTION 3.2 PRACTICE PROBLEMS

1. *scenario 1:* revenue, cost, number of CDs (n)
   scenario 2: height and time (t)

2. (a) *scenario 1:* dependent: revenue, cost
   independent: number of CDs
   scenario 2: dependent: height
   independent: time

   (b) answers will vary

3. (a) dependent: Area, A
   independent: width, w

$A(w) = 1.6w^2$

(b) dependent: Area, A
independent: width, w
$A(w) = w(w+5)$

(c) dependent: Mileage, M
independent: speed, s
$M(s) = -\frac{1}{28}s^2 + 3s - 31$

(d) dependent: Effectiveness, E
independent: number of viewings, n
$E(n) = \frac{2}{3}n - \frac{1}{90}n^2$

4. answers will vary

5. answers will vary

SECTION 3.2 EXERCISES:

(Answers are found on page 153.)

Determine the following for each of the given functions.

a. $f(2)$
b. $f(-2)$
c. $f(1/2)$
d. $f(-1/2)$
e. $f(3x)$
f. $3f(x)$
g. $f(x-3)$
h. $f(x) - 3$

1. $f(x) = 3x - 2$
2. $f(x) = \frac{1}{4}x - \frac{1}{8}$
3. $f(x) = 2x^2 - 3x - 1$
4. $f(x) = 1 - x^2$
5. $f(x) = \frac{x^2 + 3x - 1}{x + 5}$
6. $f(x) = \sqrt{1 - x}$
7. $f(x) = |x - 4|$
8. The height of a ball is measured by the function $h(t) = -16t^2 + 112t - 64$, where $h(t)$ is measured in feet and t is time in seconds.
   (a) Evaluate $h(2)$ and explain what the dependent and independent variables represent.
   (b) What is the height of the ball at 3 seconds?
   (c) When is the ball at a height of 128 feet?
9. The height of a small rocket fired straight upward is measured by the function $h(t) = -16t^2 + 800t$, where $h(t)$ is measured in feet and t is time in seconds.
   (a) Evaluate h(7) and explain what the dependent and independent variables represent.
   (b) What is the height of the rocket after 6 seconds?
   (c) When is the rocket at a height of 6400 ft?

10. The height of a ball thrown straight upward is measured by the function $h(t) = -16t^2 + 40t$, where $h(t)$ is measured in feet and t is time in seconds.

    (a) Evaluate $h(1)$ and explain what the dependent and independent variables represent.

    (b) What is the height of the ball after 2.5 seconds?

    (c) When is the ball at a height of 48 ft?

    (d) When will the ball hit the ground?

11. A ball is thrown vertically upward from the top of a building 96 feet tall with and initial velocity of 80 feet per second. The height of the ball is measured by the function $h(t) = -16t^2 + 80t + 96$, where $h(t)$ is measured in feet and t is time in seconds.

    (a) Evaluate $h(3)$ and explain what the dependent and independent variables represent.

    (b) What is the height of the ball after 2.5 seconds?

    (c) When will the ball pass the top of the building on the way down?

    (d) When will the ball hit the ground?

12. A ball is dropped from a building 96 ft tall. The height of the ball after t seconds is measured by the function $h(t) = -16t^2 + 96$, where $h(t)$ is measured in feet and t is time in seconds.

    (a) Evaluate $h(0)$ and explain what the dependent and independent variables represent.

    (b) What is the height of the ball after 1.25 seconds?

    (c) How long will it take the ball to fall to 32 feet?

13. The height of a small rocket fired straight upward is measured by the function $h(t) = -16t^2 + 512t$, where $h(t)$ is measured in feet and t is time in seconds.

    (a) Evaluate $h(11)$ and explain what the dependent and independent variables represent.

    (b) What is the height of the rocket after 9 seconds?

    (c) When is the rocket at a height of 960 ft?

14. The area of a rectangle is measured by the function $A(x) = 60x - x^2$, where x is the length of the rectangle and is measured in yards.
    (a) Evaluate $A(2)$ and explain what the dependent and independent variables represent.
    (b) What is the area of the rectangle when its length is 6 yards?
    (c) What is the rectangle's length when its area is 800 square yards?
15. The area of a rectangle is measured by the function $A(l) = 70l - l^2$, where l is the length of the rectangle and is measured in feet.
    (a) Evaluate $A(5)$ and explain what the dependent and independent variables represent.
    (b) What is the area of the rectangle when its length is 4 feet?
    (c) What is the rectangle's length when its area is 1000 square feet?
16. The area of a rectangle is measured by the function $A(w) = 55w - w^2$, where w is the width of the rectangle and is measured in inches.
    (a) Evaluate $A(3)$ and explain what the dependent and independent variables represent.
    (b) What is the area of the rectangle when its width is 3 inches?
    (c) What is the rectangle's width when its area is 450 square inches?
17. The area of a rectangle is measured by the function $A(y) = -y^2 + 30y$, where y is the width of the rectangle and is measured in yards.
    (a) Evaluate $A(13)$ and explain what the dependent and independent variables represent.
    (b) What is the area of the rectangle when its width is 35 yards?
    (c) What is the rectangle's width when its area is 200 square yards?
18. The area of a rectangle is measured by the function $A(x) = -x^2 + 8x$, where x is the length of the rectangle and is measured in feet.
    (a) Evaluate 7) and explain what the dependent and independent variables represent.
    (b) What is the area of the rectangle when its length is 21 feet?
    (c) What is the rectangle's length when its area is 12 square feet?

19. The area of a rectangle is measured by the function $A(x) = -x^2 + 60x$, where x is the length of the rectangle and is measured in yards.

    (a) Evaluate $A(5)$ and explain what the dependent and independent variables represent.

    (b) What is the area of the rectangle when its length is 6 feet?

    (c) What is the rectangle's length when its area is 900 square yards?

20. The cost of producing x units of a product is $C(x) = 4x + 2$. The revenue earned by producing and selling x units of the product is $R(x) = -x^2 + 20x$.

    (a) Determine a function that measures the profit earned for producing and selling x units.

    (b) What will be the profit earned if 5 units are produced and sold?

    (c) How many units must be produced and sold to earn a profit of \$58?

21. The cost of producing x units of a product is $C(x) = 35x + 650$. The revenue earned by producing and selling x units of the product is $R(x) = -x^2 + 135x$.

    (a) Determine a function that measures the profit earned for producing and selling x units.

    (b) What will be the profit earned if 13 units are produced and sold?

22. A car dealer finds that his costs can be represented (or modeled) by the function $C(x) = 14400x + 6000$ and the revenue he makes can be represented (or modeled) by the function $R(x) = -600x^2 + 26400x + 85000$ where x is the number of cars sold per day.

    (a) Determine a function that measures the profit earned for selling x cars.

    (b) What will be the profit earned if 6 cars are sold?

    (c) How many cars must be sold to earn a profit of \$139,000?

23. A store that sells bicycles has found that its costs can be represented (or modeled) by the function $C(x) = 4000 + 400x$ and its revenue can be represented (or modeled) by the function $R(x) = 8000 + 600x - 20x^2$.

(a) Determine a function that measures the profit earned for selling x bicycles.

(b) What will be the profit earned if 12 bicycles are sold?

(c) How many bicycles must be sold to earn a profit of $4,500?

24. A computer dealer has found that her revenue for selling personal computers can be modeled by the function $R(x) = 4800 + 560x - 40x^2$. Her costs can be modeled by the function $C(x) = 200x + 1400$.

    (a) Determine a function that measures the profit earned for selling x computers.

    (b) What will be the profit earned if 30 computers are sold?

    (c) How many computers must be sold to earn a profit of $2520?

25. A store that sells telephone answering machines has determined its costs to be $C(x) = 100x + 700$ and its revenue to be $R(x) = 2400 + 280x - 20x^2$.

    (a) Determine a function that measures the profit earned for selling x answering machines.

    (b) What will be the profit earned if 25 answering machines are sold?

    (c) How many answering machines must be sold to earn a profit of $1,260?

26. A company's cost function is $C(x) = 15x + 45$ and its revenue function is $R(x) = -x^2 + 11x + 175$, where x is in hundreds of units and $C(x)$ and $R(x)$ are in thousands of dollars.

    (a) Determine a function that measures the profit earned.

    (b) What will be the profit earned if 500 units are produced and sold? (Remember, x is in hundreds of units)

    (c) How many units must be produced and sold to earn a profit of $34,000? (Remember $C(x)$ and $R(x)$ are in thousands of dollars)

27. A company's cost function is $C(x) = 3x + 20$ and its revenue function is $R(x) = -2x^2 + 5x + 85$, where x is in hundreds of units and $C(x)$ and $R(x)$ are in thousands of dollars.

    (a) Determine a function that measures the profit earned.

(b) What will be the profit earned if 500 units are produced and sold? (Remember, x is in hundreds of units)

(c) How many units must be produced and sold to earn a profit of \$5,000? (Remember $C(x)$ and $R(x)$ are in thousands of dollars)

28. A company's cost function is $C(x) = 10x + 100$ and its revenue function is $R(x) = -x^2 + 30x + 500$, where x is in hundreds of units and $C(x)$ and $R(x)$ are in thousands of dollars.

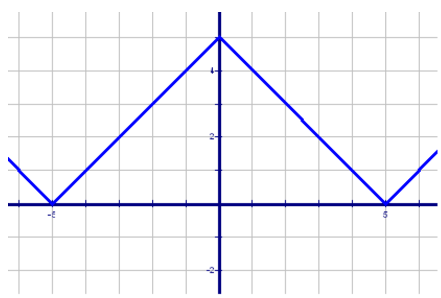
    (a) Determine a function that measures the profit earned.

    (b) What will be the profit earned if 500 units are produced and sold? (Remember, x is in hundreds of units)

    (c) How many units must be produced and sold to earn a profit of \$500,000? (Remember $C(x)$ and $R(x)$ are in thousands of dollars)

Worked examples like the following exercises can be found at www.math.kent.edu/ebooks/FUNMATHV/more2_1b.htm.

29. Answer each of the following questions using the given graph.



(a) $f(-5) =$

(b) $f(-2) =$

(c) $f(0) =$

(d) $f(1) =$

(e) $f(5) =$

(f) x-intercept(s):

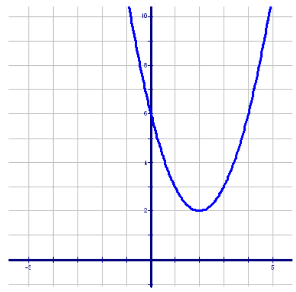
(g) y-intercept(s):

(h) Find an integer $x \geq 0$ so that $f(x) = 3$.

(i) Find an integer $x \leq 0$ so that $f(x) = 2$.

(j) How many times does the graph of $y = 1$ intersect the graph of $y = f(x)$?

30. Answer each of the following questions using the given graph.



(a) $f(0) =$

(b) $f(1) =$

(c) $f(2) =$

(d) How many times does the graph of $y = 6$ intersect the graph of $y = f(x)$?

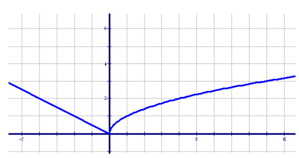
(e) x-intercept(s):

(f) y-intercept(s):

(g) Determine the minimum or maximum value of $y = f(x)$.

