

**STEWART**  
JAMES

CENGAGE **brain**.com



*Calculus*  
**7E**

Licensed to:

CENGAGE **brain**.com

**ALGEBRA****Arithmetic Operations**

$$a(b + c) = ab + ac$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

**Exponents and Radicals**

$$x^m x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x}$$

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

**Factoring Special Polynomials**

$$x^2 - y^2 = (x + y)(x - y)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

**Binomial Theorem**

$$(x + y)^2 = x^2 + 2xy + y^2 \quad (x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2$$

$$+ \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

$$\text{where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}$$

**Quadratic Formula**

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Inequalities and Absolute Value**

If  $a < b$  and  $b < c$ , then  $a < c$ .

If  $a < b$ , then  $a + c < b + c$ .

If  $a < b$  and  $c > 0$ , then  $ca < cb$ .

If  $a < b$  and  $c < 0$ , then  $ca > cb$ .

If  $a > 0$ , then

$$|x| = a \text{ means } x = a \text{ or } x = -a$$

$$|x| < a \text{ means } -a < x < a$$

$$|x| > a \text{ means } x > a \text{ or } x < -a$$

**GEOMETRY****Geometric Formulas**

Formulas for area  $A$ , circumference  $C$ , and volume  $V$ :

Triangle

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}ab \sin \theta$$

Circle

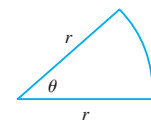
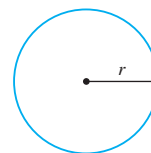
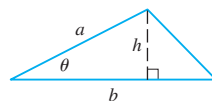
$$A = \pi r^2$$

$$C = 2\pi r$$

Sector of Circle

$$A = \frac{1}{2}r^2\theta$$

$$s = r\theta \quad (\theta \text{ in radians})$$



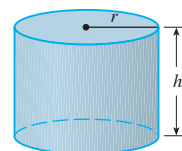
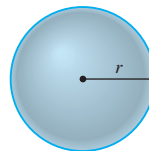
Sphere

$$V = \frac{4}{3}\pi r^3$$

$$A = 4\pi r^2$$

Cylinder

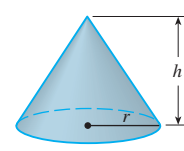
$$V = \pi r^2 h$$



Cone

$$V = \frac{1}{3}\pi r^2 h$$

$$A = \pi r \sqrt{r^2 + h^2}$$

**Distance and Midpoint Formulas**

Distance between  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{Midpoint of } \overline{P_1P_2}: \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

**Lines**

Slope of line through  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ :

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through  $P_1(x_1, y_1)$  with slope  $m$ :

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope  $m$  and y-intercept  $b$ :

$$y = mx + b$$

**Circles**

Equation of the circle with center  $(h, k)$  and radius  $r$ :

$$(x - h)^2 + (y - k)^2 = r^2$$

## TRIGONOMETRY

### Angle Measurement

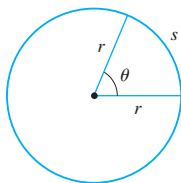
$$\pi \text{ radians} = 180^\circ$$

$$1^\circ = \frac{\pi}{180} \text{ rad}$$

$$1 \text{ rad} = \frac{180^\circ}{\pi}$$

$$s = r\theta$$

( $\theta$  in radians)



### Right Angle Trigonometry

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

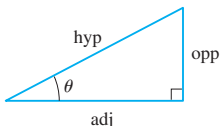
$$\csc \theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}}$$

$$\sec \theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\cot \theta = \frac{\text{adj}}{\text{opp}}$$



### Trigonometric Functions

$$\sin \theta = \frac{y}{r}$$

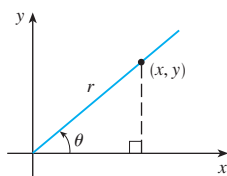
$$\csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r}$$

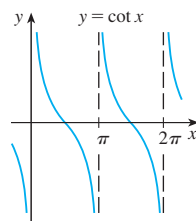
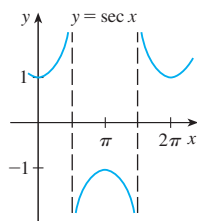
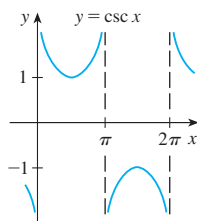
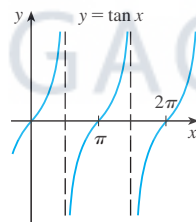
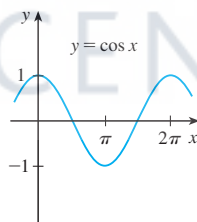
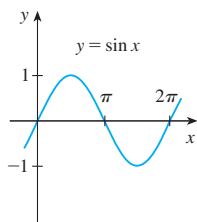
$$\sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x}$$

$$\cot \theta = \frac{x}{y}$$



### Graphs of Trigonometric Functions



### Trigonometric Functions of Important Angles

$\theta$	radians	$\sin \theta$	$\cos \theta$	$\tan \theta$
$0^\circ$	0	0	1	0
$30^\circ$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45^\circ$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60^\circ$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90^\circ$	$\pi/2$	1	0	—

### Fundamental Identities

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

### The Law of Sines

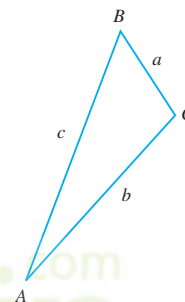
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

### The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



### Addition and Subtraction Formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

### Double-Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

### Half-Angle Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$



# CALCULUS

SEVENTH EDITION

JAMES STEWART

McMASTER UNIVERSITY

AND

UNIVERSITY OF TORONTO

CENGAGE **brain**.com



Australia · Brazil · Japan · Korea · Mexico · Singapore · Spain · United Kingdom · United States

This is an electronic version of the print textbook. Due to electronic rights restrictions, some third party content may be suppressed. Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. The publisher reserves the right to remove content from this title at any time if subsequent rights restrictions require it. For valuable information on pricing, previous editions, changes to current editions, and alternate formats, please visit [www.cengage.com/highered](http://www.cengage.com/highered) to search by ISBN#, author, title, or keyword for materials in your areas of interest.

CENGAGE **brain**

**Calculus, Seventh Edition****James Stewart**

Executive Editor: Liz Covello

Assistant Editor: Liza Neustaetter

Editorial Assistant: Jennifer Staller

Media Editor: Maureen Ross

Marketing Manager: Jennifer Jones

Marketing Coordinator: Michael Ledesma

Marketing Communications Manager: Mary Anne Payumo

Content Project Manager: Cheryll Linthicum

Art Director: Vernon T. Boes

Print Buyer: Becky Cross

Rights Acquisitions Specialist: Don Schlotman

Production Service: TECH·arts

Text Designer: TECH·arts

Photo Researcher: Terri Wright, [www.terriwright.com](http://www.terriwright.com)

Copy Editor: Kathi Townes

Cover Designer: Irene Morris

Cover Illustration: Irene Morris

Compositor: Stephanie Kuhns, TECH·arts

© 2012, 2008 Brooks/Cole, Cengage Learning

ALL RIGHTS RESERVED. No part of this work covered by the copyright herein may be reproduced, transmitted, stored, or used in any form or by any means graphic, electronic, or mechanical, including but not limited to photocopying, recording, scanning, digitizing, taping, Web distribution, information networks, or information storage and retrieval systems, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without the prior written permission of the publisher.

For product information and technology assistance, contact us at  
**Cengage Learning Customer & Sales Support, 1-800-354-9706.**

For permission to use material from this text or product,  
 submit all requests online at [www.cengage.com/permissions](http://www.cengage.com/permissions).

Further permissions questions can be e-mailed to  
[permissionrequest@cengage.com](mailto:permissionrequest@cengage.com).

Library of Congress Control Number: 2010936608

Student Edition:

ISBN-13: 978-0-538-49781-7

ISBN-10: 0-538-49781-5

Loose-leaf Edition:

ISBN-13: 978-0-8400-5818-8

ISBN-10: 0-8400-5818-7

**Brooks/Cole**
 20 Davis Drive  
 Belmont, CA 94002-3098  
 USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil, and Japan. Locate your local office at [www.cengage.com/global](http://www.cengage.com/global).

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

To learn more about Brooks/Cole, visit  
[www.cengage.com/brookscole](http://www.cengage.com/brookscole).

Purchase any of our products at your local college store or at our preferred online store [www.cengagebrain.com](http://www.cengagebrain.com).

**Trademarks**

ExamView® and ExamViewPro® are registered trademarks of FSCreations, Inc.

Windows is a registered trademark of the Microsoft Corporation and used herein under license.

Macintosh and Power Macintosh are registered trademarks of Apple Computer, Inc. Used herein under license.

Derive is a registered trademark of Soft Warehouse, Inc.

Maple is a registered trademark of Waterloo Maple, Inc.

Mathematica is a registered trademark of Wolfram Research, Inc.

Tools for Enriching is a trademark used herein under license.

Printed in the United States of America

1 2 3 4 5 6 7 14 13 12 11 10

# Contents

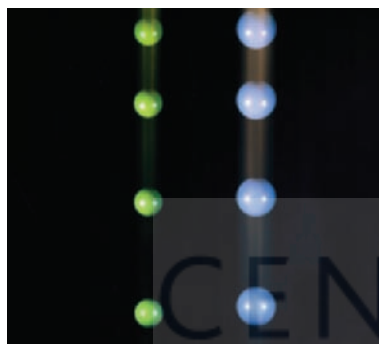
Preface	xi
To the Student	xxiii
Diagnostic Tests	xxiv

## A Preview of Calculus 1

---

## 1 Functions and Limits 9

---



1.1	Four Ways to Represent a Function	10
1.2	Mathematical Models: A Catalog of Essential Functions	23
1.3	New Functions from Old Functions	36
1.4	The Tangent and Velocity Problems	44
1.5	The Limit of a Function	50
1.6	Calculating Limits Using the Limit Laws	62
1.7	The Precise Definition of a Limit	72
1.8	Continuity	81
	Review	93

### Principles of Problem Solving 97

## 2 Derivatives 103

---



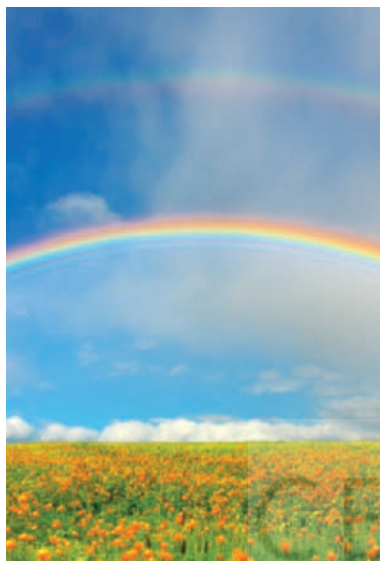
2.1	Derivatives and Rates of Change	104
	Writing Project ■ Early Methods for Finding Tangents	114
2.2	The Derivative as a Function	114
2.3	Differentiation Formulas	126
	Applied Project ■ Building a Better Roller Coaster	140
2.4	Derivatives of Trigonometric Functions	140
2.5	The Chain Rule	148
	Applied Project ■ Where Should a Pilot Start Descent?	156
2.6	Implicit Differentiation	157
	Laboratory Project ■ Families of Implicit Curves	163



- 2.7 Rates of Change in the Natural and Social Sciences 164
- 2.8 Related Rates 176
- 2.9 Linear Approximations and Differentials 183
  - Laboratory Project ■ Taylor Polynomials 189
  - Review 190

Problems Plus 194

### 3 Applications of Differentiation 197



- 3.1 Maximum and Minimum Values 198
  - Applied Project ■ The Calculus of Rainbows 206
- 3.2 The Mean Value Theorem 208
- 3.3 How Derivatives Affect the Shape of a Graph 213
- 3.4 Limits at Infinity; Horizontal Asymptotes 223
- 3.5 Summary of Curve Sketching 237
- 3.6 Graphing with Calculus *and* Calculators 244
- 3.7 Optimization Problems 250
  - Applied Project ■ The Shape of a Can 262
- 3.8 Newton's Method 263
- 3.9 Antiderivatives 269
  - Review 275

Problems Plus 279

### 4 Integrals 283



- 4.1 Areas and Distances 284
- 4.2 The Definite Integral 295
  - Discovery Project ■ Area Functions 309
- 4.3 The Fundamental Theorem of Calculus 310
- 4.4 Indefinite Integrals and the Net Change Theorem 321
  - Writing Project ■ Newton, Leibniz, and the Invention of Calculus 329
- 4.5 The Substitution Rule 330
  - Review 337

Problems Plus 341

## 5 Applications of Integration 343

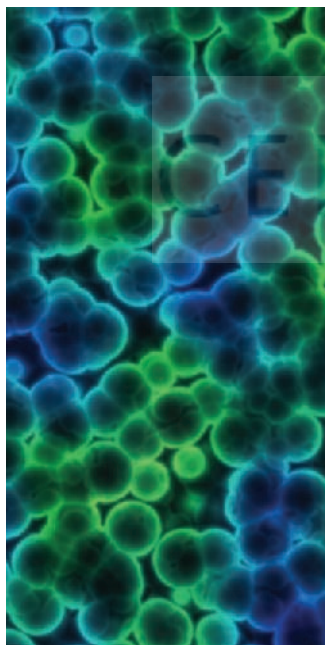


- 5.1 Areas Between Curves 344
  - Applied Project ■ The Gini Index 351
- 5.2 Volumes 352
- 5.3 Volumes by Cylindrical Shells 363
- 5.4 Work 368
- 5.5 Average Value of a Function 373
  - Applied Project ■ Calculus and Baseball 376
- Review 378

Problems Plus 380

## 6 Inverse Functions: 383

### Exponential, Logarithmic, and Inverse Trigonometric Functions



- 6.1 Inverse Functions 384

Instructors may cover either Sections 6.2–6.4 or Sections 6.2\*–6.4\*. See the Preface.

6.2 Exponential Functions and Their Derivatives 391	6.2* The Natural Logarithmic Function 421
6.3 Logarithmic Functions 404	6.3* The Natural Exponential Function 429
6.4 Derivatives of Logarithmic Functions 410	6.4* General Logarithmic and Exponential Functions 437

- 6.5 Exponential Growth and Decay 446
- 6.6 Inverse Trigonometric Functions 453
  - Applied Project ■ Where to Sit at the Movies 461
- 6.7 Hyperbolic Functions 462
- 6.8 Indeterminate Forms and l'Hospital's Rule 469
  - Writing Project ■ The Origins of l'Hospital's Rule 480
- Review 480

Problems Plus 485

## 7 Techniques of Integration 487



- 7.1 Integration by Parts 488
- 7.2 Trigonometric Integrals 495
- 7.3 Trigonometric Substitution 502
- 7.4 Integration of Rational Functions by Partial Fractions 508
- 7.5 Strategy for Integration 518
- 7.6 Integration Using Tables and Computer Algebra Systems 524
  - Discovery Project ■ Patterns in Integrals 529
- 7.7 Approximate Integration 530
- 7.8 Improper Integrals 543
  - Review 553

Problems Plus 557

## 8 Further Applications of Integration 561



- 8.1 Arc Length 562
  - Discovery Project ■ Arc Length Contest 569
- 8.2 Area of a Surface of Revolution 569
  - Discovery Project ■ Rotating on a Slant 575
- 8.3 Applications to Physics and Engineering 576
  - Discovery Project ■ Complementary Coffee Cups 586
- 8.4 Applications to Economics and Biology 587
- 8.5 Probability 592
  - Review 599

Problems Plus 601

## 9 Differential Equations 603

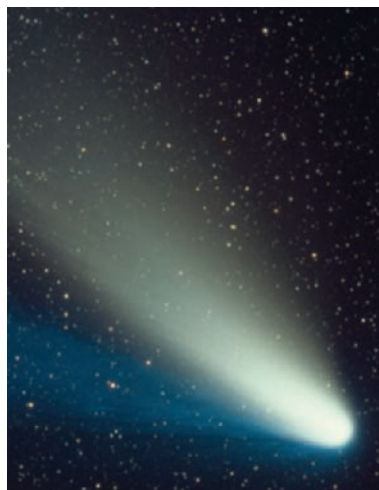


- 9.1 Modeling with Differential Equations 604
- 9.2 Direction Fields and Euler's Method 609
- 9.3 Separable Equations 618
  - Applied Project ■ How Fast Does a Tank Drain? 627
  - Applied Project ■ Which Is Faster, Going Up or Coming Down? 628
- 9.4 Models for Population Growth 629
- 9.5 Linear Equations 640

- 9.6 Predator-Prey Systems 646  
 Review 653

Problems Plus 657

## 10 Parametric Equations and Polar Coordinates 659



- 10.1 Curves Defined by Parametric Equations 660  
 Laboratory Project ■ Running Circles around Circles 668
- 10.2 Calculus with Parametric Curves 669  
 Laboratory Project ■ Bézier Curves 677
- 10.3 Polar Coordinates 678  
 Laboratory Project ■ Families of Polar Curves 688
- 10.4 Areas and Lengths in Polar Coordinates 689
- 10.5 Conic Sections 694
- 10.6 Conic Sections in Polar Coordinates 702  
 Review 709

Problems Plus 712

## 11 Infinite Sequences and Series 713



- 11.1 Sequences 714  
 Laboratory Project ■ Logistic Sequences 727
- 11.2 Series 727
- 11.3 The Integral Test and Estimates of Sums 738
- 11.4 The Comparison Tests 746
- 11.5 Alternating Series 751
- 11.6 Absolute Convergence and the Ratio and Root Tests 756
- 11.7 Strategy for Testing Series 763
- 11.8 Power Series 765
- 11.9 Representations of Functions as Power Series 770
- 11.10 Taylor and Maclaurin Series 777  
 Laboratory Project ■ An Elusive Limit 791  
 Writing Project ■ How Newton Discovered the Binomial Series 791
- 11.11 Applications of Taylor Polynomials 792  
 Applied Project ■ Radiation from the Stars 801  
 Review 802

Problems Plus 805

**12 Vectors and the Geometry of Space 809**

- 12.1** Three-Dimensional Coordinate Systems 810
- 12.2** Vectors 815
- 12.3** The Dot Product 824
- 12.4** The Cross Product 832
  - Discovery Project ■ The Geometry of a Tetrahedron 840
- 12.5** Equations of Lines and Planes 840
  - Laboratory Project ■ Putting 3D in Perspective 850
- 12.6** Cylinders and Quadric Surfaces 851
  - Review 858

Problems Plus 861

**13 Vector Functions 863**

- 13.1** Vector Functions and Space Curves 864
- 13.2** Derivatives and Integrals of Vector Functions 871
- 13.3** Arc Length and Curvature 877
- 13.4** Motion in Space: Velocity and Acceleration 886
  - Applied Project ■ Kepler's Laws 896
- Review 897

Problems Plus 900

**14 Partial Derivatives 901**

- 14.1** Functions of Several Variables 902
- 14.2** Limits and Continuity 916
- 14.3** Partial Derivatives 924
- 14.4** Tangent Planes and Linear Approximations 939
- 14.5** The Chain Rule 948
- 14.6** Directional Derivatives and the Gradient Vector 957
- 14.7** Maximum and Minimum Values 970
  - Applied Project ■ Designing a Dumpster 980
  - Discovery Project ■ Quadratic Approximations and Critical Points 980

- 14.8** Lagrange Multipliers 981  
     Applied Project ■ Rocket Science 988  
     Applied Project ■ Hydro-Turbine Optimization 990  
     Review 991

Problems Plus 995

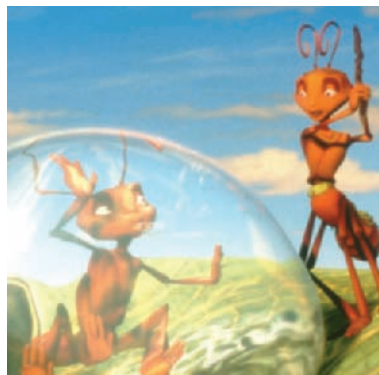
## 15 Multiple Integrals 997



- 15.1** Double Integrals over Rectangles 998  
**15.2** Iterated Integrals 1006  
**15.3** Double Integrals over General Regions 1012  
**15.4** Double Integrals in Polar Coordinates 1021  
**15.5** Applications of Double Integrals 1027  
**15.6** Surface Area 1037  
**15.7** Triple Integrals 1041  
     Discovery Project ■ Volumes of Hyperspheres 1051  
**15.8** Triple Integrals in Cylindrical Coordinates 1051  
     Discovery Project ■ The Intersection of Three Cylinders 1056  
**15.9** Triple Integrals in Spherical Coordinates 1057  
     Applied Project ■ Roller Derby 1063  
**15.10** Change of Variables in Multiple Integrals 1064  
     Review 1073

Problems Plus 1077

## 16 Vector Calculus 1079



- 16.1** Vector Fields 1080  
**16.2** Line Integrals 1087  
**16.3** The Fundamental Theorem for Line Integrals 1099  
**16.4** Green's Theorem 1108  
**16.5** Curl and Divergence 1115  
**16.6** Parametric Surfaces and Their Areas 1123  
**16.7** Surface Integrals 1134  
**16.8** Stokes' Theorem 1146  
     Writing Project ■ Three Men and Two Theorems 1152

<b>16.9</b>	The Divergence Theorem	1152
<b>16.10</b>	Summary	1159
	Review	1160

Problems Plus 1163

## **17** Second-Order Differential Equations 1165

---



<b>17.1</b>	Second-Order Linear Equations	1166
<b>17.2</b>	Nonhomogeneous Linear Equations	1172
<b>17.3</b>	Applications of Second-Order Differential Equations	1180
<b>17.4</b>	Series Solutions	1188
	Review	1193

## Appendixes A1

---

<b>A</b>	Numbers, Inequalities, and Absolute Values	A2
<b>B</b>	Coordinate Geometry and Lines	A10
<b>C</b>	Graphs of Second-Degree Equations	A16
<b>D</b>	Trigonometry	A24
<b>E</b>	Sigma Notation	A34
<b>F</b>	Proofs of Theorems	A39
<b>G</b>	Graphing Calculators and Computers	A48
<b>H</b>	Complex Numbers	A55
<b>I</b>	Answers to Odd-Numbered Exercises	A63

## Index A135

---

# Preface

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.

GEORGE POLYA

The art of teaching, Mark Van Doren said, is the art of assisting discovery. I have tried to write a book that assists students in discovering calculus—both for its practical power and its surprising beauty. In this edition, as in the first six editions, I aim to convey to the student a sense of the utility of calculus and develop technical competence, but I also strive to give some appreciation for the intrinsic beauty of the subject. Newton undoubtedly experienced a sense of triumph when he made his great discoveries. I want students to share some of that excitement.

The emphasis is on understanding concepts. I think that nearly everybody agrees that this should be the primary goal of calculus instruction. In fact, the impetus for the current calculus reform movement came from the Tulane Conference in 1986, which formulated as their first recommendation:

*Focus on conceptual understanding.*

I have tried to implement this goal through the *Rule of Three*: “Topics should be presented geometrically, numerically, and algebraically.” Visualization, numerical and graphical experimentation, and other approaches have changed how we teach conceptual reasoning in fundamental ways. The Rule of Three has been expanded to become the *Rule of Four* by emphasizing the verbal, or descriptive, point of view as well.

In writing the seventh edition my premise has been that it is possible to achieve conceptual understanding and still retain the best traditions of traditional calculus. The book contains elements of reform, but within the context of a traditional curriculum.

## Alternative Versions

I have written several other calculus textbooks that might be preferable for some instructors. Most of them also come in single variable and multivariable versions.

- *Calculus*, Seventh Edition, Hybrid Version, is similar to the present textbook in content and coverage except that all end-of-section exercises are available only in Enhanced WebAssign. The printed text includes all end-of-chapter review material.
- *Calculus: Early Transcendentals*, Seventh Edition, is similar to the present textbook except that the exponential, logarithmic, and inverse trigonometric functions are covered in the first semester.



- *Calculus: Early Transcendentals*, Seventh Edition, Hybrid Version, is similar to *Calculus: Early Transcendentals*, Seventh Edition, in content and coverage except that all end-of-section exercises are available only in Enhanced WebAssign. The printed text includes all end-of-chapter review material.
- *Essential Calculus* is a much briefer book (800 pages), though it contains almost all of the topics in *Calculus*, Seventh Edition. The relative brevity is achieved through briefer exposition of some topics and putting some features on the website.
- *Essential Calculus: Early Transcendentals* resembles *Essential Calculus*, but the exponential, logarithmic, and inverse trigonometric functions are covered in Chapter 3.
- *Calculus: Concepts and Contexts*, Fourth Edition, emphasizes conceptual understanding even more strongly than this book. The coverage of topics is not encyclopedic and the material on transcendental functions and on parametric equations is woven throughout the book instead of being treated in separate chapters.
- *Calculus: Early Vectors* introduces vectors and vector functions in the first semester and integrates them throughout the book. It is suitable for students taking Engineering and Physics courses concurrently with calculus.
- *Brief Applied Calculus* is intended for students in business, the social sciences, and the life sciences.

## What's New in the Seventh Edition?

The changes have resulted from talking with my colleagues and students at the University of Toronto and from reading journals, as well as suggestions from users and reviewers. Here are some of the many improvements that I've incorporated into this edition:

- Some material has been rewritten for greater clarity or for better motivation. See, for instance, the introduction to maximum and minimum values on page 198, the introduction to series on page 727, and the motivation for the cross product on page 832.
- New examples have been added (see Example 4 on page 1045 for instance). And the solutions to some of the existing examples have been amplified. A case in point: I added details to the solution of Example 1.6.11 because when I taught Section 1.6 from the sixth edition I realized that students need more guidance when setting up inequalities for the Squeeze Theorem.
- Chapter 1, *Functions and Limits*, consists of most of the material from Chapters 1 and 2 of the sixth edition. The section on Graphing Calculators and Computers is now Appendix G.
- The art program has been revamped: New figures have been incorporated and a substantial percentage of the existing figures have been redrawn.
- The data in examples and exercises have been updated to be more timely.
- Three new projects have been added: *The Gini Index* (page 351) explores how to measure income distribution among inhabitants of a given country and is a nice application of areas between curves. (I thank Klaus Volpert for suggesting this project.) *Families of Implicit Curves* (page 163) investigates the changing shapes of implicitly defined curves as parameters in a family are varied. *Families of Polar Curves* (page 688) exhibits the fascinating shapes of polar curves and how they evolve within a family.

- The section on the surface area of the graph of a function of two variables has been restored as Section 15.6 for the convenience of instructors who like to teach it after double integrals, though the full treatment of surface area remains in Chapter 16.
- I continue to seek out examples of how calculus applies to so many aspects of the real world. On page 933 you will see beautiful images of the earth's magnetic field strength and its second vertical derivative as calculated from Laplace's equation. I thank Roger Watson for bringing to my attention how this is used in geophysics and mineral exploration.
- More than 25% of the exercises are new. Here are some of my favorites: 2.2.13–14, 2.4.56, 2.5.67, 2.6.53–56, 2.7.22, 3.3.70, 3.4.43, 4.2.51–53, 5.4.30, 6.3.58, 11.2.49–50, 11.10.71–72, 12.1.44, 12.4.43–44, and Problems 4, 5, and 8 on pages 861–62.

## Technology Enhancements

- The media and technology to support the text have been enhanced to give professors greater control over their course, to provide extra help to deal with the varying levels of student preparedness for the calculus course, and to improve support for conceptual understanding. New Enhanced WebAssign features including a customizable Cengage YouBook, *Just in Time* review, *Show Your Work*, Answer Evaluator, Personalized Study Plan, Master Its, solution videos, lecture video clips (with associated questions), and *Visualizing Calculus* (TEC animations with associated questions) have been developed to facilitate improved student learning and flexible classroom teaching.
- *Tools for Enriching Calculus* (TEC) has been completely redesigned and is accessible in Enhanced WebAssign, CourseMate, and PowerLecture. Selected Visuals and Modules are available at [www.stewartcalculus.com](http://www.stewartcalculus.com).

