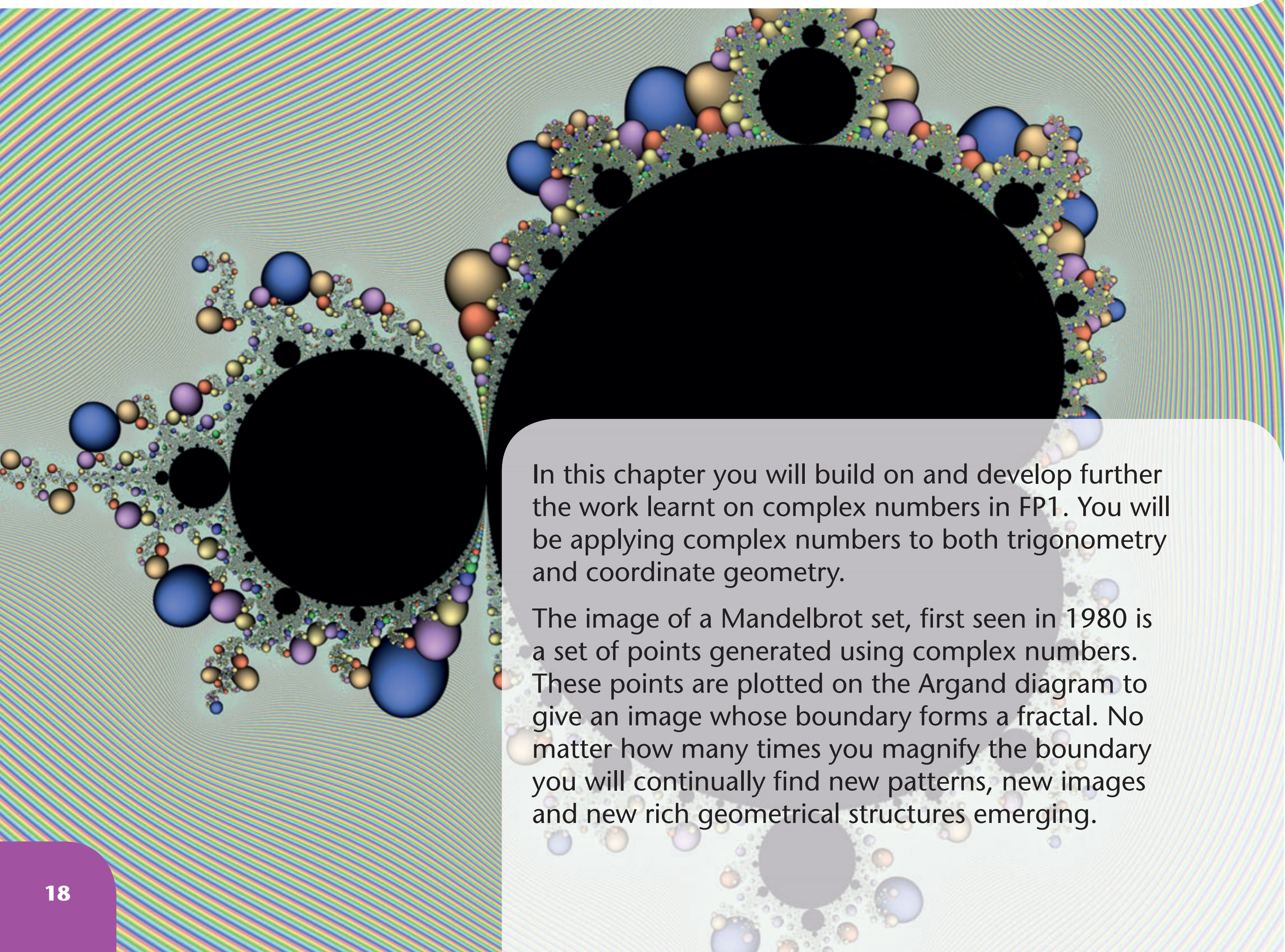
3

After completing this chapter you should be able to:

- Write down a complex number, z, in modulus–argument form as either $z = r(\cos\theta + \text{i}\sin\theta)$ or $z = r\text{e}^{\text{i}\theta}$,
  where r is the modulus of z and θ is the argument of z, and $-\pi < \theta \leqslant \pi$
- Apply de Moivre's theorem
  - to find trigonometric identities
  - to find roots of a complex number
- Represent loci and regions in the Argand diagram
- Apply transformations from the z-plane to the w-plane

Further complex numbers



In this chapter you will build on and develop further the work learnt on complex numbers in FP1. You will be applying complex numbers to both trigonometry and coordinate geometry.

The image of a Mandelbrot set, first seen in 1980 is a set of points generated using complex numbers. These points are plotted on the Argand diagram to give an image whose boundary forms a fractal. No matter how many times you magnify the boundary you will continually find new patterns, new images and new rich geometrical structures emerging.

3.1 You can express a complex number in the form $z = r(\cos\theta + i\sin\theta)$

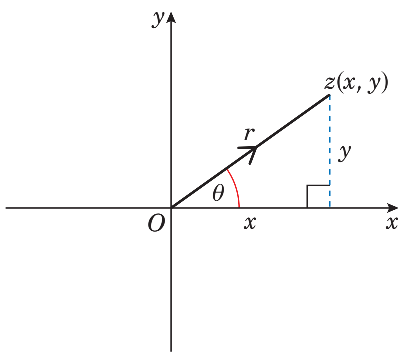
The **modulus–argument form** of the complex number $z = x + iy$ is

$$z = r(\cos\theta + i\sin\theta)$$

It is important for you to remember this formula.

where

- **r, a positive real number, is called the modulus and**
- **θ, an angle such that when $-\pi < \theta \leqslant \pi$, θ is called the principal argument.**



From the right-angled triangle,

$x = r\cos\theta$ and and $y = r\sin\theta$.

$r = |z| = \sqrt{x^2 + y^2}$

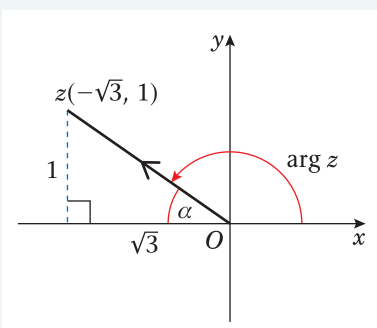
Note that θ, the argument, is not unique. The argument of z could also be $\theta \pm 2\pi$, $\theta \pm 4\pi$, etc.

To avoid duplication of θ, we usually quote θ in the range $-\pi < \theta \leqslant \pi$ and refer to it as the principal argument, 'arg', ie. $\theta = \arg z$.

$z = r(\cos\theta + i\sin\theta)$ is correct for a complex number in any of the Argand diagram quadrants.

Example 1

Express $z = -\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.

Here z is in the second quadrant so the required argument is $(\pi - \alpha)$.

$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

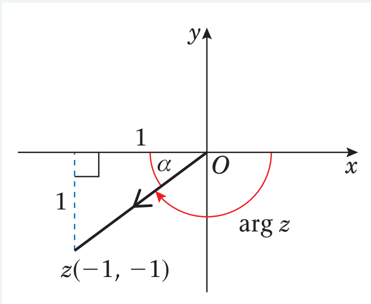
Find r and θ.

Therefore, $z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Apply $z = r(\cos\theta + i\sin\theta)$

Example 2

Express $z = -1 - \mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.

Here z is in the third quadrant so the required argument is $-\pi + \alpha$.

$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \arg z = -\pi + \tan^{-1}\left(\frac{1}{1}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

Find r and θ.

Therefore, $z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$

Apply $z = r(\cos\theta + \mathrm{i}\sin\theta)$

3.2 You can express a complex number in the form $z = r\mathrm{e}^{\mathrm{i}\theta}$.

In chapter 6, (and in your formula book), you will find the series expansions of $\cos\theta$ and $\cos\theta$. They are

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots + \frac{(-1)^r\,\theta^{2r}}{(2r)!} + \ldots\ldots \quad ①$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\ldots + \frac{(-1)^r\,\theta^{2r+1}}{(2r+1)!} + \ldots\ldots \quad ②$$

Also, for $x \in$!, the series expansion of e^x is

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots + \frac{x^r}{r!} + \ldots\ldots$$

It can be proved that the series expansion for e^x is also true if x is replaced by a complex number. If you replace x in e^x by $\mathrm{i}\theta$ the series expansion becomes

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \frac{(\mathrm{i}\theta)^6}{6!} + \ldots\ldots$$

$$= 1 + \mathrm{i}\theta + \frac{\mathrm{i}^2\theta^2}{2!} + \frac{\mathrm{i}^3\theta^3}{3!} + \frac{\mathrm{i}^4\theta^4}{4!} + \frac{\mathrm{i}^5\theta^5}{5!} + \frac{\mathrm{i}^6\theta^6}{6!} + \ldots\ldots$$

$$= 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathrm{i}\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots\ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

By comparing this series expansion with those of ① and ② you can write $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula is known as Euler's relation. It is important for you to remember this result.

You can now use Euler's relation to rewrite $z = r(\cos\theta + i\sin\theta)$ as

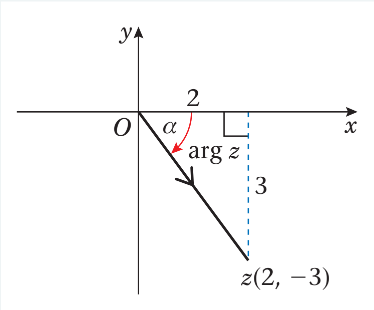
$$z = re^{i\theta}$$

This is the exponential form of the complex number z.

where $r = |z|$ and $\theta = \arg z$.

Example 3

Express $z = 2 - 3i$ in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.

Here z is in the fourth quadrant so the required argument is $-\alpha$.

$r = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$

$\theta = \arg z = -\tan^{-1}\left(\frac{3}{2}\right) = -0.98$ (2 d.p.)

Find r and θ.

Therefore, $z = \sqrt{13}\, e^{-0.98i}$

Apply $z = re^{i\theta}$.

In Chapter 8 of Core 2, you learnt the following properties which will be helpful to you in this chapter:

$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Example 4

Express

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$ **b** $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$ in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

Compare with $r(\cos\theta + i\sin\theta)$.

So, $r = \sqrt{2}$ and $\theta = \frac{\pi}{10}$.

Therefore, $z = \sqrt{2}\, e^{\frac{\pi}{10}i}$

Apply $z = re^{i\theta}$.

b $z = 5\left(\cos \frac{\pi}{8} - \text{i} \sin \frac{\pi}{8}\right)$ — Apply $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

$z = 5\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right)$ — Compare with $r(\cos\theta + \text{i}\sin\theta)$.

So, $r = 5$ and $\theta = -\frac{\pi}{8}$.

Therefore, $z = 5e^{-\frac{\pi i}{8}}$ — Apply $z = re^{\text{i}\theta}$.

Example 5

Express $z = \sqrt{2}\,\text{e}^{\frac{3\pi\text{i}}{4}}$ in the form $x + \text{i}y$, where $x \in \square$ and $y \in \square$.

$z = \sqrt{2}\,e^{\frac{3\pi i}{4}}$ — Compare with $re^{\text{i}\theta}$.

So, $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$.

$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right)$ — Apply $r(\cos\theta + \text{i}\sin\theta)$.

$= \sqrt{2}\; -\frac{1}{\sqrt{2}} + \text{i}\frac{1}{\sqrt{2}}$ — Apply $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$.

Therefore, $z = -1 + \text{i}$ — Simplify.

Example 6

Express $z = 2\text{e}^{\frac{23\pi\text{i}}{5}}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

$z = 2e^{\frac{23\pi i}{5}}$ — Compare with $re^{\text{i}\theta}$.

So, $r = 2$ and $\theta = \frac{23\pi}{5}$.

$\theta = \frac{23\pi}{5} \rightarrow \frac{13\pi}{5} \rightarrow \frac{3\pi}{5}$ — Continue to subtract 2π from θ until $2\pi < \theta \leqslant \pi$.

$z = \sqrt{2}\left(\cos\frac{3\pi}{5} + \text{i}\sin\frac{3\pi}{5}\right)$ — Apply $z = r(\cos\theta + \text{i}\sin\theta)$.

Example 7

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

$e^{i\theta} = \cos\theta + i\sin\theta$ ①

$e^{i(-\theta)} = e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

So, $e^{-i\theta} = \cos\theta - i\sin\theta$ ②

Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Adding ① and ② gives

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

Divide both sides by 2.

Hence, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, as required.

Exercise 3A

1 Express the following in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ where possible, or values to 2 dp otherwise.

a 7 **b** $-5i$ **c** $\sqrt{3} + i$ **d** $2 + 2i$ **e** $1 - i$

f -8 **g** $3 - 4i$ **h** $-8 + 6i$ **i** $2 - \sqrt{3}i$

2 Express the following in the form $x + iy$, where $x \in \square$ and $y \in \square$.

a $5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

b $\frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

c $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

d $3\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

e $2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

f $-4\left(\cos\frac{7\pi}{6} + i\sin - \frac{7\pi}{6}\right)$

3 Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ where possible, or values to 2 dp otherwise.

a -3 **b** $6i$ **c** $-2\sqrt{3} - 2i$

d $-8 + i$ **e** $2 - 5i$ **f** $-2\sqrt{3} + 2\sqrt{3}i$

g $\sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ **h** $8\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ **i** $2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

4 Express the following in the form $x + iy$ where $x \in \square$ and $y \in \square$.

a $e^{\frac{\pi}{3}i}$ **b** $4e^{\pi i}$ **c** $3\sqrt{2}\,e^{\frac{\pi i}{4}}$

d $8e^{\frac{\pi i}{6}}$ **e** $3e^{-\frac{\pi i}{3}}$ **f** $e^{i\frac{5\pi}{6}}$

g $e^{-\pi i}$ **h** $3\sqrt{2}e^{-\frac{3\pi}{4i}}$ **i** $8e^{-\frac{4\pi i}{3}}$

5 Express the following in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$

a $e^{\frac{16\pi}{13}i}$ **b** $4e^{\frac{17\pi}{5}i}$ **c** $5e^{-\frac{9\pi}{8}i}$

6 Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

3.3 You need to know how multiplying and dividing affects both the modulus and argument of the resulting complex number.

For the following proofs you need to apply the following identities found in the Core 2 and Core 3 sections of your formula book:

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \quad ④$$
$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \quad ⑤$$
$$\cos^2\theta_2 + \sin^2\theta_2 = 1 \quad ⑥$$

Multiplying complex numbers z_1 and z_2

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)\\ &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)\\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))\\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))\\ &= r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)), \text{ using identities ④ and ⑤.} \end{aligned}$$

Therefore the complex number $z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ is in a **modulus–argument form** and has modulus r_1r_2 and argument $\theta_1 + \theta_2$.