(h) Find an integer $x > 2$ so that $f(x) = 3$.

31. Answer each of the following questions using the given graph.



(a) $f(-4) =$

(b) $f(-2) =$

(c) $f(1) =$

(d) $f(9) =$

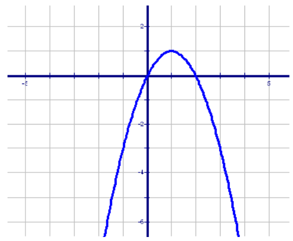
(e) x-intercept(s):

(f) y-intercept(s):

(g) Find an integer $x \geq 0$ so that $f(x) = 2$.

(h) How many times does the graph of $y = 1$ intersect the graph of $y = f(x)$?

32. Answer each of the following questions using the given graph.



(a) $f(0) =$

(b) $f(1) =$

(c) $f(2) =$

(d) How many times does the graph of $y = -6$ intersect the graph of $y = f(x)$?

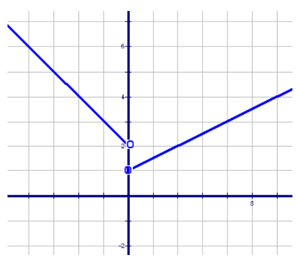
(e) x-intercept(s):

(f) y-intercept(s):

(g) Determine the minimum or maximum value of $y = f(x)$.

(h) Find an integer $x > 0$ so that $f(x) = -3$.

33. Answer each of the following questions using the given graph.



(a) $f(-4) =$

(b) $f(-2) =$

(c) $f(0) =$

(d) $f(2) =$

(e) $f(4) =$

(f) x-intercept(s):

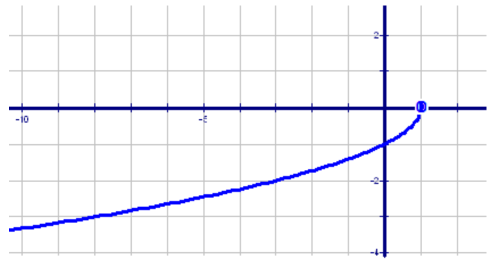
(g) y-intercept(s):

(h) Find an integer $x \geq 0$ so that $f(x) = 6$.

(i) Determine the minimum or maximum value of $y = f(x)$.

(j) How many times does the graph of $y = 2$ intersect the graph of $y = f(x)$?

34. Answer each of the following questions using the given graph.



(a) $f(-3) =$

(b) $f(0) =$

(c) $f(1) =$

(d) How many times does the graph of $y = 1$ intersect the graph of $y = f(x)$?

(e) x-intercept(s):

(f) y-intercept(s):

(g) Find an integer x so that $f(x) = -3$.

3.3 Completing the Square

In this course, we assume you know how to solve quadratic equations by factoring.

Suppose you want to solve a quadratic like $x^2 + 2x - 5 = 0$. Try to factor it. Can you? It does *not* factor over the rationals. We need to come up with another method for solving such an equation. There are actually a few different ways of doing this. We are now going to look at one method, a method called completing the square.

Example 1. *Solve* $x^2 + 2x - 5 = 0$ *by completing the square.*

Solution. We can solve this problem geometrically or algebraically. Here's the algebraic solution.

$$\begin{aligned}
\text{Move the constant term to the right side:} \quad & x^2 + 2x = 5 \\
\text{Find half the coefficient of the middle term:} \quad & \frac{1}{2} \cdot 2 = 1 \\
\text{Square it:} \quad & 1^2 = 1 \\
\text{Add this result to both sides of the equation:} \quad & x^2 + 2x + 1 = 5 + 1 \\
\text{Factor the left hand side:} \quad & (x+1)^2 = 6 \\
\text{Take the square root of both sides:} \quad & (x+1) = \pm\sqrt{6} \\
\text{Solve for x:} \quad & x = -1 \pm \sqrt{6}.
\end{aligned}$$

Check. We can check these answers by substituting them back into the original equation.

$$\begin{aligned}
\left(-1+\sqrt{6}\right)^2 + 2\left(-1+\sqrt{6}\right) - 5 &= 1 - 2\sqrt{6} + 6 + (-2) + 2\sqrt{6} - 5 \\
&= 7 - 2 - 5 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left(-1-\sqrt{6}\right)^2 + 2\left(-1-\sqrt{6}\right) - 5 &= 1 + 2\sqrt{6} + 6 + (-2) - 2\sqrt{6} - 5 \\
&= 7 - 2 - 5 \\
&= 0
\end{aligned}$$

Note that we checked each of the two answers separately. ■

Example 2. *Solve $x^2 + 10x - 3 = 0$ by completing the square.*

Solution.

$$\begin{aligned}
\text{Move the constant term to the right side:} \quad & x^2 + 10x = 3 \\
\text{Find half the coefficient of the middle term:} \quad & \frac{1}{2} \cdot 10 = 5 \\
\text{Square it:} \quad & 5^2 = 25 \\
\text{Add this result to both sides of the equation:} \quad & x^2 + 10x + 25 = 3 + 25 \\
\text{Factor the left hand side:} \quad & (x+5)^2 = 28 \\
\text{Take the square root of both sides:} \quad & (x+5) = \pm\sqrt{28} \\
& x + 5 = \pm 2\sqrt{7} \\
\text{Solve for x:} \quad & x = -5 \pm 2\sqrt{7}.
\end{aligned}$$

Check. Check both answers by substituting them back into the original equation.

$$\begin{aligned}
\left(-5+2\sqrt{7}\right)^2 + 10\left(-5+2\sqrt{7}\right) - 3 &= 25 - 20\sqrt{7} + 28 + (-50) + 20\sqrt{7} - 3 \\
&= 53 + (-50) - 3 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left(-5-2\sqrt{7}\right)^2 + 10\left(-5-2\sqrt{7}\right) - 3 &= 25 + 20\sqrt{7} + 28 + (-50) - 20\sqrt{7} - 3 \\
&= 53 + (-50) - 3 \\
&= 0
\end{aligned}$$

■

Practice 1. *(Answers on page 117.) Solve each quadratic equation by completing the square.*

a. $x^2 - 8x = 3$ *b.* $x^2 + 4x = -11$ *c.* $x^2 - 12x + 7 = 0$

Example 3. *Solve $x^2 - 3x - 5 = 0$ by completing the square.*

Solution.

Move the constant term to the right side:	$x^2 - 3x = 5$
Find half the coefficient of the middle term:	$\frac{1}{2} \cdot (-3) = -\frac{3}{2}$
Square it:	$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$
Add this result to both sides of the equation:	$x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4}$
Factor the left hand side:	$\left(x - \frac{3}{2}\right)^2 = \frac{20}{4} + \frac{9}{4}$
	$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$
Take the square root of both sides:	$\left(x - \frac{3}{2}\right) = \pm\sqrt{\frac{29}{4}}$
	$x - \frac{3}{2} = \pm\frac{\sqrt{29}}{2}$
Solve for x:	$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$
	$x = \frac{3 \pm \sqrt{29}}{2}.$

The checks are left to you. ■

Practice 2. *(Answers on page 117.) Solve each quadratic equation by completing the square.*

a. $x^2 + 5x - 9 = 0$ *b.* $x^2 - 3x + 6 = 0$

Example 4. *Solve* $2x^2 - 12x + 3 = 0$ *by completing the square.*

Solution. Note that this equation has a leading coefficient that is *not* equal to 1. Before we complete the square, we need to divide both sides by the leading coefficient.

$$
\begin{aligned}
\text{Divide both sides by 2:} \quad & \frac{1}{2}\left(2x^2 - 12x + 3\right) = \frac{1}{2}\cdot 0 \\
& \frac{2}{2}x^2 - \frac{12}{2}x + \frac{3}{2} = 0 \\
& x^2 - 6x + \frac{3}{2} = 0 \\
\text{Constant term to right side:} \quad & x^2 - 6x = -\frac{3}{2} \\
\text{Halve the middle coefficient:} \quad & \frac{1}{2}\cdot(-6) = -3 \\
\text{Square it:} \quad & (-3)^2 = 9 \\
\text{Add to both sides:} \quad & x^2 - 6x + 9 = -\frac{3}{2} + 9 \\
& x^2 - 6x + 9 = -\frac{3}{2} + \frac{18}{2} \\
& x^2 - 6x + 9 = \frac{15}{2} \\
\text{Factor the left-hand side:} \quad & (x-3)^2 = \frac{15}{2} \\
\text{Take square roots:} \quad & (x-3) = \pm\sqrt{\frac{15}{2}} \\
& x - 3 = \pm\frac{\sqrt{15}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} \\
& x - 3 = \pm\frac{\sqrt{30}}{2} \\
\text{Solve for x:} \quad & x = 3 \pm \frac{\sqrt{30}}{2} \\
& x = \frac{6 \pm \sqrt{30}}{2}.
\end{aligned}
$$

The checks are left to you. ■

Practice 3. *(Answers on the facing page.) Solve each quadratic equation by completing the square.*

a. $4x^2 + 40x - 3 = 0$ *b.* $3x^2 - 6x + 2 = 0$ *c.* $2x^2 + 16x - 5 = 0$

ANSWERS TO SECTION 3.3 PRACTICE PROBLEMS

1. (a) $x = 4 \pm \sqrt{19}$
   (b) no real solutions
   (c) $x = 6 \pm \sqrt{29}$
2. (a) $x = \frac{-5 \pm \sqrt{61}}{2}$
   (b) no real solutions
3. (a) $x = -5 \pm \frac{\sqrt{103}}{2}$
   (b) $x = 1 \pm \frac{1}{\sqrt{3}}$
   (c) $x = -4 \pm \sqrt{\frac{37}{2}}$

SECTION 3.3 EXERCISES:

(Answers are found on page 155.)

Solve each of the following by completing the square.

1. $x^2 + 14x + 7 = 0$
2. $m^2 + 6m - 5 = 0$
3. $p^2 + 4p - 6 = 0$
4. $x^2 + 2x - 2 = 0$
5. $x^2 - 9x + 3 = 0$
6. $y^2 + 5y + 1 = 0$
7. $y^2 - 22y + 9 = 0$
8. $w^2 - 18w = 7$
9. $3x^2 - 15x = 6$
10. $5y^2 - 10y = 10$
11. $4x^2 + 32x - 8 = 0$
12. $2m^2 + 8m + 1 = 0$
13. $-3x^2 + 6x - 3 = 0$
14. $-2z^2 - 20z + 4 = 0$
15. $-x^2 - 14x + 5 = 0$
16. $-4m^2 + 16m + 2 = 0$
17. $x^2 - x - 4 = 0$
18. $x^2 - 3x - 1 = 0$
19. $\frac{1}{4}y^2 + \frac{1}{2}y - 3 = 0$
20. $\frac{1}{6}x^2 - \frac{2}{3}x + 1 = 0$
21. $\frac{2}{5}m^2 + \frac{6}{5}m - \frac{1}{10} = 0$
22. $\frac{3}{7}a^2 + \frac{1}{7}a + 3 = 0$
23. $x^2 + 9x + 7 = 0$
24. $w^2 + 5w - 3 = 0$
25. $2z^2 - 6z - 11 = 0$
26. $8x^2 + 16x - 32 = 0$
27. $p^2 - 2p - 12 = 0$
28. $\frac{1}{25}x^2 + \frac{1}{5}x - 1 = 0$
29. $n^2 + 11n - 12 = 0$
30. $3m^2 + 11m + 5 = 0$
31. $y^2 - 4y - 4 = 0$
32. $-x^2 - 5x + 7 = 0$

3.4 The Quadratic Formula

The Quadratic Formula

If we generalize the completing the square procedure that we worked on in section 3.3, the result is a formula which we can use to solve *any* quadratic equation.