## Features

**CONCEPTUAL EXERCISES** The most important way to foster conceptual understanding is through the problems that we assign. To that end I have devised various types of problems. Some exercise sets begin with requests to explain the meanings of the basic concepts of the section. (See, for instance, the first few exercises in Sections 1.5, 1.8, 11.2, 14.2, and 14.3.) Similarly, all the review sections begin with a Concept Check and a True-False Quiz. Other exercises test conceptual understanding through graphs or tables (see Exercises 2.1.17, 2.2.33–38, 2.2.41–44, 9.1.11–13, 10.1.24–27, 11.10.2, 13.2.1–2, 13.3.33–39, 14.1.1–2, 14.1.32–42, 14.3.3–10, 14.6.1–2, 14.7.3–4, 15.1.5–10, 16.1.11–18, 16.2.17–18, and 16.3.1–2).

Another type of exercise uses verbal description to test conceptual understanding (see Exercises 1.8.10, 2.2.56, 3.3.51–52, and 7.8.67). I particularly value problems that combine and compare graphical, numerical, and algebraic approaches (see Exercises 3.4.31–32, 2.7.25, and 9.4.2).

**GRADED EXERCISE SETS** Each exercise set is carefully graded, progressing from basic conceptual exercises and skill-development problems to more challenging problems involving applications and proofs.

**REAL-WORLD DATA** My assistants and I spent a great deal of time looking in libraries, contacting companies and government agencies, and searching the Internet for interesting real-world data to introduce, motivate, and illustrate the concepts of calculus. As a result, many of the examples and exercises deal with functions defined by such numerical data or graphs. See, for instance, Figure 1 in Section 1.1 (seismograms from the Northridge earthquake), Exercise

2.2.34 (percentage of the population under age 18), Exercise 4.1.16 (velocity of the space shuttle *Endeavour*), and Figure 4 in Section 4.4 (San Francisco power consumption). Functions of two variables are illustrated by a table of values of the wind-chill index as a function of air temperature and wind speed (Example 2 in Section 14.1). Partial derivatives are introduced in Section 14.3 by examining a column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity. This example is pursued further in connection with linear approximations (Example 3 in Section 14.4). Directional derivatives are introduced in Section 14.6 by using a temperature contour map to estimate the rate of change of temperature at Reno in the direction of Las Vegas. Double integrals are used to estimate the average snowfall in Colorado on December 20–21, 2006 (Example 4 in Section 15.1). Vector fields are introduced in Section 16.1 by depictions of actual velocity vector fields showing San Francisco Bay wind patterns.

**PROJECTS** One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed. I have included four kinds of projects: *Applied Projects* involve applications that are designed to appeal to the imagination of students. The project after Section 9.3 asks whether a ball thrown upward takes longer to reach its maximum height or to fall back to its original height. (The answer might surprise you.) The project after Section 14.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired velocity. *Laboratory Projects* involve technology; the one following Section 10.2 shows how to use Bézier curves to design shapes that represent letters for a laser printer. *Writing Projects* ask students to compare present-day methods with those of the founders of calculus—Fermat’s method for finding tangents, for instance. Suggested references are supplied. *Discovery Projects* anticipate results to be discussed later or encourage discovery through pattern recognition (see the one following Section 7.6). Others explore aspects of geometry: tetrahedra (after Section 12.4), hyperspheres (after Section 15.7), and intersections of three cylinders (after Section 15.8). Additional projects can be found in the *Instructor’s Guide* (see, for instance, Group Exercise 4.1: Position from Samples).

**PROBLEM SOLVING** Students usually have difficulties with problems for which there is no single well-defined procedure for obtaining the answer. I think nobody has improved very much on George Polya’s four-stage problem-solving strategy and, accordingly, I have included a version of his problem-solving principles following Chapter 1. They are applied, both explicitly and implicitly, throughout the book. After the other chapters I have placed sections called *Problems Plus*, which feature examples of how to tackle challenging calculus problems. In selecting the varied problems for these sections I kept in mind the following advice from David Hilbert: “A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts.” When I put these challenging problems on assignments and tests I grade them in a different way. Here I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant.

**DUAL TREATMENT OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS** There are two possible ways of treating the exponential and logarithmic functions and each method has its passionate advocates. Because one often finds advocates of both approaches teaching the same course, I include full treatments of both methods. In Sections 6.2, 6.3, and 6.4 the exponential function is defined first, followed by the logarithmic function as its inverse. (Students have seen these functions introduced this way since high school.) In the alternative approach, presented in Sections 6.2\*, 6.3\*, and 6.4\*, the logarithm is defined as an integral and the exponential function is its inverse. This latter method is, of course, less intuitive but more elegant. You can use whichever treatment you prefer.

If the first approach is taken, then much of Chapter 6 can be covered before Chapters 4 and 5, if desired. To accommodate this choice of presentation there are specially identified

problems involving integrals of exponential and logarithmic functions at the end of the appropriate sections of Chapters 4 and 5. This order of presentation allows a faster-paced course to teach the transcendental functions and the definite integral in the first semester of the course.

For instructors who would like to go even further in this direction I have prepared an alternate edition of this book, called *Calculus, Early Transcendentals, Seventh Edition*, in which the exponential and logarithmic functions are introduced in the first chapter. Their limits and derivatives are found in the second and third chapters at the same time as polynomials and the other elementary functions.

### TOOLS FOR ENRICHING™ CALCULUS

TEC is a companion to the text and is intended to enrich and complement its contents. (It is now accessible in Enhanced WebAssign, CourseMate, and PowerLecture. Selected Visuals and Modules are available at [www.stewartcalculus.com](http://www.stewartcalculus.com).) Developed by Harvey Keynes, Dan Clegg, Hubert Hohn, and myself, TEC uses a discovery and exploratory approach. In sections of the book where technology is particularly appropriate, marginal icons direct students to TEC modules that provide a laboratory environment in which they can explore the topic in different ways and at different levels. **Visuals are animations of figures in text; Modules are more elaborate activities and include exercises.** Instructors can choose to become involved at several different levels, ranging from simply encouraging students to use the Visuals and Modules for independent exploration, to assigning specific exercises from those included with each Module, or to creating additional exercises, labs, and projects that make use of the Visuals and Modules.

### HOMEWORK HINTS

Homework Hints presented in the form of questions try to imitate an effective teaching assistant by functioning as a silent tutor. Hints for representative exercises (usually odd-numbered) are included in every section of the text, indicated by printing the exercise number in red. They are constructed so as not to reveal any more of the actual solution than is minimally necessary to make further progress, and are available to students at [stewartcalculus.com](http://stewartcalculus.com) and in CourseMate and Enhanced WebAssign.

### ENHANCED WEBASSIGN

Technology is having an impact on the way homework is assigned to students, particularly in large classes. The use of online homework is growing and its appeal depends on ease of use, grading precision, and reliability. With the seventh edition we have been working with the calculus community and WebAssign to develop a more robust online homework system. Up to 70% of the exercises in each section are assignable as online homework, including free response, multiple choice, and multi-part formats.

The system also includes Active Examples, in which students are guided in step-by-step tutorials through text examples, with links to the textbook and to video solutions. New enhancements to the system include a customizable eBook, a *Show Your Work* feature, *Just in Time* review of precalculus prerequisites, an improved Assignment Editor, and an Answer Evaluator that accepts more mathematically equivalent answers and allows for homework grading in much the same way that an instructor grades.

### [www.stewartcalculus.com](http://www.stewartcalculus.com)

This site includes the following.

- Homework Hints
- Algebra Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to the better historical websites
- Additional Topics (complete with exercise sets): Fourier Series, Formulas for the Remainder Term in Taylor Series, Rotation of Axes
- Archived Problems (Drill exercises that appeared in previous editions, together with their solutions)
- Challenge Problems (some from the Problems Plus sections from prior editions)

- Links, for particular topics, to outside web resources
- Selected Tools for Enriching Calculus (TEC) Modules and Visuals

## Content

- Diagnostic Tests** The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.
- A Preview of Calculus** This is an overview of the subject and includes a list of questions to motivate the study of calculus.
- 1 Functions and Limits** From the beginning, multiple representations of functions are stressed: verbal, numerical, visual, and algebraic. A discussion of mathematical models leads to a review of the standard functions from these four points of view. The material on limits is motivated by a prior discussion of the tangent and velocity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. Section 1.7, on the precise epsilon-delta definition of a limit, is an optional section.
- 2 Derivatives** The material on derivatives is covered in two sections in order to give students more time to get used to the idea of a derivative as a function. The examples and exercises explore the meanings of derivatives in various contexts. Higher derivatives are introduced in Section 2.2.
- 3 Applications of Differentiation** The basic facts concerning extreme values and shapes of curves are deduced from the Mean Value Theorem. Graphing with technology emphasizes the interaction between calculus and calculators and the analysis of families of curves. Some substantial optimization problems are provided, including an explanation of why you need to raise your head  $42^\circ$  to see the top of a rainbow.
- 4 Integrals** The area problem and the distance problem serve to motivate the definite integral, with sigma notation introduced as needed. (Full coverage of sigma notation is provided in Appendix E.) Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables.
- 5 Applications of Integration** Here I present the applications of integration—area, volume, work, average value—that can reasonably be done without specialized techniques of integration. General methods are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral.
- 6 Inverse Functions: Exponential, Logarithmic, and Inverse Trigonometric Functions** As discussed more fully on page xiv, only one of the two treatments of these functions need be covered. Exponential growth and decay are covered in this chapter.
- 7 Techniques of Integration** All the standard methods are covered but, of course, the real challenge is to be able to recognize which technique is best used in a given situation. Accordingly, in Section 7.5, I present a strategy for integration. The use of computer algebra systems is discussed in Section 7.6.
- 8 Further Applications of Integration** Here are the applications of integration—arc length and surface area—for which it is useful to have available all the techniques of integration, as well as applications to biology, economics, and physics (hydrostatic force and centers of mass). I have also included a section on probability. There are more applications here than can realistically be covered in a given course. Instructors should select applications suitable for their students and for which they themselves have enthusiasm.

- 9 Differential Equations** Modeling is the theme that unifies this introductory treatment of differential equations. Direction fields and Euler's method are studied before separable and linear equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal consideration. These methods are applied to the exponential, logistic, and other models for population growth. The first four or five sections of this chapter serve as a good introduction to first-order differential equations. An optional final section uses predator-prey models to illustrate systems of differential equations.
- 10 Parametric Equations and Polar Coordinates** This chapter introduces parametric and polar curves and applies the methods of calculus to them. Parametric curves are well suited to laboratory projects; the three presented here involve families of curves and Bézier curves. A brief treatment of conic sections in polar coordinates prepares the way for Kepler's Laws in Chapter 13.
- 11 Infinite Sequences and Series** The convergence tests have intuitive justifications (see page 738) as well as formal proofs. Numerical estimates of sums of series are based on which test was used to prove convergence. The emphasis is on Taylor series and polynomials and their applications to physics. Error estimates include those from graphing devices.
- 12 Vectors and The Geometry of Space** The material on three-dimensional analytic geometry and vectors is divided into two chapters. Chapter 12 deals with vectors, the dot and cross products, lines, planes, and surfaces.
- 13 Vector Functions** This chapter covers vector-valued functions, their derivatives and integrals, the length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler's laws.
- 14 Partial Derivatives** Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, I introduce partial derivatives by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity.
- 15 Multiple Integrals** Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute probabilities, surface areas, and (in projects) volumes of hyperspheres and volumes of intersections of three cylinders. Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals.
- 16 Vector Calculus** Vector fields are introduced through pictures of velocity fields showing San Francisco Bay wind patterns. The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.
- 17 Second-Order Differential Equations** Since first-order differential equations are covered in Chapter 9, this final chapter deals with second-order linear differential equations, their application to vibrating springs and electric circuits, and series solutions. $\pi$

## Ancillaries

*Calculus*, Seventh Edition, is supported by a complete set of ancillaries developed under my direction. Each piece has been designed to enhance student understanding and to facilitate creative instruction. With this edition, new media and technologies have been developed that help students to visualize calculus and instructors to customize content to better align with the way they teach their course. The tables on pages xxi–xxii describe each of these ancillaries.

## Acknowledgments

The preparation of this and previous editions has involved much time spent reading the reasoned (but sometimes contradictory) advice from a large number of astute reviewers. I greatly appreciate the time they spent to understand my motivation for the approach taken. I have learned something from each of them.

### SEVENTH EDITION REVIEWERS

Amy Austin, *Texas A&M University*  
 Anthony J. Bevelacqua, *University of North Dakota*  
 Zhen-Qing Chen, *University of Washington—Seattle*  
 Jenna Carpenter, *Louisiana Tech University*  
 Le Baron O. Ferguson, *University of California—Riverside*  
 Shari Harris, *John Wood Community College*  
 Amer Iqbal, *University of Washington—Seattle*  
 Akhtar Khan, *Rochester Institute of Technology*  
 Marianne Korten, *Kansas State University*  
 Joyce Longman, *Villanova University*  
 Richard Millsbaugh, *University of North Dakota*  
 Lon H. Mitchell, *Virginia Commonwealth University*  
 Ho Kuen Ng, *San Jose State University*  
 Norma Ortiz-Robinson, *Virginia Commonwealth University*  
 Qin Sheng, *Baylor University*  
 Magdalena Toda, *Texas Tech University*  
 Ruth Trygstad, *Salt Lake Community College*  
 Klaus Volpert, *Villanova University*  
 Peiyong Wang, *Wayne State University*

### TECHNOLOGY REVIEWERS

Maria Andersen, <i>Muskegon Community College</i>	Brian Karasek, <i>South Mountain Community College</i>
Eric Aurand, <i>Eastfield College</i>	Jason Kozinski, <i>University of Florida</i>
Joy Becker, <i>University of Wisconsin—Stout</i>	Carole Krueger, <i>The University of Texas at Arlington</i>
Przemyslaw Bogacki, <i>Old Dominion University</i>	Ken Kubota, <i>University of Kentucky</i>
Amy Elizabeth Bowman, <i>University of Alabama in Huntsville</i>	John Mitchell, <i>Clark College</i>
Monica Brown, <i>University of Missouri—St. Louis</i>	Donald Paul, <i>Tulsa Community College</i>
Roxanne Byrne, <i>University of Colorado at Denver and Health Sciences Center</i>	Chad Pierson, <i>University of Minnesota, Duluth</i>
Teri Christiansen, <i>University of Missouri—Columbia</i>	Lanita Presson, <i>University of Alabama in Huntsville</i>
Bobby Dale Daniel, <i>Lamar University</i>	Karin Reinhold, <i>State University of New York at Albany</i>
Jennifer Daniel, <i>Lamar University</i>	Thomas Riedel, <i>University of Louisville</i>
Andras Domokos, <i>California State University, Sacramento</i>	Christopher Schroeder, <i>Morehead State University</i>
Timothy Flaherty, <i>Carnegie Mellon University</i>	Angela Sharp, <i>University of Minnesota, Duluth</i>
Lee Gibson, <i>University of Louisville</i>	Patricia Shaw, <i>Mississippi State University</i>
Jane Golden, <i>Hillsborough Community College</i>	Carl Spitznagel, <i>John Carroll University</i>
Semion Gutman, <i>University of Oklahoma</i>	Mohammad Tabanjeh, <i>Virginia State University</i>
Diane Hoffoss, <i>University of San Diego</i>	Capt. Koichi Takagi, <i>United States Naval Academy</i>
Lorraine Hughes, <i>Mississippi State University</i>	Lorna TenEyck, <i>Chemeketa Community College</i>
Jay Jahangiri, <i>Kent State University</i>	Roger Werbylo, <i>Pima Community College</i>
John Jernigan, <i>Community College of Philadelphia</i>	David Williams, <i>Clayton State University</i>
	Zhuan Ye, <i>Northern Illinois University</i>

## PREVIOUS EDITION REVIEWERS

- B. D. Aggarwala, *University of Calgary*  
 John Alberghini, *Manchester Community College*  
 Michael Albert, *Carnegie-Mellon University*  
 Daniel Anderson, *University of Iowa*  
 Donna J. Bailey, *Northeast Missouri State University*  
 Wayne Barber, *Chemeketa Community College*  
 Marilyn Belkin, *Villanova University*  
 Neil Berger, *University of Illinois, Chicago*  
 David Berman, *University of New Orleans*  
 Richard Biggs, *University of Western Ontario*  
 Robert Blumenthal, *Oglethorpe University*  
 Martina Bode, *Northwestern University*  
 Barbara Bohannon, *Hofstra University*  
 Philip L. Bowers, *Florida State University*  
 Amy Elizabeth Bowman, *University of Alabama in Huntsville*  
 Jay Bourland, *Colorado State University*  
 Stephen W. Brady, *Wichita State University*  
 Michael Breen, *Tennessee Technological University*  
 Robert N. Bryan, *University of Western Ontario*  
 David Buchthal, *University of Akron*  
 Jorge Cassio, *Miami-Dade Community College*  
 Jack Ceder, *University of California, Santa Barbara*  
 Scott Chapman, *Trinity University*  
 James Choike, *Oklahoma State University*  
 Barbara Cortzen, *DePaul University*  
 Carl Cowen, *Purdue University*  
 Philip S. Croke, *Vanderbilt University*  
 Charles N. Curtis, *Missouri Southern State College*  
 Daniel Cyphert, *Armstrong State College*  
 Robert Dahlin  
 M. Hilary Davies, *University of Alaska Anchorage*  
 Gregory J. Davis, *University of Wisconsin–Green Bay*  
 Elias Deeba, *University of Houston–Downtown*  
 Daniel DiMaria, *Suffolk Community College*  
 Seymour Ditor, *University of Western Ontario*  
 Greg Dresden, *Washington and Lee University*  
 Daniel Drucker, *Wayne State University*  
 Kenn Dunn, *Dalhousie University*  
 Dennis Dunninger, *Michigan State University*  
 Bruce Edwards, *University of Florida*  
 David Ellis, *San Francisco State University*  
 John Ellison, *Grove City College*  
 Martin Erickson, *Truman State University*  
 Garret Etgen, *University of Houston*  
 Theodore G. Faticoni, *Fordham University*  
 Laurene V. Fausett, *Georgia Southern University*  
 Norman Feldman, *Sonoma State University*  
 Newman Fisher, *San Francisco State University*  
 José D. Flores, *The University of South Dakota*  
 William Francis, *Michigan Technological University*  
 James T. Franklin, *Valencia Community College, East*  
 Stanley Friedlander, *Bronx Community College*  
 Patrick Gallagher, *Columbia University–New York*  
 Paul Garrett, *University of Minnesota–Minneapolis*  
 Frederick Gass, *Miami University of Ohio*  
 Bruce Gilligan, *University of Regina*  
 Matthias K. Gobbert, *University of Maryland, Baltimore County*  
 Gerald Goff, *Oklahoma State University*  
 Stuart Goldenberg, *California Polytechnic State University*  
 John A. Graham, *Buckingham Browne & Nichols School*  
 Richard Grassl, *University of New Mexico*  
 Michael Gregory, *University of North Dakota*  
 Charles Groetsch, *University of Cincinnati*  
 Paul Triantafilos Hadavas, *Armstrong Atlantic State University*  
 Salim M. Haïdar, *Grand Valley State University*  
 D. W. Hall, *Michigan State University*  
 Robert L. Hall, *University of Wisconsin–Milwaukee*  
 Howard B. Hamilton, *California State University, Sacramento*  
 Darel Hardy, *Colorado State University*  
 Gary W. Harrison, *College of Charleston*  
 Melvin Hausner, *New York University/Courant Institute*  
 Curtis Herink, *Mercer University*  
 Russell Herman, *University of North Carolina at Wilmington*  
 Allen Hesse, *Rochester Community College*  
 Randall R. Holmes, *Auburn University*  
 James F. Hurley, *University of Connecticut*  
 Matthew A. Isom, *Arizona State University*  
 Gerald Janusz, *University of Illinois at Urbana-Champaign*  
 John H. Jenkins, *Embry-Riddle Aeronautical University, Prescott Campus*  
 Clement Jeske, *University of Wisconsin, Platteville*  
 Carl Jockusch, *University of Illinois at Urbana-Champaign*  
 Jan E. H. Johansson, *University of Vermont*  
 Jerry Johnson, *Oklahoma State University*  
 Zsuzsanna M. Kadas, *St. Michael's College*  
 Nets Katz, *Indiana University Bloomington*  
 Matt Kaufman  
 Matthias Kawski, *Arizona State University*  
 Frederick W. Keene, *Pasadena City College*  
 Robert L. Kelley, *University of Miami*  
 Virgil Kowalik, *Texas A&I University*  
 Kevin Kreider, *University of Akron*  
 Leonard Krop, *DePaul University*  
 Mark Krusemeyer, *Carleton College*  
 John C. Lawlor, *University of Vermont*  
 Christopher C. Leary, *State University of New York at Geneseo*  
 David Leeming, *University of Victoria*  
 Sam Lesseig, *Northeast Missouri State University*  
 Phil Locke, *University of Maine*  
 Joan McCarter, *Arizona State University*  
 Phil McCartney, *Northern Kentucky University*  
 James McKinney, *California State Polytechnic University, Pomona*  
 Igor Malyshev, *San Jose State University*  
 Larry Mansfield, *Queens College*  
 Mary Martin, *Colgate University*  
 Nathaniel F. G. Martin, *University of Virginia*  
 Gerald Y. Matsumoto, *American River College*  
 Tom Metzger, *University of Pittsburgh*



Michael Montañó, *Riverside Community College*  
 Teri Jo Murphy, *University of Oklahoma*  
 Martin Nakashima, *California State Polytechnic University, Pomona*  
 Richard Nowakowski, *Dalhousie University*  
 Hussain S. Nur, *California State University, Fresno*  
 Wayne N. Palmer, *Utica College*  
 Vincent Panico, *University of the Pacific*  
 F. J. Papp, *University of Michigan–Dearborn*  
 Mike Penna, *Indiana University–Purdue University Indianapolis*  
 Mark Pinsky, *Northwestern University*  
 Lothar Redlin, *The Pennsylvania State University*  
 Joel W. Robbin, *University of Wisconsin–Madison*  
 Lila Roberts, *Georgia College and State University*  
 E. Arthur Robinson, Jr., *The George Washington University*  
 Richard Rockwell, *Pacific Union College*  
 Rob Root, *Lafayette College*  
 Richard Ruedemann, *Arizona State University*  
 David Ryeburn, *Simon Fraser University*  
 Richard St. Andre, *Central Michigan University*  
 Ricardo Salinas, *San Antonio College*  
 Robert Schmidt, *South Dakota State University*  
 Eric Schreiner, *Western Michigan University*  
 Mihr J. Shah, *Kent State University–Trumbull*  
 Theodore Shifrin, *University of Georgia*

Wayne Skrapek, *University of Saskatchewan*  
 Larry Small, *Los Angeles Pierce College*  
 Teresa Morgan Smith, *Blinn College*  
 William Smith, *University of North Carolina*  
 Donald W. Solomon, *University of Wisconsin–Milwaukee*  
 Edward Spitznagel, *Washington University*  
 Joseph Stampfli, *Indiana University*  
 Kristin Stoley, *Blinn College*  
 M. B. Tavakoli, *Chaffey College*  
 Paul Xavier Uhlig, *St. Mary's University, San Antonio*  
 Stan Ver Nooy, *University of Oregon*  
 Andrei Verona, *California State University–Los Angeles*  
 Russell C. Walker, *Carnegie Mellon University*  
 William L. Walton, *McCallie School*  
 Jack Weiner, *University of Guelph*  
 Alan Weinstein, *University of California, Berkeley*  
 Theodore W. Wilcox, *Rochester Institute of Technology*  
 Steven Willard, *University of Alberta*  
 Robert Wilson, *University of Wisconsin–Madison*  
 Jerome Wolbert, *University of Michigan–Ann Arbor*  
 Dennis H. Wortman, *University of Massachusetts, Boston*  
 Mary Wright, *Southern Illinois University–Carbondale*  
 Paul M. Wright, *Austin Community College*  
 Xian Wu, *University of South Carolina*

In addition, I would like to thank Jordan Bell, George Bergman, Leon Gerber, Mary Pugh, and Simon Smith for their suggestions; Al Shenk and Dennis Zill for permission to use exercises from their calculus texts; COMAP for permission to use project material; George Bergman, David Bleecker, Dan Clegg, Victor Kaftal, Anthony Lam, Jamie Lawson, Ira Rosenholtz, Paul Sally, Lowell Smylie, and Larry Wallen for ideas for exercises; Dan Drucker for the roller derby project; Thomas Banchoff, Tom Farmer, Fred Gass, John Ramsay, Larry Riddle, Philip Traffin, and Klaus Volpert for ideas for projects; Dan Anderson, Dan Clegg, Jeff Cole, Dan Drucker, and Barbara Frank for solving the new exercises and suggesting ways to improve them; Marv Riedesel and Mary Johnson for accuracy in proofreading; and Jeff Cole and Dan Clegg for their careful preparation and proofreading of the answer manuscript.

In addition, I thank those who have contributed to past editions: Ed Barbeau, Fred Brauer, Andy Bulman-Fleming, Bob Burton, David Cusick, Tom DiCiccio, Garret Etgen, Chris Fisher, Stuart Goldenberg, Arnold Good, Gene Hecht, Harvey Keynes, E.L. Koh, Zdislav Kovarik, Kevin Kreider, Emile LeBlanc, David Leep, Gerald Leibowitz, Larry Peterson, Lothar Redlin, Carl Riehm, John Ringland, Peter Rosenthal, Doug Shaw, Dan Silver, Norton Starr, Saleem Watson, Alan Weinstein, and Gail Wolkowicz.

I also thank Kathi Townes, Stephanie Kuhns, and Rebekah Million of TECHarts for their production services and the following Brooks/Cole staff: Cheryll Linthicum, content project manager; Liza Neustaetter, assistant editor; Maureen Ross, media editor; Sam Subity, managing media editor; Jennifer Jones, marketing manager; and Vernon Boes, art director. They have all done an outstanding job.

I have been very fortunate to have worked with some of the best mathematics editors in the business over the past three decades: Ron Munro, Harry Campbell, Craig Barth, Jeremy Hayhurst, Gary Ostedt, Bob Pirtle, Richard Stratton, and now Liz Covello. All of them have contributed greatly to the success of this book.