Also, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then

$$\begin{aligned} z_1z_2 &= \left(r_1e^{i\theta_1}\right)\left(r_2e^{i\theta_2}\right)\\ &= r_1r_2e^{i\theta_1 + i\theta_2}\\ &= r_1r_2e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Therefore the complex number $z_1z_2 = r_1r_2e^{i(\theta_1 + \theta_2)}$ is in an **exponential form** and has modulus r_1r_2 and argument $\theta_1 + \theta_2$.

Dividing a complex number z_1 by a complex number z_2

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)}\\ &= \frac{r_1(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{r_2(\cos\theta_2\cos\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 - i^2\sin\theta_2\sin\theta_2)}\\ &= \frac{r_1((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}\\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)), \text{ using identities ④, ⑤ and ⑥.} \end{aligned}$$

Therefore the complex number $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \text{i}\sin(\theta_1 - \theta_2))$ is in **modulus–argument form** and has modulus $\frac{r_1}{r_2}$ and argument $\theta_1 - \theta_2$.

Also, if $z_1 = r_1 e^{\text{i}\theta_1}$ and $z_2 = r_2 e^{\text{i}\theta_2}$ then

$$\frac{z_1}{z_2} = \frac{r_1 e^{\text{i}\theta_1}}{r_2 e^{\text{i}\theta_2}}$$

$$= \frac{r_1}{r_2} e^{\text{i}\theta_1} e^{-\text{i}\theta_2}$$

$$= \frac{r_1}{r_2} e^{\text{i}\theta_1 - \text{i}\theta_2}$$

$$= \frac{r_1}{r_2} e^{\text{i}(\theta_1 - \theta_2)}$$

Therefore the complex number $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{\text{i}(\theta_1 - \theta_2)}$ is in an **exponential form** and has modulus $\frac{r_1}{r_2}$ and argument $\theta_1 = \theta_2$.

In summary, you need to learn and apply the following results for complex numbers z_1 and z_2:

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

When you multiply z_1 by z_2
- you multiply their moduli and
- add their arguments as shown.

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

When you divide z_1 by z_2
- you divide their moduli and
- subtract their arguments as shown.

Example 8

Express $3\left(\cos\frac{5\pi}{12} + \text{i}\sin\frac{5\pi}{12}\right) \times 4\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right)$ in the form $x + \text{i}y$.

$$3\left(\cos\frac{5\pi}{12} + \text{i}\sin\frac{5\pi}{12}\right) \times 4\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right)$$

$$= 3(4)\left(\cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) + \text{i}\sin\left(\frac{5\pi}{12} + \frac{\pi}{12}\right)\right)$$

Apply the result, $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + \text{i}\sin(\theta_1 + \theta_2))$.

$$= 12\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$$

Simplify.

$$= 12(0 + \text{i}(1))$$

Apply $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

$$= 12\text{i}$$

Example 9

Express $2\left(\cos\frac{\pi}{15} + \mathrm{i}\sin\frac{\pi}{15}\right) \times 3\left(\cos\frac{2\pi}{5} - \mathrm{i}\sin\frac{2\pi}{5}\right)$ in the form $x + \mathrm{i}y$.

$$2\left(\cos\frac{\pi}{15} + \mathrm{i}\sin\frac{\pi}{15}\right) \times 3\left(\cos\frac{2\pi}{5} - \mathrm{i}\sin\frac{2\pi}{5}\right)$$

$z_2 = 3\left(\cos\frac{2\pi}{5} - \mathrm{i}\sin\frac{2\pi}{5}\right)$ must be written in the form $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$.

$$= 2\left(\cos\frac{\pi}{15} + \mathrm{i}\sin\frac{\pi}{15}\right) \times 3\left(\cos\left(-\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{5}\right)\right)$$

Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

$$= 2(3)\left(\cos\left(\frac{\pi}{15} - \frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi}{15} - \frac{2\pi}{5}\right)\right)$$

Apply the result, $z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$.

$$= 6\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$= 6\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right)$$

Apply $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

$$= 3 - 3\sqrt{3}\mathrm{i}$$

Example 10

Express $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}$ in the form $x + \mathrm{i}y$.

$$\frac{\sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}$$

$$= \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{12} - \frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{12} - \frac{5\pi}{6}\right)\right)$$

By applying the result, $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2))$.

$$= \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$$

Simplify.

$$= \frac{\sqrt{2}}{2}\left(-\frac{1}{\sqrt{2}} + \mathrm{i}\left(-\frac{1}{\sqrt{2}}\right)\right)$$

Apply $\cos\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

$$= -\frac{1}{2} - \frac{1}{2}\mathrm{i}$$

Exercise 3B

1 Express in the following in the form $x + iy$.

a $(\cos 2\theta + i \sin 2\theta)(\cos 3\theta + i \sin 3\theta)$

b $\left(\cos \frac{3\pi}{11} + i \sin \frac{3\pi}{11}\right)\left(\cos \frac{8\pi}{11} + i \sin \frac{8\pi}{11}\right)$

c $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \times 2\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$

d $\sqrt{6}\left(\cos \left(\frac{-\pi}{12}\right) + i \sin \left(\frac{-\pi}{12}\right)\right) \times \sqrt{3}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

e $4\left(\cos \left(\frac{-5\pi}{9}\right) + i \sin \left(\frac{-5\pi}{9}\right)\right) \times \frac{1}{2}\left(\cos \left(\frac{-5\pi}{18}\right) + i \sin \left(\frac{-5\pi}{18}\right)\right)$

f $6\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right) \times 5\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)$

g $(\cos 4\theta + i \sin 4\theta)(\cos \theta - i \sin \theta)$

h $3\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right) \times \sqrt{2}\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)$

2 Express the following in the form $x + iy$.

a $\dfrac{\cos 5\theta + i \sin 5\theta}{\cos 2\theta + i \sin 2\theta}$

b $\dfrac{\sqrt{2}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)}$

c $\dfrac{3\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)}{4\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)}$

d $\dfrac{\cos 2\theta - i \sin 2\theta}{\cos 3\theta + i \sin 3\theta}$

3 z and w are two complex numbers where

$$z = -9 + 3\sqrt{3}i, \ |w| = \sqrt{3} \text{ and } \arg w = \frac{7\pi}{12}.$$

Express the following in the form $r(\cos \theta + i \sin \theta)$,

a z, **b** w, **c** zw, **d** $\frac{z}{w}$,

where $-\pi < \theta \leq \pi$.

3.4 You need to be able to prove that $[r\cos\theta + \text{i}\sin\theta)]^n = r^n(\cos n\theta + \text{i}\sin n\theta)$ for any integer n.

If $z = r(\cos\theta + \text{i}\sin\theta)$, then

$$z^2 = z \times z = r(\cos\theta + \text{i}\sin\theta) \times r(\cos\theta + \text{i}\sin\theta)$$
$$= rr(\cos\theta + \theta) + \text{i}\sin(\theta + \theta))$$
$$= r^2(\cos 2\theta + \text{i}\sin 2\theta),$$

By applying the result, $z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + \text{i}\sin(\theta_1 + \theta_2))$

$$z^3 = z^2 \times z = r^2(\cos 2\theta + \text{i}\sin 2\theta) \times r(\cos\theta + \text{i}\sin\theta)$$
$$= r^2r(\cos 2\theta + \theta) + \text{i}\sin(2\theta + \theta))$$
$$= r^3(\cos 3\theta + \text{i}\sin 3\theta)$$

By applying the result, $z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + \text{i}\sin(\theta_1 + \theta_2))$

$$z^4 = z^3 \times z = r^3(\cos 3\theta + \text{i}\sin 3\theta) \times r(\cos\theta + \text{i}\sin\theta)$$
$$= r^3r(\cos 3\theta + \theta) + \text{i}\sin(3\theta + \theta))$$
$$= r^4(\cos 4\theta + \text{i}\sin 4\theta)$$

By applying the result, $z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + \text{i}\sin(\theta_1 + \theta_2))$

The above results show that

- $z^1 = [r(\cos\theta + \text{i}\sin\theta)]^1 = r(\cos\theta + \text{i}\sin\theta)$
- $z^2 = [r(\cos\theta + \text{i}\sin\theta)]^2 = r^2(\cos 2\theta + \text{i}\sin 2\theta)$
- $z^3 = [r(\cos\theta + \text{i}\sin\theta)]^3 = r^3(\cos 3\theta + \text{i}\sin 3\theta)$
- $z^4 = [r(\cos\theta + \text{i}\sin\theta)]^4 = r^4(\cos 4\theta + \text{i}\sin 4\theta)$

Therefore it follows that the **general statement** for any positive integer, n is

■ $z^n = [r(\cos\theta + \text{i}\sin\theta)]^n = r^n(\cos n\theta + \text{i}\sin n\theta)$

This is de Moivre's theorem. It is important for you to remember this result.

It will be shown that de Moivre's theorem is true for any integer n.

Proof of de Moivre's theorem when n is a positive integer

You can use the method of proof by induction (found in FP1) to prove that $[r(\cos\theta + \text{i}\sin\theta)]^n = r^n(\cos n\theta + \text{i}\sin n\theta)$ is true for all positive integers.

1. Basis Step

$n = 1$; LHS $= [r(\cos\theta + \text{i}\sin\theta)]^1 = r(\cos\theta + \text{i}\sin\theta)$

RHS $= r^1(\cos 1\theta + \text{i}\sin 1\theta) = r(\cos\theta + \text{i}\sin\theta)$

As LHS = RHS, de Moivre's theorem is true for $n = 1$.

2. Assumption Step

Assume that de Moivre's theorem is true for $n = k$, $k \in \square^+$.

ie. $[r(\cos\theta + \text{i}\sin\theta)]^k = r^k(\cos k\theta + \text{i}\sin k\theta)$

3. Inductive Step

With $n = k + 1$, de Moivre's theorem becomes:

$$[r(\cos\theta + \text{i}\sin\theta)]^{k+1} = [r(\cos\theta + \text{i}\sin\theta)]^k \times r(\cos\theta + \text{i}\sin\theta)$$
$$= r^k(\cos k\theta + \text{i}\sin k\theta) \times r(\cos\theta + \text{i}\sin\theta)$$ (by assumption step)
$$= r^{k+1}(\cos k\theta + \text{i}\sin k\theta)(\cos\theta + \text{i}\sin\theta)$$
$$= r^{k+1}(\cos k\theta + \theta) + \text{i}\sin(k\theta + \theta))$$ (from section 3.3)
$$= r^{k+1}(\cos(k+1)\theta + \text{i}\sin(k+1)\theta)$$

Therefore, de Moivre's theorem is true when $n = k + 1$.

If de Moivre's theorem is true for $n = k$, then it has been shown to be true for $n = k + 1$.

4. Conclusion Step

As de Moivre's theorem is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Proof of de Moivre's theorem when *n* is a negative integer

We will now prove that $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ is true for all negative integers.

If n is a negative integer, it can then be written in the form $n = -m$, where m is a positive integer.

$$\text{LHS} = [r(\cos\theta + i\sin\theta)]^n = [r(\cos\theta + i\sin\theta)]^{-m}$$

$$= \frac{1}{[r(\cos\theta + i\sin\theta)]^m}$$

$$= \frac{1}{r^m(\cos m\theta + i\sin m\theta)}$$ — Applying de Moivre's theorem for positive integer m.

$$= \frac{1}{r^m(\cos m\theta + i\sin m\theta)} \times \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{r^m(\cos^2 m\theta - i^2\sin^2 m\theta)}$$ — Difference of two squares.

$$= \frac{1}{r^m}\frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta + \sin^2 m\theta)}$$ — Identity $\cos^2 m\theta + \sin^2 m\theta = 1$

$$= r^{-m}(\cos m\theta - i\sin m\theta)$$ — $\frac{1}{r^m} = r^{-m}$

$$= r^{-m}(\cos(-m\theta) + i\sin(-m\theta))$$ — Using $\cos\theta = \cos(-\theta)$ and $\sin\theta = -\sin(-\theta)$.

$$= r^n(\cos n\theta + i\sin n\theta) = \text{RHS}$$ — Applying $m = -n$.

Therefore, we have proved that de Moivre's theorem is true when n is a negative integer.

Also for,

$n = 0$; $\text{LHS} = [r(\cos\theta + i\sin\theta)]^0 = 1$

$\text{RHS} = r^0(\cos 0 + i\sin 0) = 1(1) = 1$

As LHS = RHS, de Moivre's theorem is true for $n = 0$.

Therefore we have proved that de Moivre's theorem is true for any integer n.

de Moivre's theorem can also be written in exponential form.

If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$z^n = [r(\cos\theta + i\sin\theta)]^n = [re^{i\theta}]^n$$

$$= r^n(e^{i\theta})^n = r^n e^{in\theta}$$ — Applying $(x^a)^b = x^{ab}$.

$$= r^n(\cos n\theta + i\sin n\theta)$$

Therefore $[re^{i\theta}]^n = r^n e^{in\theta}$ — This is de Moivre's theorem, stated in an exponential form.

Example 11

Simplify $\dfrac{\left(\cos\dfrac{9\pi}{17} + \mathrm{i}\sin\dfrac{9\pi}{17}\right)^5}{\left(\cos\dfrac{2\pi}{17} - \mathrm{i}\sin\dfrac{2\pi}{17}\right)^3}$.

$$\frac{\left(\cos\frac{9\pi}{17} + \mathrm{i}\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - \mathrm{i}\sin\frac{2\pi}{17}\right)^3}$$

$$= \frac{\left(\cos\frac{9\pi}{17} + \mathrm{i}\sin\frac{9\pi}{17}\right)^5}{\left(\cos\left(-\frac{2\pi}{17}\right) - \mathrm{i}\sin\left(-\frac{2\pi}{17}\right)\right)^3}$$

Apply $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ to the denominator.

$$= \frac{\cos\frac{45\pi}{17} + \mathrm{i}\sin\frac{45\pi}{17}}{\cos\left(-\frac{6\pi}{17}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{17}\right)}$$

Apply de Moivre's theorem to both the numerator and the denominator.

$$= \cos\left(\frac{45\pi}{17} - -\frac{6\pi}{17}\right) + \mathrm{i}\sin\left(\frac{45\pi}{17} - -\frac{6\pi}{17}\right)$$

By applying the result, $\frac{z_1}{z_2} = \cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)$.

$$= \cos\frac{51\pi}{17} + \mathrm{i}\sin\frac{51\pi}{17}$$

Simplify.

$$= \cos 3\pi + \mathrm{i}\sin 3\pi$$

$$= \cos\pi + \mathrm{i}\sin\pi$$

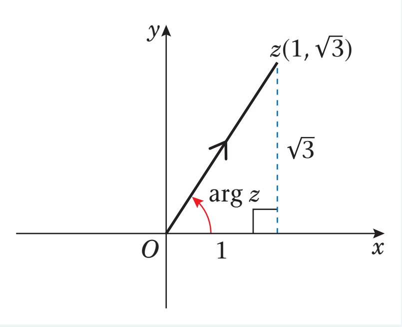
Subtract 2π from the argument.