Example 1. *Solve $ax^2 + bx + c = 0$ by completing the square.*

Solution.

Divide both sides by the leading coefficient:

$$\begin{aligned} \frac{1}{a}\left(ax^2 + bx + c\right) &= \frac{1}{a} \cdot 0 \\ \frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \end{aligned}$$

Take the constant term to right side:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Halve the coefficient of the middle term:

$$\frac{1}{2} \cdot \frac{b}{a} = \frac{b}{2a}$$

Square it:

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Add the result to both sides:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Notice that now the left-hand side is a perfect square and can be factored:

$$\begin{aligned}
x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2}
\end{aligned}$$

To add the fractions on the right-hand side, we note that the LCD is $4a^2$.

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac + b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}
\end{aligned}$$

Take the square root of each side:

$$\begin{aligned}
\left(x + \frac{b}{2a}\right) &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Solve for x:

$$\begin{aligned}
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

We call this last equation the ***quadratic formula***. We can use it to solve *any* quadratic equation, even those we can factor. Note that we need to get the right side equal to zero before we identify a, b, and c. ■

We'll now solve the same equations we did by completing the square in the previous section, except now we'll use the formula.

Example 2. *Solve* $x^2 + 2x - 5 = 0$.

Solution. Since the right-hand side equals zero, we can identify a, b, and c:

$$a = 1, \qquad b = 2, \qquad \text{and} \qquad c = -5.$$

We now apply the quadratic formula and simplify.

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1} \\ &= \frac{-2 \pm \sqrt{4 + 4 \cdot 5}}{2} \\ &= \frac{-2 \pm \sqrt{4(1 + 5)}}{2} \\ &= \frac{-2 \pm \sqrt{4 \cdot 6}}{2} \\ &= \frac{-2 \pm 2\sqrt{6}}{2} \\ &= \frac{2\left(-1 \pm \sqrt{6}\right)}{2} \\ &= -1 \pm \sqrt{6}. \end{aligned}$$

Note that this is the same answer we obtained when we solved this same equation by completing the square in Section 3.3. ■

Example 3. *Solve* $x^2 + 10x - 3 = 0$.

Solution. Since the right-hand side equals zero, we can identify a, b, and c:

$$a = 1, \qquad b = 10, \qquad \text{and} \qquad c = -3.$$

We now apply the quadratic formula and simplify.

$$\begin{aligned} x &= \frac{-10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} \\ &= \frac{-10 \pm \sqrt{100 + 4 \cdot 3}}{2} \\ &= \frac{-10 \pm \sqrt{4(25+3)}}{2} \\ &= \frac{-10 \pm 2\sqrt{28}}{2} \\ &= \frac{-10 \pm 2\sqrt{4 \cdot 7}}{2} \\ &= \frac{-10 \pm 2 \cdot 2\sqrt{7}}{2} \\ &= \frac{2\left(-5 \pm 2\sqrt{7}\right)}{2} \\ &= -5 \pm 2\sqrt{7}. \end{aligned}$$

Compare this to the answer we obtained in Section 3.3. ■

Example 4. *Solve* $x^2 - 3x - 5 = 0$*.*

Solution. Since the right-hand side equals zero, we can identify a, b, and c:

$$a = 1, \qquad b = -3, \qquad \text{and} \qquad c = -5.$$

We now apply the quadratic formula and simplify.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1} \\ &= \frac{-3 \pm \sqrt{9 - (-20)}}{2} \\ &= \frac{-3 \pm \sqrt{29}}{2}. \end{aligned}$$

Compare this to the answer we obtained in Section 3.3. ■

Example 5. *Solve* $2x^2 - 12x + 3 = 0$*.*

Solution. Since the right-hand side equals zero, we can identify a, b, and c:

$$a = 2, \qquad b = -12, \qquad \text{and} \qquad c = 3.$$

We now apply the quadratic formula and simplify.

$$\begin{aligned} x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} \\ &= \frac{12 \pm \sqrt{4^2 \cdot 3^2 - 4 \cdot 6}}{4} \\ &= \frac{12 \pm \sqrt{4(4 \cdot 9 - 6)}}{4} \\ &= \frac{12 \pm \sqrt{4(36 - 6)}}{4} \\ &= \frac{12 \pm 2\sqrt{30}}{4} \\ &= \frac{2(6 \pm \sqrt{30})}{4} \\ &= \frac{6 \pm \sqrt{30}}{2}. \end{aligned}$$

Compare this to the answer we obtained in Section 3.3. ■

The Discriminant

Example 6. *Solve each of the following quadratic equations using the quadratic formula and make note of* how many *solutions you obtain. Then graph each of the quadratic functions given by the expressions on the left sides of the equations.*

a. $x^2 - 7x + 12 = 0$ *b.* $4x^2 - 12x + 9 = 0$ *c.* $x^2 + x + 1 = 0$

Solution.

a. Since the right-hand side equals zero, we can identify a, b, and c:

$$a = 1, \qquad b = -7, \qquad \text{and} \qquad c = 12.$$

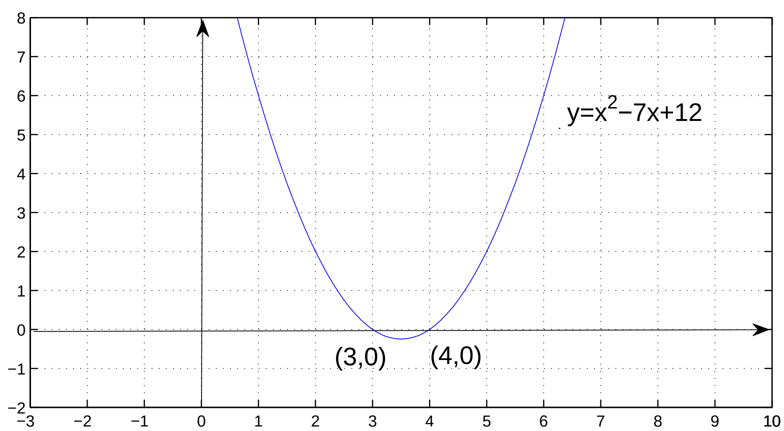
We now apply the quadratic formula and simplify.

$$
\begin{aligned}
x &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 12}}{2 \cdot 1} \\
&= \frac{7 \pm \sqrt{49 - 48}}{2} \\
&= \frac{7 \pm \sqrt{1}}{2} \\
&= \frac{7 \pm 1}{2}
\end{aligned}
$$

We split the $\pm$ sign into its separate parts:

$$
\begin{aligned}
x &= \frac{7+1}{2} &\quad \text{or} \quad && x &= \frac{7-1}{2} \\
x &= \frac{8}{2} &\quad \text{or} \quad && x &= \frac{6}{2} \\
x &= 4 &\quad \text{or} \quad && x &= 3.
\end{aligned}
$$

This particular equation has 2 distinct solutions. These solutions are the x-intercepts of the graph of the function given by the expression on the left side, as illustrated below.



b. Since the right-hand side equals zero, we can identify a, b, and c:

$$a = 4, \qquad b = -12, \qquad \text{and} \qquad c = 9.$$

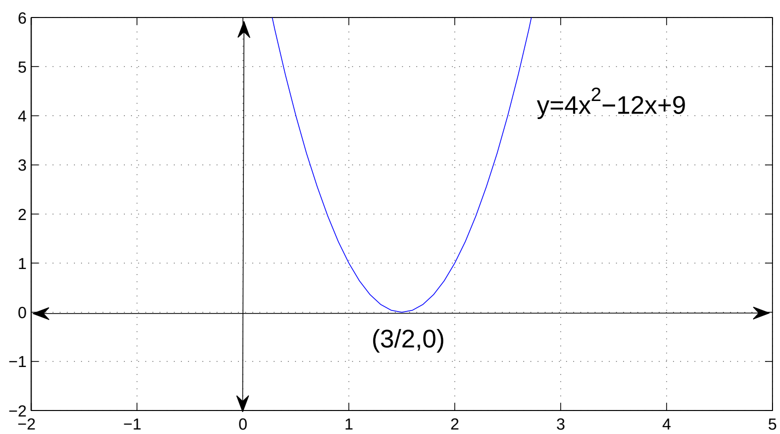
We now apply the quadratic formula and simplify.

$$\begin{aligned} x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} \\ &= \frac{12 \pm \sqrt{144 - 144}}{8} \\ &= \frac{12 \pm \sqrt{0}}{8} \\ &= \frac{12 \pm 0}{8} \end{aligned}$$

We split the $\pm$ sign into its separate parts:

$$\begin{aligned} x &= \frac{12+0}{8} \qquad \text{or} \qquad x = \frac{12-0}{8} \\ x &= \frac{3}{2} \qquad\quad \text{or} \qquad x = \frac{3}{2}. \end{aligned}$$

Since adding zero or subtracting zero gives the same result, we have one distinct solution. Note that the parabola just sits on the $x$axis at $x = \frac{3}{2}$.



c. We identify a, b, and c:

$$a = 1, \qquad b = 1, \qquad \text{and} \qquad c = 1.$$

We now apply the quadratic formula and simplify.

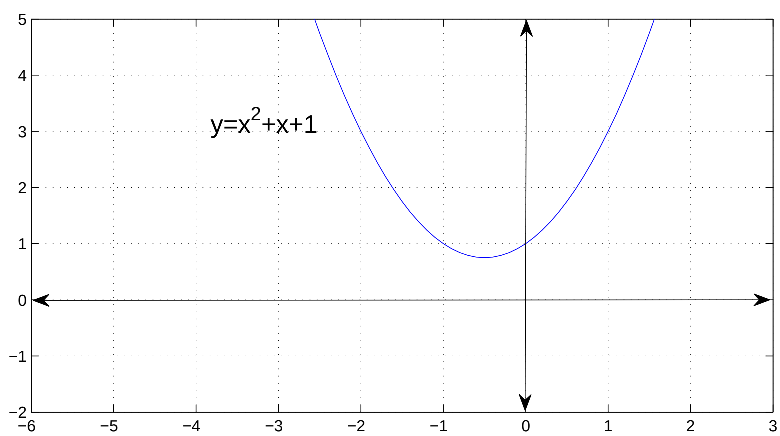
$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 4}}{2} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \end{aligned}$$

We split the $\pm$ sign into its separate parts:

$$x = \frac{-1 + \sqrt{-3}}{2} \qquad \text{or} \qquad x = \frac{-1 - \sqrt{-3}}{2},$$

Since these solutions have *negatives under the radical sign*, these solutions are *not real*. We say there are *no real solutions* to this equation. Notice that the parabola representing the quadratic expression on the left side of the equation has no x-intercepts.