JAMES STEWART

## Ancillaries for Instructors

### PowerLecture

ISBN 0-8400-5414-9

*This comprehensive DVD contains all art from the text in both jpeg and PowerPoint formats, key equations and tables from the text, complete pre-built PowerPoint lectures, an electronic version of the Instructor's Guide, Solution Builder, ExamView testing software, Tools for Enriching Calculus, video instruction, and JoinIn on TurningPoint clicker content.*

### Instructor's Guide

by Douglas Shaw

ISBN 0-8400-5407-6

*Each section of the text is discussed from several viewpoints. The Instructor's Guide contains suggested time to allot, points to stress, text discussion topics, core materials for lecture, workshop/discussion suggestions, group work exercises in a form suitable for handout, and suggested homework assignments. An electronic version of the Instructor's Guide is available on the PowerLecture DVD.*

### Complete Solutions Manual

#### Single Variable

By Daniel Anderson, Jeffery A. Cole, and Daniel Drucker

ISBN 0-8400-5302-9

#### Multivariable

By Dan Clegg and Barbara Frank

ISBN 0-8400-4947-1

*Includes worked-out solutions to all exercises in the text.*

### Solution Builder

[www.cengage.com/solutionbuilder](http://www.cengage.com/solutionbuilder)

*This online instructor database offers complete worked out solutions to all exercises in the text. Solution Builder allows you to create customized, secure solutions printouts (in PDF format) matched exactly to the problems you assign in class.*

### Printed Test Bank

By William Steven Harmon

ISBN 0-8400-5408-4

*Contains text-specific multiple-choice and free response test items.*

### ExamView Testing

*Create, deliver, and customize tests in print and online formats with ExamView, an easy-to-use assessment and tutorial software. ExamView contains hundreds of multiple-choice and free response test items. ExamView testing is available on the PowerLecture DVD.*

## Ancillaries for Instructors and Students

### Stewart Website

[www.stewartcalculus.com](http://www.stewartcalculus.com)

Contents: *Homework Hints* ■ *Algebra Review* ■ *Additional Topics* ■ *Drill exercises* ■ *Challenge Problems* ■ *Web Links* ■ *History of Mathematics* ■ *Tools for Enriching Calculus (TEC)*

### TEC Tools for Enriching™ Calculus

By James Stewart, Harvey Keynes, Dan Clegg, and developer Hu Hohn

*Tools for Enriching Calculus (TEC) functions as both a powerful tool for instructors, as well as a tutorial environment in which students can explore and review selected topics. The Flash simulation modules in TEC include instructions, written and audio explanations of the concepts, and exercises. TEC is accessible in CourseMate, WebAssign, and PowerLecture. Selected Visuals and Modules are available at [www.stewartcalculus.com](http://www.stewartcalculus.com).*

### WebAssign Enhanced WebAssign

[www.webassign.net](http://www.webassign.net)

*WebAssign's homework delivery system lets instructors deliver, collect, grade, and record assignments via the web. Enhanced WebAssign for Stewart's Calculus now includes opportunities for students to review prerequisite skills and content both at the start of the course and at the beginning of each section. In addition, for selected problems, students can get extra help in the form of "enhanced feedback" (rejoinders) and video solutions. **Other key features include:** thousands of problems from Stewart's Calculus, a customizable Cengage YouBook, Personal Study Plans, Show Your Work, Just in Time Review, Answer Evaluator, Visualizing Calculus animations and modules, quizzes, lecture videos (with associated questions), and more!*

### Cengage Customizable YouBook

*YouBook is a Flash-based eBook that is interactive and customizable! Containing all the content from Stewart's Calculus, YouBook features a text edit tool that allows instructors to modify the textbook narrative as needed. With YouBook, instructors can quickly re-order entire sections and chapters or hide any content they don't teach to create an eBook that perfectly matches their syllabus. Instructors can further customize the text by adding instructor-created or YouTube video links. Additional media assets include: animated figures, video clips, highlighting, notes, and more! YouBook is available in Enhanced WebAssign.*



www.cengagebrain.com

*CourseMate is a perfect self-study tool for students, and requires no set up from instructors. CourseMate brings course concepts to life with interactive learning, study, and exam preparation tools that support the printed textbook. CourseMate for Stewart's Calculus includes: an interactive eBook, Tools for Enriching Calculus, videos, quizzes, flashcards, and more! For instructors, CourseMate includes Engagement Tracker, a first-of-its-kind tool that monitors student engagement.*

### Maple CD-ROM

*Maple provides an advanced, high performance mathematical computation engine with fully integrated numerics & symbolics, all accessible from a WYSIWYG technical document environment.*

### CengageBrain.com

*To access additional course materials and companion resources, please visit [www.cengagebrain.com](http://www.cengagebrain.com). At the CengageBrain.com home page, search for the ISBN of your title (from the back cover of your book) using the search box at the top of the page. This will take you to the product page where free companion resources can be found.*

## Ancillaries for Students

### Student Solutions Manual

#### Single Variable

By Daniel Anderson, Jeffery A. Cole, and Daniel Drucker  
ISBN 0-8400-4949-8

#### Multivariable

By Dan Clegg and Barbara Frank  
ISBN 0-8400-4945-5

*Provides completely worked-out solutions to all odd-numbered exercises in the text, giving students a chance to check their answers and ensure they took the correct steps to arrive at an answer.*

### Study Guide

#### Single Variable

By Richard St. Andre  
ISBN 0-8400-5409-2

#### Multivariable

By Richard St. Andre  
ISBN 0-8400-5410-6

*For each section of the text, the Study Guide provides students with a brief introduction, a short list of concepts to master, as*

*well as summary and focus questions with explained answers. The Study Guide also contains "Technology Plus" questions, and multiple-choice "On Your Own" exam-style questions.*

### CalcLabs with Maple

**Single Variable** By Philip B. Yasskin and Robert Lopez  
ISBN 0-8400-5811-X

**Multivariable** By Philip B. Yasskin and Robert Lopez  
ISBN 0-8400-5812-8

### CalcLabs with Mathematica

**Single Variable** By Selwyn Hollis  
ISBN 0-8400-5814-4

**Multivariable** By Selwyn Hollis  
ISBN 0-8400-5813-6

*Each of these comprehensive lab manuals will help students learn to use the technology tools available to them. CalcLabs contain clearly explained exercises and a variety of labs and projects to accompany the text.*

### A Companion to Calculus

By Dennis Ebersole, Doris Schattschneider, Alicia Sevilla, and Kay Somers  
ISBN 0-495-01124-X

*Written to improve algebra and problem-solving skills of students taking a Calculus course, every chapter in this companion is keyed to a calculus topic, providing conceptual background and specific algebra techniques needed to understand and solve calculus problems related to that topic. It is designed for calculus courses that integrate the review of precalculus concepts or for individual use.*

### Linear Algebra for Calculus

by Konrad J. Heuvers, William P. Francis, John H. Kuisti, Deborah F. Lockhart, Daniel S. Moak, and Gene M. Ortner  
ISBN 0-534-25248-6

*This comprehensive book, designed to supplement the calculus course, provides an introduction to and review of the basic ideas of linear algebra.*





# To the Student

Reading a calculus textbook is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.


Some students start by trying their homework problems and read the text only if they get stuck on an exercise. I suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, I suggest that you cover up the solution and try solving the problem yourself. You'll get a lot more from looking at the solution if you do so.


Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix I. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from mine, don't immediately assume you're wrong. For example, if the answer given in the back of the book is  $\sqrt{2} - 1$  and you obtain  $1/(1 + \sqrt{2})$ , then you're right and rationalizing the denominator will show that the answers are equivalent.

The icon  indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software. (Appendix G discusses the use of these graphing devices and some of the pitfalls that you may encounter.) But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol  is

reserved for problems in which the full resources of a computer algebra system (like Derive, Maple, Mathematica, or the TI-89/92) are required.

You will also encounter the symbol , which warns you against committing an error. I have placed this symbol in the margin in situations where I have observed that a large proportion of my students tend to make the same mistake.

*Tools for Enriching Calculus*, which is a companion to this text, is referred to by means of the symbol  and can be accessed in Enhanced WebAssign and CourseMate (selected Visuals and Modules are available at [www.stewartcalculus.com](http://www.stewartcalculus.com)). It directs you to modules in which you can explore aspects of calculus for which the computer is particularly useful.

*Homework Hints* for representative exercises are indicated by printing the exercise number in red: **5**. These hints can be found on [stewartcalculus.com](http://stewartcalculus.com) as well as Enhanced WebAssign and CourseMate. The homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. You need to pursue each hint in an active manner with pencil and paper to work out the details. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint.

I recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. I hope you will discover that it is not only useful but also intrinsically beautiful.

JAMES STEWART

# Diagnostic Tests

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following tests are intended to diagnose weaknesses that you might have in these areas. After taking each test you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

## A Diagnostic Test: Algebra

1. Evaluate each expression without using a calculator.

(a)  $(-3)^4$       (b)  $-3^4$       (c)  $3^{-4}$   
 (d)  $\frac{5^{23}}{5^{21}}$       (e)  $\left(\frac{2}{3}\right)^{-2}$       (f)  $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

(a)  $\sqrt{200} - \sqrt{32}$   
 (b)  $(3a^3b^3)(4ab^2)^2$   
 (c)  $\left(\frac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

(a)  $3(x + 6) + 4(2x - 5)$       (b)  $(x + 3)(4x - 5)$   
 (c)  $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$       (d)  $(2x + 3)^2$   
 (e)  $(x + 2)^3$

4. Factor each expression.

(a)  $4x^2 - 25$       (b)  $2x^2 + 5x - 12$   
 (c)  $x^3 - 3x^2 - 4x + 12$       (d)  $x^4 + 27x$   
 (e)  $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$       (f)  $x^3y - 4xy$

5. Simplify the rational expression.

(a)  $\frac{x^2 + 3x + 2}{x^2 - x - 2}$       (b)  $\frac{2x^2 - x - 1}{x^2 - 9} \cdot \frac{x + 3}{2x + 1}$   
 (c)  $\frac{x^2}{x^2 - 4} - \frac{x + 1}{x + 2}$       (d)  $\frac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{y} - \frac{1}{x}}$

6. Rationalize the expression and simplify.

(a)  $\frac{\sqrt{10}}{\sqrt{5} - 2}$

(b)  $\frac{\sqrt{4+h} - 2}{h}$

7. Rewrite by completing the square.

(a)  $x^2 + x + 1$

(b)  $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a)  $x + 5 = 14 - \frac{1}{2}x$

(b)  $\frac{2x}{x+1} = \frac{2x-1}{x}$

(c)  $x^2 - x - 12 = 0$

(d)  $2x^2 + 4x + 1 = 0$

(e)  $x^4 - 3x^2 + 2 = 0$

(f)  $3|x - 4| = 10$

(g)  $2x(4-x)^{-1/2} - 3\sqrt{4-x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a)  $-4 < 5 - 3x \leq 17$

(b)  $x^2 < 2x + 8$

(c)  $x(x-1)(x+2) > 0$

(d)  $|x - 4| < 3$

(e)  $\frac{2x-3}{x+1} \leq 1$

10. State whether each equation is true or false.

(a)  $(p+q)^2 = p^2 + q^2$

(b)  $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(c)  $\sqrt{a^2 + b^2} = a + b$

(d)  $\frac{1+TC}{C} = 1 + T$

(e)  $\frac{1}{x-y} = \frac{1}{x} - \frac{1}{y}$

(f)  $\frac{1/x}{a/x - b/x} = \frac{1}{a-b}$

## Answers to Diagnostic Test A: Algebra

1. (a) 81

(b) -81

(c)  $\frac{1}{81}$

(d) 25

(e)  $\frac{9}{4}$

(f)  $\frac{1}{8}$

2. (a)  $6\sqrt{2}$

(b)  $48a^5b^7$

(c)  $\frac{x}{9y^7}$

3. (a)  $11x - 2$

(b)  $4x^2 + 7x - 15$

(c)  $a - b$

(d)  $4x^2 + 12x + 9$

(e)  $x^3 + 6x^2 + 12x + 8$

4. (a)  $(2x - 5)(2x + 5)$

(b)  $(2x - 3)(x + 4)$

(c)  $(x - 3)(x - 2)(x + 2)$

(d)  $x(x + 3)(x^2 - 3x + 9)$

(e)  $3x^{-1/2}(x - 1)(x - 2)$

(f)  $xy(x - 2)(x + 2)$

5. (a)  $\frac{x+2}{x-2}$

(b)  $\frac{x-1}{x-3}$

(c)  $\frac{1}{x-2}$

(d)  $-(x + y)$

6. (a)  $5\sqrt{2} + 2\sqrt{10}$

(b)  $\frac{1}{\sqrt{4+h} + 2}$

7. (a)  $(x + \frac{1}{2})^2 + \frac{3}{4}$

(b)  $2(x - 3)^2 - 7$

8. (a) 6

(b) 1

(c) -3, 4

(d)  $-1 \pm \frac{1}{2}\sqrt{2}$

(e)  $\pm 1, \pm\sqrt{2}$

(f)  $\frac{2}{3}, \frac{22}{3}$

(g)  $\frac{12}{5}$

9. (a)  $[-4, 3)$

(b)  $(-2, 4)$

(c)  $(-2, 0) \cup (1, \infty)$

(d)  $(1, 7)$

(e)  $(-1, 4]$

10. (a) False

(b) True

(c) False

(d) False

(e) False

(f) True

If you have had difficulty with these problems, you may wish to consult the Review of Algebra on the website [www.stewartcalculus.com](http://www.stewartcalculus.com)

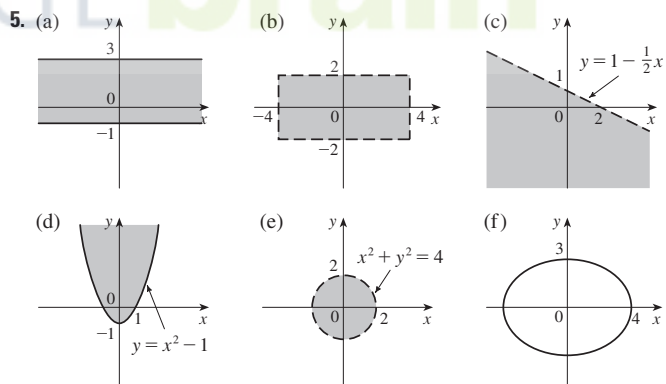
## B Diagnostic Test: Analytic Geometry

1. Find an equation for the line that passes through the point  $(2, -5)$  and
  - (a) has slope  $-3$
  - (b) is parallel to the  $x$ -axis
  - (c) is parallel to the  $y$ -axis
  - (d) is parallel to the line  $2x - 4y = 3$
2. Find an equation for the circle that has center  $(-1, 4)$  and passes through the point  $(3, -2)$ .
3. Find the center and radius of the circle with equation  $x^2 + y^2 - 6x + 10y + 9 = 0$ .
4. Let  $A(-7, 4)$  and  $B(5, -12)$  be points in the plane.
  - (a) Find the slope of the line that contains  $A$  and  $B$ .
  - (b) Find an equation of the line that passes through  $A$  and  $B$ . What are the intercepts?
  - (c) Find the midpoint of the segment  $AB$ .
  - (d) Find the length of the segment  $AB$ .
  - (e) Find an equation of the perpendicular bisector of  $AB$ .
  - (f) Find an equation of the circle for which  $AB$  is a diameter.
5. Sketch the region in the  $xy$ -plane defined by the equation or inequalities.
 

(a) $-1 \leq y \leq 3$	(b) $ x  < 4$ and $ y  < 2$
(c) $y < 1 - \frac{1}{2}x$	(d) $y \geq x^2 - 1$
(e) $x^2 + y^2 < 4$	(f) $9x^2 + 16y^2 = 144$

## Answers to Diagnostic Test B: Analytic Geometry

1. (a)  $y = -3x + 1$   
(c)  $x = 2$
2.  $(x + 1)^2 + (y - 4)^2 = 52$
3. Center  $(3, -5)$ , radius 5
4. (a)  $-\frac{4}{3}$   
(b)  $4x + 3y + 16 = 0$ ;  $x$ -intercept  $-4$ ,  $y$ -intercept  $-\frac{16}{3}$   
(c)  $(-1, -4)$   
(d) 20  
(e)  $3x - 4y = 13$   
(f)  $(x + 1)^2 + (y + 4)^2 = 100$



If you have had difficulty with these problems, you may wish to consult the review of analytic geometry in Appendixes B and C.

**C Diagnostic Test: Functions**

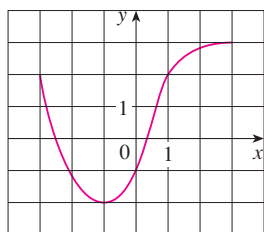


FIGURE FOR PROBLEM 1

- The graph of a function  $f$  is given at the left.
  - State the value of  $f(-1)$ .
  - Estimate the value of  $f(2)$ .
  - For what values of  $x$  is  $f(x) = 2$ ?
  - Estimate the values of  $x$  such that  $f(x) = 0$ .
  - State the domain and range of  $f$ .
- If  $f(x) = x^3$ , evaluate the difference quotient  $\frac{f(2+h) - f(2)}{h}$  and simplify your answer.
- Find the domain of the function.
  - $f(x) = \frac{2x + 1}{x^2 + x - 2}$
  - $g(x) = \frac{\sqrt[3]{x}}{x^2 + 1}$
  - $h(x) = \sqrt{4-x} + \sqrt{x^2 - 1}$
- How are graphs of the functions obtained from the graph of  $f$ ?
  - $y = -f(x)$
  - $y = 2f(x) - 1$
  - $y = f(x - 3) + 2$
- Without using a calculator, make a rough sketch of the graph.
  - $y = x^3$
  - $y = (x + 1)^3$
  - $y = (x - 2)^3 + 3$
  - $y = 4 - x^2$
  - $y = \sqrt{x}$
  - $y = 2\sqrt{x}$
  - $y = -2^x$
  - $y = 1 + x^{-1}$
- Let  $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$ 
  - Evaluate  $f(-2)$  and  $f(1)$ .
  - Sketch the graph of  $f$ .
- If  $f(x) = x^2 + 2x - 1$  and  $g(x) = 2x - 3$ , find each of the following functions.
  - $f \circ g$
  - $g \circ f$
  - $g \circ g \circ g$

**Answers to Diagnostic Test C: Functions**

- 2
  - 2.8
  - 3, 1
  - 2.5, 0.3
  - $[-3, 3], [-2, 3]$
- $12 + 6h + h^2$
- $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
  - $(-\infty, \infty)$
  - $(-\infty, -1] \cup [1, 4]$
- Reflect about the  $x$ -axis
  - Stretch vertically by a factor of 2, then shift 1 unit downward
  - Shift 3 units to the right and 2 units upward
- - 
  -
- 3, 3
  -
- $(f \circ g)(x) = 4x^2 - 8x + 2$
  - $(g \circ f)(x) = 2x^2 + 4x - 5$
  - $(g \circ g \circ g)(x) = 8x - 21$

If you have had difficulty with these problems, you should look at Sections 1.1–1.3 of this book.



## D Diagnostic Test: Trigonometry

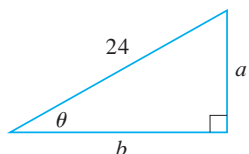


FIGURE FOR PROBLEM 5

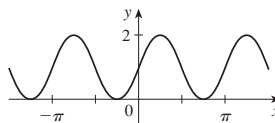
- Convert from degrees to radians.
  - $300^\circ$
  - $-18^\circ$
- Convert from radians to degrees.
  - $5\pi/6$
  - 2
- Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of  $30^\circ$ .
- Find the exact values.
  - $\tan(\pi/3)$
  - $\sin(7\pi/6)$
  - $\sec(5\pi/3)$
- Express the lengths  $a$  and  $b$  in the figure in terms of  $\theta$ .
- If  $\sin x = \frac{1}{3}$  and  $\sec y = \frac{5}{4}$ , where  $x$  and  $y$  lie between 0 and  $\pi/2$ , evaluate  $\sin(x + y)$ .
- Prove the identities.
  - $\tan \theta \sin \theta + \cos \theta = \sec \theta$
  - $\frac{2 \tan x}{1 + \tan^2 x} = \sin 2x$
- Find all values of  $x$  such that  $\sin 2x = \sin x$  and  $0 \leq x \leq 2\pi$ .
- Sketch the graph of the function  $y = 1 + \sin 2x$  without using a calculator.

## Answers to Diagnostic Test D: Trigonometry

- $5\pi/3$
  - $-\pi/10$
- $150^\circ$
  - $360^\circ/\pi \approx 114.6^\circ$
- $2\pi$  cm
- $\sqrt{3}$
  - $-\frac{1}{2}$
  - 2
- $24 \sin \theta$
  - $24 \cos \theta$

- $\frac{1}{15}(4 + 6\sqrt{2})$
- $0, \pi/3, \pi, 5\pi/3, 2\pi$

9.



If you have had difficulty with these problems, you should look at Appendix D of this book.

# A Preview of Calculus



© Ziga Camernik / Shutterstock



© Pichugin Dmitry / Shutterstock

By the time you finish this course, you will be able to estimate the number of laborers needed to build a pyramid, explain the formation and location of rainbows, design a roller coaster for a smooth ride, and calculate the force on a dam.



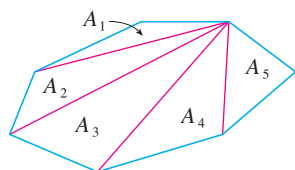
© Brett Mulcahy / Shutterstock



© iofoto / Shutterstock

Calculus is fundamentally different from the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study. Here we give a glimpse of some of the main ideas of calculus by showing how the concept of a limit arises when we attempt to solve a variety of problems.

2 A PREVIEW OF CALCULUS



$$A = A_1 + A_2 + A_3 + A_4 + A_5$$

FIGURE 1

**The Area Problem**

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the “method of exhaustion.” They knew how to find the area  $A$  of any polygon by dividing it into triangles as in Figure 1 and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

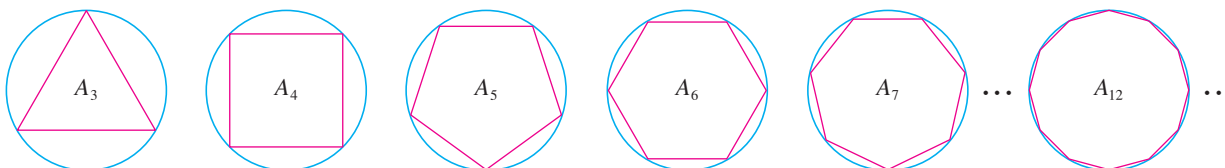


FIGURE 2

Let  $A_n$  be the area of the inscribed polygon with  $n$  sides. As  $n$  increases, it appears that  $A_n$  becomes closer and closer to the area of the circle. We say that the area of the circle is the *limit* of the areas of the inscribed polygons, and we write

$$A = \lim_{n \rightarrow \infty} A_n$$

**TEC** In the Preview Visual, you can see how areas of inscribed and circumscribed polygons approximate the area of a circle.

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century BC) used exhaustion to prove the familiar formula for the area of a circle:  $A = \pi r^2$ .

We will use a similar idea in Chapter 4 to find areas of regions of the type shown in Figure 3. We will approximate the desired area  $A$  by areas of rectangles (as in Figure 4), let the width of the rectangles decrease, and then calculate  $A$  as the limit of these sums of areas of rectangles.

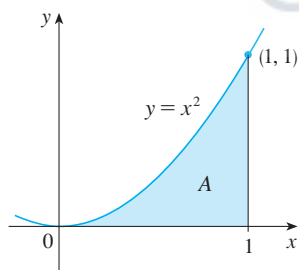


FIGURE 3

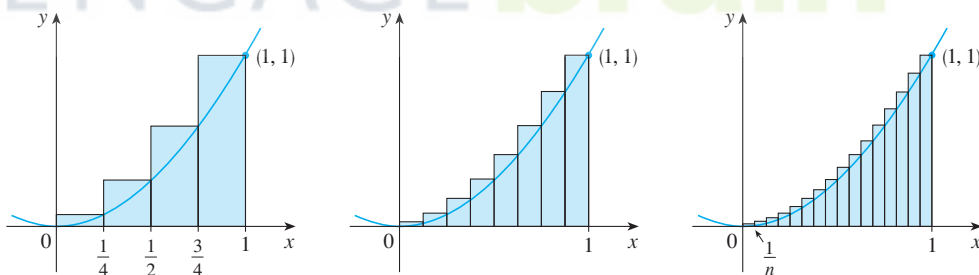
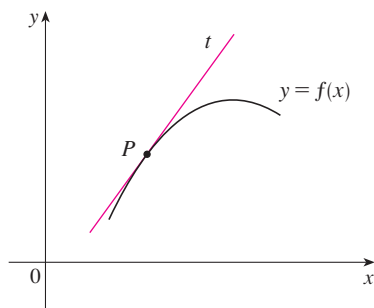


FIGURE 4

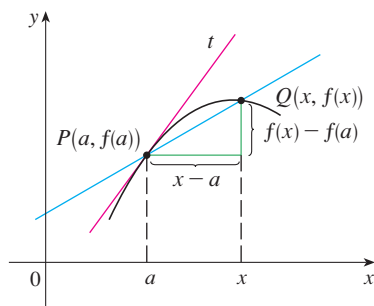
The area problem is the central problem in the branch of calculus called *integral calculus*. The techniques that we will develop in Chapter 4 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.

**The Tangent Problem**

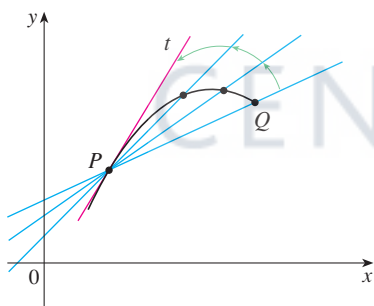
Consider the problem of trying to find an equation of the tangent line  $t$  to a curve with equation  $y = f(x)$  at a given point  $P$ . (We will give a precise definition of a tangent line in



**FIGURE 5**  
The tangent line at  $P$



**FIGURE 6**  
The secant line  $PQ$



**FIGURE 7**  
Secant lines approaching the tangent line

Chapter 1. For now you can think of it as a line that touches the curve at  $P$  as in Figure 5.) Since we know that the point  $P$  lies on the tangent line, we can find the equation of  $t$  if we know its slope  $m$ . The problem is that we need two points to compute the slope and we know only one point,  $P$ , on  $t$ . To get around the problem we first find an approximation to  $m$  by taking a nearby point  $Q$  on the curve and computing the slope  $m_{PQ}$  of the secant line  $PQ$ . From Figure 6 we see that

$$\boxed{1} \quad m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Now imagine that  $Q$  moves along the curve toward  $P$  as in Figure 7. You can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope  $m_{PQ}$  of the secant line becomes closer and closer to the slope  $m$  of the tangent line. We write

$$m = \lim_{Q \rightarrow P} m_{PQ}$$

and we say that  $m$  is the limit of  $m_{PQ}$  as  $Q$  approaches  $P$  along the curve. Since  $x$  approaches  $a$  as  $Q$  approaches  $P$ , we could also use Equation 1 to write

$$\boxed{2} \quad m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Specific examples of this procedure will be given in Chapter 1.

The tangent problem has given rise to the branch of calculus called *differential calculus*, which was not invented until more than 2000 years after integral calculus. The main ideas behind differential calculus are due to the French mathematician Pierre Fermat (1601–1665) and were developed by the English mathematicians John Wallis (1616–1703), Isaac Barrow (1630–1677), and Isaac Newton (1642–1727) and the German mathematician Gottfried Leibniz (1646–1716).

The two branches of calculus and their chief problems, the area problem and the tangent problem, appear to be very different, but it turns out that there is a very close connection between them. The tangent problem and the area problem are inverse problems in a sense that will be described in Chapter 4.

### Velocity

When we look at the speedometer of a car and read that the car is traveling at 48 mi/h, what does that information indicate to us? We know that if the velocity remains constant, then after an hour we will have traveled 48 mi. But if the velocity of the car varies, what does it mean to say that the velocity at a given instant is 48 mi/h?

In order to analyze this question, let's examine the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in feet) at 1-second intervals as in the following chart:

$t =$ Time elapsed (s)	0	1	2	3	4	5
$d =$ Distance (ft)	0	2	9	24	42	71

As a first step toward finding the velocity after 2 seconds have elapsed, we find the average velocity during the time interval  $2 \leq t \leq 4$ :

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{42 - 9}{4 - 2} \\ &= 16.5 \text{ ft/s} \end{aligned}$$

Similarly, the average velocity in the time interval  $2 \leq t \leq 3$  is

$$\text{average velocity} = \frac{24 - 9}{3 - 2} = 15 \text{ ft/s}$$

We have the feeling that the velocity at the instant  $t = 2$  can't be much different from the average velocity during a short time interval starting at  $t = 2$ . So let's imagine that the distance traveled has been measured at 0.1-second time intervals as in the following chart:

$t$	2.0	2.1	2.2	2.3	2.4	2.5
$d$	9.00	10.02	11.16	12.45	13.96	15.80

Then we can compute, for instance, the average velocity over the time interval  $[2, 2.5]$ :

$$\text{average velocity} = \frac{15.80 - 9.00}{2.5 - 2} = 13.6 \text{ ft/s}$$

The results of such calculations are shown in the following chart:

Time interval	$[2, 3]$	$[2, 2.5]$	$[2, 2.4]$	$[2, 2.3]$	$[2, 2.2]$	$[2, 2.1]$
Average velocity (ft/s)	15.0	13.6	12.4	11.5	10.8	10.2

The average velocities over successively smaller intervals appear to be getting closer to a number near 10, and so we expect that the velocity at exactly  $t = 2$  is about 10 ft/s. In Chapter 1 we will define the instantaneous velocity of a moving object as the limiting value of the average velocities over smaller and smaller time intervals.

In Figure 8 we show a graphical representation of the motion of the car by plotting the distance traveled as a function of time. If we write  $d = f(t)$ , then  $f(t)$  is the number of feet traveled after  $t$  seconds. The average velocity in the time interval  $[2, t]$  is

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t) - f(2)}{t - 2}$$

which is the same as the slope of the secant line  $PQ$  in Figure 8. The velocity  $v$  when  $t = 2$  is the limiting value of this average velocity as  $t$  approaches 2; that is,

$$v = \lim_{t \rightarrow 2} \frac{f(t) - f(2)}{t - 2}$$

and we recognize from Equation 2 that this is the same as the slope of the tangent line to the curve at  $P$ .

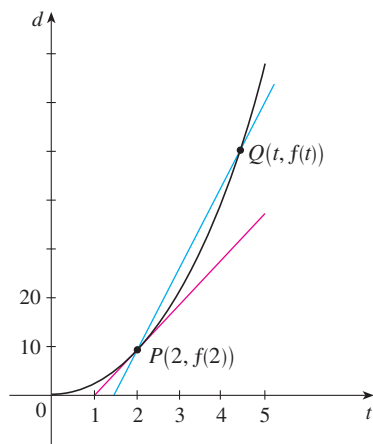


FIGURE 8

Thus, when we solve the tangent problem in differential calculus, we are also solving problems concerning velocities. The same techniques also enable us to solve problems involving rates of change in all of the natural and social sciences.