Apply $\cos\pi = -1$ and $\sin\pi = 0$.

Therefore, $\dfrac{\left(\cos\dfrac{9\pi}{17} + \mathrm{i}\sin\dfrac{9\pi}{17}\right)^5}{\left(\cos\dfrac{2\pi}{17} - \mathrm{i}\sin\dfrac{2\pi}{17}\right)^3} = -1.$

Example 12

Express $(1 + \sqrt{3}\mathrm{i})^7$ in the form $x + \mathrm{i}y$ where $x \in \square$ and $y \in \square$.



Firstly you need to find the modulus and argument of $1 + \sqrt{3}\mathrm{i}$.

You may want to draw an Argand diagram to help you.

$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \arg z = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

So, $1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$(1 + \sqrt{3}i)^7 = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^7$

$= 2^7\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)$

$= 512\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 512\left(\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$

Therefore $(1 + \sqrt{3}i)^9 = 256 + 256\sqrt{3}i$.

Find r and θ.

Apply $z = r(\cos\theta + i\sin\theta)$.

Apply de Moivre's theorem.

Subtract 2π from the argument.

Apply $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Exercise 3C

1 Use de Moivre's theorem to simplify each of the following:

a $(\cos\theta + i\sin\theta)^6$

b $(\cos 3\theta + i\sin 3\theta)^4$

c $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

d $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

e $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$

f $\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)^{15}$

g $\frac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta + i\sin 2\theta}$

h $\frac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

i $\frac{1}{(\cos 2\theta + i\sin 2\theta)^3}$

j $\frac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3}$

k $\frac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta + i\sin 3\theta}$

l $\frac{\cos\theta - i\sin\theta}{\cos 2\theta - i\sin 2\theta}$

2 Evaluate $\frac{\left(\cos\frac{7\pi}{13} + i\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13} - i\sin\frac{4\pi}{13}\right)^6}$.

3 Express the following in the form $x + iy$ where $x \in \square$ and $y \in \square$.

a $(1 + i)^5$

b $(-2 + 2i)^8$

c $(1 - i)^6$

d $(1 - \sqrt{3}i)^6$

e $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$

f $\left(-2\sqrt{3} - 2i\right)^5$

4 Express $(3 + \sqrt{3}i)^5$ in the form $a + b\sqrt{3}i$ where a and b are integers.

3.5 You can apply de Moivre's theorem to trigonometric identities.

You need to be able to apply the following binomial expansion found in the Core 2 section of your formula book.

$$(a + b)^n = a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}\, b^3 + \ldots\ldots + b^n$$

Example 13

Express $\cos 3\theta$ in terms of powers of $\cos\theta$.

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ — de Moivre's theorem.

$= \cos^3\theta + {}^3C_1\cos^2\theta\,(i\sin\theta) + {}^3C_2\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$ — Applying the binomial expansion to $(\cos\theta + i\sin\theta)^3$ where $a = \cos\theta$ and $b = i\sin\theta$.

$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$ — Simplify.

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ — Applying $i^2 = -1$ and $i^3 = -i$.

Equating the real parts gives

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ — Note for the LHS that the real part of $\cos 3\theta + i\sin 3\theta$ is $\cos 3\theta$.

$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta)$ — Apply $\sin^2\theta = 1 - \cos^2\theta$.

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$ — Multiplying out brackets.

$= 4\cos^3\theta - 3\cos\theta$ — Simplify.

Therefore, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Example 14

Express

a $\cos 6\theta$ in terms of powers of $\cos\theta$,

b $\dfrac{\sin 6\theta}{\sin\theta}$, $\theta \neq n\pi$, $n \in \square$, in terms of powers of $\cos\theta$.

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$ — de Moivre's theorem.

$= \cos^6\theta + {}^6C_1\cos^5\theta(i\sin\theta) + {}^6C_2\cos^4\theta(i\sin\theta)^2 + {}^6C_3\cos^3\theta(i\sin\theta)^3 + {}^6C_4\cos^2\theta(i\sin\theta)^4 + {}^6C_5\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$ — Applying the binomial expansion to $(\cos\theta + i\sin\theta)^3$ where $a = \cos\theta$ and $b = i\sin\theta$.

$= \cos^6\theta + 6i\cos^5\theta\sin\theta + 15i^2\cos^4\theta\sin^2\theta + 20i^3\cos^3\theta\sin^3\theta + 15i^4\cos^2\theta\sin^4\theta + 6i^5\cos\theta\sin^5\theta + i^6\sin^6\theta$ — Simplify.

$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$ — Applying $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and $i^6 = -i$.

a Equating the real parts gives

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

Note for the LHS that the real part of $\cos 6\theta + i\sin 6\theta$ is $\cos 6\theta$.

$$= \cos^6\theta - 15\cos^4\theta\,(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$\sin^2\theta = 1 - \cos^2\theta$, $\sin^4\theta = (\sin^2\theta)^2$ and $\sin^6\theta = (\sin^2\theta)^3$.

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta) - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

Multiplying out brackets.

Applying a cubic binomial expansion.

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$$

Expand brackets.

$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

Simplify.

Therefore, $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

b Equating the imaginary parts gives

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

Note for the LHS that the imaginary part of $\cos 6\theta + i\sin 6\theta$ is $\sin 6\theta$.

So $$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta\sin^2\theta + 6\cos\theta\sin^4\theta$$

Dividing both sides by $\sin\theta$ gives

$$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$$

$\sin^4\theta = (\sin^2\theta)^2$ and $\sin^2\theta = 1 - \cos^2\theta$.

$$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$$

Multiplying out brackets.

$$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta - 12\cos^3\theta + 6\cos^5\theta$$

Expand brackets.

$$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$$

Simplify.

Therefore,

$$\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta.$$

Now we will investigate finding trigonometric identities for $\sin^n \theta$ and $\cos^n \theta$ where n is a positive integer.

If $z = \cos\theta + i\sin\theta$, then

$$\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1} = (\cos(-1\theta) + i\sin(-1\theta)) = \cos\theta - i\sin\theta$$

Applying de Moivre's theorem.

Using $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$.

It follows that

$$z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta$$

Also,

$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Applying de Moivre's theorem.

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n} = (\cos(-n\theta) + i\sin(-n\theta)) = \cos n\theta - i\sin n\theta$$

Applying de Moivre's theorem.

Using $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$.

It follows that

$$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta) = 2i\sin n\theta$$

To summarise, you need to be able to apply these results:

$$z + \frac{1}{z} = 2\cos\theta \qquad z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z - \frac{1}{z} = 2i\sin\theta \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Example 15

Express $\cos^5\theta$ in the form $a\cos 5\theta + b\sin 3\theta + C\cos\theta$, where a, b and c are constants.

$$\left(z + \frac{1}{z}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$$

Applying $z + \frac{1}{z} = 2\cos\theta$.

$$= z^5 + {}^5C_1 z^4\left(\frac{1}{z}\right) + {}^5C_2 z^3\left(\frac{1}{z}\right)^2 + {}^5C_3 z^2\left(\frac{1}{z}\right)^3 + {}^5C_4 z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5$$

Applying the binomial expansion to $\left(z + \frac{1}{z}\right)^5$ where $a = z$ and $b = \frac{1}{z}$.

$$= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^5}\right)$$

Simplify.

$$= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

Simplify further.

$$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

Group z^n and $\frac{1}{z^n}$ terms.

$$= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$$

Applying $z^n + \frac{1}{z^n} = 2\cos n\theta$.

So, $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

Put LHS $= 32\cos^5\theta =$ RHS.

and $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$

$a = \frac{1}{16}$, $b = \frac{5}{16}$ and $c = \frac{5}{8}$.

Example 16

Prove that $\sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$.

$\left(z-\frac{1}{z}\right)^3 = (2i\sin\theta)^3 = 8i^3\sin^3\theta = -8i\sin^3\theta$	Applying $z - \frac{1}{z} = 2i\sin\theta$.
$= z^3 + {}^3C_1 z^2\left(-\frac{1}{z}\right) + {}^3C_2 z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3$	Applying the binomial expansion to $\left(z-\frac{1}{z}\right)^3$ where $a = z$ and $b = -\frac{1}{z}$.
$= z^3 + 3z^2\left(-\frac{1}{z}\right) + 3z\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z^3}\right)$	Simplify.
$= z^3 - 3z + \frac{3}{z^2} - \frac{1}{z^3}$	Simplify further.
$= \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$	Group z^n and $\frac{1}{z^n}$ terms.
$= 2i\sin 3\theta - 3(2i\sin\theta)$	Applying $z^n - \frac{1}{z^n} = 2i\sin n\theta$.
So, $-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$	Put LHS $= -8i\sin^3\theta =$ RHS.
and $\sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$	Divide both sides by -8

Example 17

a Express $\sin^4\theta$ in the form $d\cos 4\theta + e\cos 2\theta + f$, where d, e and f are constants.

b Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$.

a $\left(z-\frac{1}{z}\right)^4 = (2i\sin\theta)4 = 16i4\sin4\theta = 16\sin4\theta$	Applying $z - \frac{1}{z} = 2i\sin\theta$.
$= z^4 + {}^4C_1 z^3\left(-\frac{1}{z}\right) + {}^4C_2 z^2\left(-\frac{1}{z}\right)^2 + {}^4C_3 z^1\left(-\frac{1}{z}\right)^3 + \left(-\frac{1}{z}\right)^4$	Applying the binomial expansion to $\left(z-\frac{1}{z}\right)^4$ where $a = z$ and $b = -\frac{1}{z}$.
$= z^4 + 4z^3\left(-\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(-\frac{1}{z^3}\right) + \left(\frac{1}{z^4}\right)$	Simplify.
$= z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$	Simplify further.
$= \left(z^4 - \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$	Group z^n and $\frac{1}{z^n}$ terms.
$= 2\cos 4\theta - 4(2\cos 2\theta) + 6$	Applying $z^n + \frac{1}{z^n} = 2\cos n\theta$.
So, $16i\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$	Put LHS $= 16\sin^4\theta =$ RHS.
and $\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$	$d = \frac{1}{8}$, $e = -\frac{1}{2}$ and $f = \frac{3}{8}$.

b $\int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}\right) d\theta$

$= \left[\frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{1}{32}\sin 2\pi - \frac{1}{6}\sin \pi + \frac{3}{8}\left(\frac{\pi}{2}\right)\right) - (0)$

$= \left(0 - 0 + \frac{3\pi}{16}\right) - (0)$

So $\int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta = \frac{3\pi}{16}$.

Use your answer from part **a**.

$\cos k\theta$ integrates to $\frac{1}{k}\sin k\theta$.

Insert in limits and subtract.

Exercise 3D

Use applications of de Moivre's theorem to prove the following trigonometric identities:

1 $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

2 $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

3 $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

4 $\cos^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

5 $\cos^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

6 **a** Show that $32\cos^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta + 10$.

b Hence find $\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are constants.

7 **a** Use de Moivre's theorem to show that $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

b Hence, or otherwise, show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

c Use your answer to part **b** to find, to 2 dp, the four solutions of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

3.6 You can use de Moivre's theorem to find the n^{th} roots of a complex number.

In this section we will apply the following results:

1 As the argument θ is not unique, the complex number $z = r(\cos\theta + i\sin\theta)$ can also be expressed in the form $z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$, where $k = \square$.

2 de Moivre's theorem states that $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$, where $n \in \square$.

Note that it can be proved (but is beyond the scope of the FP2 unit) that de Moivre's theorem is also valid when n is a rational number (i.e. $n \in \square$).

Example 18

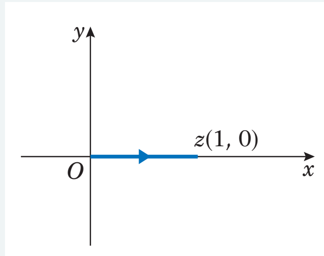
a Solve the equation $z^3 = 1$.

b Represent your solutions to part **a** on an Argand diagram.

a Simply $r = 1$ and $\theta = \arg z = 0$

Firstly we need to find both the modulus and argument of 1.

Find r and θ.



Apply $r(\cos\theta + \mathrm{i}\sin\theta)$.

So, $z^3 = 1(\cos 0 + \mathrm{i}\sin 0)$

$z^3 = (\cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi))$

$r(\cos(\theta + 2k\pi) + \mathrm{i}\sin(\theta + 2k\pi))$

Hence, $z = [(\cos(2k\pi) + \mathrm{i}\sin(0 + 2k\pi))]^{\frac{1}{3}}$

Take the n^{th} root.

So, $z = \cos\left(\frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{3}\right)$

Apply de Moivre's theorem. Now we need to find the three roots.

$k = 0$, $z_1 = \cos 0 + \mathrm{i}\sin 0 = 1$

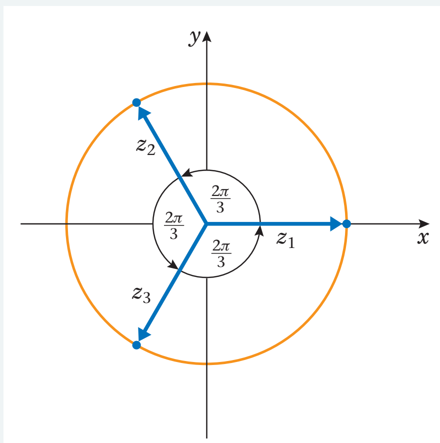
$k = 1$, $z_2 = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3} = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$

$k = -1$, $z_3 = \cos\left(\frac{-2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-2\pi}{3}\right)$

$= -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$

Change the value of k to find the three roots where the argument lies in the interval $-\pi < \theta \leqslant \pi$.

b Plot the plots $z_1 = 1$, $z_2 = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$ and $z_2 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$ on the Argand diagram.



z_1, z_2 and z_3 are the cube roots of unity.

The points z_1, z_2 and z_3 lie on a circle of radius 1 unit.

The angles between each of the vectors z_1, z_2 and z_3 are $\frac{2\pi}{3}$, as shown on the Argand diagram.