■



Practice 1. *Discuss with a colleague: Which part of the quadratic formula determines how many solutions there are? How can you determine the number of solutions without solving the equation?*

The radicand of the quadratic formula tells us how many solutions there are to the equation. If

$$\begin{aligned} b^2 - 4ac > 0, &\qquad \text{then there are two real solutions;} \\ b^2 - 4ac = 0, &\qquad \text{then there is one real solutions;} \\ b^2 - 4ac < 0, &\qquad \text{then there are no real solutions.} \end{aligned}$$

The expression $b^2 - 4ac$ is called the ***discriminant*** of the quadratic equation $ax^2 + bx + c = 0$.

Example 7. *Use the discriminant to determine how many solutions there are to each of the following quadratic equations.*

a. $x^2 - 7x + 2 = 0$ *b.* $2x^2 - 3x + 5 = 0$ *c.* $9x^2 - 24x + 16 = 0$

Solution.

a. For $x^2 - 7x + 2 = 0$,

$$a = 1, \qquad b = -7, \qquad \text{and} \qquad c = 2.$$

So the discriminant is

$$\begin{aligned} b^2 - 4ac &= (-7)^2 - 4(1)(2) \\ &= 49 - 8 \\ &= 41 \\ &> 0. \end{aligned}$$

Therefore, there are two real solutions.

b. For $2x^2 - 3x + 5 = 0$,

$$a = 2, \qquad b = -3, \qquad \text{and} \qquad c = 5.$$

So the discriminant is

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4(2)(5) \\ &= 9 - 40 \\ &= -31 \\ &< 0. \end{aligned}$$

Therefore, there are no real solutions.

c. For $9x^2 - 24x + 16 = 0$,

$$a = 9, \qquad b = -24, \qquad \text{and} \qquad c = 16.$$

So the discriminant is

$$\begin{aligned} b^2 - 4ac &= (-24)^2 - 4(9)(16) \\ &= 576 - 576 \\ &= 0. \end{aligned}$$

Therefore, there is one real solutions.

The graphs are left for you. ■

SECTION 3.4 EXERCISES:
(Answers are found on page 155.)

Solve each of the following using the quadratic formula. Be sure to simplify your answers.

1. $x^2 + 14x + 7 = 0$
2. $m^2 + 6m - 5 = 0$
3. $p^2 + 4p - 6 = 0$
4. $x^2 + 2x - 2 = 0$
5. $x^2 - 12x = 3$
6. $y^2 + 8y = -1$
7. $y^2 = 22y - 9$
8. $w^2 = 18w + 7$
9. $3x^2 - 18x = 6$
10. $5y^2 - 10y = 10$
11. $4x^2 + 32x - 8 = 0$
12. $2m^2 + 8m + 1 = 0$
13. $-3x^2 + 6x = 3$
14. $-2z^2 - 20z = -4$
15. $-x^2 - 14x + 5 = 0$
16. $-4m^2 + 16m + 2 = 0$
17. $x^2 = x + 4$
18. $x^2 = 3x - 1$
19. $\frac{1}{4}y^2 + \frac{1}{2}y - 3 = 0$
20. $\frac{1}{6}x^2 - \frac{2}{3}x + 1 = 0$
21. $\frac{2}{5}m^2 + \frac{4}{5}m = \frac{1}{10}$

22. $\frac{3}{7}a^2 + \frac{1}{7}a = -3$
23. $x^2 + 9x + 7 = 0$
24. $\frac{1}{15}w^2 + \frac{1}{3}w = \frac{1}{5}$
25. $z^2 - 6z - 11 = 0$
26. $8x^2 = -16x + 32$
27. $2p^2 = 4p + 24$
28. $\frac{1}{25}x^2 + \frac{1}{5}x = 1$
29. $x^2 + 4x - 32 = 0$
30. $3m^2 + 11m = -5$
31. $9x^2 = 12x + 14$
32. $-x^2 - 5x + 7 = 0$

Use the discriminant to determine the number of real solutions to each of the following quadratic equations.

33. $x^2 - 6x + 1 = 0$
34. $m^2 + 3m - 2 = 0$
35. $y^2 - 6y = -9$
36. $x^2 + 5x = 1$
37. $-3w^2 + 6w - 3 = 0$
38. $5y^2 - 2y + 3 = 0$
39. $8z^2 = -5z - 3$
40. $-4p^2 = 12p + 9$
41. $2x^2 - \sqrt{5}\,x + 1 = 0$
42. $\sqrt{6}\,y^2 + 2y - \sqrt{\frac{3}{2}} = 0$

43. $x^2 + bx - c = 0$, where $c > 0$
44. $x^2 - bx + c = 0$, where $c > 0$ and $b > 2\sqrt{c}$

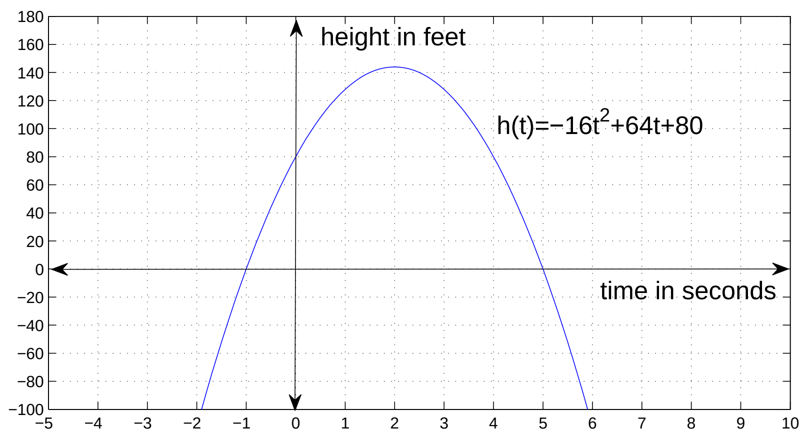
3.5 Maximum and Minimum Values: Problem Solving With Quadratic Functions and Equations

Example 1. *The height $h(t)$ of a fireworks display t seconds after having been launched is given by*

$$h(t) = -16t^2 + 64t + 80.$$

Suppose we wanted to know the maximum height reached by the fireworks. Where on the graph of the function would the maximum height be represented?

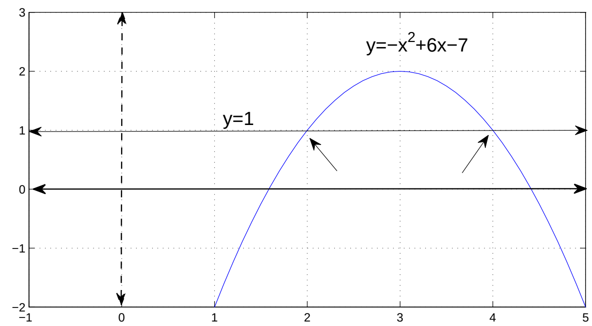
Here is the graph $y = h(t)$.



The highest point on the graph of a parabola is called the ***vertex***. In this section, we explore the vertex of quadratic functions and develop an algebraic method for finding it.

Activity 1.

Consider the graphs of $f(x) = -x^2 + 6x - 7$ and the horizontal line $y = 1$.

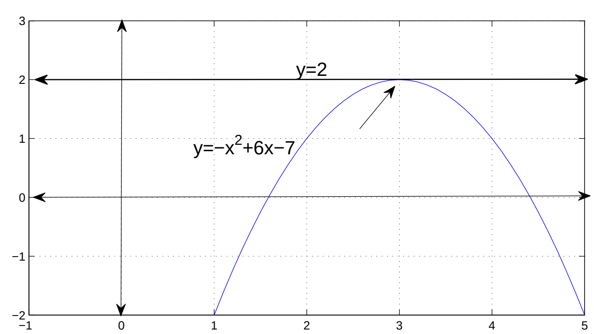


As the graphs intersect at two distinct points, we know that there must be two distinct values for x such that $-x^2 + 6x - 7 = 1$. Use the quadratic formula to solve this equation and find these values for x. (We know that the equation will factor, but please use the formula anyway. Youll see something really neat in just a bit.)

Activity 2.

Now consider the case where $f(x) = -x^2 + 6x - 7 = 2$. Use the quadratic formula to solve this equation, too.

You should have found that there is only one distinct solution. This means that the graphs of $f(x) = -x^2 + 6x - 7$ and the horizontal line $y = 2$ intersect at only one distinct point and that the line $y = 2$ is ***tangent*** to the graph of $f(x) = -x^2 + 6x - 7$ at the point $(3, 2)$ as is shown in the following graph.

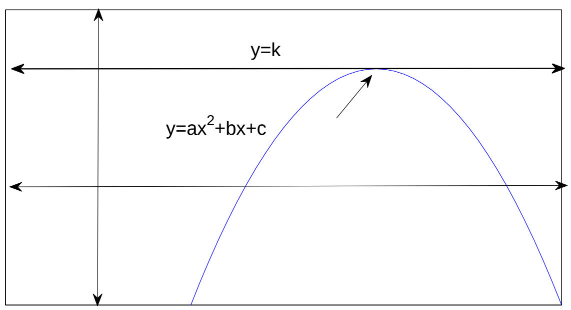


What is the maximum value for the output of this function?

The maximum value of$f(x)$ is 2 and this maximum value occurs when $x = 3$. Note that the term "maximum value of" refers to the *output* of the function. The point is called the ***vertex***.

Now let us consider the general case. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. We will assume that the graph is a parabola that "opens down." We wish to find the maximum value of $f(x)$. Please note that the following will work for quadratics whose graphs are parabolas that "open up" and can be used to find minimum values.

Consider the graphs of $f(x) = ax^2 + bx + c$ (again where $a, b, c \in \mathbb{R}$ and $a \neq 0$) and the horizontal line $y = k$ where $k \in \mathbb{R}$. The line $y = k$ is tangent to $f(x)$ at the vertex of the parabola. Note that there can only be one such horizontal tangent line to a parabola intersecting at exactly one point.



Now, there can only be *one* solution to the equation $ax^2 + bx + c = k$, as the graphs intersect at only one point. Using the quadratic formula, we have:

$$ax^2 + bx + c = k$$
$$ax^2 + bx + (c - k) = 0.$$

Note that $(c - k) \in \mathbb{R}$, as $c \in \mathbb{R}$ and $k \in \mathbb{R}$. Now,

$$x = \frac{-b \pm \sqrt{b^2 - 4(a)(c-k)}}{2a}.$$

However, as there can only be one solution we know that

$$\frac{-b + \sqrt{b^2 - 4(a)(c-k)}}{2a} = \frac{-b - \sqrt{b^2 - 4(a)(c-k)}}{2a}.$$

The only way this could be is if the discriminant is zero. In other words,

$$b^2 - 4(a)(c - k) = 0.$$

Thus, the maximum value for the output occurs if

$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}.$$

This means that for any function $f(x) = ax^2 + bx + c$ ($a, b, c \in \mathbb{R}$, $a \neq 0$), its maximum value (or minimum value) will occur when $x = \dfrac{-b}{2a}$ and in fact, its maximum value (or minimum value) will be $f\left(\dfrac{-b}{2a}\right)$. Graphically this means that the coordinates of the vertex of any such quadratic function will be $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$.

Example 2. *Determine the minimum value of* $f(x) = x^2 + 8x + 24$.