### The Limit of a Sequence

In the fifth century BC the Greek philosopher Zeno of Elea posed four problems, now known as *Zeno's paradoxes*, that were intended to challenge some of the ideas concerning space and time that were held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise that has been given a head start. Zeno argued, as follows, that Achilles could never pass the tortoise: Suppose that Achilles starts at position  $a_1$  and the tortoise starts at position  $t_1$ . (See Figure 9.) When Achilles reaches the point  $a_2 = t_1$ , the tortoise is farther ahead at position  $t_2$ . When Achilles reaches  $a_3 = t_2$ , the tortoise is at  $t_3$ . This process continues indefinitely and so it appears that the tortoise will always be ahead! But this defies common sense.



FIGURE 9

One way of explaining this paradox is with the idea of a *sequence*. The successive positions of Achilles ( $a_1, a_2, a_3, \dots$ ) or the successive positions of the tortoise ( $t_1, t_2, t_3, \dots$ ) form what is known as a sequence.

In general, a sequence  $\{a_n\}$  is a set of numbers written in a definite order. For instance, the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$$

can be described by giving the following formula for the  $n$ th term:

$$a_n = \frac{1}{n}$$

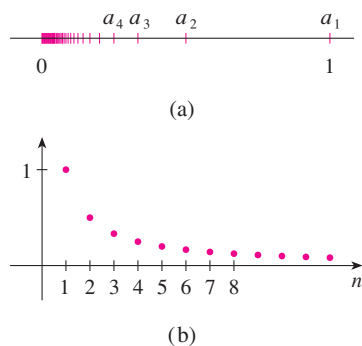


FIGURE 10

We can visualize this sequence by plotting its terms on a number line as in Figure 10(a) or by drawing its graph as in Figure 10(b). Observe from either picture that the terms of the sequence  $a_n = 1/n$  are becoming closer and closer to 0 as  $n$  increases. In fact, we can find terms as small as we please by making  $n$  large enough. We say that the limit of the sequence is 0, and we indicate this by writing

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

In general, the notation

$$\lim_{n \rightarrow \infty} a_n = L$$

is used if the terms  $a_n$  approach the number  $L$  as  $n$  becomes large. This means that the numbers  $a_n$  can be made as close as we like to the number  $L$  by taking  $n$  sufficiently large.

The concept of the limit of a sequence occurs whenever we use the decimal representation of a real number. For instance, if

$$\begin{aligned} a_1 &= 3.1 \\ a_2 &= 3.14 \\ a_3 &= 3.141 \\ a_4 &= 3.1415 \\ a_5 &= 3.14159 \\ a_6 &= 3.141592 \\ a_7 &= 3.1415926 \\ &\vdots \\ &\vdots \end{aligned}$$

then 
$$\lim_{n \rightarrow \infty} a_n = \pi$$

The terms in this sequence are rational approximations to  $\pi$ .

Let's return to Zeno's paradox. The successive positions of Achilles and the tortoise form sequences  $\{a_n\}$  and  $\{t_n\}$ , where  $a_n < t_n$  for all  $n$ . It can be shown that both sequences have the same limit:

$$\lim_{n \rightarrow \infty} a_n = p = \lim_{n \rightarrow \infty} t_n$$

It is precisely at this point  $p$  that Achilles overtakes the tortoise.

### The Sum of a Series

Another of Zeno's paradoxes, as passed on to us by Aristotle, is the following: "A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended." (See Figure 11.)

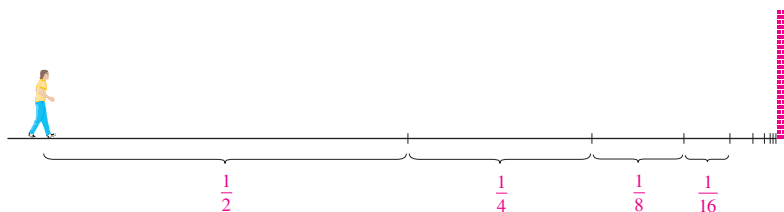


FIGURE 11

Of course, we know that the man can actually reach the wall, so this suggests that perhaps the total distance can be expressed as the sum of infinitely many smaller distances as follows:

$$\boxed{3} \quad 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

Zeno was arguing that it doesn't make sense to add infinitely many numbers together. But there are other situations in which we implicitly use infinite sums. For instance, in decimal notation, the symbol  $0.\overline{3} = 0.3333\dots$  means

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \dots$$

and so, in some sense, it must be true that

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \dots = \frac{1}{3}$$

More generally, if  $d_n$  denotes the  $n$ th digit in the decimal representation of a number, then

$$0.d_1d_2d_3d_4\dots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \dots + \frac{d_n}{10^n} + \dots$$

Therefore some infinite sums, or infinite series as they are called, have a meaning. But we must define carefully what the sum of an infinite series is.

Returning to the series in Equation 3, we denote by  $s_n$  the sum of the first  $n$  terms of the series. Thus

$$s_1 = \frac{1}{2} = 0.5$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = 0.75$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$$

$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$$

$$s_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.96875$$

$$s_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.984375$$

$$s_7 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 0.9921875$$

⋮

$$s_{10} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{1024} \approx 0.99902344$$

⋮

$$s_{16} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{16}} \approx 0.99998474$$

Observe that as we add more and more terms, the partial sums become closer and closer to 1. In fact, it can be shown that by taking  $n$  large enough (that is, by adding sufficiently many terms of the series), we can make the partial sum  $s_n$  as close as we please to the number 1. It therefore seems reasonable to say that the sum of the infinite series is 1 and to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1$$



In other words, the reason the sum of the series is 1 is that

$$\lim_{n \rightarrow \infty} s_n = 1$$

In Chapter 11 we will discuss these ideas further. We will then use Newton's idea of combining infinite series with differential and integral calculus.

### Summary

We have seen that the concept of a limit arises in trying to find the area of a region, the slope of a tangent to a curve, the velocity of a car, or the sum of an infinite series. In each case the common theme is the calculation of a quantity as the limit of other, easily calculated quantities. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits.

After Sir Isaac Newton invented his version of calculus, he used it to explain the motion of the planets around the sun. Today calculus is used in calculating the orbits of satellites and spacecraft, in predicting population sizes, in estimating how fast oil prices rise or fall, in forecasting weather, in measuring the cardiac output of the heart, in calculating life insurance premiums, and in a great variety of other areas. We will explore some of these uses of calculus in this book.

In order to convey a sense of the power of the subject, we end this preview with a list of some of the questions that you will be able to answer using calculus:

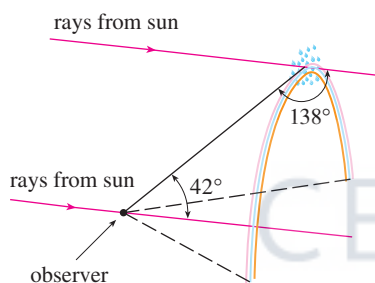


FIGURE 12

1. How can we explain the fact, illustrated in Figure 12, that the angle of elevation from an observer up to the highest point in a rainbow is  $42^\circ$ ? (See page 206.)
2. How can we explain the shapes of cans on supermarket shelves? (See page 262.)
3. Where is the best place to sit in a movie theater? (See page 461.)
4. How can we design a roller coaster for a smooth ride? (See page 140.)
5. How far away from an airport should a pilot start descent? (See page 156.)
6. How can we fit curves together to design shapes to represent letters on a laser printer? (See page 677.)
7. How can we estimate the number of workers that were needed to build the Great Pyramid of Khufu in ancient Egypt? (See page 373.)
8. Where should an infielder position himself to catch a baseball thrown by an outfielder and relay it to home plate? (See page 658.)
9. Does a ball thrown upward take longer to reach its maximum height or to fall back to its original height? (See page 628.)
10. How can we explain the fact that planets and satellites move in elliptical orbits? (See page 892.)
11. How can we distribute water flow among turbines at a hydroelectric station so as to maximize the total energy production? (See page 990.)
12. If a marble, a squash ball, a steel bar, and a lead pipe roll down a slope, which of them reaches the bottom first? (See page 1063.)



# Appendixes

- A** Numbers, Inequalities, and Absolute Values
- B** Coordinate Geometry and Lines
- C** Graphs of Second-Degree Equations
- D** Trigonometry
- E** Sigma Notation
- F** Proofs of Theorems
- G** Graphing Calculators and Computers
- H** Complex Numbers
- I** Answers to Odd-Numbered Exercises

CENGAGE **brain**.com

## A Numbers, Inequalities, and Absolute Values

Calculus is based on the real number system. We start with the **integers**:

$$\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots$$

Then we construct the **rational numbers**, which are ratios of integers. Thus any rational number  $r$  can be expressed as

$$r = \frac{m}{n} \quad \text{where } m \text{ and } n \text{ are integers and } n \neq 0$$

Examples are

$$\frac{1}{2} \quad -\frac{3}{7} \quad 46 = \frac{46}{1} \quad 0.17 = \frac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like  $\frac{3}{0}$  and  $\frac{0}{0}$  are undefined.) Some real numbers, such as  $\sqrt{2}$ , can't be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that the following are also irrational numbers:

$$\sqrt{3} \quad \sqrt{5} \quad \sqrt[3]{2} \quad \pi \quad \sin 1^\circ \quad \log_{10} 2$$

The set of all real numbers is usually denoted by the symbol  $\mathbb{R}$ . When we use the word *number* without qualification, we mean “real number.”

Every number has a decimal representation. If the number is rational, then the corresponding decimal is repeating. For example,

$$\frac{1}{2} = 0.5000\dots = 0.5\bar{0} \quad \frac{2}{3} = 0.6666\dots = 0.\bar{6}$$

$$\frac{157}{495} = 0.31717171\dots = 0.31\bar{7} \quad \frac{9}{7} = 1.285714285714\dots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.) On the other hand, if the number is irrational, the decimal is nonrepeating:

$$\sqrt{2} = 1.414213562373095\dots \quad \pi = 3.141592653589793\dots$$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol  $\approx$  is read “is approximately equal to.” The more decimal places we retain, the better the approximation we get.

The real numbers can be represented by points on a line as in Figure 1. The positive direction (to the right) is indicated by an arrow. We choose an arbitrary reference point  $O$ , called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number  $x$  is represented by the point on the line a distance of  $x$  units to the right of the origin, and each negative number  $-x$  is represented by the point  $x$  units to the left of the origin. Thus every real number is represented by a point on the line, and every point  $P$  on the line corresponds to exactly one real number. The number associated with the point  $P$  is called the **coordinate** of  $P$  and the line is then called a **coordinate**

**line**, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

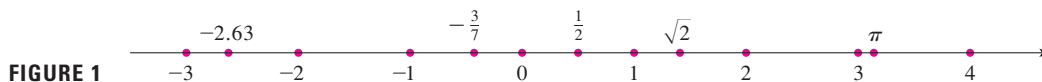


FIGURE 1

The real numbers are ordered. We say *a is less than b* and write  $a < b$  if  $b - a$  is a positive number. Geometrically this means that *a* lies to the left of *b* on the number line. (Equivalently, we say *b is greater than a* and write  $b > a$ .) The symbol  $a \leq b$  (or  $b \geq a$ ) means that either  $a < b$  or  $a = b$  and is read “*a* is less than or equal to *b*.” For instance, the following are true inequalities:

$$7 < 7.4 < 7.5 \quad -3 > -\pi \quad \sqrt{2} < 2 \quad \sqrt{2} \leq 2 \quad 2 \leq 2$$

In what follows we need to use *set notation*. A **set** is a collection of objects, and these objects are called the **elements** of the set. If *S* is a set, the notation  $a \in S$  means that *a* is an element of *S*, and  $a \notin S$  means that *a* is not an element of *S*. For example, if *Z* represents the set of integers, then  $-3 \in Z$  but  $\pi \notin Z$ . If *S* and *T* are sets, then their **union**  $S \cup T$  is the set consisting of all elements that are in *S* or *T* (or in both *S* and *T*). The **intersection** of *S* and *T* is the set  $S \cap T$  consisting of all elements that are in both *S* and *T*. In other words,  $S \cap T$  is the common part of *S* and *T*. The empty set, denoted by  $\emptyset$ , is the set that contains no element.

Some sets can be described by listing their elements between braces. For instance, the set *A* consisting of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write *A* in *set-builder notation* as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read “*A* is the set of *x* such that *x* is an integer and  $0 < x < 7$ .”

### Intervals

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. For example, if  $a < b$ , the **open interval** from *a* to *b* consists of all numbers between *a* and *b* and is denoted by the symbol  $(a, b)$ . Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$

Notice that the endpoints of the interval—namely, *a* and *b*—are excluded. This is indicated by the round brackets  $()$  and by the open dots in Figure 2. The **closed interval** from *a* to *b* is the set

$$[a, b] = \{x \mid a \leq x \leq b\}$$

Here the endpoints of the interval are included. This is indicated by the square brackets  $[\ ]$  and by the solid dots in Figure 3. It is also possible to include only one endpoint in an interval, as shown in Table 1.












FIGURE 2  
Open interval  $(a, b)$



FIGURE 3  
Closed interval  $[a, b]$

## 1 Table of Intervals

Table 1 lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that  $a < b$ .

Notation	Set description	Picture
$(a, b)$	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	
$[a, b)$	$\{x \mid a \leq x < b\}$	
$(a, b]$	$\{x \mid a < x \leq b\}$	
$(a, \infty)$	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geq a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leq b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

We also need to consider infinite intervals such as

$$(a, \infty) = \{x \mid x > a\}$$


This does not mean that  $\infty$  (“infinity”) is a number. The notation  $(a, \infty)$  stands for the set of all numbers that are greater than  $a$ , so the symbol  $\infty$  simply indicates that the interval extends indefinitely far in the positive direction.

## Inequalities

When working with inequalities, note the following rules.

### 2 Rules for Inequalities

1. If  $a < b$ , then  $a + c < b + c$ .
2. If  $a < b$  and  $c < d$ , then  $a + c < b + d$ .
3. If  $a < b$  and  $c > 0$ , then  $ac < bc$ .
4. If  $a < b$  and  $c < 0$ , then  $ac > bc$ .
5. If  $0 < a < b$ , then  $1/a > 1/b$ .

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but  Rule 4 says that **if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality**. For example, if we take the inequality  $3 < 5$  and multiply by 2, we get  $6 < 10$ , but if we multiply by  $-2$ , we get  $-6 > -10$ . Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

**EXAMPLE 1** Solve the inequality  $1 + x < 7x + 5$ .

**SOLUTION** The given inequality is satisfied by some values of  $x$  but not by others. To *solve* an inequality means to determine the set of numbers  $x$  for which the inequality is true. This is called the *solution set*.

First we subtract 1 from each side of the inequality (using Rule 1 with  $c = -1$ ):

$$x < 7x + 4$$

Then we subtract  $7x$  from both sides (Rule 1 with  $c = -7x$ ):

$$-6x < 4$$

Now we divide both sides by  $-6$  (Rule 4 with  $c = -\frac{1}{6}$ ):

$$x > -\frac{4}{6} = -\frac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than  $-\frac{2}{3}$ . In other words, the solution of the inequality is the interval  $(-\frac{2}{3}, \infty)$ .

**EXAMPLE 2** Solve the inequalities  $4 \leq 3x - 2 < 13$ .

**SOLUTION** Here the solution set consists of all values of  $x$  that satisfy both inequalities. Using the rules given in [2], we see that the following inequalities are equivalent:

$$4 \leq 3x - 2 < 13$$

$$6 \leq 3x < 15 \quad (\text{add } 2)$$

$$2 \leq x < 5 \quad (\text{divide by } 3)$$

Therefore the solution set is  $[2, 5)$ .

**EXAMPLE 3** Solve the inequality  $x^2 - 5x + 6 \leq 0$ .

**SOLUTION** First we factor the left side:

$$(x - 2)(x - 3) \leq 0$$

We know that the corresponding equation  $(x - 2)(x - 3) = 0$  has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \quad (2, 3) \quad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \Rightarrow x < 2 \Rightarrow x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	-	-	+
$2 < x < 3$	+	-	-
$x > 3$	+	+	+

Another method for obtaining the information in the chart is to use *test values*. For instance, if we use the test value  $x = 1$  for the interval  $(-\infty, 2)$ , then substitution in  $x^2 - 5x + 6$  gives

$$1^2 - 5(1) + 6 = 2$$

A visual method for solving Example 3 is to use a graphing device to graph the parabola  $y = x^2 - 5x + 6$  (as in Figure 4) and observe that the curve lies on or below the  $x$ -axis when  $2 \leq x \leq 3$ .

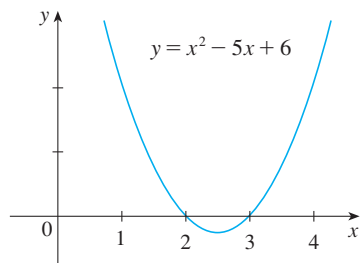


FIGURE 4

The polynomial  $x^2 - 5x + 6$  doesn't change sign inside any of the three intervals, so we conclude that it is positive on  $(-\infty, 2)$ .

Then we read from the chart that  $(x - 2)(x - 3)$  is negative when  $2 < x < 3$ . Thus the solution of the inequality  $(x - 2)(x - 3) \leq 0$  is

$$\{x \mid 2 \leq x \leq 3\} = [2, 3]$$

Notice that we have included the endpoints 2 and 3 because we are looking for values of  $x$  such that the product is either negative or zero. The solution is illustrated in Figure 5.

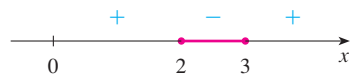


FIGURE 5

**EXAMPLE 4** Solve  $x^3 + 3x^2 > 4x$ .

**SOLUTION** First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \quad \text{or} \quad x(x - 1)(x + 4) > 0$$

As in Example 3 we solve the corresponding equation  $x(x - 1)(x + 4) = 0$  and use the solutions  $x = -4$ ,  $x = 0$ , and  $x = 1$  to divide the real line into four intervals  $(-\infty, -4)$ ,  $(-4, 0)$ ,  $(0, 1)$ , and  $(1, \infty)$ . On each interval the product keeps a constant sign as shown in the following chart:

Interval	$x$	$x - 1$	$x + 4$	$x(x - 1)(x + 4)$
$x < -4$	-	-	-	-
$-4 < x < 0$	-	-	+	+
$0 < x < 1$	+	-	+	-
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$



FIGURE 6

The solution is illustrated in Figure 6.

### Absolute Value

The **absolute value** of a number  $a$ , denoted by  $|a|$ , is the distance from  $a$  to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \quad \text{for every number } a$$

For example,

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1 \quad |3 - \pi| = \pi - 3$$

In general, we have

**3**

$$|a| = a \quad \text{if } a \geq 0$$

$$|a| = -a \quad \text{if } a < 0$$

Remember that if  $a$  is negative, then  $-a$  is positive.

**EXAMPLE 5** Express  $|3x - 2|$  without using the absolute-value symbol.

**SOLUTION**

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geq 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geq \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$

Recall that the symbol  $\sqrt{\phantom{x}}$  means “the positive square root of.” Thus  $\sqrt{r} = s$  means  $s^2 = r$  and  $s \geq 0$ . Therefore the equation  $\sqrt{a^2} = a$  is not always true. It is true only when  $a \geq 0$ . If  $a < 0$ , then  $-a > 0$ , so we have  $\sqrt{a^2} = -a$ . In view of [3], we then have the equation

**4**

$$\sqrt{a^2} = |a|$$

which is true for all values of  $a$ .

Hints for the proofs of the following properties are given in the exercises.

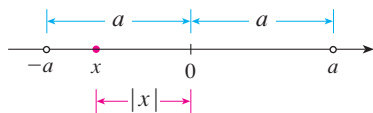
**5 Properties of Absolute Values** Suppose  $a$  and  $b$  are any real numbers and  $n$  is an integer. Then

1.  $|ab| = |a||b|$
2.  $\frac{|a|}{|b|} = \frac{|a|}{|b|} \quad (b \neq 0)$
3.  $|a^n| = |a|^n$

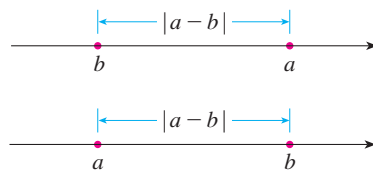
For solving equations or inequalities involving absolute values, it’s often very helpful to use the following statements.

**6** Suppose  $a > 0$ . Then

4.  $|x| = a$  if and only if  $x = \pm a$
5.  $|x| < a$  if and only if  $-a < x < a$
6.  $|x| > a$  if and only if  $x > a$  or  $x < -a$



**FIGURE 7**



**FIGURE 8**

Length of a line segment =  $|a - b|$

**EXAMPLE 6** Solve  $|2x - 5| = 3$ .

**SOLUTION** By Property 4 of [6],  $|2x - 5| = 3$  is equivalent to

$$2x - 5 = 3 \quad \text{or} \quad 2x - 5 = -3$$

So  $2x = 8$  or  $2x = 2$ . Thus  $x = 4$  or  $x = 1$ .



**EXAMPLE 7** Solve  $|x - 5| < 2$ .

**SOLUTION 1** By Property 5 of [6],  $|x - 5| < 2$  is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

and the solution set is the open interval  $(3, 7)$ .

**SOLUTION 2** Geometrically the solution set consists of all numbers  $x$  whose distance from 5 is less than 2. From Figure 9 we see that this is the interval  $(3, 7)$ . ■

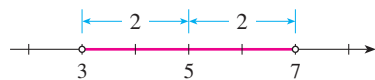


FIGURE 9

**EXAMPLE 8** Solve  $|3x + 2| \geq 4$ .

**SOLUTION** By Properties 4 and 6 of [6],  $|3x + 2| \geq 4$  is equivalent to

$$3x + 2 \geq 4 \quad \text{or} \quad 3x + 2 \leq -4$$

In the first case  $3x \geq 2$ , which gives  $x \geq \frac{2}{3}$ . In the second case  $3x \leq -6$ , which gives  $x \leq -2$ . So the solution set is

$$\{x \mid x \leq -2 \text{ or } x \geq \frac{2}{3}\} = (-\infty, -2] \cup [\frac{2}{3}, \infty)$$
 ■

Another important property of absolute value, called the Triangle Inequality, is used frequently not only in calculus but throughout mathematics in general.

**7 The Triangle Inequality** If  $a$  and  $b$  are any real numbers, then

$$|a + b| \leq |a| + |b|$$

Observe that if the numbers  $a$  and  $b$  are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if  $a$  and  $b$  have opposite signs, the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, but we can prove it as follows.

Notice that

$$-|a| \leq a \leq |a|$$

is always true because  $a$  equals either  $|a|$  or  $-|a|$ . The corresponding statement for  $b$  is

$$-|b| \leq b \leq |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

If we now apply Properties 4 and 5 (with  $x$  replaced by  $a + b$  and  $a$  by  $|a| + |b|$ ), we obtain

$$|a + b| \leq |a| + |b|$$

which is what we wanted to show.

**EXAMPLE 9** If  $|x - 4| < 0.1$  and  $|y - 7| < 0.2$ , use the Triangle Inequality to estimate  $|(x + y) - 11|$ .

**SOLUTION** In order to use the given information, we use the Triangle Inequality with  $a = x - 4$  and  $b = y - 7$ :

$$\begin{aligned} |(x + y) - 11| &= |(x - 4) + (y - 7)| \\ &\leq |x - 4| + |y - 7| \\ &< 0.1 + 0.2 = 0.3 \end{aligned}$$

Thus  $|(x + y) - 11| < 0.3$

## A Exercises

**1–12** Rewrite the expression without using the absolute value symbol.

- |                         |                         |
|-------------------------|-------------------------|
| 1. $ 5 - 23 $           | 2. $ 5  -  -23 $        |
| 3. $ -π $               | 4. $ \pi - 2 $          |
| 5. $ \sqrt{5} - 5 $     | 6. $  -2  -  -3  $      |
| 7. $ x - 2 $ if $x < 2$ | 8. $ x - 2 $ if $x > 2$ |
| 9. $ x + 1 $            | 10. $ 2x - 1 $          |
| 11. $ x^2 + 1 $         | 12. $ 1 - 2x^2 $        |

**13–38** Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

- |                                    |                               |
|------------------------------------|-------------------------------|
| 13. $2x + 7 > 3$                   | 14. $3x - 11 < 4$             |
| 15. $1 - x \leq 2$                 | 16. $4 - 3x \geq 6$           |
| 17. $2x + 1 < 5x - 8$              | 18. $1 + 5x > 5 - 3x$         |
| 19. $-1 < 2x - 5 < 7$              | 20. $1 < 3x + 4 \leq 16$      |
| 21. $0 \leq 1 - x < 1$             | 22. $-5 \leq 3 - 2x \leq 9$   |
| 23. $4x < 2x + 1 \leq 3x + 2$      | 24. $2x - 3 < x + 4 < 3x - 2$ |
| 25. $(x - 1)(x - 2) > 0$           | 26. $(2x + 3)(x - 1) \geq 0$  |
| 27. $2x^2 + x \leq 1$              | 28. $x^2 < 2x + 8$            |
| 29. $x^2 + x + 1 > 0$              | 30. $x^2 + x > 1$             |
| 31. $x^2 < 3$                      | 32. $x^2 \geq 5$              |
| 33. $x^3 - x^2 \leq 0$             |                               |
| 34. $(x + 1)(x - 2)(x + 3) \geq 0$ |                               |
| 35. $x^3 > x$                      | 36. $x^3 + 3x < 4x^2$         |
| 37. $\frac{1}{x} < 4$              | 38. $-3 < \frac{1}{x} \leq 1$ |

**39.** The relationship between the Celsius and Fahrenheit temperature scales is given by  $C = \frac{5}{9}(F - 32)$ , where  $C$  is the temper-

ature in degrees Celsius and  $F$  is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range  $50 \leq F \leq 95$ ?

- 40.** Use the relationship between  $C$  and  $F$  given in Exercise 39 to find the interval on the Fahrenheit scale corresponding to the temperature range  $20 \leq C \leq 30$ .
- 41.** As dry air moves upward, it expands and in so doing cools at a rate of about  $1^\circ\text{C}$  for each 100-m rise, up to about 12 km.
- (a) If the ground temperature is  $20^\circ\text{C}$ , write a formula for the temperature at height  $h$ .
- (b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?
- 42.** If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height  $h$  above the ground  $t$  seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

**43–46** Solve the equation for  $x$ .

- |                          |   |
|--------------------------|---|
| 43. $ 2x  = 3$           | 44. $ 3x + 5  = 1$                            |
| 45. $ x + 3  =  2x + 1 $ | 46. $\left  \frac{2x - 1}{x + 1} \right  = 3$ |

**47–56** Solve the inequality.

- |                         |                                 |
|-------------------------|---------------------------------|
| 47. $ x  < 3$           | 48. $ x  \geq 3$                |
| 49. $ x - 4  < 1$       | 50. $ x - 6  < 0.1$             |
| 51. $ x + 5  \geq 2$    | 52. $ x + 1  \geq 3$            |
| 53. $ 2x - 3  \leq 0.4$ | 54. $ 5x - 2  < 6$              |
| 55. $1 \leq  x  \leq 4$ | 56. $0 <  x - 5  < \frac{1}{2}$ |

**A10 APPENDIX A** NUMBERS, INEQUALITIES, AND ABSOLUTE VALUES**57–58** Solve for  $x$ , assuming  $a$ ,  $b$ , and  $c$  are positive constants.