We can also write the three cube roots of 1 as 1, ω and ω^2, where

- $\omega = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$
- $\omega^2 = \omega \times \omega = \left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{4} - \frac{1}{4}\sqrt{3}\mathrm{i} - \frac{1}{4}\sqrt{3}\mathrm{i} - \frac{3}{4}$

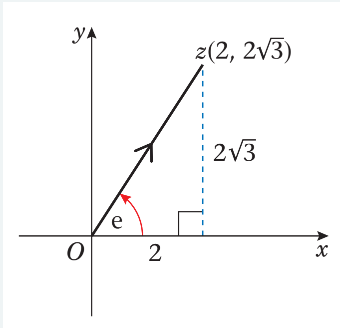
$= -\frac{1}{2} - \frac{1}{2}\sqrt{3}\mathrm{i}$

Also further note that

- $\omega^* = \omega^2$ and
- $1 + \omega + \omega^2 = 1 + \left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - \frac{1}{2}\sqrt{3}\mathrm{i}\right) = 0$

Example 19

Solve the equation $z^4 = 2 + 2\sqrt{3}\text{i}$.



Firstly we need to find both the modulus and argument of $2 + 2\sqrt{3}\text{i}$.

Find r and θ.

$$r = \sqrt{(2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4$$

$$\theta = \arg z = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Apply $r(\cos\theta + \text{i}\sin\theta)$.

$$\text{So, } z^4 = 4\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right).$$

$r(\cos(\theta + 2k\pi) + \text{i}\sin(\theta + 2k\pi))$.

$$z^4 = 4\left(\cos\frac{\pi}{3} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{3} + 2k\pi\right)$$

Take the n^{th} root.

$$\text{Hence, } = \left[4\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{3} + 2k\pi\right)\right)\right]^{\frac{1}{4}}$$

Apply de Moivre's theorem.

$$= (4)^{\frac{1}{4}}\left(\cos\left(\frac{\frac{\pi}{3} + 2k\pi}{4}\right) + \text{i}\sin\left(\frac{\frac{\pi}{3} + 2\,k\pi}{4}\right)\right)$$

Simplify and note $\sqrt{2}\,\sqrt{2}\,\sqrt{2}\,\sqrt{2} = 4$.

$$= \sqrt{2}\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right) + \text{i}\sin\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right)\right)$$

$$\text{So, } z = \sqrt{2}\left(\cos\left(\frac{\pi}{12} + \frac{k\pi}{2}\right) + \text{i}\sin\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$$

Now we need to find the four roots.

Change the value of k to find the four roots where the argument lies in the interval $-\pi < \theta \leqslant \pi$.

$$k = 0,\ z = \sqrt{2}\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right)$$

$$k = 1,\ z = \sqrt{2}\left(\cos\frac{7\pi}{12} + \text{i}\sin\frac{7\pi}{12}\right)$$

$$k = -1,\ z = \sqrt{2}\left(\cos\left(\frac{-5\pi}{12}\right) + \text{i}\sin\left(\frac{-5\pi}{12}\right)\right)$$

$$k = -1,\ z = \sqrt{2}\left(\cos\left(\frac{-11\pi}{12}\right) + \text{i}\sin\left(\frac{-11\pi}{12}\right)\right)$$

Solutions in the form $r(\cos\theta + \text{i}\sin\theta)$.

$$\text{So } z = \sqrt{2}\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right), \sqrt{2}\left(\cos\frac{7\pi}{12} + \text{i}\sin\frac{7\pi}{12}\right),$$

$$\sqrt{2}\left(\cos\left(\frac{-5\pi}{12}\right) + \text{i}\sin\left(\frac{-5\pi}{12}\right)\right)$$

$$\text{and } \sqrt{2}\left(\cos\left(\frac{-11\pi}{12}\right) + \text{i}\sin\left(\frac{-11\pi}{12}\right)\right)$$

Solutions in the form $re^{\text{i}\theta}$.

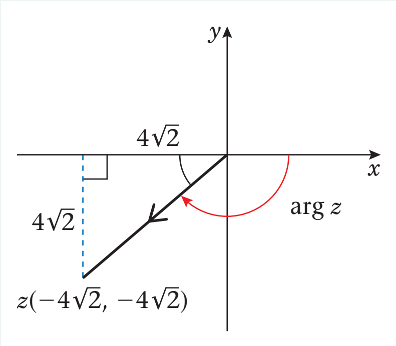
$$\text{or } z = \sqrt{2}e^{\frac{\pi i}{12}}, \sqrt{2}e^{\frac{7\pi i}{12}}, \sqrt{2}e^{\frac{-5\pi i}{12}} \text{ and } \sqrt{2}e^{\frac{-11\pi i}{12}}.$$

Example 20

Solve the equation $z^3 + 4\sqrt{2} + 4\sqrt{2}i = 0$.

$z^3 + 4\sqrt{2} + 4\sqrt{2}i = 0.$

$z^3 = -4\sqrt{2} - 4\sqrt{2}i$ — Make z^3 the subject.



First we need to find both the modulus and argument of $2 + 2\sqrt{3}i$.

$r = \sqrt{(-4\sqrt{2})^2 + (-4\sqrt{2})^2} = \sqrt{32 + 32} = \sqrt{64} = 8$ — Find r and θ.

$\theta = \arg z = -\pi + \tan^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

So, $z^3 = 8\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right).$ — Apply $r(\cos\theta + i\sin\theta)$.

$z^3 = 8\left(\cos\left(-\frac{3\pi}{4} + 2k\pi\right) + i\sin\left(-\frac{3\pi}{4} + 2k\pi\right)\right)$ — $r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$.

Hence, $z = \left[8\left(\cos\left(-\frac{3\pi}{4} + 2k\pi\right) + i\sin\left(-\frac{3\pi}{4} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$ — Take the n^{th} root.

$= (8)^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{3\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{-\frac{3\pi}{4} + 2k\pi}{3}\right)\right)$ — Apply de Moivre's theorem.

So, $z = 2\left(\cos\left(-\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right)$ — Now we need to find the three roots.

$k = 0,\ z = 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$k = 1,\ z = 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

$k = -1,\ z = 2\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(\frac{-11\pi}{12}\right)\right)$

Change the value of k to find the three roots where the argument lies in the interval $-\pi < \theta \leqslant \pi$.

So $z = 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right),\ 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right),$

and $2\left(\cos\left(\frac{-11\pi}{12}\right) + i\sin\left(\frac{-11\pi}{12}\right)\right)$ — Solutions in the form $r(\cos\theta + i\sin\theta)$.

or $z = 2e^{\frac{-\pi i}{4}},\ 2e^{\frac{5\pi i}{12}}$ and $2e^{\frac{-11\pi i}{12}}$. — Solutions in the form $re^{i\theta}$.

Exercise 3E

1 Solve the following equations, expressing your answers for z in the form $x + iy$, where $x \in \square$ and $y \in \square$.

a $z^4 - 1 = 0$ **b** $z^3 - i = 0$ **c** $z^3 = 27$

d $z^4 + 64 = 0$ **e** $z^4 + 4 = 0$ **f** $z^3 + 8i = 0$

2 Solve the following equations, expressing your answers for z in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $z^7 = 1$ **b** $z^4 + 16i = 0$ **c** $z^5 + 32 = 0$

d $z^3 = 2 + 2i$ **e** $z^4 + 2\sqrt{3}i = 2$ **f** $z^3 + 16\sqrt{3} + 16i = 0$

3 Solve the following equations, expressing your answers for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give θ to 2 dp.

a $z^4 = 3 + 4i$ **b** $z^3 = \sqrt{11} - 4i$ **c** $z^4 = -\sqrt{7} + 3i$

4 **a** Find the three roots of the equation $(z + 1)^3 = -1$.
Give your answers in the form $x + iy$, where $x \in \square$ and $y \in \square$.

b Plot the points representing these three roots on an Argand diagram.

c Given that these three points lie on a circle, find its centre and radius.

5 **a** Find the five roots of the equation $z^5 - 1 = 0$.
Give your answers in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

b Given that the sum of all five roots of $z^5 - 1 = 0$ is zero, show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}.$$

6 **a** Find the modulus and argument of $-2 - 2\sqrt{3}i$.

b Hence find all the solutions of the equation $z^4 + 2 + 2\sqrt{3}i = 0$.
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

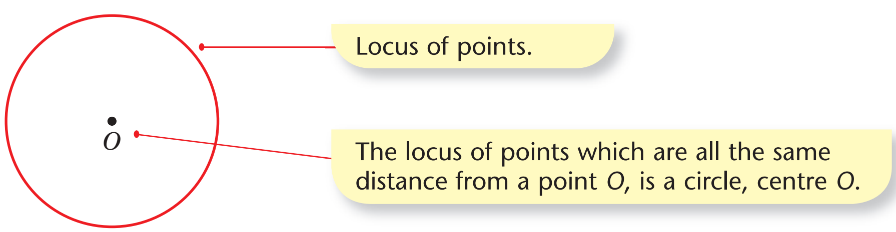
7 **a** Find the modulus and argument of $\sqrt{6} + \sqrt{2}i$.

b Solve the equation. Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

3.7 You can use complex numbers to represent a locus of points on an Argand diagram.

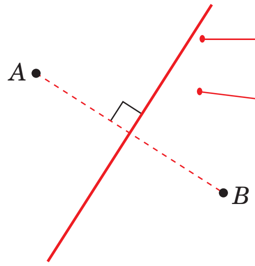
A locus of points is a set of points which obey a particular rule. Examples of loci are:

- a circle



The Cartesian equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

- a perpendicular bisector

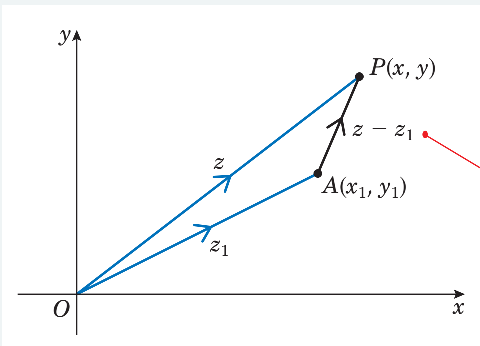


Locus of points.

The locus of points which are all equidistant from both the point A and the point B is the perpendicular bisector of the line segment AB.

Example 21

If $z = x + iy$ represents the variable point $P(x, y)$ and $z_1 = x_1 + iy_1$ represents the fixed point $A(x_1, y_1)$, what does $|z - z_1|$ represent on an Argand diagram?



$|z - z_1|$ is the modulus or length of the vector $\overrightarrow{AP}$. Therefore $|z - z_1|$ represents the length of the line joining the fixed point $A(z_1, y_1)$ to the variable point $P(x, y)$. Put simply $|z - z_1|$ represents the distance between the fixed point $A(z_1, y_1)$ and the variable point $P(x, y)$.

From the Argand diagram, $\overrightarrow{OP} = z$ and $\overrightarrow{OA} = z_1$.

Hence, $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$
$= -z_1 + z$
$= z - z_1$

So $z - z_1$ represents the vector $\overrightarrow{AP}$.

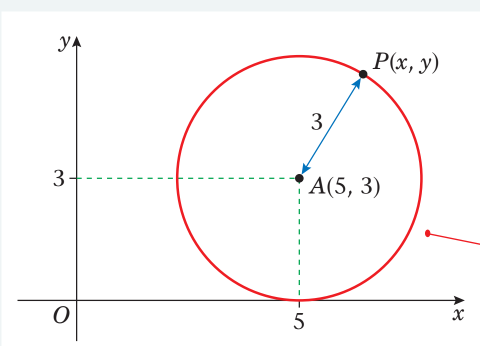
Example 22

If $|z - 5 - 3i| = 3$,

a sketch the locus of $P(x, y)$ which is represented by z on an Argand diagram,

b use an algebraic method to find the Cartesian equation of this locus.

a



$|z - 5 - 3i|$ can be written as $|z - (5 + 3i)|$ and this represents the distance between the fixed point $A(5, 3)$ and the variable point $P(x, y)$.

As this distance is always equal to 3, then the locus of P is a circle centre $(5, 3)$, radius 3.

b $|z - 5 - 3i| = 3$

$\Rightarrow \quad |x + iy - 5 - 3i| = 3$

$\Rightarrow \quad |(x - 5) + i(y - 3)| = 3$

$\Rightarrow \quad (x - 5)^2 + (y - 3)^2 = 3^2$

Hence the Cartesian equation of the locus of P is $(z - 5)^2 + (y - 3)^2 = 9$.

z can be rewritten as $z = x + iy$.

Group the real and imaginary parts.

Apply Pythagoras' theorem.

This equation could have been written down.

It follows that

$|z - z_1| = r$ is represented by a circle centre (x_1, y_1) with radius r, where $z_1 = x_1 + iy_1$.

Example 23

Give a geometrical interpretation of the locus of points z, represented by

a $|z - 3i| = 4$ **b** $|z - (2 + 3i)| = 5$ **c** $|z - 3 + 5i| = 2$ **d** $|2 - 5i - z| = 3$

a $z - 3i = 4$ is a circle centre $(0, 3)$, radius 4.

b $|z - (2 + 3i)| = 5$ is a circle centre $(2, 3)$ and radius 5.

c $|z - 3 + 5i| = 2$ is a circle centre $(3, -5)$ and radius 2.

d $|2 - 5i - z| = 3$ can be rewritten as

$|(-1)(z - 2 + 5i)| = 3$

$\Rightarrow \quad |(-1)||z - 2 + 5i| = 3 \quad \Rightarrow \quad |z - 2 + 5i| = 3$

So $|2 - 5i - z| = 3$ is a circle centre $(2, -5)$ and radius 3.

Applying $|z - z_1| = r$, where z_1 is the centre of the circle and r is the radius of the circle.

Apply $|z_1 z_2| = |z_1||z_2|$.

Apply $|-1| = 1$.

Applying $|z - z_1| = r$.

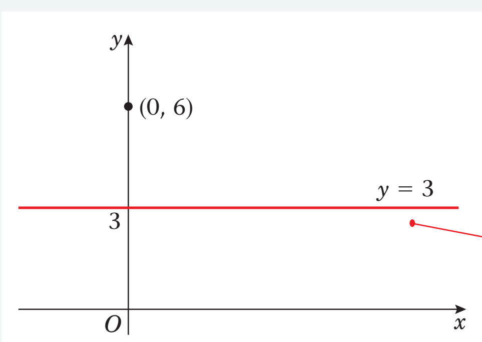
Example 24

If $|z| = |z - 6i|$,

a sketch the locus of $P(x, y)$ which is represented by z on an Argand diagram,

b use an algebraic method to find the Cartesian equation of this locus.

a



$|z|$ represents the distance from the origin to P. $|z - 6i|$ represents the distance from the point $(0, 6)$ to P. As $|z| = |z - 6i|$, then P is the locus of points which are equidistant from the points $(0, 0)$ and $(0, 6)$. Therefore the locus of P is the perpendicular bisector of the line joining $(0, 0)$ and $(0, 6)$, which has equation $y = 3$.

b $|z| = |z - 6\text{i}|$

$\Rightarrow \quad |x + \text{i}y| = |x + \text{i}y - 6\text{i}|$ — z can be rewritten as $z = x + \text{i}y$.