Solution. We have $a = 1$, $b = 8$, and $c = 24$. The minimum (or maximum) value occurs when

$$x = \frac{-b}{2a} = \frac{-8}{2(1)} = -4.$$

Find the output, which is the minimum (or maximum) value by substituting this value for x into the function formula.

$$\begin{aligned} f(-4) &= (-4)^2 + 8(-4) + 24 \\ &= 16 - 32 + 24 \\ &= 8. \end{aligned}$$

The vertex is therefore $(-4, 8)$.

To determine whether this is a maximum or minimum value we could either graph $f(x)$ or choose an arbitrary value for x and compare. Let $x = 2$. Then

$$\begin{aligned} f(2) &= (2)^2 + 8(2) + 24 \\ &= 4 + 16 + 24 \\ &= 44. \end{aligned}$$

We know that 16 is either a maximum or minimum value and that $44 > 16$, so 16 must be the minimum value of the function.

You might make note of the effect of the *leading coefficient* on the orientation of the graph (turned upward or turned downward). ■

SECTION 3.5 EXERCISES:

(Answers are found on page 156.)

Determine the coordinates of the vertex for the graphs of the following functions.

1. $f(x) = x^2 - 8x + 18$
2. $g(x) = -x^2 - 2x + 2$
3. $h(x) = 9x^2 - 6x + 10$
4. $f(x) = x^2 + 10x - 25$
5. $h(x) = 2x^2 + 4x - 12$
6. $h(x) = 4x^2 + 20x - 3$
7. $g(x) = \frac{3}{5}x^2 + 25$
8. $f(x) = \frac{1}{3}x^2 - 6x + 7$
9. $3x^2 - 5x + 2$
10. $g(x) = 6x^2 - 15x$

Determine the coordinates of the vertex and the intercepts for each of the following functions. Then sketch its graph.

11. $f(x) = x^2 - x - 6$
12. $g(x) = x^2 + 6x + 10$
13. $h(x) = -16x^2 + 24x - 41$
14. $f(x) = x^2 - 10x + 16$
15. $g(x) = x^2 - 6x - 7$
16. $h(x) = 3x^2 - 14x + 8$
17. $f(x) = 5x^2 + 16x + 3$
18. $g(x) = -4x^2 + 5x + 21$
19. $h(x) = \frac{1}{2}x^2 + 6x + 20$
20. $f(x) = x^2 + 2x - 5$

21. The height above ground of a projectile is measured by the function $s(t) = -16t^2 + 544t$, where t is time in seconds and $s(t)$ is in feet. Determine the maximum height achieved by the projectile.

22. The height of a small rocket fired straight upward is measured by the function $h(t) = -16t^2 + 512t$ where $h(t)$ is measured in feet and t is measured in seconds. What is the maximum height the rocket will reach?

23. A ball is thrown vertically upward from the top of a building 96 feet tall with an initial velocity of 80 feet per second. The height of the ball is measured by the function $h(t) = -16t^2 + 80t + 96$ where $h(t)$

is measured in feet and t is time measured in seconds. How many feet will the ball travel upward?

24. The height of a ball thrown straight upward is measured by the function $h(t) = -16t^2 + 40t$ where $h(t)$ is measured in feet and t is time in seconds. When is the ball halfway in its trajectory?

25. A worker on the ground tosses a hammer to a coworker on a scaffold 15 feet above the ground. The initial velocity of the hammer can be represented by $h(t) = -16t^2 + 32t$ where h is measured in feet and t is time in seconds. Will the hammer reach the worker on the scaffold?

26. A company's revenue from selling x units of a product is modeled by the function $R(x) = -0.1x^2 + 20x$ and the cost to produce x units is modeled by the function $C(x) = 4x + 2$. Both $R(x)$ and $C(x)$ are measured in thousands of dollars. Determine the number of units the company should produce and sell to maximize its profit. Determine the maximum profit.

27. A computer dealer has found that his monthly revenue for selling personal computers can be modeled by the function $R(x) = -120x^2 + 6000x + 24000$. His costs can be modeled by $C(x) = 3600x + 14400$ How many computers should he sell monthly to achieve the maximum profit?

28. A store that sells answering machines has determined its costs to be $C(x) = 100x + 700$ and its revenue to be $R(x) = 8400 + 280 - 20x^2$. What is the maximum profit the store can make?

29. A company's cost function is $C(x) = 3x + 20$ and its revenue function is $R(x) = -2x^2 + 5x + 85$ where x is in hundreds of units and $C(x)$ and $R(x)$ are in thousands of dollars. Find its maximum profit.

30. A store selling bicycles has found that its costs can be represented by $C(x) = 4000 + 400x$ and its revenue by $R(x) = 8000 + 600x - 20x^2$. How many bicycles should be sold to maximize profit? What is the maximum profit?

31. One number is one less than twice another number. Their product equals 1. Find the numbers.

32. The sum of two numbers is 100. What is the largest product of these two numbers?

33. The average of two numbers is 9.5. What is the largest possible product of these two numbers?

34. Twice one number minus a second number is 12. Find the smallest product of these two numbers.

35. Two numbers differ by $1\frac{1}{2}$. What is the smallest their product can be?

36. A farmer has 2400 feet of fencing and wants to enclose a rectangular area. Determine the dimensions of the rectangle that giv the maximum area. What is the maximum area?

37. If the perimeter of a rectangle is 20 feet, what is the largest area it can enclose?

38. Joan purchased 100 feet of fence. She plans to create a rectangular garden using the side of her garage as one side of the garden and fencing to border the other three sides. What is the maximum area she can enclose? What dimensions should she use for the garden in order to maximize the area?

39. A boat manufacturer is constructing sails in the shape of a right triangle. The legs of the triangle must total 36 feet. What dimensions should he use for the legs in order to maximize the surface area of the sail?

40. An isosceles triangle has a perimeter of 400 and its height is $\frac{4}{5}$ of either of the equal sides. What would the dimensions need to be to maximize the area of this triangle?

41. Show that the maximum rectangular area enclosed by a finite length of fence must be a square.

3.6 Quadratic Inequalities

We have seen that the factored form of a polynomial is useful for solving equations because of the Zero Product Property. Now we will see how these ideas can be applied to the solution of inequalities. Recall that the product of two positive factors is positive, the product of two negative factors is positive, and the product of one positive with one negative factor is negative.

Example 1. *Find all x for which*

$$(x+1)(x-4) > 0.$$

Solution. We need to find all values of x for which the factors $x+1$ and $x-4$ are either both positive or both negative. We will first find where each of these linear factors is equal to zero, since that is where it will change sign.

$$\begin{aligned} x+1 &= 0 & x-4 &= 0 \\ x &= -1 & x &= 4. \end{aligned}$$

Thus, -1 and 4 are the ***boundary points*** where the value of the product $(x+1)(x-4)$ can change sign. These two boundary points divide the real line into three intervals (reading from left to right on the real line):

$$(-\infty, -1), \qquad (-1, 4), \qquad \text{and} \qquad (4, \infty).$$

We will first determine the sign of each linear factor on each of these three intervals. We construct a ***sign chart***. This is a table with the intervals determined by the roots of the linear factors labeling the columns (from left to right on the real line) and the linear factors labeling the first two rows, while the product itself labels the last row.

	$(-\infty, -1)$	$(-1, 4)$	$(4, \infty)$
$x+1$			
$x-4$			
$(x+1)(x-4)$			

Each cell in the body of the sign chart will be filled with "+" or "−", depending on the sign of the corresponding expression for x in the corresponding interval. Note that if x is in $(-\infty, -1)$, then x is less than -1 and so $x+1$ is less than 0. If x is greater than -1, then $x+1$ is greater than 0. (If this is not apparent, the reader can test values of x from each interval in the linear factor.) Thus, we fill in the first row of the sign chart. The second row is filled in similarly.

	$(-\infty,-1)$	$(-1,4)$	$(4,\infty)$
$x+1$	$-$	$+$	$+$
$x-4$	$-$	$-$	$+$
$(x+1)(x-4)$			

Next, we wish to find the sign of the product $(x+1)(x-4)$ on each of the three intervals. To do this, we may simply "multiply down" the signs in each column. For example, for $x \in (-\infty,-1)$, both $x+1$ and $x-4$ are negative, so their product is positive. We complete the rest of the third row in the same way to obtain:

	$(-\infty,-1)$	$(-1,4)$	$(4,\infty)$
$x+1$	$-$	$+$	$+$
$x-4$	$-$	$-$	$+$
$(x+1)(x-4)$	$+$	$-$	$+$

Now the problem was to find the values of x for which $(x+1)(x-4) > 0$. From the sign chart, we see that the product is positive for x in the first and third interval in our sign chart. Since it is a strict inequality, the boundary points are not included in the solution set. Thus, the solution set consists of the union of these intervals, that is, $(-\infty,-1)\cup(4,\infty)$. ■

Practice 1. *Solve for the variable:* $(2x-1)(x+4) < 0$ *Express your solution set in interval notation.* *(Answers on page 139.)*

Example 2. *Solve for t:* $\quad t(t+5) \leq 50.$

Solution. Our method enables us to find where a factored expression is positive or negative, that is, greater than or less than 0. Therefore, we must perform some algebra to get 0 on one side and a factored polynomial on the other side.

$$\begin{aligned} t(t+5) &\leq 50 \\ t^2+5t &\leq 50 \\ t^2+5t-50 &\leq 0 \\ (t+10)(t-5) &\leq 0 \end{aligned}$$

We want the product of $t+10$ and $t-5$ to be negative or equal to 0, so we need to find all values of t for which one the factors is negative and the other is positive or where one of these factors is equal to zero. We first determine where each is 0.

$$\begin{aligned} t+10 &= 0 \quad & t-5 &= 0 \\ t &= -10 \quad & t &= 5. \end{aligned}$$

Thus, -10 and 5 are the boundary points where the value of the product $(t+10)(t-5)$ can change sign. These two boundary points divide the real line into three intervals (reading from left to right on the real line):

$$(-\infty,-10), \qquad (-10,5), \qquad \text{and} \qquad (5,\infty).$$

Next, we construct a sign chart. The labels are as follows:

	$(-\infty,-10)$	$(-10,5)$	$(5,\infty)$
$t+10$			
$t-5$			
$(t+10)(t-5)$			

Now we fill in the body of the sign chart. If t is in $(-\infty,-10)$, then t is less than -10 and so $t+10$ is less than 0. If t is greater than -10, then $t+10$ is greater than 0. (Again, the reader can test values of t from each interval in the linear factor.) Thus, we fill in the first row of the sign chart. The second row is filled in similarly.

	$(-\infty,-10)$	$(-10,5)$	$(5,\infty)$
$t+10$	$-$	$+$	$+$
$t-5$	$-$	$-$	$+$
$(t+10)(t-5)$			

Next, we multiply down each column to find the sign of the product $(t+10)(t-5)$ on each of the three intervals.

	$(-\infty,-10)$	$(-10,5)$	$(5,\infty)$
$t+10$	$-$	$+$	$+$
$t-5$	$-$	$-$	$+$
$(t+10)(t-5)$	$+$	$-$	$+$

Now the problem was to find the values of t for which $(t+10)(t-5) \leq 0$. From the sign chart, we see that the product is negative for t in the middle interval in our sign chart. The product is 0 at the boundary points. Thus, the solution set is $[-10,5]$. ■

Practice 2. *Solve for the variable:* $t(t-2) \geq 2t-3$ *Express your solution set in interval notation. (Answers below.)*

ANSWERS TO SECTION 3.6 PRACTICE PROBLEMS

1. $\left(-4, \frac{1}{2}\right)$
2. $(-\infty,1] \cup [3,\infty)$

SECTION 3.6 EXERCISES:

(Answers are found on page 156.)