**57.**  $a(bx - c) \geq bc$

**58.**  $a \leq bx + c < 2a$

**59–60** Solve for  $x$ , assuming  $a$ ,  $b$ , and  $c$  are negative constants.

**59.**  $ax + b < c$

**60.**  $\frac{ax + b}{c} \leq b$

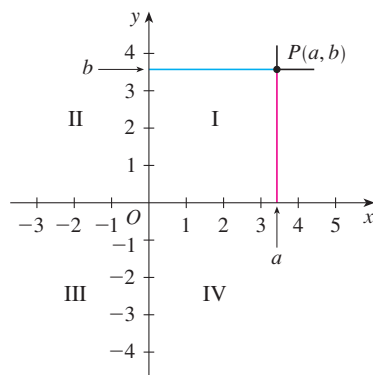
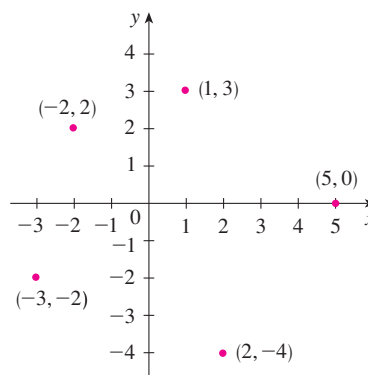
**61.** Suppose that  $|x - 2| < 0.01$  and  $|y - 3| < 0.04$ . Use the Triangle Inequality to show that  $|(x + y) - 5| < 0.05$ .**62.** Show that if  $|x + 3| < \frac{1}{2}$ , then  $|4x + 13| < 3$ .**63.** Show that if  $a < b$ , then  $a < \frac{a + b}{2} < b$ .**64.** Use Rule 3 to prove Rule 5 of [2].**65.** Prove that  $|ab| = |a||b|$ . [Hint: Use Equation 4.]**66.** Prove that  $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ .**67.** Show that if  $0 < a < b$ , then  $a^2 < b^2$ .**68.** Prove that  $|x - y| \geq |x| - |y|$ . [Hint: Use the Triangle Inequality with  $a = x - y$  and  $b = y$ .]**69.** Show that the sum, difference, and product of rational numbers are rational numbers.**70.** (a) Is the sum of two irrational numbers always an irrational number?

(b) Is the product of two irrational numbers always an irrational number?

**B** Coordinate Geometry and Lines

Just as the points on a line can be identified with real numbers by assigning them coordinates, as described in Appendix A, so the points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin  $O$  on each line. Usually one line is horizontal with positive direction to the right and is called the  $x$ -axis; the other line is vertical with positive direction upward and is called the  $y$ -axis.

Any point  $P$  in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through  $P$  perpendicular to the  $x$ - and  $y$ -axes. These lines intersect the axes in points with coordinates  $a$  and  $b$  as shown in Figure 1. Then the point  $P$  is assigned the ordered pair  $(a, b)$ . The first number  $a$  is called the  **$x$ -coordinate** of  $P$ ; the second number  $b$  is called the  **$y$ -coordinate** of  $P$ . We say that  $P$  is the point with coordinates  $(a, b)$ , and we denote the point by the symbol  $P(a, b)$ . Several points are labeled with their coordinates in Figure 2.

**FIGURE 1****FIGURE 2**

By reversing the preceding process we can start with an ordered pair  $(a, b)$  and arrive at the corresponding point  $P$ . Often we identify the point  $P$  with the ordered pair  $(a, b)$  and refer to “the point  $(a, b)$ .” [Although the notation used for an open interval  $(a, b)$  is the

same as the notation used for a point  $(a, b)$ , you will be able to tell from the context which meaning is intended.]

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the French mathematician René Descartes (1596–1650), even though another Frenchman, Pierre Fermat (1601–1665), invented the principles of analytic geometry at about the same time as Descartes. The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane** and is denoted by  $\mathbb{R}^2$ .

The  $x$ - and  $y$ -axes are called the **coordinate axes** and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose  $x$ - and  $y$ -coordinates are both positive.

**EXAMPLE 1** Describe and sketch the regions given by the following sets.

- (a)  $\{(x, y) \mid x \geq 0\}$       (b)  $\{(x, y) \mid y = 1\}$       (c)  $\{(x, y) \mid |y| < 1\}$

**SOLUTION**

(a) The points whose  $x$ -coordinates are 0 or positive lie on the  $y$ -axis or to the right of it as indicated by the shaded region in Figure 3(a).

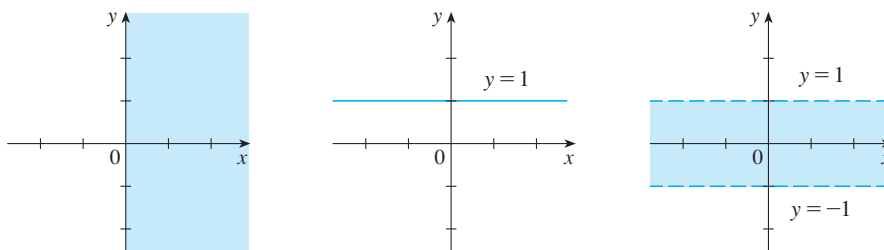


FIGURE 3

(a)  $x \geq 0$

(b)  $y = 1$

(c)  $|y| < 1$

(b) The set of all points with  $y$ -coordinate 1 is a horizontal line one unit above the  $x$ -axis [see Figure 3(b)].

(c) Recall from Appendix A that

$$|y| < 1 \quad \text{if and only if} \quad -1 < y < 1$$

The given region consists of those points in the plane whose  $y$ -coordinates lie between  $-1$  and  $1$ . Thus the region consists of all points that lie between (but not on) the horizontal lines  $y = 1$  and  $y = -1$ . [These lines are shown as dashed lines in Figure 3(c) to indicate that the points on these lines don't lie in the set.]

Recall from Appendix A that the distance between points  $a$  and  $b$  on a number line is  $|a - b| = |b - a|$ . Thus the distance between points  $P_1(x_1, y_1)$  and  $P_3(x_2, y_1)$  on a horizontal line must be  $|x_2 - x_1|$  and the distance between  $P_2(x_2, y_2)$  and  $P_3(x_2, y_1)$  on a vertical line must be  $|y_2 - y_1|$ . (See Figure 4.)

To find the distance  $|P_1P_2|$  between any two points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ , we note that triangle  $P_1P_2P_3$  in Figure 4 is a right triangle, and so by the Pythagorean Theorem we have

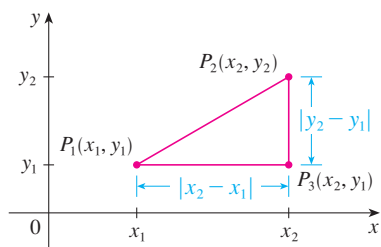


FIGURE 4

$$\begin{aligned} |P_1P_2| &= \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

**1 Distance Formula** The distance between the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**EXAMPLE 2** The distance between  $(1, -2)$  and  $(5, 3)$  is

$$\sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

## Lines

We want to find an equation of a given line  $L$ ; such an equation is satisfied by the coordinates of the points on  $L$  and by no other point. To find the equation of  $L$  we use its *slope*, which is a measure of the steepness of the line.

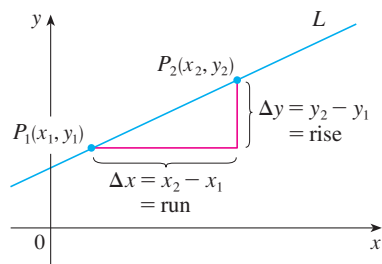


FIGURE 5

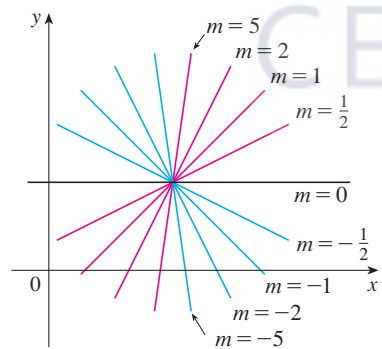


FIGURE 6

**2 Definition** The **slope** of a nonvertical line that passes through the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Thus the slope of a line is the ratio of the change in  $y$ ,  $\Delta y$ , to the change in  $x$ ,  $\Delta x$ . (See Figure 5.) The slope is therefore the rate of change of  $y$  with respect to  $x$ . The fact that the line is straight means that the rate of change is constant.

Figure 6 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0.

Now let's find an equation of the line that passes through a given point  $P_1(x_1, y_1)$  and has slope  $m$ . A point  $P(x, y)$  with  $x \neq x_1$  lies on this line if and only if the slope of the line through  $P_1$  and  $P$  is equal to  $m$ ; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when  $x = x_1$  and  $y = y_1$ . Therefore it is an equation of the given line.

**3 Point-Slope Form of the Equation of a Line** An equation of the line passing through the point  $P_1(x_1, y_1)$  and having slope  $m$  is

$$y - y_1 = m(x - x_1)$$

**EXAMPLE 3** Find an equation of the line through  $(1, -7)$  with slope  $-\frac{1}{2}$ .

**SOLUTION** Using [3] with  $m = -\frac{1}{2}$ ,  $x_1 = 1$ , and  $y_1 = -7$ , we obtain an equation of the line as

$$y + 7 = -\frac{1}{2}(x - 1)$$

which we can rewrite as

$$2y + 14 = -x + 1 \quad \text{or} \quad x + 2y + 13 = 0$$

**EXAMPLE 4** Find an equation of the line through the points  $(-1, 2)$  and  $(3, -4)$ .

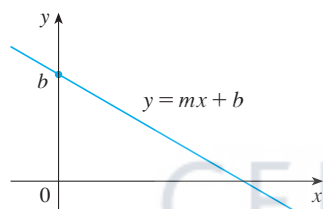
**SOLUTION** By Definition 2 the slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with  $x_1 = -1$  and  $y_1 = 2$ , we obtain

$$y - 2 = -\frac{3}{2}(x + 1)$$

which simplifies to  $3x + 2y = 1$



**FIGURE 7**

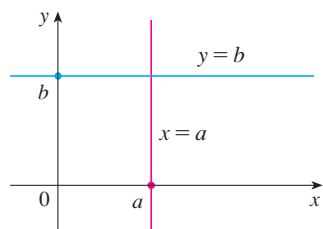
Suppose a nonvertical line has slope  $m$  and  $y$ -intercept  $b$ . (See Figure 7.) This means it intersects the  $y$ -axis at the point  $(0, b)$ , so the point-slope form of the equation of the line, with  $x_1 = 0$  and  $y_1 = b$ , becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

**4 Slope-Intercept Form of the Equation of a Line** An equation of the line with slope  $m$  and  $y$ -intercept  $b$  is

$$y = mx + b$$



**FIGURE 8**

In particular, if a line is horizontal, its slope is  $m = 0$ , so its equation is  $y = b$ , where  $b$  is the  $y$ -intercept (see Figure 8). A vertical line does not have a slope, but we can write its equation as  $x = a$ , where  $a$  is the  $x$ -intercept, because the  $x$ -coordinate of every point on the line is  $a$ .

Observe that the equation of every line can be written in the form

**5**

$$Ax + By + C = 0$$

because a vertical line has the equation  $x = a$  or  $x - a = 0$  ( $A = 1$ ,  $B = 0$ ,  $C = -a$ ) and a nonvertical line has the equation  $y = mx + b$  or  $-mx + y - b = 0$  ( $A = -m$ ,  $B = 1$ ,  $C = -b$ ). Conversely, if we start with a general first-degree equation, that is, an equation of the form [5], where  $A$ ,  $B$ , and  $C$  are constants and  $A$  and  $B$  are not both 0, then we can show that it is the equation of a line. If  $B = 0$ , the equation becomes  $Ax + C = 0$  or  $x = -C/A$ , which represents a vertical line with  $x$ -intercept  $-C/A$ . If  $B \neq 0$ , the equation

can be rewritten by solving for  $y$ :

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and we recognize this as being the slope-intercept form of the equation of a line ( $m = -A/B$ ,  $b = -C/B$ ). Therefore an equation of the form [5] is called a **linear equation** or the **general equation of a line**. For brevity, we often refer to “the line  $Ax + By + C = 0$ ” instead of “the line whose equation is  $Ax + By + C = 0$ .”

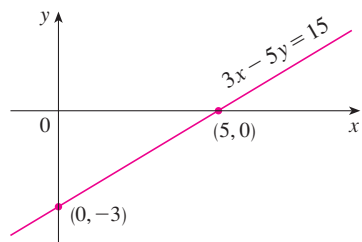


FIGURE 9

**EXAMPLE 5** Sketch the graph of the equation  $3x - 5y = 15$ .

**SOLUTION** Since the equation is linear, its graph is a line. To draw the graph, we can simply find two points on the line. It's easiest to find the intercepts. Substituting  $y = 0$  (the equation of the  $x$ -axis) in the given equation, we get  $3x = 15$ , so  $x = 5$  is the  $x$ -intercept. Substituting  $x = 0$  in the equation, we see that the  $y$ -intercept is  $-3$ . This allows us to sketch the graph as in Figure 9.

**EXAMPLE 6** Graph the inequality  $x + 2y > 5$ .

**SOLUTION** We are asked to sketch the graph of the set  $\{(x, y) \mid x + 2y > 5\}$  and we begin by solving the inequality for  $y$ :

$$\begin{aligned} x + 2y &> 5 \\ 2y &> -x + 5 \\ y &> -\frac{1}{2}x + \frac{5}{2} \end{aligned}$$

Compare this inequality with the equation  $y = -\frac{1}{2}x + \frac{5}{2}$ , which represents a line with slope  $-\frac{1}{2}$  and  $y$ -intercept  $\frac{5}{2}$ . We see that the given graph consists of points whose  $y$ -coordinates are *larger* than those on the line  $y = -\frac{1}{2}x + \frac{5}{2}$ . Thus the graph is the region that lies *above* the line, as illustrated in Figure 10.

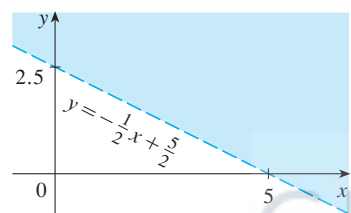


FIGURE 10

## Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus, Sixth Edition* by Stewart, Redlin, and Watson (Belmont, CA, 2012).

### [6] Parallel and Perpendicular Lines

- Two nonvertical lines are parallel if and only if they have the same slope.
- Two lines with slopes  $m_1$  and  $m_2$  are perpendicular if and only if  $m_1 m_2 = -1$ ; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

**EXAMPLE 7** Find an equation of the line through the point  $(5, 2)$  that is parallel to the line  $4x + 6y + 5 = 0$ .

**SOLUTION** The given line can be written in the form

$$y = -\frac{2}{3}x - \frac{5}{6}$$

which is in slope-intercept form with  $m = -\frac{2}{3}$ . Parallel lines have the same slope, so the required line has slope  $-\frac{2}{3}$  and its equation in point-slope form is

$$y - 2 = -\frac{2}{3}(x - 5)$$

We can write this equation as  $2x + 3y = 16$ .

**EXAMPLE 8** Show that the lines  $2x + 3y = 1$  and  $6x - 4y - 1 = 0$  are perpendicular.

**SOLUTION** The equations can be written as

$$y = -\frac{2}{3}x + \frac{1}{3} \quad \text{and} \quad y = \frac{3}{2}x - \frac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\frac{2}{3} \quad \text{and} \quad m_2 = \frac{3}{2}$$

Since  $m_1 m_2 = -1$ , the lines are perpendicular.

## B Exercises

**1–6** Find the distance between the points.

- |                     |                      |
|---------------------|----------------------|
| 1. (1, 1), (4, 5)   | 2. (1, -3), (5, 7)   |
| 3. (6, -2), (-1, 3) | 4. (1, -6), (-1, -3) |
| 5. (2, 5), (4, -7)  | 6. (a, b), (b, a)    |

**7–10** Find the slope of the line through  $P$  and  $Q$ .

- |                             |                             |
|-----------------------------|-----------------------------|
| 7. $P(1, 5)$ , $Q(4, 11)$   | 8. $P(-1, 6)$ , $Q(4, -3)$  |
| 9. $P(-3, 3)$ , $Q(-1, -6)$ | 10. $P(-1, -4)$ , $Q(6, 0)$ |

- 11.** Show that the triangle with vertices  $A(0, 2)$ ,  $B(-3, -1)$ , and  $C(-4, 3)$  is isosceles.
- 12.** (a) Show that the triangle with vertices  $A(6, -7)$ ,  $B(11, -3)$ , and  $C(2, -2)$  is a right triangle using the converse of the Pythagorean Theorem.  
 (b) Use slopes to show that  $ABC$  is a right triangle.  
 (c) Find the area of the triangle.
- 13.** Show that the points  $(-2, 9)$ ,  $(4, 6)$ ,  $(1, 0)$ , and  $(-5, 3)$  are the vertices of a square.
- 14.** (a) Show that the points  $A(-1, 3)$ ,  $B(3, 11)$ , and  $C(5, 15)$  are collinear (lie on the same line) by showing that  $|AB| + |BC| = |AC|$ .  
 (b) Use slopes to show that  $A$ ,  $B$ , and  $C$  are collinear.
- 15.** Show that  $A(1, 1)$ ,  $B(7, 4)$ ,  $C(5, 10)$ , and  $D(-1, 7)$  are vertices of a parallelogram.
- 16.** Show that  $A(1, 1)$ ,  $B(11, 3)$ ,  $C(10, 8)$ , and  $D(0, 6)$  are vertices of a rectangle.

**17–20** Sketch the graph of the equation.

**17.**  $x = 3$

**18.**  $y = -2$

**19.**  $xy = 0$

**20.**  $|y| = 1$

**21–36** Find an equation of the line that satisfies the given conditions.

- 21.** Through  $(2, -3)$ , slope 6
- 22.** Through  $(-1, 4)$ , slope  $-3$
- 23.** Through  $(1, 7)$ , slope  $\frac{2}{3}$
- 24.** Through  $(-3, -5)$ , slope  $-\frac{7}{2}$
- 25.** Through  $(2, 1)$  and  $(1, 6)$
- 26.** Through  $(-1, -2)$  and  $(4, 3)$
- 27.** Slope 3,  $y$ -intercept  $-2$
- 28.** Slope  $\frac{2}{5}$ ,  $y$ -intercept 4
- 29.**  $x$ -intercept 1,  $y$ -intercept  $-3$
- 30.**  $x$ -intercept  $-8$ ,  $y$ -intercept 6
- 31.** Through  $(4, 5)$ , parallel to the  $x$ -axis
- 32.** Through  $(4, 5)$ , parallel to the  $y$ -axis
- 33.** Through  $(1, -6)$ , parallel to the line  $x + 2y = 6$
- 34.**  $y$ -intercept 6, parallel to the line  $2x + 3y + 4 = 0$
- 35.** Through  $(-1, -2)$ , perpendicular to the line  $2x + 5y + 8 = 0$
- 36.** Through  $(\frac{1}{2}, -\frac{2}{3})$ , perpendicular to the line  $4x - 8y = 1$

**37–42** Find the slope and  $y$ -intercept of the line and draw its graph.

**37.**  $x + 3y = 0$

**38.**  $2x - 5y = 0$



**A16 APPENDIX B** COORDINATE GEOMETRY AND LINES

39.  $y = -2$

40.  $2x - 3y + 6 = 0$

41.  $3x - 4y = 12$

42.  $4x + 5y = 10$

43–52 Sketch the region in the  $xy$ -plane.

43.  $\{(x, y) \mid x < 0\}$

44.  $\{(x, y) \mid y > 0\}$

45.  $\{(x, y) \mid xy < 0\}$

46.  $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$

47.  $\{(x, y) \mid |x| \leq 2\}$

48.  $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

49.  $\{(x, y) \mid 0 \leq y \leq 4 \text{ and } x \leq 2\}$

50.  $\{(x, y) \mid y > 2x - 1\}$

51.  $\{(x, y) \mid 1 + x \leq y \leq 1 - 2x\}$

52.  $\{(x, y) \mid -x \leq y < \frac{1}{2}(x + 3)\}$

53. Find a point on the  $y$ -axis that is equidistant from  $(5, -5)$  and  $(1, 1)$ .54. Show that the midpoint of the line segment from  $P_1(x_1, y_1)$  to  $P_2(x_2, y_2)$  is

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

55. Find the midpoint of the line segment joining the given points.  
(a)  $(1, 3)$  and  $(7, 15)$       (b)  $(-1, 6)$  and  $(8, -12)$ 56. Find the lengths of the medians of the triangle with vertices  $A(1, 0)$ ,  $B(3, 6)$ , and  $C(8, 2)$ . (A median is a line segment from a vertex to the midpoint of the opposite side.)57. Show that the lines  $2x - y = 4$  and  $6x - 2y = 10$  are not parallel and find their point of intersection.58. Show that the lines  $3x - 5y + 19 = 0$  and  $10x + 6y - 50 = 0$  are perpendicular and find their point of intersection.59. Find an equation of the perpendicular bisector of the line segment joining the points  $A(1, 4)$  and  $B(7, -2)$ .60. (a) Find equations for the sides of the triangle with vertices  $P(1, 0)$ ,  $Q(3, 4)$ , and  $R(-1, 6)$ .

(b) Find equations for the medians of this triangle. Where do they intersect?

61. (a) Show that if the  $x$ - and  $y$ -intercepts of a line are nonzero numbers  $a$  and  $b$ , then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.(b) Use part (a) to find an equation of the line whose  $x$ -intercept is 6 and whose  $y$ -intercept is  $-8$ .

62. A car leaves Detroit at 2:00 PM, traveling at a constant speed west along I-96. It passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.

(a) Express the distance traveled in terms of the time elapsed.

(b) Draw the graph of the equation in part (a).

(c) What is the slope of this line? What does it represent?

**C** Graphs of Second-Degree Equations

In Appendix B we saw that a first-degree, or linear, equation  $Ax + By + C = 0$  represents a line. In this section we discuss second-degree equations such as

$$x^2 + y^2 = 1 \quad y = x^2 + 1 \quad \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad x^2 - y^2 = 1$$

which represent a circle, a parabola, an ellipse, and a hyperbola, respectively.

The graph of such an equation in  $x$  and  $y$  is the set of all points  $(x, y)$  that satisfy the equation; it gives a visual representation of the equation. Conversely, given a curve in the  $xy$ -plane, we may have to find an equation that represents it, that is, an equation satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the basic principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the geometric problem.

**Circles**

As an example of this type of problem, let's find an equation of the circle with radius  $r$  and center  $(h, k)$ . By definition, the circle is the set of all points  $P(x, y)$  whose distance from

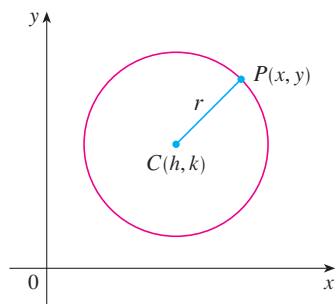


FIGURE 1

the center  $C(h, k)$  is  $r$ . (See Figure 1.) Thus  $P$  is on the circle if and only if  $|PC| = r$ . From the distance formula, we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the desired equation.

**1 Equation of a Circle** An equation of the circle with center  $(h, k)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin  $(0, 0)$ , the equation is

$$x^2 + y^2 = r^2$$

**EXAMPLE 1** Find an equation of the circle with radius 3 and center  $(2, -5)$ .

**SOLUTION** From Equation 1 with  $r = 3$ ,  $h = 2$ , and  $k = -5$ , we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$

**EXAMPLE 2** Sketch the graph of the equation  $x^2 + y^2 + 2x - 6y + 7 = 0$  by first showing that it represents a circle and then finding its center and radius.

**SOLUTION** We first group the  $x$ -terms and  $y$ -terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

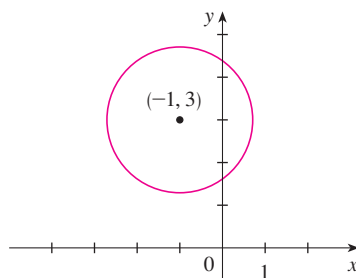
Then we complete the square within each grouping, adding the appropriate constants (the squares of half the coefficients of  $x$  and  $y$ ) to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or

$$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle **1**, we see that  $h = -1$ ,  $k = 3$ , and  $r = \sqrt{3}$ , so the given equation represents a circle with center  $(-1, 3)$  and radius  $\sqrt{3}$ . It is sketched in Figure 2.



**FIGURE 2**  
 $x^2 + y^2 + 2x - 6y + 7 = 0$

## Parabolas

The geometric properties of parabolas are reviewed in Section 10.5. Here we regard a parabola as a graph of an equation of the form  $y = ax^2 + bx + c$ .

**EXAMPLE 3** Draw the graph of the parabola  $y = x^2$ .

**SOLUTION** We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure 3.

$x$	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
$\pm 1$	1
$\pm 2$	4
$\pm 3$	9

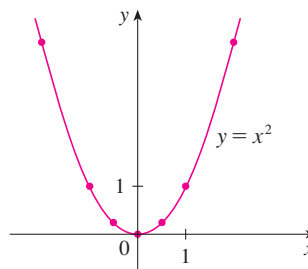


FIGURE 3

Figure 4 shows the graphs of several parabolas with equations of the form  $y = ax^2$  for various values of the number  $a$ . In each case the *vertex*, the point where the parabola changes direction, is the origin. We see that the parabola  $y = ax^2$  opens upward if  $a > 0$  and downward if  $a < 0$  (as in Figure 5).

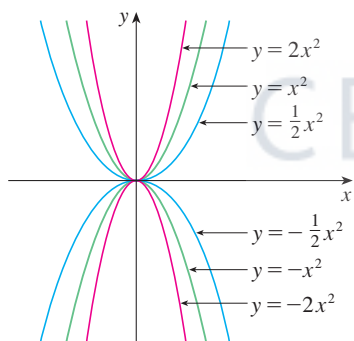
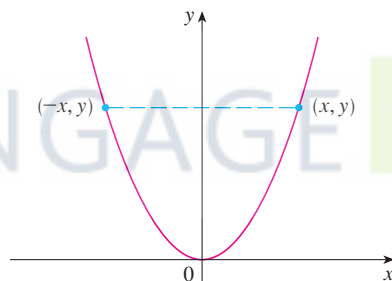
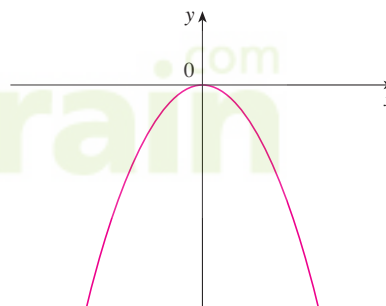


FIGURE 4



(a)  $y = ax^2$ ,  $a > 0$



(b)  $y = ax^2$ ,  $a < 0$

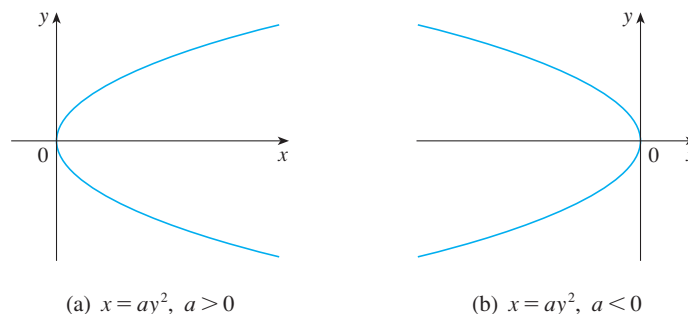
FIGURE 5

Notice that if  $(x, y)$  satisfies  $y = ax^2$ , then so does  $(-x, y)$ . This corresponds to the geometric fact that if the right half of the graph is reflected about the  $y$ -axis, then the left half of the graph is obtained. We say that the graph is **symmetric with respect to the  $y$ -axis**.

The graph of an equation is symmetric with respect to the  $y$ -axis if the equation is unchanged when  $x$  is replaced by  $-x$ .

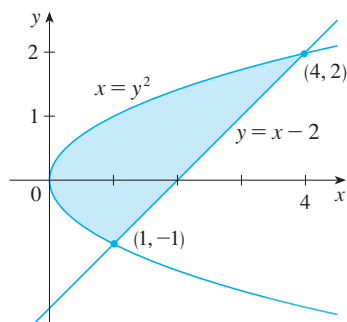
If we interchange  $x$  and  $y$  in the equation  $y = ax^2$ , the result is  $x = ay^2$ , which also represents a parabola. (Interchanging  $x$  and  $y$  amounts to reflecting about the diagonal line  $y = x$ .) The parabola  $x = ay^2$  opens to the right if  $a > 0$  and to the left if  $a < 0$ . (See

Figure 6.) This time the parabola is symmetric with respect to the  $x$ -axis because if  $(x, y)$  satisfies  $x = ay^2$ , then so does  $(x, -y)$ .



**FIGURE 6**

The graph of an equation is symmetric with respect to the  $x$ -axis if the equation is unchanged when  $y$  is replaced by  $-y$ .