$\Rightarrow \quad |x + \text{i}y| = |x + \text{i}(y - 6)|$ — Group the real and imaginary parts.

$\Rightarrow \quad x^2 + y^2 = x^2 + (y - 6)^2$ — Remove the moduli.

$\Rightarrow \quad x^2 + y^2 = x^2 + y^2 - 12y + 36$ — Expand brackets.

$\Rightarrow \quad 12y = 36$ — Simplify.

Hence the Cartesian equation of the locus of P is $y = 3$. — This equation could have been written down.

Example 25

If $|z - 3| = |z + \text{i}|$,

a use an algebraic method to find a Cartesian equation of the locus of z,

b sketch the locus of z on an Argand diagram.

a $|z - 3| = |z + \text{i}|$

$\Rightarrow \quad |x + \text{i}y - 3| = |x + \text{i}y + \text{i}|$ — z can be rewritten as $z = x + \text{i}y$.

$\Rightarrow \quad |(x - 3) + \text{i}y| = |x + (y + 1)|$ — Group the real and imaginary parts.

$\Rightarrow \quad |(x - 3)^2 + y^2 = x^2 + (y + 1)^2$ — Remove the moduli.

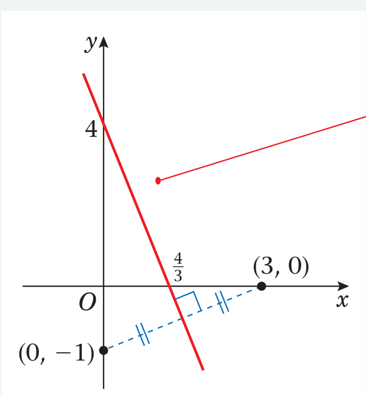
$\Rightarrow \quad x^2 - 6x + 9 + y^2 = x^2 + y^2 + 2y + 1$ — Expand brackets.

$\Rightarrow \quad -6x + 9 = 2y + 1$ — Simplify.

$\Rightarrow \quad -6x + 8 = 2y$ — Simplify further.

Hence the Cartesian equation of the locus of z is $y = -3x + 4$.

b



The locus of z has the equation $y = -3x + 4$.

When $x = 0$, $y = 4$, and when $y = 0$, $x = \frac{4}{3}$. So the locus of z goes through the points $(0, 4)$ and $\left(\frac{4}{3}, 0\right)$.

The locus of z is the perpendicular bisector of the line segment joining $(0, -1)$ to $(3, 0)$.

It follows that

$|z - z_1| = |z - z_2|$ is represented by a perpendicular bisector of the line segment joining the points z_1 to z_2.

Example 26

If $|z - 6| = 2|z + 6 - 9\mathrm{i}|$,

a use algebra to show that the locus of z is a circle, stating its centre and its radius.

b sketch the locus of z on an Argand diagram.

a $|z - 6| = 2|z + 6 - 9\mathrm{i}|$

$\Rightarrow |x + \mathrm{i}y - 6| = 2|z + \mathrm{i}y + 6 - 9\mathrm{i}|$ — z can be written as $z = x + \mathrm{i}y$.

$\Rightarrow |(x - 6) + \mathrm{i}y| = 2|(x + 6) + \mathrm{i}(y - 9)|$ — Group the real and imaginary parts.

$\Rightarrow |(x - 6) + \mathrm{i}y|^2 = 2^2|(x + 6) + \mathrm{i}(y - 9)|^2$ — Square both sides.

$\Rightarrow (x - 6)^2 + y^2 = 4[(x + 6)^2 + (y - 9)^2]$ — Remove the moduli.

$\Rightarrow x^2 - 12x + 36 + y^2 = 4[x^2 + 12x + 36 + y^2 - 18y + 81]$ — Expand brackets.

$\Rightarrow x^2 - 12x + 36 + y^2 = 4x^2 + 48x + 144 + 4y^2 - 72y + 324$ — Expand brackets on the RHS.

$\Rightarrow 3x^2 + 60x + 3y^2 - 72y + 432 = 0$ — Rearranging and collecting terms.

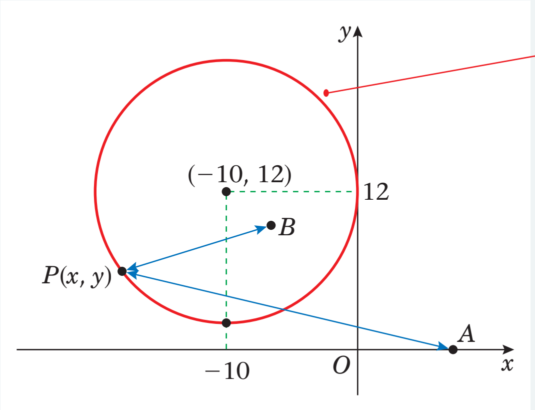
$\Rightarrow x^2 + 20x + y^2 - 24y + 144 = 0$ — Divide both sides by 3.

$\Rightarrow (x + 10)^2 - 100 + (y - 12)^2 - 144 + 144 = 0$ — Complete the square on x and on y.

$\Rightarrow (x + 10)^2 + (y - 12)^2 = 100$ — Circle: $(x - a)^2 + (y - b)^2 = r^2$ with $(a, b) = (-10, 12)$ and $r = 10$.

Hence the locus of z is a circle centre $(-10, 12)$, radius 10.

b



Locus of z as required.

$|z - 6|$ represents the distance from the point $A(6, 0)$ to $P(x, y)$. $|z + 6 - 9\mathrm{i}| = |z - (-6 + 9\mathrm{i})|$ represents the distance from the point $B(-6, 9)$ to $P(x, y)$. $|z - 6| = 2|z + 6 - 9\mathrm{i}|$ gives $AP = 2BP$. This means that P is the locus of points where the distance AP is twice the distance BP. It is not obvious, however, from the outset that the locus of points is a circle. This is why we used algebra in part **a** to show this!

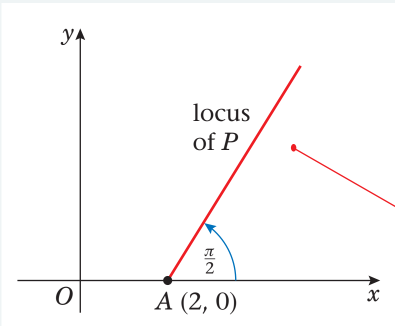
It follows that if

$|z - z_1| = \lambda|z - z_2|$, where $\lambda > 0$, $\lambda \neq 1$, then it may be more appropriate to apply an algebraic method to find the locus of points, z, represented by this equation.

Example 27

If $\arg(z - 2) = \frac{\pi}{3}$, sketch the locus of $P(x, y)$ which is represented by z on an Argand diagram.

Find the Cartesian equation of this locus.



$x - 2$ is represented by the vector from the fixed point $A(2, 0)$ to the point $P(x, y)$.

As $\arg(z - 2) = \frac{\pi}{3}$, then the locus of P is the set of points where the vector $\overrightarrow{AP}$ makes an angle of $\frac{\pi}{3}$ in an anti−clockwise sense from the positive x-axis.

The locus of P is referred to as a **half-line**.

$\arg(z - 2) = \frac{\pi}{3}$

$\Rightarrow \quad \arg(x + iy - 2) = \frac{\pi}{3}$ — z can be rewritten as $z = x + iy$.

$\Rightarrow \quad \arg((x - 2) + iy) = \frac{\pi}{3}$ — Group the real and imaginary parts.

$\Rightarrow \quad \frac{y}{(x - 2)} = \tan\left(\frac{\pi}{3}\right)$ — Remove the argument. $\left(\tan \frac{\pi}{3}\right) = \sqrt{3}$.

$\Rightarrow \quad y = \sqrt{3}(x - 2)$

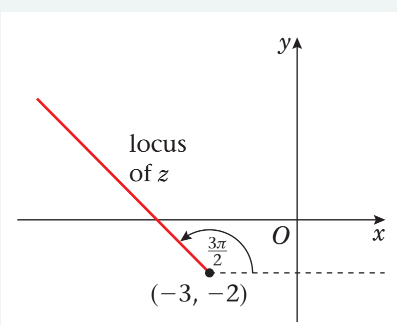
Hence the Cartesian equation of the locus of P is $y = \sqrt{3}x - 2\sqrt{3}$.

As the locus is a half-line, this equation is restricted for x … 2, y … 0.

Example 28

If $\arg(z + 3 + 2i) = \frac{3\pi}{4}$, sketch the locus of z on an Argand diagram.

Find the Cartesian equation of this locus.



$z + 3 + 2i$ can be written as $z - (-3 - 2i)$ and this represents the vector from the fixed point $(-3, -2)$ to the variable point (x, y)

As $\arg(z + 3 + 2i) = \frac{3\pi}{4}$, then the locus of z is the half-line from $(-3, -2)$ making an angle of $\frac{3\pi}{4}$ in an anti−clockwise sense from a line in the same direction as the positive x-axis.

$\arg(z + 3 + 2i) = \frac{3\pi}{4}$

$\Rightarrow \quad \arg(x + iy + 3 + 2i) = \frac{3\pi}{4}$

$\Rightarrow \quad \arg((x + 3) + i(y + 2)) = \frac{3\pi}{4}$

$\Rightarrow \quad \frac{(y + 2)}{(x + 3)} = \tan\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad y + 2 = -1(x + 3)$

Hence the Cartesian equation of the locus of P is $y = -x - 5$.

z can be rewritten as $z = x + iy$.

Group the real and imaginary parts.

Remove the argument.

$\tan\left(\frac{3\pi}{4}\right) = -1$.

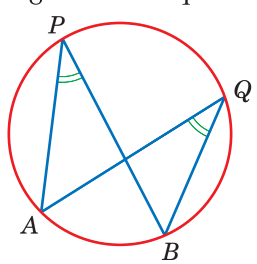
As the locus is a half-line, this equation is restricted for $x \leqslant -3$, y … -2.

It follows that

$\arg(z - z_1) = \theta$ is represented by a half-line from the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.

You need to know and be able to apply the following circle theorems:

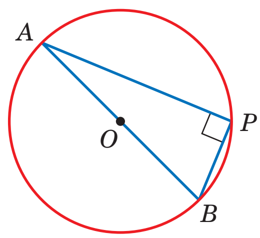
- Angles subtended at an arc in the same segment are equal.



$\angle APB = \angle AQB$

$A\hat{P}B = A\hat{Q}B$

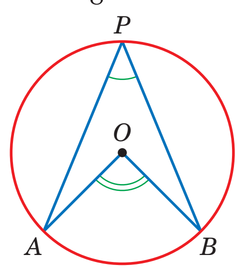
- The angle in a semi-circle is a right angle.



$\angle APB = 90°$

$A\hat{P}B = 90°$

- The angle subtended the centre of the circle is twice the angle at the circumference.



$\angle AOB = 2 \times \angle APB$

$A\hat{O}B = 2 \times A\hat{P}B$

Example 29

If $\arg\left(\frac{z - 6}{z - 2}\right) = \frac{\pi}{4}$,

a sketch the locus of $P(x, y)$ which is represented by z on an Argand diagram,

b find the Cartesian equation of this locus.

a $\arg\left(\frac{z - 6}{z - 2}\right) = \arg(z - 6) - \arg(z - 2) = \frac{\pi}{4}$

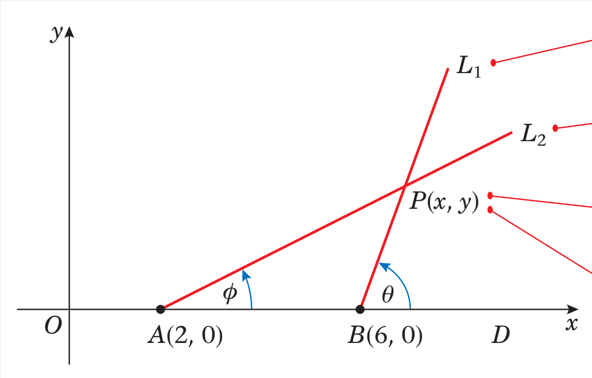
Let L_1 be the half-line satisfying $\arg(z - 6) = \theta$

and let L_2 be the half-line satisfying $\arg(z - 2) = \phi$

So it follows that $\theta - \phi = \frac{\pi}{4}$ ①

Using $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Using $\arg(z - 6) - \arg(z - 2) = \frac{\pi}{4}$.



From ΔABP, it follows that

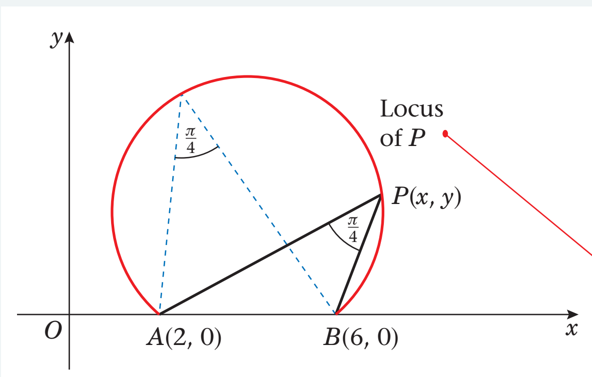
$$B\hat{P}A + P\hat{A}B = P\hat{B}D$$

$$\Rightarrow \quad B\hat{P}A + \phi = \theta$$

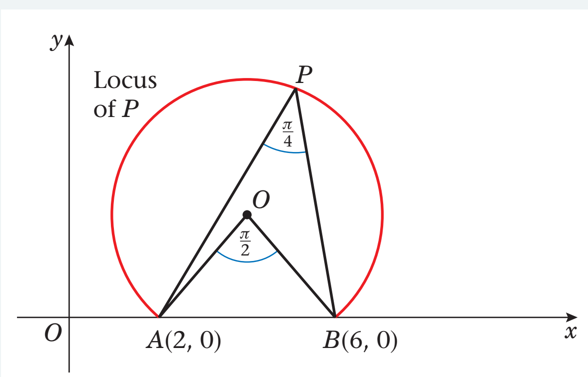
$$\Rightarrow \quad B\hat{P}A = \theta - \phi$$

$$\Rightarrow \quad B\hat{P}A = \frac{\pi}{4}$$

As θ and ϕ vary, the angle $B\hat{P}A$ is constant and is $\frac{\pi}{4}$.



b



All points on L_1 satisfy $\arg(z - 6) = \theta$.

All points on L_2 satisfy $\arg(z - 2) = \phi$.

Therefore the point P is found lying on both L_1 and L_2 where $\theta - \phi = \frac{\pi}{4}$.

As P lies on L_2 and L_2, it is found where L_1 and L_2 intersect.

The exterior angle of a triangle is the sum of the two opposite interior angles.

From diagram, $PA\hat{B} = \phi$ and $P\hat{B}D = \theta$

Using $\theta - \phi = \frac{\pi}{4}$ ①

P can vary but $B\hat{P}A$ must always be $\frac{\pi}{4}$.