Find the value(s) of x which satisfy each of the following inequalities. Express your solution in interval notation.

1. $x^2+2x+1>0$
2. $x^2-5x-14 \leq 0$
3. $m^2-m-6<0$
4. $y^2+6y+5 \geq 0$
5. $z^2+25z \leq 0$
6. $x^2-4>0$
7. $w^2 \geq 8w-12$
8. $p^2-11p<-28$
9. $h^2 \leq 16$
10. $x^2>7x$
11. $-y^2 \geq 6y-7$
12. $-z^2+4z<-45$
13. $3w^2-2w-5<0$
14. $20x^2-7x-6 \geq 0$

15. $8n^2 + 16n - 24 \leq 0$
16. $6m^2 - 13m + 5 \geq 0$
17. $3x^2 - 5x - 12 > 0$
18. $2y^2 + 5y - 3 \leq 0$
19. $-5p^2 > 7p + 2$
20. $-3w^2 \geq 10w + 8$
21. $3x^2 < 4x - 1$
22. $4x^2 - 13x \leq -10$
23. $x^2 + 9x + 20 \geq 0$
24. $-x^2 < 2x + 3$
25. $2x^2 > 6x$
26. $5x^2 - 12x + 7 > 0$
27. $4x^2 \leq 3x + 1$
28. $x^2 + 5x \geq 24$

3.7 Equations Reducible to Quadratic Form

Recall the factoring we did in Section 2.4. In this section, we will simply add one extra step, i.e., we will set these "quadratics in disguise" equal to zero and solve.

Example 1. *Solve* $(x-1)^2 + 7(x-1) + 12 = 0$.

Solution. First we factor the left-hand side as we did in Section 2.4. Notice that the binomial $(x-1)$ is in two of the terms in this polynomial and is squared in one of them. If we *substitute* u for $(x-1)$, the result will be much easier to work with.

$$\begin{aligned}
\text{Original equation:} \quad && (x-1)^2 + 7(x-1) + 12 &= 0 \\
\text{Substitute } u \text{ for } (x-1)\text{:} \quad && u^2 + 7u + 12 &= 0 \\
\text{Factor the left-hand side:} \quad && (u+3)(u+4) &= 0 \\
\text{Back-substitute:} \quad && [(x-1)+3][(x-1)+4] &= 0 \\
&& (x+2)(x+3) &= 0
\end{aligned}$$

We next use the Zero Product Property to solve the equation:

$$\begin{aligned}
x+2 &= 0 \quad \text{or} \quad & x+3 &= 0 \\
x &= -2 \quad \text{or} \quad & x &= -3.
\end{aligned}$$

Check. We can check by substituting these values back into the original equation.

Let $x = -2$. Then

$$\begin{aligned}
(x-1)^2 + 7(x-1) + 12 &= [(-2)-1]^2 + 7[(-2)-1] + 12 \\
&= (-3)^2 + 7(-3) + 12 \\
&= 9 - 21 + 12 \\
&= -12 + 12 \\
&= 0.
\end{aligned}$$

Let $x = -3$. Then

$$\begin{aligned}
(x-1)^2 + 7(x-1) + 12 &= [(-3)-1]^2 + 7[(-3)-1] + 12 \\
&= (-4)^2 + 7(-4) + 12 \\
&= 16 - 28 + 12 \\
&= -12 + 12 \\
&= 0.
\end{aligned}$$

You could also check by graphing. ■

Alternate Solution. Some people might take this problem, notice that that left-hand side is a "quadratic in disguise" and factor it directly, without using a substitution.

$$\begin{aligned} (x-1)^2 + 7(x-1) + 12 &= 0 \\ [(x-1)+3][(x-1)+4] &= 0 \\ (x+2)(x+3) &= 0 \end{aligned}$$

and then set each factor equal to zero and solve, just as in the previous solution. ■

Example 2. *Solve* $2(x-5)^2 + 3(x-5) - 5 = 0.$

Solution. First we factor the left-hand side as we did in Section 2.4. Again, substitution will make this polynomial easier to factor. Since the binomial $(x-5)$ appears twice— once as a linear term and once squared— we'll substitute u for it.

$$\begin{aligned} \text{Original equation:} &\quad & 2(x-5)^2 + 3(x-5) - 5 &= 0 \\ \text{Substitute } u \text{ for } (x-1)\text{:} &\quad & 2u^2 + 3u - 5 &= 0 \\ \text{Factor the left-hand side:} &\quad & (2u+5)(u-1) &= 0 \\ \text{Back-substitute:} &\quad & [2(x-5)+5][(x-5)-1] &= 0 \\ &\quad & [2x-10+5][x-6] &= 0 \\ &\quad & (2x-5)(x-6) &= 0 \end{aligned}$$

We next use the Zero Product Property to solve the equation:

$$\begin{aligned} 2x - 5 &= 0 &\quad \text{or} \quad & x - 6 = 0 \\ 2x &= 5 &\quad \text{or} \quad & x = 6 \\ x &= \frac{5}{2} &\quad \text{or} \quad & x = 6. \end{aligned}$$

Check. We can check by substituting these values back into the original equation.

Let $x = \frac{5}{2}$. Then

$$\begin{aligned} 2(x-5)^2 + 3(x-5) - 5 &= 2\left(\frac{5}{2} - 5\right)^2 + 3\left(\frac{5}{2} - 5\right) - 5 \\ &= 2\left(\frac{5}{2} - \frac{10}{2}\right)^2 + 3\left(\frac{5}{2} - \frac{10}{2}\right) - 5 \\ &= 2\left(-\frac{5}{2}\right)^2 + 3\left(-\frac{5}{2}\right) - 5 \\ &= 2\left(\frac{25}{4}\right) + \frac{-15}{2} - 5 \\ &= \frac{25}{2} + \frac{-15}{2} - \frac{10}{2} \\ &= \frac{25 - 15 - 10}{2} \\ &= \frac{0}{2} \\ &= 0. \end{aligned}$$

Let $x = 6$. Then

$$\begin{aligned} 2(x-5)^2 + 3(x-5) - 5 &= 2(6-5)^2 + 3(6-5) - 5 \\ &= 2(1)^2 + 3(1) - 5 \\ &= 2 + 3 - 5 \\ &= 0. \end{aligned}$$

You could also check by graphing. ■

Alternate Solution. Once again, you may prefer to factor it directly without using a substitution.

$$\begin{aligned} 2(x-5)^2 + 3(x-5) + -5 = 0 \\ [2(x-5)+5][(x-5)-1] = 0 \\ (2x-10+5)(x-6) = 0(2x-5)(x+6) \qquad &= 0 \end{aligned}$$

and then set each factor equal to zero and solve, as we did above. ■

Practice 1. *(Answers on page 145.) Solve each of the following equations.*

a. $(x+4)^2 + 3(x+4) - 18 = 0$

b. $(x+2)^2 - 2(x+2) - 15 = 0$

c. $3(x-1)^2 + 5(x-1) - 2 = 0$

Example 3. *Solve* $3x^{2/3} - 5x^{1/3} - 2 = 0.$

Solution. Again, we factor the left-hand side as we did in Section 2.4. Look at the middle term (without the coefficient): $x^{1/3}$. If we square it, we get the first term:

$$\left(x^{1/3}\right)^2 = x^{2/3}.$$

This is how we know that this is reducible to a quadratic, a "quadratic in disguise." Use substitution, letting $u = x^{1/3}$.

$$\begin{aligned}
\text{Original equation:} \quad & 3x^{2/3} - 5x^{1/3} - 2 = 0 \\
& 3\left(x^{1/3}\right)^2 - 5x^{1/3} - 2 = 0 \\
\text{Substitute } u \text{ for } x^{1/3}\text{:} \quad & 3u^2 - 5u - 2 = 0 \\
\text{Factor the left-hand side:} \quad & (3u+1)(u-2) = 0 \\
\text{Back-substitute:} \quad & \left(3x^{1/3}+1\right)\left(x^{1/3}-2\right) = 0
\end{aligned}$$

We next use the Zero Product Property to solve the equation:

$$\begin{aligned}
3x^{1/3} + 1 &= 0 \quad && \text{or} \quad & x^{1/3} - 2 &= 0 \\
3x^{1/3} &= -1 \quad && \text{or} \quad & x^{1/3} &= 2 \\
x^{1/3} &= -\frac{1}{3} \quad && \text{or} \quad & x^{1/3} &= 2
\end{aligned}$$

To eliminate the $1/3$ power, cube both sides:

$$\begin{aligned}
\left(x^{1/3}\right)^3 &= \left(-\frac{1}{3}\right)^3 \quad && \text{or} \quad & \left(x^{1/3}\right)^3 &= 2^3 \\
x &= -\frac{1}{27} \quad && \text{or} \quad & x &= 8.
\end{aligned}$$

The checks are left for you. ■

Alternate Solution. Again, you may choose to factor directly without making a substitution.

$$3x^{2/3} - 5x^{1/3} - 2 = 0$$
$$\left(3x^{1/3} + 1\right)\left(x^{1/3} - 2\right) = 0$$

and then set each factor equal to zero and solve, as we did above. ■

Practice 2. *(Answers below.) Solve each of the following equations.*

a. $x^{2/3} + 3x^{1/3} - 18 = 0$

b. $m^{1/2} - 2m^{1/4} - 15 = 0$

c. $3p^{2/5} + 5p^{1/5} - 2 = 0$

ANSWERS TO SECTION 3.7 PRACTICE PROBLEMS

1. (a) $x = -10, -1$
   (b) $x = 3, -5$
   (c) $x = \frac{4}{3}, -1$

2. (a) $x = 216, 27$
   (b) $m = 625, 81$
   (c) $p = \frac{1}{243}, 32$

SECTION 3.7 EXERCISES:

(Answers are found on page 157.)

Solve each of the following equations.