**FIGURE 7**

**EXAMPLE 4** Sketch the region bounded by the parabola  $x = y^2$  and the line  $y = x - 2$ .

**SOLUTION** First we find the points of intersection by solving the two equations. Substituting  $x = y + 2$  into the equation  $x = y^2$ , we get  $y + 2 = y^2$ , which gives

$$0 = y^2 - y - 2 = (y - 2)(y + 1)$$

so  $y = 2$  or  $-1$ . Thus the points of intersection are  $(4, 2)$  and  $(1, -1)$ , and we draw the line  $y = x - 2$  passing through these points. We then sketch the parabola  $x = y^2$  by referring to Figure 6(a) and having the parabola pass through  $(4, 2)$  and  $(1, -1)$ . The region bounded by  $x = y^2$  and  $y = x - 2$  means the finite region whose boundaries are these curves. It is sketched in Figure 7.

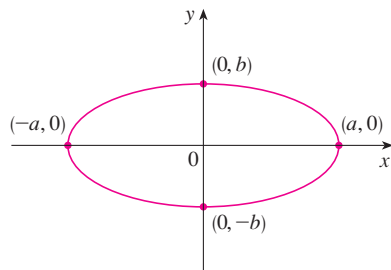
### Ellipses

The curve with equation

2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where  $a$  and  $b$  are positive numbers, is called an **ellipse** in standard position. (Geometric properties of ellipses are discussed in Section 10.5.) Observe that Equation 2 is unchanged if  $x$  is replaced by  $-x$  or  $y$  is replaced by  $-y$ , so the ellipse is symmetric with respect to both axes. As a further aid to sketching the ellipse, we find its intercepts.



**FIGURE 8**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The  **$x$ -intercepts** of a graph are the  $x$ -coordinates of the points where the graph intersects the  $x$ -axis. They are found by setting  $y = 0$  in the equation of the graph.

The  **$y$ -intercepts** are the  $y$ -coordinates of the points where the graph intersects the  $y$ -axis. They are found by setting  $x = 0$  in its equation.

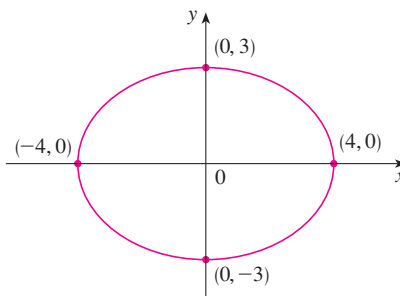
If we set  $y = 0$  in Equation 2, we get  $x^2 = a^2$  and so the  $x$ -intercepts are  $\pm a$ . Setting  $x = 0$ , we get  $y^2 = b^2$ , so the  $y$ -intercepts are  $\pm b$ . Using this information, together with symmetry, we sketch the ellipse in Figure 8. If  $a = b$ , the ellipse is a circle with radius  $a$ .

**EXAMPLE 5** Sketch the graph of  $9x^2 + 16y^2 = 144$ .

**SOLUTION** We divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse [2], so we have  $a^2 = 16$ ,  $b^2 = 9$ ,  $a = 4$ , and  $b = 3$ . The  $x$ -intercepts are  $\pm 4$ ; the  $y$ -intercepts are  $\pm 3$ . The graph is sketched in Figure 9.



**FIGURE 9**  
 $9x^2 + 16y^2 = 144$

### Hyperbolas

The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{3}$$

is called a **hyperbola** in standard position. Again, Equation 3 is unchanged when  $x$  is replaced by  $-x$  or  $y$  is replaced by  $-y$ , so the hyperbola is symmetric with respect to both axes. To find the  $x$ -intercepts we set  $y = 0$  and obtain  $x^2 = a^2$  and  $x = \pm a$ . However, if we put  $x = 0$  in Equation 3, we get  $y^2 = -b^2$ , which is impossible, so there is no  $y$ -intercept. In fact, from Equation 3 we obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$$

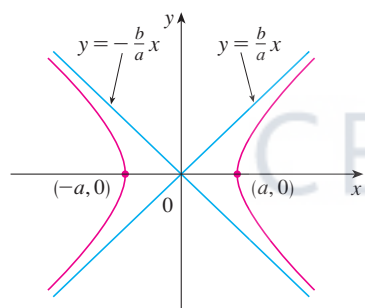
which shows that  $x^2 \geq a^2$  and so  $|x| = \sqrt{x^2} \geq a$ . Therefore we have  $x \geq a$  or  $x \leq -a$ . This means that the hyperbola consists of two parts, called its *branches*. It is sketched in Figure 10.

In drawing a hyperbola it is useful to draw first its *asymptotes*, which are the lines  $y = (b/a)x$  and  $y = -(b/a)x$  shown in Figure 10. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. This involves the idea of a limit, which is discussed in Chapter 1. (See also Exercise 57 in Section 3.5.)

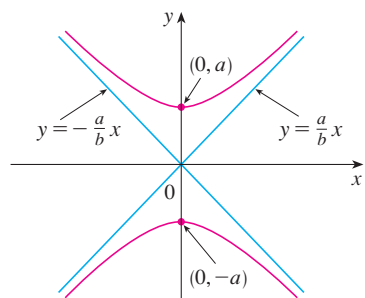
By interchanging the roles of  $x$  and  $y$  we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which also represents a hyperbola and is sketched in Figure 11.



**FIGURE 10**  
The hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



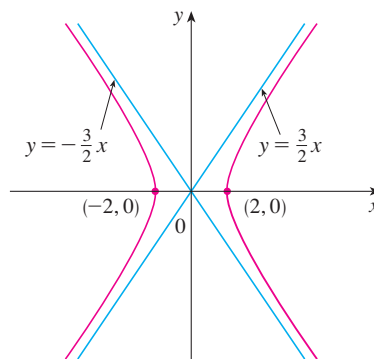
**FIGURE 11**  
The hyperbola  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

**EXAMPLE 6** Sketch the curve  $9x^2 - 4y^2 = 36$ .

**SOLUTION** Dividing both sides by 36, we obtain

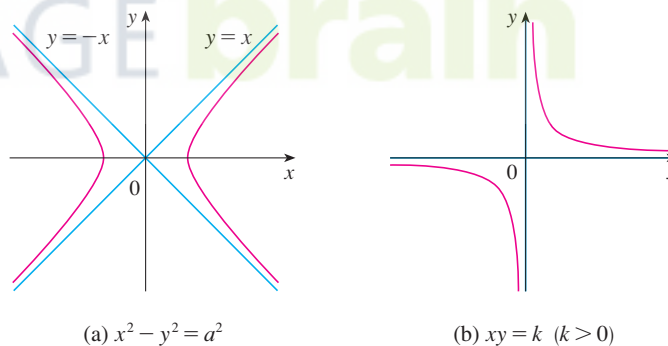
$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

which is the standard form of the equation of a hyperbola (Equation 3). Since  $a^2 = 4$ , the  $x$ -intercepts are  $\pm 2$ . Since  $b^2 = 9$ , we have  $b = 3$  and the asymptotes are  $y = \pm(\frac{3}{2})x$ . The hyperbola is sketched in Figure 12.



**FIGURE 12**  
The hyperbola  $9x^2 - 4y^2 = 36$

If  $b = a$ , a hyperbola has the equation  $x^2 - y^2 = a^2$  (or  $y^2 - x^2 = a^2$ ) and is called an *equilateral hyperbola* [see Figure 13(a)]. Its asymptotes are  $y = \pm x$ , which are perpendicular. If an equilateral hyperbola is rotated by  $45^\circ$ , the asymptotes become the  $x$ - and  $y$ -axes, and it can be shown that the new equation of the hyperbola is  $xy = k$ , where  $k$  is a constant [see Figure 13(b)].



**FIGURE 13**  
Equilateral hyperbolas

### Shifted Conics

Recall that an equation of the circle with center the origin and radius  $r$  is  $x^2 + y^2 = r^2$ , but if the center is the point  $(h, k)$ , then the equation of the circle becomes

$$(x - h)^2 + (y - k)^2 = r^2$$

Similarly, if we take the ellipse with equation

4

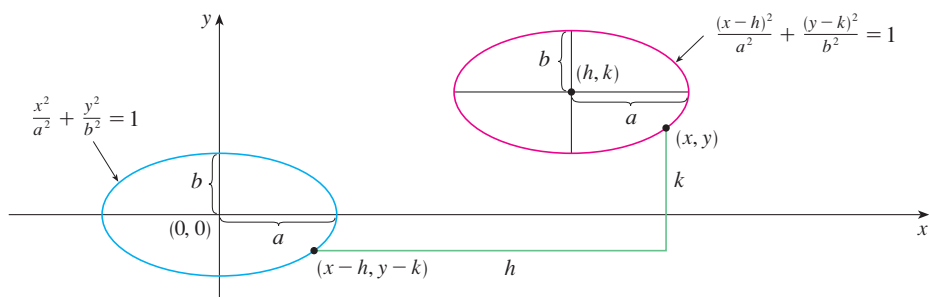
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and translate it (shift it) so that its center is the point  $(h, k)$ , then its equation becomes

**5**

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

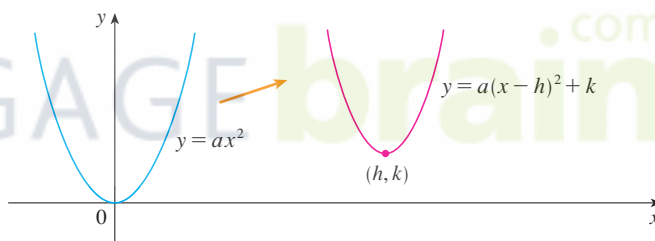
(See Figure 14.)



**FIGURE 14**

Notice that in shifting the ellipse, we replaced  $x$  by  $x - h$  and  $y$  by  $y - k$  in Equation 4 to obtain Equation 5. We use the same procedure to shift the parabola  $y = ax^2$  so that its vertex (the origin) becomes the point  $(h, k)$  as in Figure 15. Replacing  $x$  by  $x - h$  and  $y$  by  $y - k$ , we see that the new equation is

$$y - k = a(x - h)^2 \quad \text{or} \quad y = a(x - h)^2 + k$$



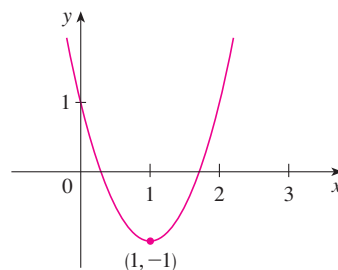
**FIGURE 15**

**EXAMPLE 7** Sketch the graph of the equation  $y = 2x^2 - 4x + 1$ .

**SOLUTION** First we complete the square:

$$y = 2(x^2 - 2x) + 1 = 2(x - 1)^2 - 1$$

In this form we see that the equation represents the parabola obtained by shifting  $y = 2x^2$  so that its vertex is at the point  $(1, -1)$ . The graph is sketched in Figure 16.

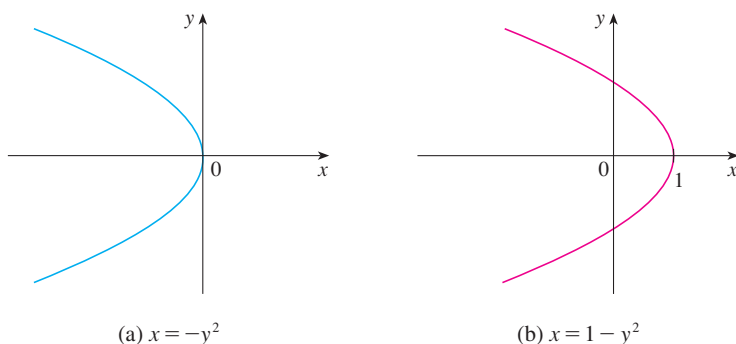


**FIGURE 16**

$$y = 2x^2 - 4x + 1$$

**EXAMPLE 8** Sketch the curve  $x = 1 - y^2$ .

**SOLUTION** This time we start with the parabola  $x = -y^2$  (as in Figure 6 with  $a = -1$ ) and shift one unit to the right to get the graph of  $x = 1 - y^2$ . (See Figure 17.)



**FIGURE 17**

## C Exercises

**1–4** Find an equation of a circle that satisfies the given conditions.

- Center  $(3, -1)$ , radius 5
- Center  $(-2, -8)$ , radius 10
- Center at the origin, passes through  $(4, 7)$
- Center  $(-1, 5)$ , passes through  $(-4, -6)$

**5–9** Show that the equation represents a circle and find the center and radius.

- $x^2 + y^2 - 4x + 10y + 13 = 0$
- $x^2 + y^2 + 6y + 2 = 0$
- $x^2 + y^2 + x = 0$
- $16x^2 + 16y^2 + 8x + 32y + 1 = 0$
- $2x^2 + 2y^2 - x + y = 1$

**10.** Under what condition on the coefficients  $a$ ,  $b$ , and  $c$  does the equation  $x^2 + y^2 + ax + by + c = 0$  represent a circle? When that condition is satisfied, find the center and radius of the circle.

**11–32** Identify the type of curve and sketch the graph. Do not plot points. Just use the standard graphs given in Figures 5, 6, 8, 10, and 11 and shift if necessary.

- $y = -x^2$
- $y^2 - x^2 = 1$
- $x^2 + 4y^2 = 16$
- $x = -2y^2$

**15.**  $16x^2 - 25y^2 = 400$

**17.**  $4x^2 + y^2 = 1$

**19.**  $x = y^2 - 1$

**21.**  $9y^2 - x^2 = 9$

**23.**  $xy = 4$

**25.**  $9(x - 1)^2 + 4(y - 2)^2 = 36$

**26.**  $16x^2 + 9y^2 - 36y = 108$

**27.**  $y = x^2 - 6x + 13$

**29.**  $x = 4 - y^2$

**31.**  $x^2 + 4y^2 - 6x + 5 = 0$

**32.**  $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

**16.**  $25x^2 + 4y^2 = 100$

**18.**  $y = x^2 + 2$

**20.**  $9x^2 - 25y^2 = 225$

**22.**  $2x^2 + 5y^2 = 10$

**24.**  $y = x^2 + 2x$

**28.**  $x^2 - y^2 - 4x + 3 = 0$

**30.**  $y^2 - 2x + 6y + 5 = 0$

**33–34** Sketch the region bounded by the curves.

**33.**  $y = 3x, \quad y = x^2$

**34.**  $y = 4 - x^2, \quad x - 2y = 2$

**35.** Find an equation of the parabola with vertex  $(1, -1)$  that passes through the points  $(-1, 3)$  and  $(3, 3)$ .

**36.** Find an equation of the ellipse with center at the origin that passes through the points  $(1, -10\sqrt{2}/3)$  and  $(-2, 5\sqrt{5}/3)$ .

**37–40** Sketch the graph of the set.

**37.**  $\{(x, y) \mid x^2 + y^2 \leq 1\}$

**38.**  $\{(x, y) \mid x^2 + y^2 > 4\}$

**39.**  $\{(x, y) \mid y \geq x^2 - 1\}$

**40.**  $\{(x, y) \mid x^2 + 4y^2 \leq 4\}$



## D Trigonometry

### Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains  $360^\circ$ , which is the same as  $2\pi$  rad. Therefore

1

$$\pi \text{ rad} = 180^\circ$$

and

2

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ \quad 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}$$

#### EXAMPLE 1

- (a) Find the radian measure of  $60^\circ$ . (b) Express  $5\pi/4$  rad in degrees.

#### SOLUTION

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by  $\pi/180$ . Therefore

$$60^\circ = 60 \left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by  $180/\pi$ . Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4} \left(\frac{180}{\pi}\right) = 225^\circ$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$	$270^\circ$	$360^\circ$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$

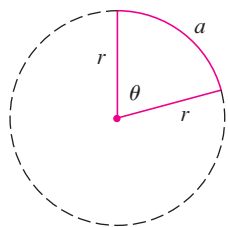


FIGURE 1

Figure 1 shows a sector of a circle with central angle  $\theta$  and radius  $r$  subtending an arc with length  $a$ . Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference  $2\pi r$  and central angle  $2\pi$ , we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for  $\theta$  and for  $a$ , we obtain

3

$$\theta = \frac{a}{r}$$

$$a = r\theta$$

Remember that Equations 3 are valid only when  $\theta$  is measured in radians.

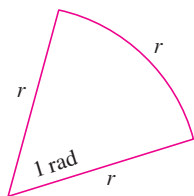


FIGURE 2

In particular, putting  $a = r$  in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

**EXAMPLE 2**

- (a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
- (b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of  $3\pi/8$  rad?

**SOLUTION**

(a) Using Equation 3 with  $a = 6$  and  $r = 5$ , we see that the angle is

$$\theta = \frac{6}{5} = 1.2 \text{ rad}$$

(b) With  $r = 3$  cm and  $\theta = 3\pi/8$  rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive  $x$ -axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4.

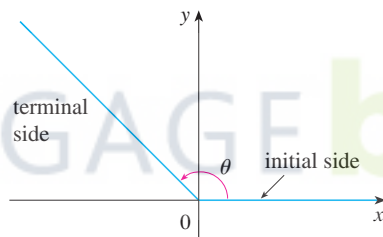


FIGURE 3  $\theta \geq 0$

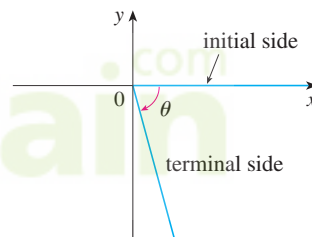


FIGURE 4  $\theta < 0$

Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles  $3\pi/4$ ,  $-5\pi/4$ , and  $11\pi/4$  have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \quad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and  $2\pi$  rad represents a complete revolution.

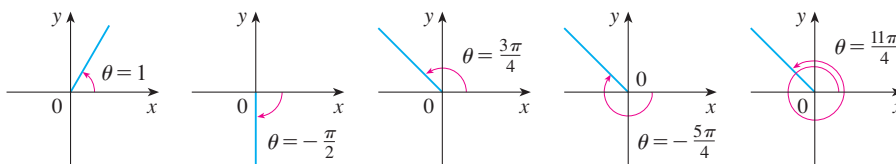


FIGURE 5  
Angles in standard position

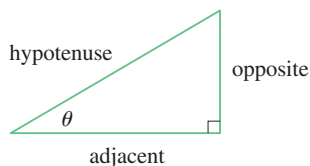


FIGURE 6

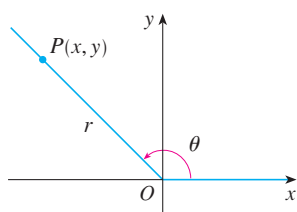


FIGURE 7

If we put  $r = 1$  in Definition 5 and draw a unit circle with center the origin and label  $\theta$  as in Figure 8, then the coordinates of  $P$  are  $(\cos \theta, \sin \theta)$ .

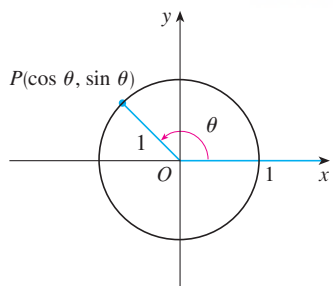


FIGURE 8

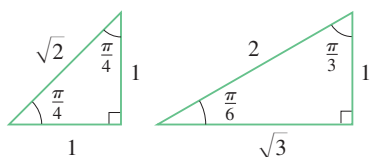


FIGURE 9

### The Trigonometric Functions

For an acute angle  $\theta$  the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

4

$$\begin{aligned} \sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}} \end{aligned}$$

This definition doesn't apply to obtuse or negative angles, so for a general angle  $\theta$  in standard position we let  $P(x, y)$  be any point on the terminal side of  $\theta$  and we let  $r$  be the distance  $|OP|$  as in Figure 7. Then we define

5

$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

Since division by 0 is not defined,  $\tan \theta$  and  $\sec \theta$  are undefined when  $x = 0$  and  $\csc \theta$  and  $\cot \theta$  are undefined when  $y = 0$ . Notice that the definitions in [4] and [5] are consistent when  $\theta$  is an acute angle.

If  $\theta$  is a number, the convention is that  $\sin \theta$  means the sine of the angle whose *radian* measure is  $\theta$ . For example, the expression  $\sin 3$  implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number, we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle  $3^\circ$  we would write  $\sin 3^\circ$  and, with our calculator in degree mode, we find that

$$\sin 3^\circ \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 9. For instance,

$$\begin{aligned} \sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \sin \frac{\pi}{6} &= \frac{1}{2} & \sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2} \\ \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2} & \cos \frac{\pi}{3} &= \frac{1}{2} \\ \tan \frac{\pi}{4} &= 1 & \tan \frac{\pi}{6} &= \frac{1}{\sqrt{3}} & \tan \frac{\pi}{3} &= \sqrt{3} \end{aligned}$$

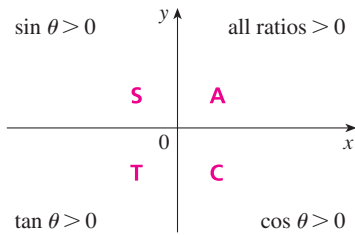


FIGURE 10

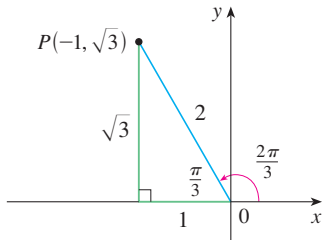


FIGURE 11

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule “All Students Take Calculus” shown in Figure 10.

**EXAMPLE 3** Find the exact trigonometric ratios for  $\theta = 2\pi/3$ .

**SOLUTION** From Figure 11 we see that a point on the terminal line for  $\theta = 2\pi/3$  is  $P(-1, \sqrt{3})$ . Therefore, taking

$$x = -1 \quad y = \sqrt{3} \quad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\begin{aligned} \sin \frac{2\pi}{3} &= \frac{\sqrt{3}}{2} & \cos \frac{2\pi}{3} &= -\frac{1}{2} & \tan \frac{2\pi}{3} &= -\sqrt{3} \\ \csc \frac{2\pi}{3} &= \frac{2}{\sqrt{3}} & \sec \frac{2\pi}{3} &= -2 & \cot \frac{2\pi}{3} &= -\frac{1}{\sqrt{3}} \end{aligned}$$

The following table gives some values of  $\sin \theta$  and  $\cos \theta$  found by the method of Example 3.

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

**EXAMPLE 4** If  $\cos \theta = \frac{2}{5}$  and  $0 < \theta < \pi/2$ , find the other five trigonometric functions of  $\theta$ .

**SOLUTION** Since  $\cos \theta = \frac{2}{5}$ , we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 12. If the opposite side has length  $x$ , then the Pythagorean Theorem gives  $x^2 + 4 = 25$  and so  $x^2 = 21$ ,  $x = \sqrt{21}$ . We can now use the diagram to write the other five trigonometric functions:

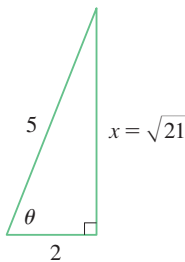


FIGURE 12

$$\sin \theta = \frac{\sqrt{21}}{5} \quad \tan \theta = \frac{\sqrt{21}}{2}$$

$$\csc \theta = \frac{5}{\sqrt{21}} \quad \sec \theta = \frac{5}{2} \quad \cot \theta = \frac{2}{\sqrt{21}}$$

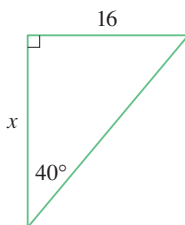


FIGURE 13

**EXAMPLE 5** Use a calculator to approximate the value of  $x$  in Figure 13.

**SOLUTION** From the diagram we see that

$$\tan 40^\circ = \frac{16}{x}$$

Therefore

$$x = \frac{16}{\tan 40^\circ} \approx 19.07$$

### Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

$$\begin{aligned} \text{6} \quad \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\ \tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \end{aligned}$$

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that  $x^2 + y^2 = r^2$ . Therefore

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

$$\text{7} \quad \sin^2 \theta + \cos^2 \theta = 1$$

If we now divide both sides of Equation 7 by  $\cos^2 \theta$  and use Equations 6, we get

$$\text{8} \quad \tan^2 \theta + 1 = \sec^2 \theta$$

Similarly, if we divide both sides of Equation 7 by  $\sin^2 \theta$ , we get

$$\text{9} \quad 1 + \cot^2 \theta = \csc^2 \theta$$

The identities

$$\text{10a} \quad \sin(-\theta) = -\sin \theta$$

$$\text{10b} \quad \cos(-\theta) = \cos \theta$$

Odd functions and even functions are discussed in Section 1.1.

show that sine is an odd function and cosine is an even function. They are easily proved by drawing a diagram showing  $\theta$  and  $-\theta$  in standard position (see Exercise 39).

Since the angles  $\theta$  and  $\theta + 2\pi$  have the same terminal side, we have

$$\text{11} \quad \sin(\theta + 2\pi) = \sin \theta \quad \cos(\theta + 2\pi) = \cos \theta$$

These identities show that the sine and cosine functions are periodic with period  $2\pi$ .

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

12a

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

12b

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

The proofs of these addition formulas are outlined in Exercises 85, 86, and 87.

By substituting  $-y$  for  $y$  in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following **subtraction formulas**:

13a

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

13b

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for  $\tan(x \pm y)$ :

14a

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

14b

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put  $y = x$  in the addition formulas [12], we get the **double-angle formulas**:

15a

$$\sin 2x = 2 \sin x \cos x$$

15b

$$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity  $\sin^2 x + \cos^2 x = 1$ , we obtain the following alternate forms of the double-angle formulas for  $\cos 2x$ :

16a

$$\cos 2x = 2 \cos^2 x - 1$$

16b

$$\cos 2x = 1 - 2 \sin^2 x$$

If we now solve these equations for  $\cos^2 x$  and  $\sin^2 x$ , we get the following **half-angle formulas**, which are useful in integral calculus:

17a

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

17b

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product formulas**, which can be deduced from Equations 12 and 13:

18a

$$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$$

18b

$$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$$

18c

$$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of the identities 13–18, remember that they can all be deduced from Equations 12a and 12b.

**EXAMPLE 6** Find all values of  $x$  in the interval  $[0, 2\pi]$  such that  $\sin x = \sin 2x$ .

**SOLUTION** Using the double-angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \quad \text{or} \quad \sin x(1 - 2 \cos x) = 0$$

Therefore there are two possibilities:

$$\sin x = 0 \quad \text{or} \quad 1 - 2 \cos x = 0$$

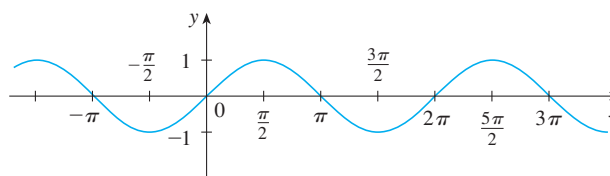
$$x = 0, \pi, 2\pi \quad \cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

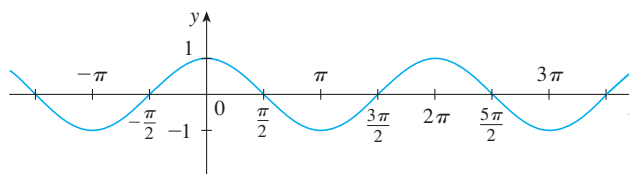
The given equation has five solutions:  $0, \pi/3, \pi, 5\pi/3,$  and  $2\pi$ .