From circle theorems, angles in the same segment of a circle are equal.

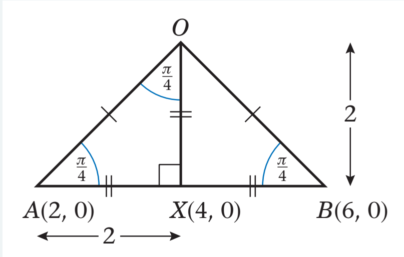
Therefore as P varies, $B\hat{P}A$ will always be equal to $\frac{\pi}{4}$.

So, it follows that P must lie on an arc of a circle cut off at $A(2, 0)$ and at $B(6, 0)$.

$B\hat{P}A = \frac{\pi}{4} \Rightarrow A\hat{O}B = \frac{\pi}{2}$ as the angle subtended at the centre of the circle is twice the angle at the circumference.

As OA and OB are both radii, then the radius, $r = OA = OB$.

This implies that ΔOAB is isosceles and $O\hat{A}B = O\hat{B}A = \frac{\pi}{4}$.



Hence the Cartesian equation of the locus of P is $(x - 4)^2 + (y - 2)^2 = 8$, where $y > 0$.

Let X be the midpoint of AB. Hence $A\hat{O}B = \frac{\pi}{2}$ and $X\hat{O}A = O\hat{A}X = \frac{\pi}{4}$. So ΔOAX is isosceles $AX = OX = 2$.

O has coordinates $(4, 2)$ and $r = \sqrt{2^2 + 2^2} = \sqrt{8}$.
The locus is the part of the circle where $y > 0$.

Example 30

If $\arg\left(\frac{z}{z - 4i}\right) = \frac{\pi}{2}$, sketch the locus of $P(x, y)$ which is represented by z on an Argand diagram.

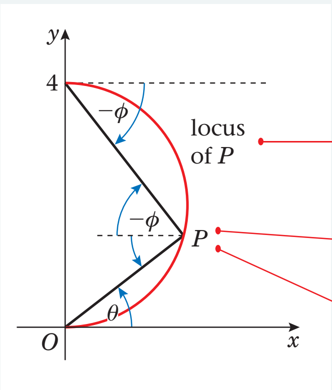
$\arg\left(\frac{z}{z - 4i}\right) = \arg z - \arg(z - 4i) = \theta - \phi$

where $\arg z = \theta$ and $\arg(z - 4i) = \phi$.

Hence,

$\theta - \phi = \frac{\pi}{2}$

as $\arg\frac{z}{z - 4i} = \frac{\pi}{2}$.



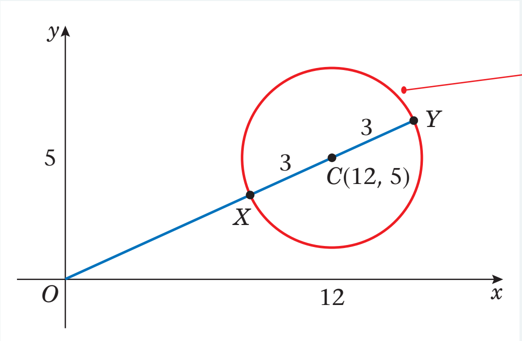
The locus of P is the arc of a semi-circle passing through the points (0, 0) and (4, 0), as shown.

The angle at P is always $\frac{\pi}{2}$, because the angle in a semi-circle is a right-angle.

It follows from the diagram that at P, $-\phi + \theta = \theta - \phi = \frac{\pi}{2}$, as required.

Example 31

Given that the complex number $z = x + iy$ satisfies the equation $|z - 12 - 5i| = 3$, find the minimum value of $|z|$ and maximum value of $|z|$.



The locus of z is a circle centre $C(12, 5)$, radius 3.

$|z|$ represents the distance from the origin to any point on this locus.
$|z|_{min}$ and $|z|_{max}$ are represented by the distances OX and OY respectively.

$|z|_{min} = OC - CX = 13 - 3 = 10.$

$|z|_{max}$ O= $OC + CY = 13 + 3 = 16.$

The minimum value of $|z|$ is 10 and the maximum value of $|z|$ is 16.

The distance, $OC = \sqrt{12^2 + 5^2} = 13.$

The radius $r = CX = CY = 3.$

Exercise 3F

1 Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z| = 6$ **b** $|z| = 10$ **c** $|z - 3| = 2$

d $|z + 3\text{i}| = 3$ **e** $|z - 4\text{i}| = 5$ **f** $|z + 1| = 1$

g $|z - 1 - \text{i}| = 5$ **h** $|z + 3 + 4\text{i}| = 4$ **i** $|z - 5 + 6\text{i}| = 5$

j $|2z + 6 - 4\text{i}| = 6$ **k** $|3z - 9 - 6\text{i}| = 12$ **l** $|3z - 9 - 6\text{i}| = 12$

2 Sketch the locus of z when:

a $\arg z = \frac{\pi}{3}$ **b** $\arg(z + 3) = \frac{\pi}{4}$ **c** $\arg(z - 2) = \frac{\pi}{2}$

d $\arg(z + 2 + 2\text{i}) = -\frac{\pi}{4}$ **e** $\arg(z - 1 - \text{i}) = \frac{3\pi}{4}$ **f** $\arg(z + 3\text{i}) = \pi$

g $\arg(z - 1 + 3\text{i}) = \frac{2\pi}{3}$ **h** $\arg(z - 3 + 4\text{i}) = -\frac{\pi}{2}$ **i** $\arg(z + 4\text{i}) = -\frac{3\pi}{4}$

3 Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z - 6| = |z - 2|$ **b** $|z + 8| = |z - 4|$

c $|z| = |z + 6\text{i}|$ **d** $|z + 3\text{i}| = |z - 8\text{i}|$

e $|z - 2 - 2\text{i}| = |z + 2 + 2\text{i}|$ **f** $|z + 4 + \text{i}| = |z + 4 + 6\text{i}|$

g $|z + 3 - 5\text{i}| = |z - 7 - 5\text{i}|$ **h** $|z + 4 - 2\text{i}| = |z - 8 + 2\text{i}|$

i $\frac{|z + 6 - \text{i}|}{|z - 10 - 5\text{i}|} = 1$ **j** $\frac{|z + 6 - \text{i}|}{|z - 10 - 5\text{i}|} = 1$

k $|z + 7 + 2\text{i}| = |z - 4 - 3\text{i}|$ **l** $|z + 1 - 6\text{i}| = |2 + 3\text{i} - z|$

4 Find the Cartesian equation of the locus of z when:

a $z - z^* = 0$ **b** $z + z^* = 0$

5 Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|2 - z| = 3$ **b** $|5\text{i} - z| = 4$ **c** $|3 - 2\text{i} - z| = 3$

6 Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z + 3| = 3|z - 5|$ **b** $|z - 3| = 4|z + 1|$

c $|z - \text{i}| = 2|z + \text{i}|$ **d** $|z + 2 - 7\text{i}| = 2|z - 10 + 2\text{i}|$

e $|z + 4 - 2\text{i}| = 2|z - 2 - 5\text{i}|$ **f** $|z| = 2|2 - z|$

7 Sketch the locus of z when:

a $\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$

b $\arg\left(\frac{z-3i}{z+4}\right) = \frac{\pi}{6}$

c $\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$

d $\arg\left(\frac{z-3i}{z-5}\right) = \frac{\pi}{4}$

e $\arg(z) - \arg(z - 2 + 3i) = \frac{\pi}{3}$

f $\arg\left(\frac{z-4i}{z+4}\right) = \frac{\pi}{2}$

8 Use the Argand diagram to find the value of z that satisfies the equations $|z| = 5$ and $\arg(z + 4) = \frac{\pi}{2}$.

9 Given that the complex number z satisfies $|z - 2 - 2i| = 2$,

a sketch, on an Argand diagram, the locus of z.

Given further that $\arg(z - 2 - 2i) = \frac{\pi}{6}$,

b find the value of z in the form $a + ib$, where $a \in \square$ and $b \in \square$.

10 Sketch on one Argand diagram the locus of points satisfying

a $|z - 2i| = z - 8i|$,

b $\arg(z - 1 - i) = \frac{\pi}{4}$.

The complex number z satisfies both $|z - 2i| = |z - 8i|$, and $\arg(z - 1 - i) = \frac{\pi}{4}$.

c Use your answers to parts **a** and **b** to find the value of z.

11 Sketch on one Argand diagram the locus of points satisfying

a $|z - 3 + 2i| = 4$

b $\arg(z - 1) = -\frac{\pi}{4}$.

The complex number z satisfies both $|z - 3 + 2i| = 4$ and $\arg(z - 1) = -\frac{\pi}{4}$.

Given that $z = a + ib$ where $a \in \square$ and $b \in \square$,

c find the exact value of a and the exact value of b.

12 On an Argand diagram the point P represents the complex number z.
Given that $|z - 4 - 3i| = 8$,

a find the Cartesian equation for the locus of P,

b sketch the locus of P,

c find the maximum and minimum values of $|z|$ for points on this locus.

13 Given that $\arg(z + 4) = \frac{\pi}{3}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

b find the minimum value of $|z|$ for points on this locus.

14 The complex number $z = x + iy$ satisfies the equation $|z + 1 + i| = 2|z + 4 - 2i|$.
The complex number z is represented by the point P in the Argand diagram.

a Show that the locus of P is a circle with centre $\left(-5, \frac{9}{2}\right)$.

b Find the exact radius of this circle.

15 If the complex number z satisfies both $\arg z = \frac{\pi}{3}$ and $\arg(z - 4) = \frac{\pi}{2}$,

a find the value of z in the form $a + ib$, where $a \in \square$ and $b \in \square$,

b Hence, find $\arg(z - 8\text{i})$.

16 The point P represents a complex number z in an Argand diagram.

Given that $|z + 2 - 2\sqrt{3}\text{i}| = 2$,

a sketch the locus of P on an Argand diagram.

b Write down the minimum value of $\arg z$.

c Find the maximum value of $\arg z$.

17 The point P represents a complex number z in an Argand diagram.

Given that $\arg(z) - \arg(z + 4) = \frac{\pi}{4}$ is a locus of points P lying on an arc of a circle C,

a sketch the locus of points P,

b find a Cartesian equation for the circle C,

c find the coordinates of the centre of C,

d find the radius of C,

e find the finite area bounded by the locus of P and the x−axis.

3.8 You can use complex numbers to represent regions on an Argand diagram.

Example 32

a Shade in, on separate Argand diagrams the region represented by

i $|z - 4 - 2\text{i}| \leqslant 2$, **ii** $|z - 4| < |z - 6|$, **iii** $0 \leqslant \arg(z - 2 - 2\text{i}) \leqslant \frac{\pi}{4}$.

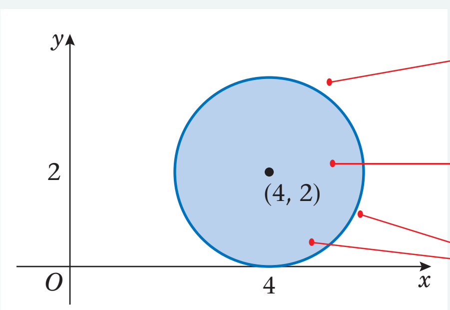
b Hence on one Argand diagram shade in the region which satisfies

$|z - 4 - 2\text{i}| \leqslant 2$,

$|z - 4| < |z - 6|$

$0 \leqslant \arg(z - 2 - 2\text{i}) \leqslant \frac{\pi}{4}$.

a i $|z - 4 - 2\text{i}| \leqslant 2$,

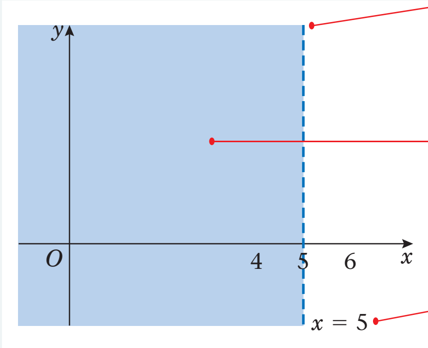


$|z - 4 - 2\text{i}| \leqslant 2$ represents a circle centre (4, 2) radius 2.

$|z - 4 - 2\text{i}| \leqslant 2$ represents the region on the inside of this circle.

$|z - 4 - 2\text{i}| \leqslant 2$ represents the boundary and the inside of this circle.

ii $|z - 4| < |z - 6|$

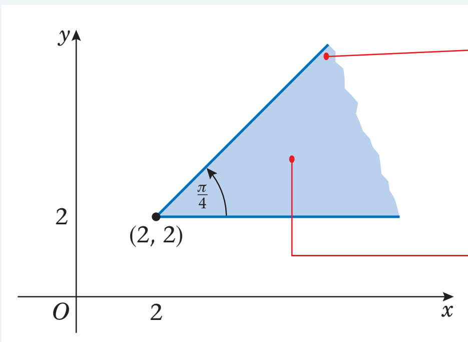


$|z - 4| = |z - 6|$ is represented by the line $x = 5$. This line is the perpendicular bisector of the line segment joining (4, 0) to (6, 0).

$|z - 4| < |z - 6|$ represents the region $x < 5$. All points in this region are closer to (4, 0) than to (6, 0).

Note this region does not include the line $x = 5$. So $x = 5$ is represented by a dashed line.

iii $0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$.

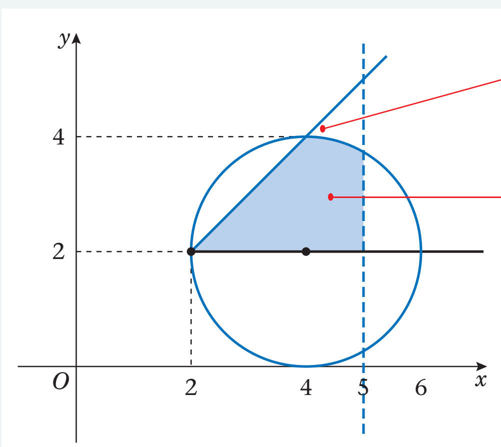


$\arg(z - 2 - 2i) = \frac{\pi}{4}$ is the half-line shown from the point (2, 2).

$\arg(z - 2 - 2i) = 0$ is the other half-line shown from the point (2, 2).

$0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$ is represented by the region in between and including these two half-lines.

b $|z - 4 - 2i| \leqslant 2$, $|z - 4| < |z - 6|$ and $0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$.



The line $\arg(z - 2 - 2i) = \frac{\pi}{4}$ and the circle $|z - 4 - 2i| \leqslant 2$, both go through the point (4, 4).

The region shaded is satisfied by all three of $|z - 4 - 2i| \leqslant 2$, $|z - 4| < |z - 6|$ and $0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$.