1. $(x+1)^2 - 2(x+1) - 3 = 0$
2. $(x+4)^2 + 7(x+4) + 12 = 0$
3. $(x-2)^2 + 5(x-2) + 6 = 0$
4. $(x-5)^2 + 6(x-5) + 5 = 0$
5. $(x+6)^2 - 8(x+6) - 9 = 0$
6. $(x+3)^2 - (x+3) - 6 = 0$
7. $\left(\frac{1}{x} + 5\right)^2 - 6\left(\frac{1}{x} + 5\right) + 8 = 0$
8. $\left(\frac{1}{x+2}\right)^2 - 2\left(\frac{1}{x+2}\right) - 8 = 0$
9. $\left(\frac{x-1}{x}\right)^2 + 4\left(\frac{x-1}{x}\right) + 3 = 0$
10. $\left(\frac{x}{x+1}\right)^2 = \frac{4x}{x+1} - 4$
11. $3(x+3)^2 - 2(x+3) - 5 = 0$
12. $6(x-1)^2 - 13(x-1) - 5 = 0$
13. $14(x-5)^2 - 9(x-5) + 1 = 0$
14. $20(x-2)^2 - 7(x-2) - 6 = 0$
15. $-(3x-1)^2 - 6(3x-1) + 7 = 0$
16. $-(5x+3)^2 + 9(5x+3) - 20 = 0$

17. $2(x+5)^2 - 3 = -5(x+5)$
18. $7(x-1)^2 = -19(x-1) + 6$
19. $x^4 - 5x^2 + 4 = 0$
20. $x^4 - 6x^2 + 8 = 0$
21. $x^4 - 13x^2 - 14 = 0$
22. $x^4 + 2x^2 - 3 = 0$
23. $x^6 - 7x^3 + 6 = 0$
24. $x^6 + 6x^3 + 5 = 0$
25. $x^6 + 6x^3 + 5 = 0$
26. $x^6 + x^3 - 6 = 0$
27. $x^{1/3} + x^{1/6} - 2 = 0$
28. $x^{2/3} + 5x^{1/3} + 4 = 0$
29. $x^{4/3} - 5x^{2/3} + 6 = 0$
30. $x - 9x^{1/2} + 8 = 0$
31. $2x^{1/2} + 3x^{1/4} - 5 = 0$
32. $8x^{2/3} + 24x^{1/3} - 32 = 0$
33. $x + 6\sqrt{x} + 5 = 0$
34. $x - 5\sqrt{x} + 6 = 0$
35. $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$
36. $\sqrt[3]{x^2} + \sqrt[3]{x} - 12 = 0$

Answers to Odd-Numbered Exercises

Chapter 0

Section 0.1 (Exercises on page 3.)

1. $xy(2x-y)(2x+y)$
3. $(1-a-b)(1+a+b)$
5. $x^2(3-x^2y)(3+x^2y)$
7. $(7a-4)(a+3)$
9. $5(x-3)(x+3)$
11. $[(x-3)^5-1][(x-3)^5+1]$
13. $(2z-w)(x-2y)$
15. $(2+x)(6-7x)$
17. $(3y-2x)(3y+2x)(9y^2+4x^2)$
19. $(4y+3z)(y-8z)$
21. $2xy(x-4y)(x+y)$
23. $3a^2(a-4)(a+4)$
25. $(xy-15)(xy+15)$
27. $x(2x-5)^2$

Chapter 1

Section 1.1 (Exercises on page 10.)

1. rational expression
3. NOT a rational expression
5. rational expression

7. rational expression
9. rational expression
11. $5 \times ? = 105$, so $? = 21$
13. $0 \times ? = 81$; no solution
15. $81 \times ? = 0$, so $? = 0$
17. $0 \times ? = 0$, so indeterminate
19. $-3 \times ? = -81$, so $? = 27$
21. (a) $r(-1) = 0$
    (b) $r(0)$ is undefined
    (c) $r(2) = \frac{3}{4}$
23. (a) $r(-5) = 0$
    (b) $r(0) = -1$
    (c) $r(5)$ is undefined
25. (a) $r(-4)$ is undefined
    (b) $r(0) = 0$
    (c) $r(2) = \frac{5}{6}$
27. (a) $r(-2) = \frac{2}{11}$
    (b) $r(0) = 0$
    (c) $r(3) = \frac{3}{4}$
29. (a) $r(-1) = -1$
    (b) $r(0) = 0$
    (c) $r(1)$ is undefined
31. defined for all real numbers
    $\text{dom}(r) = (-\infty, \infty)$
33. undefined for $x = -2, 0$
    $\text{dom}(r) =$
    $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$
35. undefined for $t = -2, 5$
    $\text{dom}(r) =$
    $(-\infty, -2) \cup (-2, 5) \cup (5, \infty)$
37. undefined for $y = 0, \frac{4}{5}$
    $\text{dom}(r) =$
    $(-\infty, 0) \cup (0, \frac{4}{5}) \cup (\frac{4}{5}, \infty)$
39. defined for all real numbers
    $\text{dom}(r) = (-\infty, \infty)$

Section 1.2 (Exercises on page 15.)

1. x-int: $(8, 0)$
   VA: $x = -3$
3. x-int: $(-5, 0)$
   VA: $x = 0$
5. x-int: $(-3, 0)$, $(3, 0)$
   VA: $x = 1/2$
7. x-int:none
   VA: $x = 6$
9. x-int: $(5, 0)$
   VA: $x = -1$
11. x-ints: $(3, 0)$,$(-2, 0)$
    VA: none
13. x-ints: $(0, 0)$, $(2, 0)$,$(-2, 0)$
    VA: $x = -3$
15. x-ints: $(-5, 0)$, $(5, 0)$
    VAs: $x = -6$, $x = 6$

Section 1.3 (Exercises on page 22.)

1. $-\frac{4x^2}{y^3}$
3. $\frac{x+8}{2}$
5. -1

7. $\frac{1}{y-5}$

9. $\frac{1}{t-7}$

11. $\frac{x-9}{x+9}$

13. $m+4$

15. $-\frac{x+2}{x}$

17. $\frac{y+4}{y+2}$

19. $\frac{x-4}{(x-2)(x+2)}$

21. $-\frac{1}{2}$

23. $\frac{x-1}{x}$

25. $\frac{x+1}{x-1}$

27. $\frac{y+2}{2(y+3)}$

29. $\frac{2a+3}{2a-3}$

31. $-\frac{12}{5}$

33. $\frac{16}{27}$

35. $\frac{3-x}{3+x}$

37. 1

39. $\frac{x}{3x+1}$

41. $-\frac{1}{7x}$

43. $-\frac{1}{4x}$

45. $\frac{2+w}{2w}$

47. $\frac{2y-x}{xy}$

49. $\frac{2x-4}{8-3x}$

Section 1.4 (Exercises on page 32.)

1. $-\frac{2}{5}$

3. $\frac{9}{32}$

5. $\frac{10}{xy}$

7. $2t$

9. $\frac{3}{5(x+1)}$

11. $\frac{m+3}{(3-m)(m+5)}$

13. $2(a-b)(a+b)$

15. $-\frac{1}{4}$

17. $-\frac{5}{3}$

19. $\frac{2}{a+2}$

21. $\frac{x^2+1}{3(x+1)}$

23. $\frac{2a-3}{2a(a-1)}$

25. 5

27. $-\frac{1}{23}$

29. $\frac{9}{4}$

31. none

33. x^6

35. $\frac{3b}{2a}$

37. $\frac{x+1}{x-1}$

39. $\frac{5y+7}{y^2-y+9}$

41. $\frac{3}{20}$

43. $3\frac{1}{4}$

45. $\frac{4x}{5}$

47. $\frac{6a}{5b}$

49. $\frac{x}{5}$

51. $\frac{t^5}{10}$

53. $\frac{x-12}{3}$

55. $-\frac{5y}{y+3}$

57. $\frac{15}{14}$

59. $\frac{a+b}{a-b}$

61. $\frac{x-2}{3x(x+1)}$

63. $\frac{y-3}{2(y+4)}$

65. $\frac{x-2}{x+4}$

Section 1.5 (Exercises on page 43.)

1. $\frac{1}{x^3}$

3. $\frac{1}{t^2}$

5. -1

7. -2

9. $\frac{1}{t}$

11. $\frac{1}{x^3(7x+1)}$

13. 2

15. $\frac{4}{x-2}$

17. $\frac{2}{a^6}$

19. $\frac{3}{x-3}$

21. $\frac{1}{2x+1}$

23. LCD: $(x+1)(x-2)$
$\frac{5(x-2)}{(x+1)(x-2)}$, $\frac{x(x+1)}{(x+1)(x-2)}$

25. LCD: $x(6-x)$
$\frac{(x+10)(6-x)}{x(6-x)}$, $\frac{x}{x(6-x)}$

27. LCD: $(5m-2)(5m+2)(m+1)$
$\frac{(m+2)(m+1)}{(5m-2)(5m+2)(m+1)}$,
$\frac{7(5m+2)}{(5m-2)(5m+2)(m+1)}$

29. LCD: $5y(y+1)$
$\frac{y+1}{5y(y+1)}$, $\frac{y}{5y(y+1)}$

31. LCD: $24y^5$
$\frac{30y^3}{24y^5}$, $\frac{22}{24y^5}$, $\frac{-9y^2}{24y^5}$

33. LCD: $9a(a+2)$
$\frac{7a+14}{9a(a+2)}$, $\frac{6a^2}{9a(a+2)}$, $\frac{9a^2+9a}{9a(a+2)}$

35. $\frac{x^2+3x+18}{9x^2}$

37. $\frac{5y+4}{(y-1)(2y+1)}$

39. $\frac{2x^2+1}{2x}$

41. $\frac{4t-45}{5(t+5)(t-5)}$

43. $\frac{2(x^2+2)}{x^2(x+4)}$

45. $\frac{-(x^2+x+8)}{(x-2)(x+2)(x-3)}$

47. $\frac{6a}{a+1}$

49. $-\frac{n}{(n+2)^2}$

51. $\frac{4ab}{(a+b)^2}$

53. $\frac{3y-4}{2y-3}$

55. $\frac{3x^2+4x-14}{84x^3}$

57. $-\frac{2}{y+5}$

59. $\frac{y+4}{y+2}$

Section 1.6 (Exercises on page 56.)

1. $t = 12$
3. $x = \frac{1}{3}$
5. all real numbers except $0, -1$
7. $x = \frac{14}{3}$
9. no solution
11. $x = \frac{6}{7}$
13. no solution
15. $y = 8$
17. all real numbers
19. $x = -1$
21. $x = \pm 2$
23. $y = 3$
25. $t = -4, 3$
27. $z = 2$
29. no solution
31. $x = \pm\sqrt{5}$
33. $m = -5, 2$
35. all real numbers except ± 3
37. $y = -1$
39. no solution

Section 1.7 (Exercises on page 70.)

1. The number is ± 4.
3. The number is $\frac{1}{2}$ or 1.
5. The number is $-\frac{3}{2}$ or 1.
7. The number is 5 or -4.
9. The number is -3 or 2.
11. The lengths of the indicated sides are $x = 12$ and $x + 8 = 20$.
13. The lengths of the indicated sides are $x = 2\frac{1}{2}$, $2x + 3 = 8$, and $3x = 7\frac{1}{2}$.
15. Thelma is 5 feet tall.
17. The man is 5 feet tall.
19. \$7.50 per hour
21. 130 boxes per girl
23. 3 feet per yard
25. The slope is $\frac{1}{4}$.
27. 3.5%
29. 94%
31. It will take them 36 minutes to shovel the driveway together.
33. It would take Todd 2 hrs. 48 min. working alone.
35. It would take Craig 9 hours working alone.
37. The train is traveling at 100 miles per hour and the truck at 60 miles per hour.
39. The tub fills in 2 hours with the leak and in 40 min. without it.
41. The boat travels at 12 mph in still water.

43. The boat travels at 25 mph in still air.

45. The plane would fly at 200 miles per hour in still air.

47. Cashews cost $10 per pound and peanuts cost $2 per pound.

Chapter 2

Section 2.2 (Exercises on page 84.)