Exercise 3G

1 On an Argand diagram shade in the regions represented by the following inequalities:

a $|z| < 3$ **b** $|z - 2i| > 2$ **c** $|z + 7|$ … $|z - 1|$ **d** $|z + 6| > |z + 2 + 8i|$

e $2 \leqslant |z| \leqslant 3$??? **f** $1 \leqslant |z + 4i| \leqslant$??? **g** $3 \leqslant |z - 3 + 5i| \leqslant$??? **h** $2|z|$ … $|z - 3|$

2 The region R in an Argand diagram is satisfied by the inequalities $|z| \leqslant 5$ and $|z| \leqslant |z - 6\text{i}|$. Draw an Argand diagram and shade in the region R.

3 Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z + 1 - \text{i}| \leqslant 1$ and $0 < \arg z \leqslant \frac{3\pi}{4}$.

4 Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z| \leqslant 3$ and $\frac{\pi}{4} \leqslant \arg(z + 3) \leqslant \pi$.

5 **a** Sketch in on the same Argand diagram:

i the locus of points representing $|z - 2| = |z - 6 - 8\text{i}|$,

ii the locus of points representing $\arg(z - 4 - 2\text{i}) = 0$,

iii the locus of points representing $\arg(z - 4 - 2\text{i}) = \frac{\pi}{2}$.

The region R is defined by the inequalities $|z - 2| \leqslant |z - 6 - 8\text{i}|$ and $\arg(z - 4 - 2\text{i}) \leqslant \frac{\pi}{2}$.

b On your sketch in part **a**, identify, by shading, the region R.

6 **a** Find the Cartesian equations of:

i the locus of points representing $|z + 10| = |z - 6 - 4\sqrt{2}\text{i}|$,

ii the locus of points representing $|z + 1| = 3$.

b Find the two values of z that satisfy both $|z + 10| = |z - 6 - 4\sqrt{2}\text{i}|$ and $|z + 1| = 3$.

c Hence shade in the region R on an Argand diagram which satisfies both $|z + 10| \leqslant |z - 6 - 4\sqrt{2}\text{i}|$ and $|z + 1| \leqslant 3$.

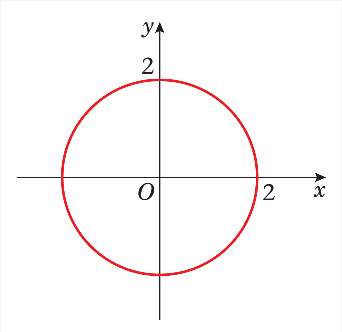
3.9 You can apply transformations that map points on the z-plane to points on the w-plane by applying a formula relating $z = x + \text{i}y$ to $w = u + \text{i}v$.

Example 33

The point P represents the complex number z on an Argand diagram where $|z| = 2$. T_1, T_2 and T_3 represent transformations from the z-plane, where $z = x + \text{i}y$, to the w-plane where $w = u + \text{i}v$. Describe the locus of the image of P under the transformations:

a T_1: $w = z - 2 + 4\text{i}$, **b** T_2: $w = 3z$, **c** T_3: $w = \frac{1}{2}z + \text{i}$.

- $|z| = 2$



Firstly the locus of P in the z-plane is a circle centre (0, 0), radius 2.

This is the locus of P in the z-plane before any transformations have been applied.

a T_1: $w = z - 2 + 4\text{i}$

$\Rightarrow\ w + 2 - 4\text{i} = z$

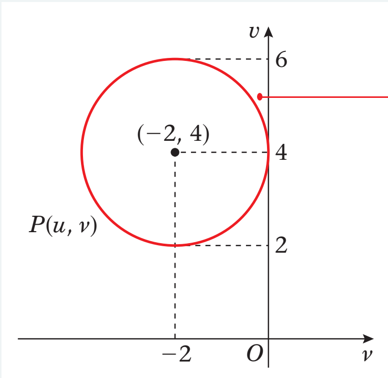
Rearrange to make z the subject.

$\Rightarrow\ |w + 2 - 4\text{i}| = |z|$

Apply the modulus to both sides of the equation.

$\Rightarrow\ |w + 2 - 4\text{i}| = 2$

Apply $|z| = 2$.



The image of P under T_1 is $|w + 2 - 4\text{i}| = 2$. This is represented by a circle centre $(-2, 4)$, radius 2.

Therefore the transformation T_1: $w = z - 2 + 4\text{i}$, represents a translation of z by the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$.

b T_2: $w = 3z$,

$\Rightarrow\ |w| = |3z|$

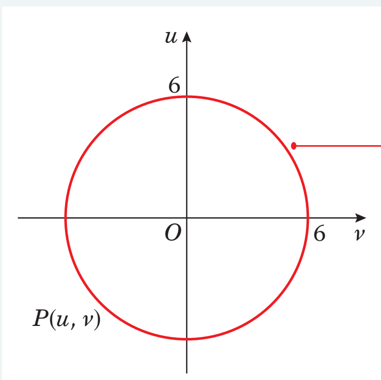
Apply the modulus to both sides of the equation.

$\Rightarrow\ |w| = |3||z|$

Apply $|z_1 z_2| = |z_1||z_2|$.

$\Rightarrow\ |w| = (3)(2) = 6$

Apply $|3| = 3$ and $|z| = 2$.



The image of P under T_2 is $|w| = 6$. This is represented by a circle centre (6, 6), radius 6.

Therefore the transformation T_2: $w = 3z$, represents an enlargement of z by scale factor 3 about the point (0, 0).

c T_3: $w = \frac{1}{2}z + \text{i}$

$\Rightarrow\ w - \text{i} = \frac{1}{2}z$

Rearrange to make $\frac{1}{2}z$ the subject.

$\Rightarrow\ |w - \text{i}| = \left|\frac{1}{2}z\right|$

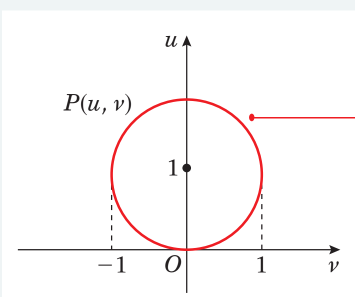
Apply the modulus to both sides of the equation.

$\Rightarrow\ |w - \text{i}| = \left|\frac{1}{2}\right||z|$

Apply $|z_1 z_2| = |z_1||z_2|$.

$\Rightarrow\ |w - \text{i}| = \left(\frac{1}{2}\right)(2) = 1$

Apply $\left|\frac{1}{2}\right| = \frac{1}{2}$ and $|z| = 2$.



The image of P under T_3 is $|w - \text{i}| = 1$. This is represented by a circle centre (0, 1) radius 1.

Therefore the transformation T_3: $w = \frac{1}{2}z + \text{i}$, represents an enlargement of z by scale factor $\frac{1}{2}$ about the point (0, 0), followed by a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

It follows that

- $w = z + a + \mathrm{i}b$ represents a translation with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$, where $a, b \in \square$.
- $w = kz$ represents an enlargement scale factor k centre (0, 0), where $k > 0$ and $k \in \square$.
- $w = kz + a + \mathrm{i}b$ represents an enlargement scale factor k centre (0, 0) followed by a translation with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$, where $k > 0$ and $a, b, k \in \square$.

Example 34

For the transformation $w = z^2$, where $z + x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, find the locus of w when z lies on a circle with equation $x^2 + y^2 = 16$.

$\lvert z \rvert = 4$	Represents $x^2 + y^2 = 16$.
$w = z^2 \quad \Rightarrow \quad \lvert w \rvert = \lvert z^2 \rvert$	Take the modulus of each side of the equation.
$\Rightarrow \quad \lvert w \rvert = \lvert z \rvert \lvert z \rvert$	Apply $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$, where $z_1 = z_2 = z$
$\Rightarrow \quad \lvert w \rvert = (4)(4)$	Apply $\lvert z \rvert = 4$.
$\Rightarrow \quad \lvert w \rvert = 16$	Circle centre (0, 0), radius 16.

Hence the locus of w is a circle centre (0, 0), radius 16.

Example 35

The transformation T from the z-plane, where $z = w + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \dfrac{5\mathrm{i}z + \mathrm{i}}{z + 1}$, $z \neq -\mathrm{i}$.

Show that the image, under T, of the circle $\lvert z \rvert = 1$ in the z-plane is a line l in the w-plane. Sketch l in on Argand diagram.

$$w = \frac{5\mathrm{i}z + \mathrm{i}}{z + 1}$$

$$\Rightarrow \quad w(z + 1) = 5\mathrm{i}z + \mathrm{i}$$

$$\Rightarrow \quad wz + w = 5\mathrm{i}z + \mathrm{i}$$

$$\Rightarrow \quad wz - 5\mathrm{i}z = \mathrm{i} - w$$

$$\Rightarrow \quad z(w - 5\mathrm{i}) = \mathrm{i} - w$$

$$\Rightarrow \quad z = \frac{\mathrm{i} - w}{w - 5\mathrm{i}}$$

Rearrange the transformation equation $w = \dfrac{5\mathrm{i}z + \mathrm{i}}{z + 1}$ to make z the subject of the equation.

So, $|z| = \left|\frac{i - w}{w - 5i}\right|$

Take the modulus of each side of the equation.

$\Rightarrow \quad 1 = \frac{|i - w|}{|w - 5i|}$

Apply $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, and $|z| = 1$.

$\Rightarrow \quad |w - 5i| = |i - w|$

Rearranging.

$\Rightarrow \quad |w - 5i| = |(-1)(w - i)|$

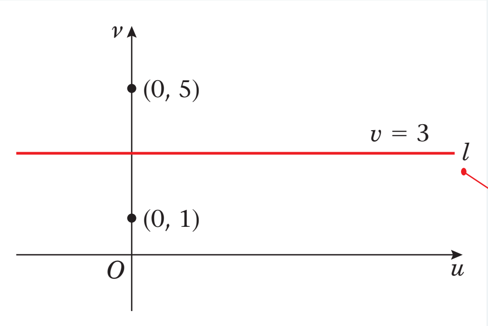
Take out a factor of -1 on the RHS.

$\Rightarrow \quad |w - 5i| = |(-1)||(w - i)|$

Apply $|z_1 z_2| = |z_1||z_2|$.

$\Rightarrow \quad |w - 5i| = |w - i|$

Apply $|-1| = 1$.



As we are working in the w-plane we plot u against v.
$|w - 5i| = |w - i|$ is in the form $|w - w_1| = |w - w_2|$ and represents the points on the perpendicular bisector of the line joining (0, 1) and (0, 5).
Therefore the line l has equation $v = 3$.

Therefore the image of $|z| = 1$, under T, is the line l with equation $v = 3$.

Example 36

The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{3z - 2}{z + 1}$, $z \neq -1$.

Show that the image, under T, of the circle $x^2 + y^2 = 4$ in the z-plane is a circle C in the w-plane, stating the centre and radius of C.

$w = \frac{3z - 2}{z + 1}$

$\Rightarrow \quad w(z + 1) = 3z - 2$

$\Rightarrow \quad wz + w = 3z - 2$

$\Rightarrow \quad w + 2 = 3z - wz$

$\Rightarrow \quad w + 2 = z(3 - w)$

$\Rightarrow \quad \frac{w + 2}{3 - w} = z$

Rearrange the transformation equation $w = \frac{3z - 2}{z + 1}$ to make z the subject of the equation.

$x^2 + y^2 = 4$ can also be written as $|z| = 2$.

$x^2 + y^2 = 4$ is an equation of a circle centre (0, 0) radius 2. So $|z| = 2$.

So, $\left|\frac{w+2}{3-w}\right| = |z|$

$\Rightarrow \frac{|w+2|}{|3-w|} = 2$

$\Rightarrow |w+2| = 2|3-w|$

$\Rightarrow |w+2| = 2|-1||w-3|$

$\Rightarrow |w+2| = 2|w-3|$

$\Rightarrow |u+iv+2| = 2|u+iv-3|$

$\Rightarrow |(u+2)+iv| = 2|(u-3)+iv|$

$\Rightarrow |(u+2)+iv|^2 = 2^2|(u-3)+iv|^2$

$\Rightarrow (u+2)^2+v^2 = 4[(u-3)^2+v^2]$

$\Rightarrow u^2+4u+4+v^2 = 4[u^2-6u+9+v^2]$

$\Rightarrow u^2+4u+4+v^2-24u+36+4v^2$

$\Rightarrow 3u^2-28u+3v^2+32 = 0$

$\Rightarrow u^2-\frac{28}{3}u+v^2+\frac{32}{3} = 0$

$\Rightarrow \left(u-\frac{14}{3}\right)^2-\frac{196}{9}+v^2+\frac{32}{3} = 0$

$\Rightarrow \left(u-\frac{14}{3}\right)^2+v^2 = \frac{100}{9}$

Therefore the image of $x^2+y^2=4$, under T, is the circle C with centre $\left(\frac{14}{3}, 0\right)$ and radius $\frac{10}{3}$.

Take the modulus of each side of the equation.

Apply $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, and and $|z| = 2$.

Rearranging.

Apply $|-1| = 1$.

w can be written as $u + iv$.

Group the real and imaginary parts.

Square both sides.

Remove the moduli.

Expand brackets.

Expand brackets on the RHS.

Rearranging and collecting terms.

Divide both sides by 3.

Complete the square on u and on v.

Circle: $(u-a)^2+(v-b)^2 = r^2$ with $(a, b) = \left(\frac{14}{3}, 0\right)$ = and $r = \frac{10}{3}$.

Example 37

A transformation T of the z-plane to the w-plane is given by $w = \frac{iz-2}{1-z}$, $z \neq 1$.

Show that as z lies on the real axis in the z-plane, then w lies on the line l on the w-plane. Sketch l on an Argand diagram.

$w = \frac{iz-2}{1-z}$

$\Rightarrow w(1-z) = iz-2$

$\Rightarrow w-wz = iz-2$

$\Rightarrow w+2 = wz+iz$

$\Rightarrow w+2 = z(w+i)$

$\Rightarrow \frac{w+2}{w+i} = z$

Rearrange the transformation equation $w = \frac{iz-2}{1-z}$ to make z the subject of the equation.

So, $z = \dfrac{u + iv + 2}{u + iv + i}$

w can be written as $u + iv$.

$\Rightarrow \quad z = \dfrac{(u + 2) + iv}{u + i(v + 1)}$

Group the real and the imaginary parts on both the numerator and denominator.

$\Rightarrow \quad z = \dfrac{[(u + 2) + iv]}{[u + i(v + 1)]} \times \dfrac{[u - i(v + 1)]}{[u - i(v + 1)]}$

Multiply the numerator and denominator by the complex conjugate of $u + i(v + 1)$.

$\Rightarrow \quad z = \dfrac{u(u + 2) - i(u + 2)(v + 1) + iuv + v(v + 1)}{u^2 + (v + 1)^2}$

Multiply out brackets using $i^2 = -1$. Apply difference of two squares.