1. $(x^2+4)(x+3)$
3. $(2x^3-5)(x-1)$
5. $(4xy+3)(y-x)$
7. $3x(2x^2-1)(x-1)$
9. $y(x-z)(x+1)$
11. $(x^2+1)(x+1)$
13. $2ab([(a-b)(a+4)]$
15. $(3-2x^2)(x-1)$
17. $(xy-z)(2x^2+5)$
19. $7x^2-3)(3x^3-1)$

Section 2.3 (Exercises on page 88.)

1. $(x-12)(x+12)$
3. $(m-\sqrt{6})(m+\sqrt{6})(m^2+6)$
5. $(3a-8)(3a+8)$
7. $(2-3y)(2+3y)$
9. $(5-11a)(5+11a)$
11. $(9-b)(9+b)$
13. $xy(x-y)(x+y)$
15. $3y^2(y-3)(y+3)$
17. $(x-3)(x+1)$
19. $(x-y+3)(x+y+3)$
21. $3m^2(3n-1)(n+1)$
23. $(a-\sqrt{13})(a+\sqrt{13})$
25. $(b-\sqrt{21})(b+\sqrt{21})$
27. $(b-\sqrt{33})(b+\sqrt{33})$
29. $(\sqrt{2}\,m-\sqrt{3})(\sqrt{2}\,m+\sqrt{3})$
31. $(\sqrt{7}\,x-\sqrt{11})(\sqrt{7}\,x+\sqrt{11})$
33. $(4b-\sqrt{5})(4b+\sqrt{5})$
35. $(1-10x)(1+10x)$
37. $(x-14)(x+14)$
39. $2(6x-5)(6x+5)$
41. $(x-\sqrt{7})(x+\sqrt{7})$
43. $(5x-3y)(5x+3y)(25x^2+9y^2$
45. $(x-6y+3)(x+6y+3)$

47. $y(2xy - 3y - 13)(2xy - 3y + 13)$

49. $(a - b - 1)(a + b - 1)$

51. $(p - q + 5)(p + q + 5)$

53. $(x - y + 2)(x + y + 2)$

55. $(x + 2)(x^2 - 2x + 4)$

57. $(2x + 3)(4x^2 - 6x + 9)$

59. $(5xy + 1)(25x^2y^2 - 5xy + 1)$

61. $(x^2 + 4y)(x^4 - 4x^2y + 16y^2)$

63. $(xyz + 7)(x^2y^2z^2 - 7xyz + 49)$

65. $(x - 1)(x^2 + x + 1)$

67. $(2 - x)(4 + 2x + x^2)$

69. $(xy - 7)(x^2y^2 + 7xy + 49)$

71. $(x^2 - 4y)(x^4 + 4x^2y + 16y^2)$

73. $(5 - 6x)(25 + 30x + 36x^2)$

Section 2.4 (Exercises on page 93.)

1. $(x^2 + 5)(x^2 + 1)$

3. $(2x^4 - 1)(x^4 + 3)$

5. $(x^5 - 2)(x^5 + 1)$

7. $(4x + 1)(x - 4)$

9. $2(x + 3)(4x + 15)(x + 1)$

11. $\left(2x^{1/4} + 3\right)\left(x^{1/4} - 6\right)$

13. $6\left(2x^{2/5} - 3\right)\left(x^{2/5} + 2\right)$

15. $\left(2x^{5/2} + 5\right)\left(x^{5/2} - 1\right)$

17. $\left((2x + 1)^{1/3} - 4\right)\left((2x + 1)^{1/3} + 2\right)$

19. $(x + 1)^{-1/2}(x + 7)(x - 3)$

Chapter 3

Section 3.2 (Exercises on page 104.)

1. (a) 4
   (b) -8
   (c) $-1/2$
   (d) $-7/2$
   (e) $9x - 2$
   (f) $9x - 6$
   (g) $3x - 11$
   (h) $3x - 5$

3. (a) 1
   (b) 13
   (c) -2
   (d) 1
   (e) $18x^2 - 9x - 1$
   (f) $6x^2 - 9x - 3$
   (g) $2x^2 - 15x_26$
   (h) $2x^2 - 3x - 4$

5. (a) $9/7$

(b) -1
(c) $3/22$
(d) $-1/2$
(e) $\frac{9x^2-9x-1}{3x+5}$
(f) $\frac{3x^2+9x-3}{x+5}$
(g) $\frac{x^2-3x-1}{x+2}$
(h) $\frac{x^2-16}{x+5}$

7. (a) 2
   (b) 6
   (c) $7/2$
   (d) $9/2$
   (e) $|3x-4|$
   (f) $3|x-4|$
   (g) $|x-7|$
   (h) $|x-4|-3$

9. (a) At 7 seconds, the rocket is 4816 feet above ground level.
   (b) 4224 ft
   (c) 40 or 10 sec

11. (a) At 3 seconds, the ball is 192 feet above ground level.
    (b) 196 ft
    (c) 5 sec
    (d) 6 sec

13. (a) At 11 seconds, the rocket is 3696 feet above ground level.
    (b) 3312 ft
    (c) 2 or 30 sec

15. (a) When the length is 5 feet, the area is 325 square feet.
    (b) 264 sq ft
    (c) 50 or 20 ft

17. (a) When the width is 13 yards, the area is 221 square yards.
    (b) impossible
    (c) 10 or 20 yd

19. (a) When the length is 5 yards, the area is 275 square yards.
    (b) 116 sq yd
    (c) 30 yd

21. (a) $P(x) = -x^2 + 100x - 650$
    (b) \$481

23. (a) $P(x) = 4000 + 200x - 20x^2$
    (b) \$3520
    (c) 5 units

25. (a) $P(x) = 1700 + 180x - 20x^2$
    (b) in debt \$6300
    (c) 11 answering machines

27. (a) $P(x) = -2x^2 + 2x + 65$
    (b) \$25,000
    (c) 600 units

29. (a) 0
    (b) 3
    (c) 5
    (d) 4
    (e) 0

(f) $x = -5, 5$
(g) $y = 5$
(h) $x = 2$
(i) $x = -3$
(j) four times

31. (a) 2.5
(b) 1
(c) 1
(d) 3
(e) $x = 0$
(f) $y = 0$
(g) $x = 4$
(h) twice

33. (a) 6
(b) 4
(c) 1
(d) 2
(e) 3
(f) none
(g) $y = 1$
(h) $x = 10$
(i) min is 1
(j) once

Section 3.3 (Exercises on page 117.)

1. $x = -7 \pm \sqrt{42}$
3. $p = -2 \pm \sqrt{10}$
5. $x = \frac{9}{2} \pm \frac{\sqrt{69}}{2}$
7. $x = 11 \pm 4\sqrt{7}$
9. $x = \frac{5}{2} \pm \frac{\sqrt{33}}{2}$
11. $x = -4 \pm 3\sqrt{2}$
13. $x = 1$
15. $x = -7 \pm 3\sqrt{6}$
17. $x = \frac{1 \pm \sqrt{17}}{2}$
19. $x = -1 \pm \sqrt{13}$
21. $m = -\frac{3}{2} \pm \frac{\sqrt{10}}{2}$
23. $x = -\frac{9}{2} \pm \frac{\sqrt{53}}{2}$
25. $z = \frac{3}{2} \pm \frac{\sqrt{31}}{2}$
27. $p = 1 \pm \sqrt{13}$
29. $n = 1, -12$
31. $y = 2 \pm 2\sqrt{2}$

Section 3.4 (Exercises on page 127.)

1. $x = -7 \pm \sqrt{42}$
3. $p = -2 \pm \sqrt{10}$
5. $x = 6 \pm \sqrt{39}$
7. $x = 11 \pm 4\sqrt{7}$
9. $x = 3 \pm \sqrt{11}$
11. $x = -4 \pm 3\sqrt{2}$
13. $x = 1$
15. $x = -7 \pm 3\sqrt{6}$
17. $x = \frac{1 \pm \sqrt{17}}{2}$

19. $y = -1 \pm \sqrt{13}$
21. $m = \dfrac{-2 \pm \sqrt{5}}{2}$
23. $x = \dfrac{-9 \pm \sqrt{53}}{2}$
25. $z = 3 \pm 2\sqrt{5}$
27. $p = 1 \pm \sqrt{13}$
29. $x = -8, 4$
31. $x = \dfrac{2 \pm 3\sqrt{2}}{3}$
33. two
35. one
37. one
39. none
41. none
43. two

Section 3.5 (Exercises on page 133.)

1. vertex: $(4, 2)$
3. vertex: $\left(\dfrac{1}{3}, 9\right)$
5. vertex: $(-1, -14)$
7. vertex: $(0, 25)$
9. vertex: $\left(\dfrac{5}{6}, -\dfrac{1}{12}\right)$
11. vertex: $\left(\dfrac{1}{2}, -\dfrac{25}{4}\right)$ x-ints: $(3, 0), (-2, 0)$ y-int: $(0, -6)$
13. vertex: $\left(\dfrac{3}{4}, -32\right)$ x-ints: none y-int: $(0, -41)$
15. vertex: $(3, -16)$ x-ints: $(7, 0), (-1, 0)$ y-int: $(0, -7)$
17. vertex: $\left(-1\dfrac{3}{5}, -9\dfrac{4}{5}\right)$ x-ints: $(-1/5, 0), (-3, 0)$ y-int: $(0, 3)$
19. vertex: $(-6, 2)$ x-ints: none y-int: $(0, 20)$
21. The maximum height is 4624 feet.
23. The ball was 96 feet above the ground wen thrown and achieves a maximum height of 196 feet. Therefore it traveled 100 feet.
25. Yes. The hammer will travel a mximum of 16 feet upwards.
27. 10 computers
29. \$65,500
31. either $-1/2, -2$ or $1, 1$
33. 90.25
35. $-9/16$
37. 25 square feet
39. 18 feet by 18 feet

Section 3.6 (Exercises on page 139.)

1. $(-\infty, -1) \cup (-1, \infty)$
3. $(-2, 3)$
5. $[-25, 0]$

7. $(-\infty, 2] \cup [6, \infty)$
9. $[-4, 4]$
11. $[-7, 1]$
13. $\left(-1, \frac{5}{3}\right)$
15. $[-3, 1]$
17. $\left(-\infty, -\frac{4}{3}\right) \cup (3, \infty)$
19. $\left(-1, -\frac{2}{5}\right)$
21. $\left(\frac{1}{3}, 1\right)$
23. $(-\infty, -5] \cup (-4, \infty)$
25. $(-\infty, 0) \cup (3, \infty)$
27. $\left[-\frac{1}{4}, 1\right]$

Section 3.7 (Exercises on page 145.)

1. $x = 2, -2$
3. $x = -1, 0$
5. $x = -7, 3$
7. $x = -1, -\frac{1}{3}$
9. $x = \frac{1}{4}, \frac{1}{2}$
11. $x = -\frac{4}{3}, -4$
13. $x = \frac{36}{7}, \frac{11}{2}$
15. $x = -2, \frac{2}{3}$
17. $x = -\frac{9}{2}, -8$
19. $x = \pm 2, \pm 1$
21. $x = \pm\sqrt{14}$
23. $x = \sqrt[3]{6}, 1$
25. $x = -\sqrt[3]{5}, -1$
27. $x = 1$
29. $x = \pm 2\sqrt{2}, \pm 3\sqrt{3}$
31. $x = 1$
33. No real solution
35. $x = 256$

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