$\Rightarrow \quad z = \left[\dfrac{u(u + 2) + v(v + 1)}{u^2 + (v + 1)^2}\right] + i\left[\dfrac{uv - (u + 2)(v + 1)}{u^2 + (v + 1)^2}\right]$

Group the real and the imaginary parts.

So,

$x + iy = \left[\dfrac{u(u + 2) + v(v + 1)}{u^2 + (v + 1)^2}\right] + i\left[\dfrac{uv - (u + 2)(v + 1)}{u^2 + (v + 1)^2}\right]$

z can be written as $x + iy$.

As z lies on the real axis then $y = 0$. So,

$x + i0 = \left[\dfrac{u(u + 2) + v(v + 1)}{u^2 + (v + 1)^2}\right] + i\left[\dfrac{uv - (u + 2)(v + 1)}{u^2 + (v + 1)^2}\right]$

Set $y = 0$, as z lies on the real axis.

Hence, $0 = \dfrac{uv - (u + 2)(v + 1)}{u^2 + (v + 1)^2}$

Equating the imaginary parts.

$\Rightarrow \quad uv - (u + 2)(v + 1) = 0$

Multiply both sides by $u^2 + (v + 1)^2$.

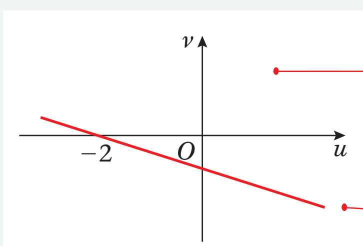
$\Rightarrow \quad uv - (uv + u + 2v + 2) = 0$

$\Rightarrow \quad uv - uv - u - 2v - 2 = 0$

$\Rightarrow \quad -u - 2 = 2v$

So w lies on the line l with equation $v = -\frac{1}{2}u - 1$.

Manipulate to make v the subject.



As we are working in the w-plane we plot u against v.

The line l has equation $v = -\frac{1}{2}u - 1$ and cuts the coordinate axes at $(-2, 0)$ and $(0, -1)$.

Exercise 3H

1 For the transformation $w = z + 4 + 3\text{i}$, sketch on separate Argand diagrams the locus of w when

a z lies on the circle $|z| = 1$,

b z lies on the half-line $z = \frac{\pi}{2}$,

c z lies on the line $y = x$.

2 A transformation T from the z-plane to the w-plane is a translation $-2 + 3\text{i}$ followed by an enlargement with centre the origin and scale factor 4. Write down the transformation T in the form $w = a\,z + b$, where $a, b \in \mathbb{C}$.

3 For the transformation $w = 3z + 2 - 5\text{i}$, find the locus of w when z lies on a circle centre O, radius 2.

4 For the transformation $w = 2z - 5 + 3\text{i}$, find the locus of w as z moves on the circle $|z - 2| = 4$.

5 For the transformation $w = z - 1 + 2\text{i}$ sketch on separate Argand diagrams the locus of w when:

a z lies on the circle $|z - 1| = 3$,

b z lies on the half-line $\arg(z - 1 + \text{i}) = \frac{\pi}{4}$,

c z lies on the line $y = x$.

6 For the transformation $w = \frac{1}{z}$, $z \neq 0$, find the locus of w when:

a z lies on the circle $|z| = 2$,

b z lies on the half-line with equation $\arg z = \frac{\pi}{4}$,

c z lies on the line with equation $y = 3x + 1$.

7 For the transformation $w = z^2$,

a show that as z moves round a circle centre $(0, 0)$ with radius 3, w moves twice round a circle centre $(0, 0)$ with radius 9,

b find the locus of w when z lies on the real axis, with equation $y = 0$,

c find the locus of w when z lies on the imaginary axis.

8 If z is any point in the region R for which $|z + 2\text{i}| < 2$,

a shade in on an Argand diagram the region R.

Find the corresponding regions for w where:

b $w = z - 2 + 5\text{i}$,

c $w = 4z + 2 + 4\text{i}$,

d $|zw + 2\text{i}w| = 1$.

9 For the transformation $w = \frac{1}{2 - z}$, $z \neq 2$, show that the image, under T, of the circle centre O and radius in the z-plane is a line l in the w-plane. Sketch l in on Argand diagram.

10 The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{16}{z}$, $z \neq 0$.

a The transformation T maps the points on the circle $|z - 4| = 4$, in the z-plane, to points on a line l in the w-plane. Find a Cartesian equation l.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane for which $|z - 4| < 4$.

11 The transformation, T, from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{3}{2 - z}$, $z \neq 2$.

Show that under T the straight line with equation $2y = x$ is transformed to a circle in the w-plane with centre $\left(\frac{3}{4}, \frac{3}{2}\right)$ and radius $\frac{5}{2}\sqrt{2}$.

12 The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{-\mathrm{i}z + \mathrm{i}}{z + 1}$, $z \neq -1$.

a The transformation T maps the points on the circle with equation $x^2 + y^2 = 1$ in the z-plane, to points on a line l in the w-plane. Find a Cartesian equation l.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane for which $|z| < 1$.

c Show that the image, under T, of the circle with equation $x^2 + y^2 = 4$ in the z-plane is a circle C in the w-plane. Find the Cartesian equation of C.

13 The transformation, T, from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{15z - 3\mathrm{i}}{3z - 1}$, $z \neq \frac{1}{3}$.

The circle with equation $|z| = 1$ is mapped by T onto the curve C.
Show that C is also a circle and find its centre and radius.

14 The transformation T from the z-plane, where $z = x = \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{1}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$.

a Show that the image, under T, of the real axis in the z-plane is a circle C_1 in the w-plane. Find the Cartesian equation of C_1.

b Show that the image, under T, of the line $x = 4$ in the z-plane is a circle C_2 in the w-plane. Find the Cartesian equation of C_2.

15 The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = z + \frac{4}{z}$, $z \neq 0$.

Show that the transformation T maps the points on the circle $|z| = 4$ to points in the interval $[-k, k]$ on the real axis, stating the value of the constant k.

16 The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \dfrac{1}{z+3}$, $z \neq -3$.

Show that the line with equation $2x - 2y = 7 = 0$ is mapped by T onto the circle C. State the centre and the exact radius of C.

Mixed exercise 3I

1 Express $\dfrac{(\cos 3x + \mathrm{i}\sin 3x)^2}{\cos x - \mathrm{i}\sin x}$ in the form $\cos nx + \mathrm{i}\sin nx$ where n is an integer to be found.

2 Use de Moivre's theorem to evaluate

a $(1 - \mathrm{i})^6$

b $\dfrac{1}{\left(\frac{1}{2} - \frac{1}{2}\mathrm{i}\right)^{16}}$

3 **a** If $z = r(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

b Express $\left(z^2 + \dfrac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$.

c Hence, or otherwise, show that $\cos^3 2\theta = a\cos 6\theta + b\cos 2\theta$, where a and b are constants.

d Hence, or otherwise, show that $\displaystyle\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, \mathrm{d}\theta = k\sqrt{3}$, where k is a constant.

4 **a** Use de Moivre's theorem to show that $\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$.

b By solving the equation $\cos 5\theta = 0$, deduce that $\cos^2\left(\dfrac{\pi}{10}\right) = \dfrac{5+\sqrt{5}}{2}$.

c Hence, or otherwise, write down the exact values of $\cos^2\left(\dfrac{3\pi}{10}\right)$, $\cos^2\left(\dfrac{7\pi}{10}\right)$ and $\cos^2\left(\dfrac{9\pi}{10}\right)$.

5 **a** Express $4 - 4\mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$, $-\pi < \theta \leqslant \pi$, where r and θ are exact values.

b Hence, or otherwise, solve the equation $z^5 = 4 - 4\mathrm{i}$ leaving your answers in the form $z = R\mathrm{e}^{\mathrm{i}k\pi}$, where R is the modulus of z and k is a rational number such that $-1 \leqslant k \leqslant 1$.

c Show on an Argand diagram the points representing your solutions.

6 **a** Find the Cartesian equations of

i the locus of points representing $|z - 3 + \mathrm{i}| = |z - 1 - \mathrm{i}|$,,

ii the locus of points representing $|z + 2| = 2\sqrt{2}$.

b Find the two values of z that satisfy both $|z - 3 + \mathrm{i}| = |z - 1 - \mathrm{i}|$ and $|z + 2| = 2\sqrt{2}$.

c Hence on one Argand diagram sketch:

i the locus of points representing $|z - 3 + \mathrm{i}| = |z - 1 - \mathrm{i}|$,

ii the locus of points representing $|z + 2| = 2\sqrt{2}$.

The region R is defined by the inequalities $|z - 3 + \mathrm{i}| \ldots |z - 1 - \mathrm{i}|$ and $|z + 2| \leqslant 2\sqrt{2}$.

d On your sketch in part **c**, identify, by shading, the region R.

7 **a** Find the Cartesian equation of the locus of points representing $|z + 2| = |2z - 1|$.

b Find the two values of z that satisfy both $|z + 2| = |2z - 1|$ and $\arg z = \frac{\pi}{4}$.

c Hence shade in the region R on an Argand diagram which satisfies both $|z + 2| \dots |2z - 1|$ and $\frac{\pi}{4} \leqslant \arg z \leqslant \pi$.

8 The point P represents a complex number z in an Argand diagram.
Given that $|z + 1 - \mathrm{i}| = 1$

a find a Cartesian equation for the locus of P,

b sketch the locus of P on an Argand diagram,

c find the greatest and least values of $|z|$,

d find the greatest and least values of $|z - 1|$.

9 Given that $\arg\left(\frac{z - 4 - 2\mathrm{i}}{z - 6\mathrm{i}}\right) = \frac{\pi}{2}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

b deduce the exact value of $|z - 2 - 4\mathrm{i}|$.

10 Given that $\arg(z - 2 + 4\mathrm{i}) = \frac{\pi}{4}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

b find the minimum value of $|z|$ for points on this locus.

11 The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u = \mathrm{i}v$, is given by

$$w = \frac{1}{z}, \quad z \neq 0.$$

a Show that the image, under T, of the line with equation $x = \frac{1}{2}$ in the z-plane is a circle C in the w-plane. Find the Cartesian equation of C.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane for which $x \dots \frac{1}{2}$.

12 The point P represents the complex number z on an Argand diagram.
Given that $|z + 4\mathrm{i}| = 2$,

a sketch the locus of P on an Argand diagram.

b Hence find the maximum value of $|z|$.

13 T_1, T_2, T_3 and T_4 represent transformations from the z-plane to the w-plane. Describe the locus of the image of P under the transformations

a T_1: $w = 2z$,

b T_2: $w = \mathrm{i}z$,

c T_3: $w = -\mathrm{i}z$,

d T_4: $w = z^*$.

14 The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \dfrac{z + 2}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$.

a Show that the image, under T, of the imaginary axis in the z-plane is a line l in the w-plane. Find the Cartesian equation of l.

b Show that the image, under T, of the line $y = x$ in the z-plane is a circle C in the w-plane. Find the centre of C and show that the radius of C is $\frac{1}{2}\sqrt{10}$.

15 The transformation, T, from the z-plane, where $z = x = \mathrm{i}y$ to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \dfrac{4 - z}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$.

The circle with equation $|z| = 1$ is mapped by T onto a line l. Show that l can be written in the form $au + bv + c = 0$, where a, b and c are integers to be determined.

16 The transformation, T, from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \dfrac{15z - 3\mathrm{i}}{3z - 1}$, $z \neq \dfrac{1}{3}$.

Show that the circle with equation $|z| = 1$ is mapped by T onto the circle C. State the centre of C and show that the radius of C can be expressed in the form $\frac{1}{2}\sqrt{k}$ where k is an integer to be determined.

17 A transformation from the z-plane to the w-plane is defined by $w = \dfrac{az + b}{z + c}$, where $a, b, c \in \square$.

Given that $w = 1$ when $z = 0$ and that $w = 3 - 2\mathrm{i}$ when $z = 2 + 3\mathrm{i}$,

a find the values of a, b and c,

b find the exact values of the two points in the complex plane which remain invariant under the transformation.

Summary of key points

1 A complex number, z, can be expressed in any one of three forms:

- $z = x = \mathrm{i}y$
- $z = r(\cos\theta + \mathrm{i}\sin\theta)$
- $z = r\mathrm{e}^{\mathrm{i}\theta}$

where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg z$.

2 For complex numbers $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$,

- $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$
- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 + z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
- $\arg(z_1 - z_2) = \arg(z_1) - \arg(z_2)$

3 If $z = r(\cos\theta + \mathrm{i}\sin\theta)$, then de Moivre's theorem states for $n \in \square$ that

$z^n = [r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$.

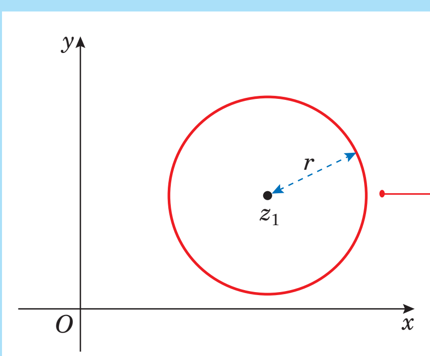
4 To express either $\sin^n\theta$ or $\cos^n\theta$ in terms of either $\cos k\theta$ or $\sin k\theta$ you need to be able to apply the following identities:

$$z + \frac{1}{z} = 2\cos\theta \qquad z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z - \frac{1}{z} = 2\mathrm{i}\sin\theta \qquad z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

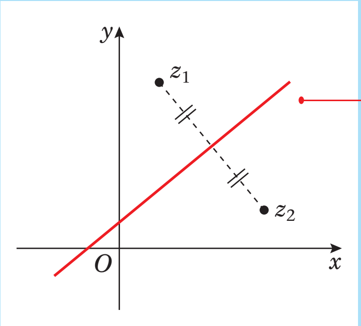
5 You need to recognise the following loci:

- $|z - z_1| = r$



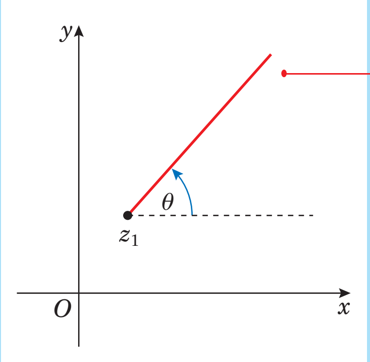
The locus of z is a circle, centre z_1, radius r.

- $|z - z_1| = |z - z_2|$



The locus of z is represented by the perpendicular bisector of the line segment joining the points z_1 to z_2.

- $\arg(z - z_1) = \theta$



The locus of z is represented by a half−line from the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.

for the fixed points $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$ and variable point $z = x + \mathrm{i}y$.