### Graphs of the Trigonometric Functions

The graph of the function  $f(x) = \sin x$ , shown in Figure 14(a), is obtained by plotting points for  $0 \leq x \leq 2\pi$  and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the



(a)  $f(x) = \sin x$



(b)  $g(x) = \cos x$

FIGURE 14

integer multiples of  $\pi$ , that is,

$$\sin x = 0 \quad \text{whenever } x = n\pi, \quad n \text{ an integer}$$

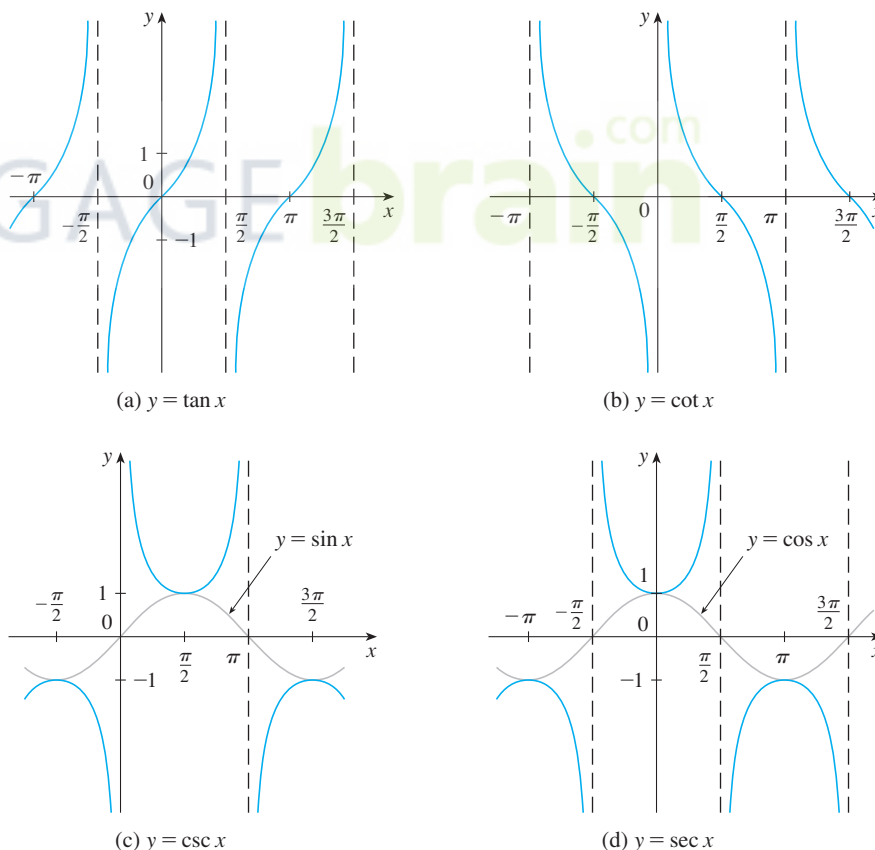
Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount  $\pi/2$  to the left [see Figure 14(b)]. Note that for both the sine and cosine functions the domain is  $(-\infty, \infty)$  and the range is the closed interval  $[-1, 1]$ . Thus, for all values of  $x$ , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

The graphs of the remaining four trigonometric functions are shown in Figure 15 and their domains are indicated there. Notice that tangent and cotangent have range  $(-\infty, \infty)$ , whereas cosecant and secant have range  $(-\infty, -1] \cup [1, \infty)$ . All four functions are periodic: tangent and cotangent have period  $\pi$ , whereas cosecant and secant have period  $2\pi$ .



**FIGURE 15**



## D Exercises

1–6 Convert from degrees to radians.

1.  $210^\circ$                       2.  $300^\circ$                       3.  $9^\circ$   
 4.  $-315^\circ$                       5.  $900^\circ$                       6.  $36^\circ$

7–12 Convert from radians to degrees.

7.  $4\pi$                       8.  $-\frac{7\pi}{2}$                       9.  $\frac{5\pi}{12}$   
 10.  $\frac{8\pi}{3}$                       11.  $-\frac{3\pi}{8}$                       12. 5

13. Find the length of a circular arc subtended by an angle of  $\pi/12$  rad if the radius of the circle is 36 cm.  
 14. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of  $72^\circ$ .  
 15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?  
 16. Find the radius of a circular sector with angle  $3\pi/4$  and arc length 6 cm.

17–22 Draw, in standard position, the angle whose measure is given.

17.  $315^\circ$                       18.  $-150^\circ$                       19.  $-\frac{3\pi}{4}$  rad  
 20.  $\frac{7\pi}{3}$  rad                      21. 2 rad                      22.  $-3$  rad

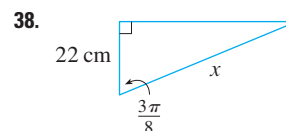
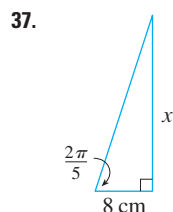
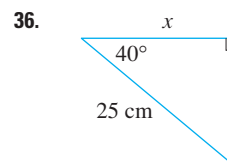
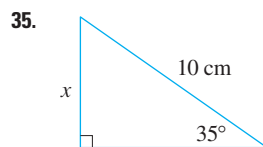
23–28 Find the exact trigonometric ratios for the angle whose radian measure is given.

23.  $\frac{3\pi}{4}$                       24.  $\frac{4\pi}{3}$                       25.  $\frac{9\pi}{2}$   
 26.  $-5\pi$                       27.  $\frac{5\pi}{6}$                       28.  $\frac{11\pi}{4}$

29–34 Find the remaining trigonometric ratios.

29.  $\sin \theta = \frac{3}{5}$ ,  $0 < \theta < \frac{\pi}{2}$   
 30.  $\tan \alpha = 2$ ,  $0 < \alpha < \frac{\pi}{2}$   
 31.  $\sec \phi = -1.5$ ,  $\frac{\pi}{2} < \phi < \pi$   
 32.  $\cos x = -\frac{1}{3}$ ,  $\pi < x < \frac{3\pi}{2}$   
 33.  $\cot \beta = 3$ ,  $\pi < \beta < 2\pi$

34.  $\csc \theta = -\frac{4}{3}$ ,  $\frac{3\pi}{2} < \theta < 2\pi$

35–38 Find, correct to five decimal places, the length of the side labeled  $x$ .

39–41 Prove each equation.

39. (a) Equation 10a                      (b) Equation 10b  
 40. (a) Equation 14a                      (b) Equation 14b  
 41. (a) Equation 18a                      (b) Equation 18b  
       (c) Equation 18c

42–58 Prove the identity.

42.  $\cos\left(\frac{\pi}{2} - x\right) = \sin x$   
 43.  $\sin\left(\frac{\pi}{2} + x\right) = \cos x$                       44.  $\sin(\pi - x) = \sin x$   
 45.  $\sin \theta \cot \theta = \cos \theta$                       46.  $(\sin x + \cos x)^2 = 1 + \sin 2x$   
 47.  $\sec y - \cos y = \tan y \sin y$   
 48.  $\tan^2 \alpha - \sin^2 \alpha = \tan^2 \alpha \sin^2 \alpha$   
 49.  $\cot^2 \theta + \sec^2 \theta = \tan^2 \theta + \csc^2 \theta$   
 50.  $2 \csc 2t = \sec t \csc t$   
 51.  $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$   
 52.  $\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \sec^2 \theta$   
 53.  $\sin x \sin 2x + \cos x \cos 2x = \cos x$   
 54.  $\sin^2 x - \sin^2 y = \sin(x + y) \sin(x - y)$   
 55.  $\frac{\sin \phi}{1 - \cos \phi} = \csc \phi + \cot \phi$

56.  $\tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}$

57.  $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta$

58.  $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

59–64 If  $\sin x = \frac{1}{3}$  and  $\sec y = \frac{5}{4}$ , where  $x$  and  $y$  lie between 0 and  $\pi/2$ , evaluate the expression.

59.  $\sin(x+y)$

60.  $\cos(x+y)$

61.  $\cos(x-y)$

62.  $\sin(x-y)$

63.  $\sin 2y$

64.  $\cos 2y$

65–72 Find all values of  $x$  in the interval  $[0, 2\pi]$  that satisfy the equation.

65.  $2 \cos x - 1 = 0$

66.  $3 \cot^2 x = 1$

67.  $2 \sin^2 x = 1$

68.  $|\tan x| = 1$

69.  $\sin 2x = \cos x$

70.  $2 \cos x + \sin 2x = 0$

71.  $\sin x = \tan x$

72.  $2 + \cos 2x = 3 \cos x$

73–76 Find all values of  $x$  in the interval  $[0, 2\pi]$  that satisfy the inequality.

73.  $\sin x \leq \frac{1}{2}$

74.  $2 \cos x + 1 > 0$

75.  $-1 < \tan x < 1$

76.  $\sin x > \cos x$

77–82 Graph the function by starting with the graphs in Figures 14 and 15 and applying the transformations of Section 1.3 where appropriate.

77.  $y = \cos\left(x - \frac{\pi}{3}\right)$

78.  $y = \tan 2x$

79.  $y = \frac{1}{3} \tan\left(x - \frac{\pi}{2}\right)$

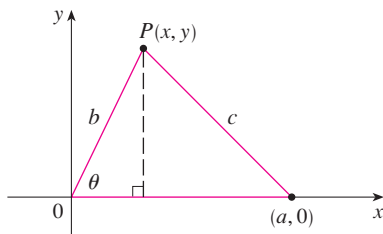
80.  $y = 1 + \sec x$

81.  $y = |\sin x|$

82.  $y = 2 + \sin\left(x + \frac{\pi}{4}\right)$

83. Prove the **Law of Cosines**: If a triangle has sides with lengths  $a$ ,  $b$ , and  $c$ , and  $\theta$  is the angle between the sides with lengths  $a$  and  $b$ , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



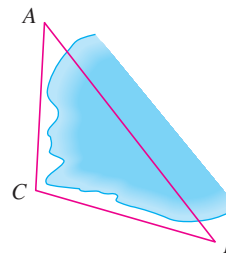
[Hint: Introduce a coordinate system so that  $\theta$  is in standard

position, as in the figure. Express  $x$  and  $y$  in terms of  $\theta$  and then use the distance formula to compute  $c$ .]

84. In order to find the distance  $|AB|$  across a small inlet, a point  $C$  was located as in the figure and the following measurements were recorded:

$$\angle C = 103^\circ \quad |AC| = 820 \text{ m} \quad |BC| = 910 \text{ m}$$

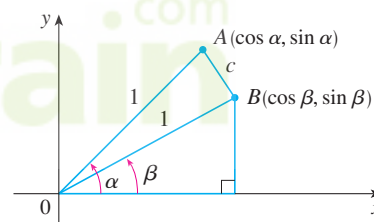
Use the Law of Cosines from Exercise 83 to find the required distance.



85. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

[Hint: Compute  $c^2$  in two ways (using the Law of Cosines from Exercise 83 and also using the distance formula) and compare the two expressions.]



86. Use the formula in Exercise 85 to prove the addition formula for cosine (12b).

87. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

to prove the subtraction formula (13a) for the sine function.

88. Show that the area of a triangle with sides of lengths  $a$  and  $b$  and with included angle  $\theta$  is

$$A = \frac{1}{2} ab \sin \theta$$

89. Find the area of triangle  $ABC$ , correct to five decimal places, if

$$|AB| = 10 \text{ cm} \quad |BC| = 3 \text{ cm} \quad \angle ABC = 107^\circ$$

## E Sigma Notation

A convenient way of writing sums uses the Greek letter  $\Sigma$  (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with  $i = n$ .

This tells us to add.

This tells us to start with  $i = m$ .

$$\sum_{i=m}^n a_i$$

**1 Definition** If  $a_m, a_{m+1}, \dots, a_n$  are real numbers and  $m$  and  $n$  are integers such that  $m \leq n$ , then

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^n f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus the symbol  $\sum_{i=m}^n$  indicates a summation in which the letter  $i$  (called the **index of summation**) takes on consecutive integer values beginning with  $m$  and ending with  $n$ , that is,  $m, m+1, \dots, n$ . Other letters can also be used as the index of summation.

**EXAMPLE 1**

(a)  $\sum_{i=1}^4 i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b)  $\sum_{i=3}^n i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c)  $\sum_{j=0}^5 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d)  $\sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e)  $\sum_{i=1}^3 \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f)  $\sum_{i=1}^4 2 = 2 + 2 + 2 + 2 = 8$

**EXAMPLE 2** Write the sum  $2^3 + 3^3 + \cdots + n^3$  in sigma notation.

**SOLUTION** There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^n i^3$$

or 
$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or 
$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$

The following theorem gives three simple rules for working with sigma notation.

**2 Theorem** If  $c$  is any constant (that is, it does not depend on  $i$ ), then

$$(a) \sum_{i=m}^n ca_i = c \sum_{i=m}^n a_i \qquad (b) \sum_{i=m}^n (a_i + b_i) = \sum_{i=m}^n a_i + \sum_{i=m}^n b_i$$

$$(c) \sum_{i=m}^n (a_i - b_i) = \sum_{i=m}^n a_i - \sum_{i=m}^n b_i$$

**PROOF** To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$\begin{aligned} (a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n) \\ = (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n) \end{aligned}$$

Rule (c) is proved similarly. ■

**EXAMPLE 3** Find  $\sum_{i=1}^n 1$ .

**SOLUTION** 
$$\sum_{i=1}^n 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$
 ■

**EXAMPLE 4** Prove the formula for the sum of the first  $n$  positive integers:

$$\sum_{i=1}^n i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

**SOLUTION** This formula can be proved by mathematical induction (see page 98) or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum  $S$  twice, once in the usual order and once in reverse order:

$$S = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$S = n + (n-1) + (n-2) + \cdots + 2 + 1$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are  $n$  terms, each of which is  $n+1$ , so

$$2S = n(n+1) \quad \text{or} \quad S = \frac{n(n+1)}{2}$$
 ■

**EXAMPLE 5** Prove the formula for the sum of the squares of the first  $n$  positive integers:

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**SOLUTION 1** Let  $S$  be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\begin{aligned}\sum_{i=1}^n [(1+i)^3 - i^3] &= (\cancel{2^3} - 1^3) + (3^3 - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + [(n+1)^3 - \cancel{n^3}] \\ &= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n\end{aligned}$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\begin{aligned}\sum_{i=1}^n [(1+i)^3 - i^3] &= \sum_{i=1}^n [3i^2 + 3i + 1] = 3 \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + \sum_{i=1}^n 1 \\ &= 3S + 3 \frac{n(n+1)}{2} + n = 3S + \frac{3}{2}n^2 + \frac{5}{2}n\end{aligned}$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \frac{3}{2}n^2 + \frac{5}{2}n$$

Solving this equation for  $S$ , we obtain

$$3S = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$$

or

$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

### Principle of Mathematical Induction

Let  $S_n$  be a statement involving the positive integer  $n$ . Suppose that

1.  $S_1$  is true.
2. If  $S_k$  is true, then  $S_{k+1}$  is true.

Then  $S_n$  is true for all positive integers  $n$ .

**SOLUTION 2** Let  $S_n$  be the given formula.

1.  $S_1$  is true because  $1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$
2. Assume that  $S_k$  is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

$$\begin{aligned}1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1) \frac{k(2k+1) + 6(k+1)}{6} \\ &= (k+1) \frac{2k^2 + 7k + 6}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}\end{aligned}$$

So  $S_{k+1}$  is true.

By the Principle of Mathematical Induction,  $S_n$  is true for all  $n$ .

See pages 98 and 100 for a more thorough discussion of mathematical induction.

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas and evaluating integrals in Chapter 4.

**3 Theorem** Let  $c$  be a constant and  $n$  a positive integer. Then

$$(a) \sum_{i=1}^n 1 = n$$

$$(b) \sum_{i=1}^n c = nc$$

$$(c) \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$(d) \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(e) \sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

**EXAMPLE 6** Evaluate  $\sum_{i=1}^n i(4i^2 - 3)$ .

**SOLUTION** Using Theorems 2 and 3, we have

$$\begin{aligned} \sum_{i=1}^n i(4i^2 - 3) &= \sum_{i=1}^n (4i^3 - 3i) = 4 \sum_{i=1}^n i^3 - 3 \sum_{i=1}^n i \\ &= 4 \left[ \frac{n(n+1)}{2} \right]^2 - 3 \frac{n(n+1)}{2} \\ &= \frac{n(n+1)[2n(n+1) - 3]}{2} \\ &= \frac{n(n+1)(2n^2 + 2n - 3)}{2} \end{aligned}$$

**EXAMPLE 7** Find  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[ \left( \frac{i}{n} \right)^2 + 1 \right]$ .

The type of calculation in Example 7 arises in Chapter 4 when we compute areas.

**SOLUTION**

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[ \left( \frac{i}{n} \right)^2 + 1 \right] &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{3}{n^3} i^2 + \frac{3}{n} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{3}{n^3} \sum_{i=1}^n i^2 + \frac{3}{n} \sum_{i=1}^n 1 \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{3}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot n \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{2} \cdot \frac{n}{n} \cdot \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right) + 3 \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{2} \cdot 1 \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right) + 3 \right] \\ &= \frac{1}{2} \cdot 1 \cdot 1 \cdot 2 + 3 = 4 \end{aligned}$$

**E** Exercises

1–10 Write the sum in expanded form.

1.  $\sum_{i=1}^5 \sqrt{i}$

2.  $\sum_{i=1}^6 \frac{1}{i+1}$

3.  $\sum_{i=4}^6 3^i$

4.  $\sum_{i=4}^6 i^3$

5.  $\sum_{k=0}^4 \frac{2k-1}{2k+1}$

6.  $\sum_{k=5}^8 x^k$

7.  $\sum_{i=1}^n i^{10}$

8.  $\sum_{j=n}^{n+3} j^2$

9.  $\sum_{j=0}^{n-1} (-1)^j$

10.  $\sum_{i=1}^n f(x_i) \Delta x_i$

11–20 Write the sum in sigma notation.

11.  $1 + 2 + 3 + 4 + \dots + 10$

12.  $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13.  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots + \frac{19}{20}$

14.  $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \dots + \frac{23}{27}$

15.  $2 + 4 + 6 + 8 + \dots + 2n$

16.  $1 + 3 + 5 + 7 + \dots + (2n - 1)$

17.  $1 + 2 + 4 + 8 + 16 + 32$

18.  $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19.  $x + x^2 + x^3 + \dots + x^n$

20.  $1 - x + x^2 - x^3 + \dots + (-1)^n x^n$

21–35 Find the value of the sum.

21.  $\sum_{i=4}^8 (3i - 2)$

22.  $\sum_{i=3}^6 i(i + 2)$

23.  $\sum_{j=1}^6 3^{j+1}$

24.  $\sum_{k=0}^8 \cos k\pi$

25.  $\sum_{n=1}^{20} (-1)^n$

26.  $\sum_{i=1}^{100} 4$

27.  $\sum_{i=0}^4 (2^i + i^2)$

28.  $\sum_{i=-2}^4 2^{3-i}$

29.  $\sum_{i=1}^n 2i$

30.  $\sum_{i=1}^n (2 - 5i)$

31.  $\sum_{i=1}^n (i^2 + 3i + 4)$

32.  $\sum_{i=1}^n (3 + 2i)^2$

33.  $\sum_{i=1}^n (i + 1)(i + 2)$

34.  $\sum_{i=1}^n i(i + 1)(i + 2)$

35.  $\sum_{i=1}^n (i^3 - i - 2)$

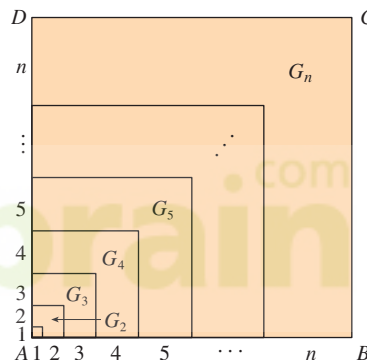
36. Find the number  $n$  such that  $\sum_{i=1}^n i = 78$ .

37. Prove formula (b) of Theorem 3.

38. Prove formula (e) of Theorem 3 using mathematical induction.

39. Prove formula (e) of Theorem 3 using a method similar to that of Example 5, Solution 1 [start with  $(1 + i)^4 - i^4$ ].

40. Prove formula (e) of Theorem 3 using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about AD 1010. The figure shows a square  $ABCD$  in which sides  $AB$  and  $AD$  have been divided into segments of lengths 1, 2, 3, . . . ,  $n$ . Thus the side of the square has length  $n(n + 1)/2$  so the area is  $[n(n + 1)/2]^2$ . But the area is also the sum of the areas of the  $n$  “gnomons”  $G_1, G_2, \dots, G_n$  shown in the figure. Show that the area of  $G_i$  is  $i^3$  and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a)  $\sum_{i=1}^n [i^4 - (i - 1)^4]$

(b)  $\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c)  $\sum_{i=3}^{99} \left( \frac{1}{i} - \frac{1}{i+1} \right)$

(d)  $\sum_{i=1}^n (a_i - a_{i-1})$

42. Prove the generalized triangle inequality:

$$\left| \sum_{i=1}^n a_i \right| \leq \sum_{i=1}^n |a_i|$$

43–46 Find the limit.

43.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left( \frac{i}{n} \right)^2$

44.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left[ \left( \frac{i}{n} \right)^3 + 1 \right]$

45.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left[ \left( \frac{2i}{n} \right)^3 + 5 \left( \frac{2i}{n} \right) \right]$

$$46. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[ \left(1 + \frac{3i}{n}\right)^3 - 2 \left(1 + \frac{3i}{n}\right) \right]$$

47. Prove the formula for the sum of a finite geometric series with first term  $a$  and common ratio  $r \neq 1$ :

$$\sum_{i=1}^n ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

$$48. \text{ Evaluate } \sum_{i=1}^n \frac{3}{2^{i-1}}.$$

$$49. \text{ Evaluate } \sum_{i=1}^n (2i + 2^i).$$

$$50. \text{ Evaluate } \sum_{i=1}^m \left[ \sum_{j=1}^n (i + j) \right].$$

## F Proofs of Theorems

In this appendix we present proofs of several theorems that are stated in the main body of the text. The sections in which they occur are indicated in the margin.

### Section 1.6

**Limit Laws** Suppose that  $c$  is a constant and the limits

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = L + M \qquad 2. \lim_{x \rightarrow a} [f(x) - g(x)] = L - M$$

$$3. \lim_{x \rightarrow a} [cf(x)] = cL \qquad 4. \lim_{x \rightarrow a} [f(x)g(x)] = LM$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{if } M \neq 0$$

**PROOF OF LAW 4** Let  $\varepsilon > 0$  be given. We want to find  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain  $|f(x) - L|$  and  $|g(x) - M|$ , we add and subtract  $Lg(x)$  as follows:

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |[f(x) - L]g(x) + L[g(x) - M]| \\ &\leq |[f(x) - L]g(x)| + |L[g(x) - M]| \quad (\text{Triangle Inequality}) \\ &= |f(x) - L||g(x)| + |L||g(x) - M| \end{aligned}$$

We want to make each of these terms less than  $\varepsilon/2$ .

Since  $\lim_{x \rightarrow a} g(x) = M$ , there is a number  $\delta_1 > 0$  such that

$$\text{if } 0 < |x - a| < \delta_1 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number  $\delta_2 > 0$  such that if  $0 < |x - a| < \delta_2$ , then

$$|g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$



Since  $\lim_{x \rightarrow a} f(x) = L$ , there is a number  $\delta_3 > 0$  such that

$$\text{if } 0 < |x - a| < \delta_3 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let  $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ . If  $0 < |x - a| < \delta$ , then we have  $0 < |x - a| < \delta_1$ ,  $0 < |x - a| < \delta_2$ , and  $0 < |x - a| < \delta_3$ , so we can combine the inequalities to obtain

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L| \frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that  $\lim_{x \rightarrow a} [f(x)g(x)] = LM$ . ■

**PROOF OF LAW 3** If we take  $g(x) = c$  in Law 4, we get

$$\begin{aligned} \lim_{x \rightarrow a} [cf(x)] &= \lim_{x \rightarrow a} [g(x)f(x)] = \lim_{x \rightarrow a} g(x) \cdot \lim_{x \rightarrow a} f(x) \\ &= \lim_{x \rightarrow a} c \cdot \lim_{x \rightarrow a} f(x) \\ &= c \lim_{x \rightarrow a} f(x) \quad (\text{by Law 7}) \end{aligned}$$
■

**PROOF OF LAW 2** Using Law 1 and Law 3 with  $c = -1$ , we have

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - g(x)] &= \lim_{x \rightarrow a} [f(x) + (-1)g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} (-1)g(x) \\ &= \lim_{x \rightarrow a} f(x) + (-1) \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) \end{aligned}$$
■

**PROOF OF LAW 5** First let us show that

$$\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$$

To do this we must show that, given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Observe that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|}$$

We know that we can make the numerator small. But we also need to know that the denominator is not small when  $x$  is near  $a$ . Since  $\lim_{x \rightarrow a} g(x) = M$ , there is a number  $\delta_1 > 0$  such that, whenever  $0 < |x - a| < \delta_1$ , we have

$$|g(x) - M| < \frac{|M|}{2}$$

and therefore

$$\begin{aligned} |M| &= |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| \\ &< \frac{|M|}{2} + |g(x)| \end{aligned}$$

This shows that

$$\text{if } 0 < |x - a| < \delta_1 \quad \text{then} \quad |g(x)| > \frac{|M|}{2}$$

and so, for these values of  $x$ ,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M||g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there exists  $\delta_2 > 0$  such that

$$\text{if } 0 < |x - a| < \delta_2 \quad \text{then} \quad |g(x) - M| < \frac{M^2}{2} \varepsilon$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then, for  $0 < |x - a| < \delta$ , we have

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon$$

It follows that  $\lim_{x \rightarrow a} 1/g(x) = 1/M$ . Finally, using Law 4, we obtain

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} f(x) \left( \frac{1}{g(x)} \right) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$

**2 Theorem** If  $f(x) \leq g(x)$  for all  $x$  in an open interval that contains  $a$  (except possibly at  $a$ ) and

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M$$

then  $L \leq M$ .

**PROOF** We use the method of proof by contradiction. Suppose, if possible, that  $L > M$ . Law 2 of limits says that

$$\lim_{x \rightarrow a} [g(x) - f(x)] = M - L$$

Therefore, for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |[g(x) - f(x)] - (M - L)| < \varepsilon$$

In particular, taking  $\varepsilon = L - M$  (noting that  $L - M > 0$  by hypothesis), we have a number  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |[g(x) - f(x)] - (M - L)| < L - M$$

Since  $a \leq |a|$  for any number  $a$ , we have

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad [g(x) - f(x)] - (M - L) < L - M$$

which simplifies to

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad g(x) < f(x)$$

But this contradicts  $f(x) \leq g(x)$ . Thus the inequality  $L > M$  must be false. Therefore  $L \leq M$ .

**3 The Squeeze Theorem** If  $f(x) \leq g(x) \leq h(x)$  for all  $x$  in an open interval that contains  $a$  (except possibly at  $a$ ) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

**PROOF** Let  $\varepsilon > 0$  be given. Since  $\lim_{x \rightarrow a} f(x) = L$ , there is a number  $\delta_1 > 0$  such that

$$\text{if } 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \varepsilon$$

that is,

$$\text{if } 0 < |x - a| < \delta_1 \quad \text{then} \quad L - \varepsilon < f(x) < L + \varepsilon$$

Since  $\lim_{x \rightarrow a} h(x) = L$ , there is a number  $\delta_2 > 0$  such that

$$\text{if } 0 < |x - a| < \delta_2 \quad \text{then} \quad |h(x) - L| < \varepsilon$$

that is,

$$\text{if } 0 < |x - a| < \delta_2 \quad \text{then} \quad L - \varepsilon < h(x) < L + \varepsilon$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ . If  $0 < |x - a| < \delta$ , then  $0 < |x - a| < \delta_1$  and  $0 < |x - a| < \delta_2$ , so

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

In particular,

$$L - \varepsilon < g(x) < L + \varepsilon$$

and so  $|g(x) - L| < \varepsilon$ . Therefore  $\lim_{x \rightarrow a} g(x) = L$ .

### Section 1.8

**8 Theorem** If  $f$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$ , then

$$\lim_{x \rightarrow a} f(g(x)) = f(b)$$

**PROOF** Let  $\varepsilon > 0$  be given. We want to find a number  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(g(x)) - f(b)| < \varepsilon$$

Since  $f$  is continuous at  $b$ , we have

$$\lim_{y \rightarrow b} f(y) = f(b)$$

and so there exists  $\delta_1 > 0$  such that

$$\text{if } 0 < |y - b| < \delta_1 \quad \text{then} \quad |f(y) - f(b)| < \varepsilon$$

Since  $\lim_{x \rightarrow a} g(x) = b$ , there exists  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |g(x) - b| < \delta_1$$

Combining these two statements, we see that whenever  $0 < |x - a| < \delta$  we have  $|g(x) - b| < \delta_1$ , which implies that  $|f(g(x)) - f(b)| < \epsilon$ . Therefore we have proved that  $\lim_{x \rightarrow a} f(g(x)) = f(b)$ .

**Section 2.4**

The proof of the following result was promised when we proved that  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ .

**Theorem** If  $0 < \theta < \pi/2$ , then  $\theta \leq \tan \theta$ .

**PROOF** Figure 1 shows a sector of a circle with center  $O$ , central angle  $\theta$ , and radius 1. Then

$$|AD| = |OA| \tan \theta = \tan \theta$$

We approximate the arc  $AB$  by an inscribed polygon consisting of  $n$  equal line segments and we look at a typical segment  $PQ$ . We extend the lines  $OP$  and  $OQ$  to meet  $AD$  in the points  $R$  and  $S$ . Then we draw  $RT \parallel PQ$  as in Figure 2. Observe that

$$\angle RTO = \angle PQO < 90^\circ$$

and so  $\angle RTS > 90^\circ$ . Therefore we have

$$|PQ| < |RT| < |RS|$$

If we add  $n$  such inequalities, we get

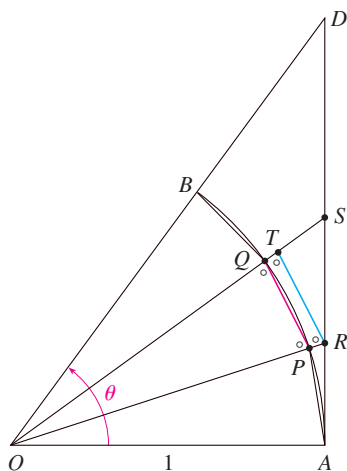
$$L_n < |AD| = \tan \theta$$

where  $L_n$  is the length of the inscribed polygon. Thus, by Theorem 1.6.2, we have

$$\lim_{n \rightarrow \infty} L_n \leq \tan \theta$$

But the arc length is defined in Equation 8.1.1 as the limit of the lengths of inscribed polygons, so

$$\theta = \lim_{n \rightarrow \infty} L_n \leq \tan \theta$$

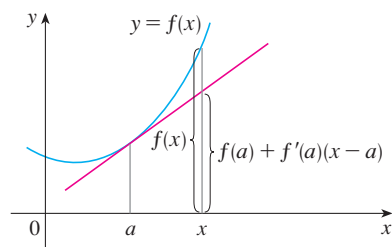


**FIGURE 1**

**Section 3.3**

**Concavity Test**

- (a) If  $f''(x) > 0$  for all  $x$  in  $I$ , then the graph of  $f$  is concave upward on  $I$ .
- (b) If  $f''(x) < 0$  for all  $x$  in  $I$ , then the graph of  $f$  is concave downward on  $I$ .



**FIGURE 2**

**PROOF OF (a)** Let  $a$  be any number in  $I$ . We must show that the curve  $y = f(x)$  lies above the tangent line at the point  $(a, f(a))$ . The equation of this tangent is

$$y = f(a) + f'(a)(x - a)$$

So we must show that

$$f(x) > f(a) + f'(a)(x - a)$$

whenever  $x \in I$  ( $x \neq a$ ). (See Figure 2.)

First let us take the case where  $x > a$ . Applying the Mean Value Theorem to  $f$  on the interval  $[a, x]$ , we get a number  $c$ , with  $a < c < x$ , such that

$$\boxed{1} \quad f(x) - f(a) = f'(c)(x - a)$$

Since  $f'' > 0$  on  $I$ , we know from the Increasing/Decreasing Test that  $f'$  is increasing on  $I$ . Thus, since  $a < c$ , we have

$$f'(a) < f'(c)$$

and so, multiplying this inequality by the positive number  $x - a$ , we get

$$\boxed{2} \quad f'(a)(x - a) < f'(c)(x - a)$$

Now we add  $f(a)$  to both sides of this inequality:

$$f(a) + f'(a)(x - a) < f(a) + f'(c)(x - a)$$

But from Equation 1 we have  $f(x) = f(a) + f'(c)(x - a)$ . So this inequality becomes

$$\boxed{3} \quad f(x) > f(a) + f'(a)(x - a)$$

which is what we wanted to prove.

For the case where  $x < a$  we have  $f'(c) < f'(a)$ , but multiplication by the negative number  $x - a$  reverses the inequality, so we get  $\boxed{2}$  and  $\boxed{3}$  as before.  

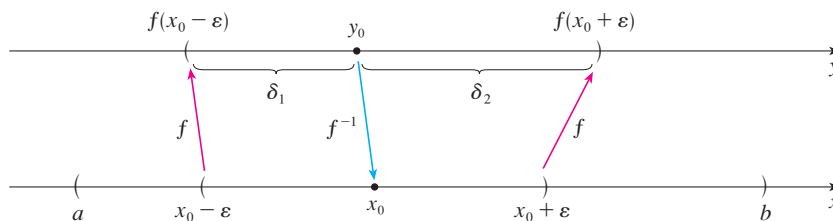
### Section 6.1

**Theorem** If  $f$  is a one-to-one continuous function defined on an interval  $(a, b)$ , then its inverse function  $f^{-1}$  is also continuous.

**PROOF** First we show that if  $f$  is both one-to-one and continuous on  $(a, b)$ , then it must be either increasing or decreasing on  $(a, b)$ . If it were neither increasing nor decreasing, then there would exist numbers  $x_1, x_2$ , and  $x_3$  in  $(a, b)$  with  $x_1 < x_2 < x_3$  such that  $f(x_2)$  does not lie between  $f(x_1)$  and  $f(x_3)$ . There are two possibilities: either (1)  $f(x_3)$  lies between  $f(x_1)$  and  $f(x_2)$  or (2)  $f(x_1)$  lies between  $f(x_2)$  and  $f(x_3)$ . (Draw a picture.) In case (1) we apply the Intermediate Value Theorem to the continuous function  $f$  to get a number  $c$  between  $x_1$  and  $x_2$  such that  $f(c) = f(x_3)$ . In case (2) the Intermediate Value Theorem gives a number  $c$  between  $x_2$  and  $x_3$  such that  $f(c) = f(x_1)$ . In either case we have contradicted the fact that  $f$  is one-to-one.

Let us assume, for the sake of definiteness, that  $f$  is increasing on  $(a, b)$ . We take any number  $y_0$  in the domain of  $f^{-1}$  and we let  $f^{-1}(y_0) = x_0$ ; that is,  $x_0$  is the number in  $(a, b)$  such that  $f(x_0) = y_0$ . To show that  $f^{-1}$  is continuous at  $y_0$  we take any  $\varepsilon > 0$  such that the interval  $(x_0 - \varepsilon, x_0 + \varepsilon)$  is contained in the interval  $(a, b)$ . Since  $f$  is increasing, it maps the numbers in the interval  $(x_0 - \varepsilon, x_0 + \varepsilon)$  onto the numbers in the interval  $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$  and  $f^{-1}$  reverses the correspondence. If we let  $\delta$  denote the smaller of the numbers  $\delta_1 = y_0 - f(x_0 - \varepsilon)$  and  $\delta_2 = f(x_0 + \varepsilon) - y_0$ , then the interval  $(y_0 - \delta, y_0 + \delta)$  is contained in the interval  $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$  and so is mapped into the interval  $(x_0 - \varepsilon, x_0 + \varepsilon)$  by  $f^{-1}$ . (See the arrow diagram in Figure 3.) We have therefore found a number  $\delta > 0$  such that

$$\text{if } |y - y_0| < \delta \quad \text{then} \quad |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$$



**FIGURE 3**

This shows that  $\lim_{y \rightarrow y_0} f^{-1}(y) = f^{-1}(y_0)$  and so  $f^{-1}$  is continuous at any number  $y_0$  in its domain.

**Section 11.8**

In order to prove Theorem 11.8.3, we first need the following results.

**Theorem**

1. If a power series  $\sum c_n x^n$  converges when  $x = b$  (where  $b \neq 0$ ), then it converges whenever  $|x| < |b|$ .
2. If a power series  $\sum c_n x^n$  diverges when  $x = d$  (where  $d \neq 0$ ), then it diverges whenever  $|x| > |d|$ .

**PROOF OF 1** Suppose that  $\sum c_n b^n$  converges. Then, by Theorem 11.2.6, we have  $\lim_{n \rightarrow \infty} c_n b^n = 0$ . According to Definition 11.1.2 with  $\epsilon = 1$ , there is a positive integer  $N$  such that  $|c_n b^n| < 1$  whenever  $n \geq N$ . Thus, for  $n \geq N$ , we have

$$|c_n x^n| = \left| \frac{c_n b^n x^n}{b^n} \right| = |c_n b^n| \left| \frac{x}{b} \right|^n < \left| \frac{x}{b} \right|^n$$

If  $|x| < |b|$ , then  $|x/b| < 1$ , so  $\sum |x/b|^n$  is a convergent geometric series. Therefore, by the Comparison Test, the series  $\sum_{n=N}^{\infty} |c_n x^n|$  is convergent. Thus the series  $\sum c_n x^n$  is absolutely convergent and therefore convergent.

**PROOF OF 2** Suppose that  $\sum c_n d^n$  diverges. If  $x$  is any number such that  $|x| > |d|$ , then  $\sum c_n x^n$  cannot converge because, by part 1, the convergence of  $\sum c_n x^n$  would imply the convergence of  $\sum c_n d^n$ . Therefore  $\sum c_n x^n$  diverges whenever  $|x| > |d|$ .

**Theorem** For a power series  $\sum c_n x^n$  there are only three possibilities:

1. The series converges only when  $x = 0$ .
2. The series converges for all  $x$ .
3. There is a positive number  $R$  such that the series converges if  $|x| < R$  and diverges if  $|x| > R$ .

**PROOF** Suppose that neither case 1 nor case 2 is true. Then there are nonzero numbers  $b$  and  $d$  such that  $\sum c_n x^n$  converges for  $x = b$  and diverges for  $x = d$ . Therefore the set  $S = \{x \mid \sum c_n x^n \text{ converges}\}$  is not empty. By the preceding theorem, the series diverges if  $|x| > |d|$ , so  $|x| \leq |d|$  for all  $x \in S$ . This says that  $|d|$  is an upper bound for the set  $S$ . Thus, by the Completeness Axiom (see Section 11.1),  $S$  has a least upper bound  $R$ . If  $|x| > R$ , then  $x \notin S$ , so  $\sum c_n x^n$  diverges. If  $|x| < R$ , then  $|x|$  is not an upper bound for  $S$  and so there exists  $b \in S$  such that  $b > |x|$ . Since  $b \in S$ ,  $\sum c_n b^n$  converges, so by the preceding theorem  $\sum c_n x^n$  converges.

**3 Theorem** For a power series  $\sum c_n(x - a)^n$  there are only three possibilities:

1. The series converges only when  $x = a$ .
2. The series converges for all  $x$ .
3. There is a positive number  $R$  such that the series converges if  $|x - a| < R$  and diverges if  $|x - a| > R$ .

**PROOF** If we make the change of variable  $u = x - a$ , then the power series becomes  $\sum c_n u^n$  and we can apply the preceding theorem to this series. In case 3 we have convergence for  $|u| < R$  and divergence for  $|u| > R$ . Thus we have convergence for  $|x - a| < R$  and divergence for  $|x - a| > R$ . ■

### Section 14.3

**Clairaut's Theorem** Suppose  $f$  is defined on a disk  $D$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , then  $f_{xy}(a, b) = f_{yx}(a, b)$ .

**PROOF** For small values of  $h$ ,  $h \neq 0$ , consider the difference

$$\Delta(h) = [f(a + h, b + h) - f(a + h, b)] - [f(a, b + h) - f(a, b)]$$

Notice that if we let  $g(x) = f(x, b + h) - f(x, b)$ , then

$$\Delta(h) = g(a + h) - g(a)$$

By the Mean Value Theorem, there is a number  $c$  between  $a$  and  $a + h$  such that

$$g(a + h) - g(a) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)]$$

Applying the Mean Value Theorem again, this time to  $f_x$ , we get a number  $d$  between  $b$  and  $b + h$  such that

$$f_x(c, b + h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If  $h \rightarrow 0$ , then  $(c, d) \rightarrow (a, b)$ , so the continuity of  $f_{xy}$  at  $(a, b)$  gives

$$\lim_{h \rightarrow 0} \frac{\Delta(h)}{h^2} = \lim_{(c, d) \rightarrow (a, b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and using the Mean Value Theorem twice and the continuity of  $f_{yx}$  at  $(a, b)$ , we obtain

$$\lim_{h \rightarrow 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that  $f_{xy}(a, b) = f_{yx}(a, b)$ . ■

## Section 14.4

**8 Theorem** If the partial derivatives  $f_x$  and  $f_y$  exist near  $(a, b)$  and are continuous at  $(a, b)$ , then  $f$  is differentiable at  $(a, b)$ .

**PROOF** Let

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

According to (14.4.7), to prove that  $f$  is differentiable at  $(a, b)$  we have to show that we can write  $\Delta z$  in the form

$$\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where  $\varepsilon_1$  and  $\varepsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

Referring to Figure 4, we write

$$\mathbf{1} \quad \Delta z = [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)]$$

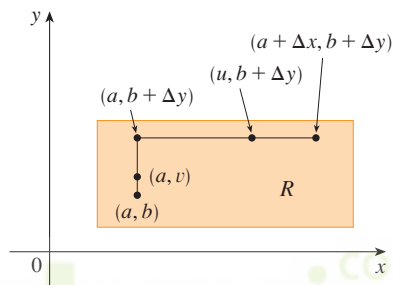


FIGURE 4

Observe that the function of a single variable

$$g(x) = f(x, b + \Delta y)$$

is defined on the interval  $[a, a + \Delta x]$  and  $g'(x) = f_x(x, b + \Delta y)$ . If we apply the Mean Value Theorem to  $g$ , we get

$$g(a + \Delta x) - g(a) = g'(u) \Delta x$$

where  $u$  is some number between  $a$  and  $a + \Delta x$ . In terms of  $f$ , this equation becomes

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = f_x(u, b + \Delta y) \Delta x$$

This gives us an expression for the first part of the right side of Equation 1. For the second part we let  $h(y) = f(a, y)$ . Then  $h$  is a function of a single variable defined on the interval  $[b, b + \Delta y]$  and  $h'(y) = f_y(a, y)$ . A second application of the Mean Value Theorem then gives

$$h(b + \Delta y) - h(b) = h'(v) \Delta y$$

where  $v$  is some number between  $b$  and  $b + \Delta y$ . In terms of  $f$ , this becomes

$$f(a, b + \Delta y) - f(a, b) = f_y(a, v) \Delta y$$



We now substitute these expressions into Equation 1 and obtain

$$\begin{aligned}\Delta z &= f_x(u, b + \Delta y) \Delta x + f_y(a, v) \Delta y \\ &= f_x(a, b) \Delta x + [f_x(u, b + \Delta y) - f_x(a, b)] \Delta x + f_y(a, b) \Delta y \\ &\quad + [f_y(a, v) - f_y(a, b)] \Delta y \\ &= f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y\end{aligned}$$

where

$$\varepsilon_1 = f_x(u, b + \Delta y) - f_x(a, b)$$

$$\varepsilon_2 = f_y(a, v) - f_y(a, b)$$

Since  $(u, b + \Delta y) \rightarrow (a, b)$  and  $(a, v) \rightarrow (a, b)$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$  and since  $f_x$  and  $f_y$  are continuous at  $(a, b)$ , we see that  $\varepsilon_1 \rightarrow 0$  and  $\varepsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

Therefore  $f$  is differentiable at  $(a, b)$ .  

## G Graphing Calculators and Computers

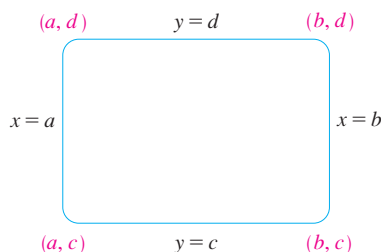
In this appendix we assume that you have access to a graphing calculator or a computer with graphing software. We see that the use of such a device enables us to graph more complicated functions and to solve more complex problems than would otherwise be possible. We also point out some of the pitfalls that can occur with these machines.

Graphing calculators and computers can give very accurate graphs of functions. But we will see in Chapter 3 that only through the use of calculus can we be sure that we have uncovered all the interesting aspects of a graph.

A graphing calculator or computer displays a rectangular portion of the graph of a function in a **display window** or **viewing screen**, which we refer to as a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care. If we choose the  $x$ -values to range from a minimum value of  $Xmin = a$  to a maximum value of  $Xmax = b$  and the  $y$ -values to range from a minimum of  $Ymin = c$  to a maximum of  $Ymax = d$ , then the visible portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

shown in Figure 1. We refer to this rectangle as the  $[a, b]$  by  $[c, d]$  *viewing rectangle*.



**FIGURE 1**

The viewing rectangle  $[a, b]$  by  $[c, d]$

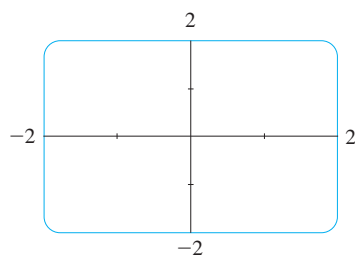
The machine draws the graph of a function  $f$  much as you would. It plots points of the form  $(x, f(x))$  for a certain number of equally spaced values of  $x$  between  $a$  and  $b$ . If an  $x$ -value is not in the domain of  $f$ , or if  $f(x)$  lies outside the viewing rectangle, it moves on to the next  $x$ -value. The machine connects each point to the preceding plotted point to form a representation of the graph of  $f$ .

**EXAMPLE 1** Draw the graph of the function  $f(x) = x^2 + 3$  in each of the following viewing rectangles.

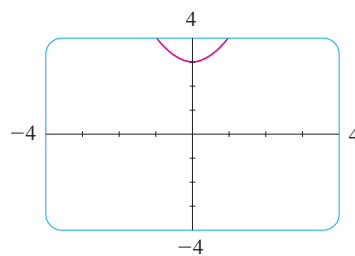
- (a)  $[-2, 2]$  by  $[-2, 2]$                       (b)  $[-4, 4]$  by  $[-4, 4]$   
 (c)  $[-10, 10]$  by  $[-5, 30]$                       (d)  $[-50, 50]$  by  $[-100, 1000]$

**SOLUTION** For part (a) we select the range by setting  $Xmin = -2$ ,  $Xmax = 2$ ,  $Ymin = -2$ , and  $Ymax = 2$ . The resulting graph is shown in Figure 2(a). The display window is blank! A moment's thought provides the explanation: Notice that  $x^2 \geq 0$  for all  $x$ , so  $x^2 + 3 \geq 3$  for all  $x$ . Thus the range of the function  $f(x) = x^2 + 3$  is  $[3, \infty)$ . This means that the graph of  $f$  lies entirely outside the viewing rectangle  $[-2, 2]$  by  $[-2, 2]$ .

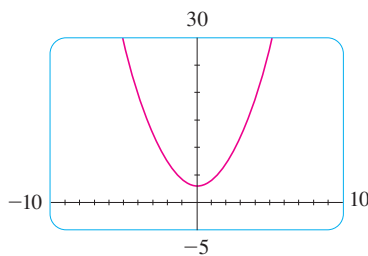
The graphs for the viewing rectangles in parts (b), (c), and (d) are also shown in Figure 2. Observe that we get a more complete picture in parts (c) and (d), but in part (d) it is not clear that the  $y$ -intercept is 3.



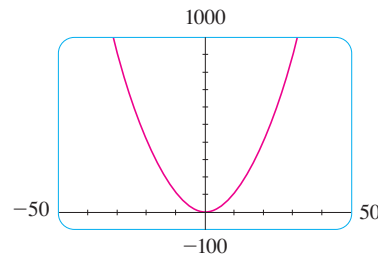
(a)  $[-2, 2]$  by  $[-2, 2]$



(b)  $[-4, 4]$  by  $[-4, 4]$



(c)  $[-10, 10]$  by  $[-5, 30]$



(d)  $[-50, 50]$  by  $[-100, 1000]$

**FIGURE 2** Graphs of  $f(x) = x^2 + 3$

We see from Example 1 that the choice of a viewing rectangle can make a big difference in the appearance of a graph. Often it's necessary to change to a larger viewing rectangle to obtain a more complete picture, a more global view, of the graph. In the next example we see that knowledge of the domain and range of a function sometimes provides us with enough information to select a good viewing rectangle.

**EXAMPLE 2** Determine an appropriate viewing rectangle for the function  $f(x) = \sqrt{8 - 2x^2}$  and use it to graph  $f$ .

**SOLUTION** The expression for  $f(x)$  is defined when

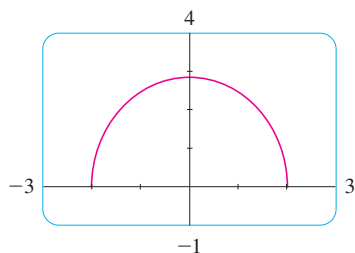
$$\begin{aligned} 8 - 2x^2 \geq 0 &\iff 2x^2 \leq 8 \iff x^2 \leq 4 \\ &\iff |x| \leq 2 \iff -2 \leq x \leq 2 \end{aligned}$$

Therefore the domain of  $f$  is the interval  $[-2, 2]$ . Also,

$$0 \leq \sqrt{8 - 2x^2} \leq \sqrt{8} = 2\sqrt{2} \approx 2.83$$

so the range of  $f$  is the interval  $[0, 2\sqrt{2}]$ .

We choose the viewing rectangle so that the  $x$ -interval is somewhat larger than the domain and the  $y$ -interval is larger than the range. Taking the viewing rectangle to be  $[-3, 3]$  by  $[-1, 4]$ , we get the graph shown in Figure 3.



**FIGURE 3**  
 $f(x) = \sqrt{8 - 2x^2}$

**EXAMPLE 3** Graph the function  $y = x^3 - 150x$ .

**SOLUTION** Here the domain is  $\mathbb{R}$ , the set of all real numbers. That doesn't help us choose a viewing rectangle. Let's experiment. If we start with the viewing rectangle  $[-5, 5]$  by

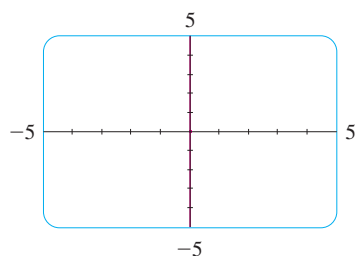


FIGURE 4

$[-5, 5]$ , we get the graph in Figure 4. It appears blank, but actually the graph is so nearly vertical that it blends with the  $y$ -axis.

If we change the viewing rectangle to  $[-20, 20]$  by  $[-20, 20]$ , we get the picture shown in Figure 5(a). The graph appears to consist of vertical lines, but we know that can't be correct. If we look carefully while the graph is being drawn, we see that the graph leaves the screen and reappears during the graphing process. This indicates that we need to see more in the vertical direction, so we change the viewing rectangle to  $[-20, 20]$  by  $[-500, 500]$ . The resulting graph is shown in Figure 5(b). It still doesn't quite reveal all the main features of the function, so we try  $[-20, 20]$  by  $[-1000, 1000]$  in Figure 5(c). Now we are more confident that we have arrived at an appropriate viewing rectangle. In Chapter 4 we will be able to see that the graph shown in Figure 5(c) does indeed reveal all the main features of the function.

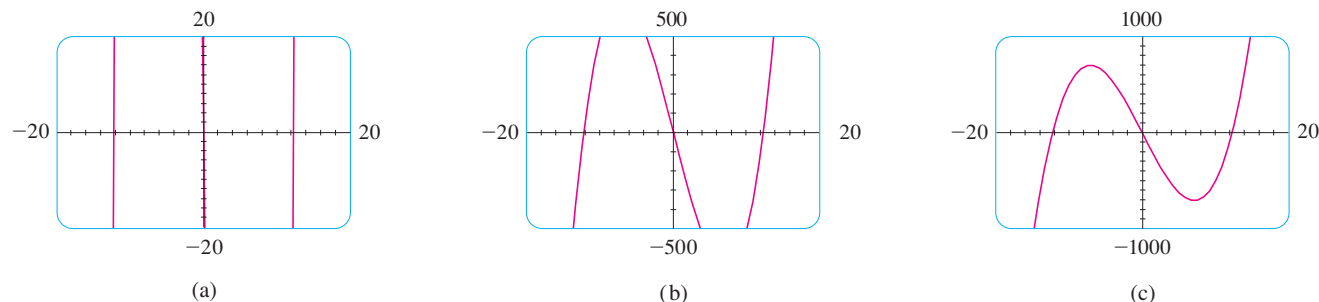


FIGURE 5 Graphs of  $y = x^3 - 150x$

**V EXAMPLE 4** Graph the function  $f(x) = \sin 50x$  in an appropriate viewing rectangle.

**SOLUTION** Figure 6(a) shows the graph of  $f$  produced by a graphing calculator using the viewing rectangle  $[-12, 12]$  by  $[-1.5, 1.5]$ . At first glance the graph appears to be reasonable. But if we change the viewing rectangle to the ones shown in the following parts of Figure 6, the graphs look very different. Something strange is happening.

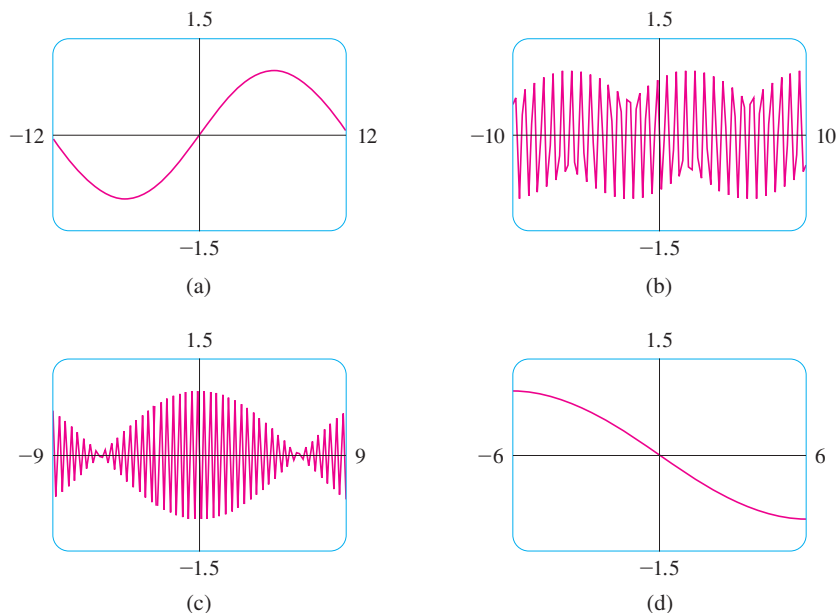
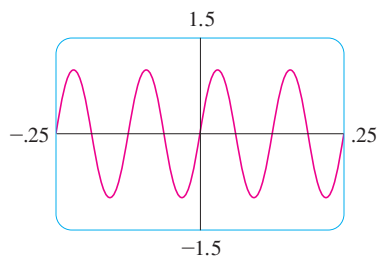


FIGURE 6  
Graphs of  $f(x) = \sin 50x$   
in four viewing rectangles

The appearance of the graphs in Figure 6 depends on the machine used. The graphs you get with your own graphing device might not look like these figures, but they will also be quite inaccurate.



**FIGURE 7**  
 $f(x) = \sin 50x$

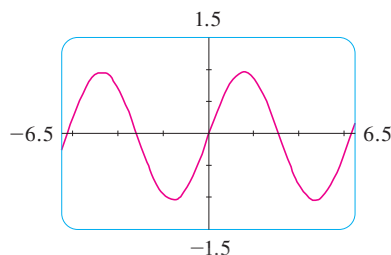
In order to explain the big differences in appearance of these graphs and to find an appropriate viewing rectangle, we need to find the period of the function  $y = \sin 50x$ . We know that the function  $y = \sin x$  has period  $2\pi$  and the graph of  $y = \sin 50x$  is shrunk horizontally by a factor of 50, so the period of  $y = \sin 50x$  is

$$\frac{2\pi}{50} = \frac{\pi}{25} \approx 0.126$$

This suggests that we should deal only with small values of  $x$  in order to show just a few oscillations of the graph. If we choose the viewing rectangle  $[-0.25, 0.25]$  by  $[-1.5, 1.5]$ , we get the graph shown in Figure 7.

Now we see what went wrong in Figure 6. The oscillations of  $y = \sin 50x$  are so rapid that when the calculator plots points and joins them, it misses most of the maximum and minimum points and therefore gives a very misleading impression of the graph.

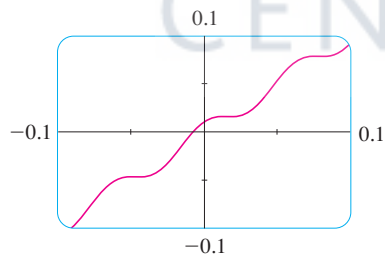
We have seen that the use of an inappropriate viewing rectangle can give a misleading impression of the graph of a function. In Examples 1 and 3 we solved the problem by changing to a larger viewing rectangle. In Example 4 we had to make the viewing rectangle smaller. In the next example we look at a function for which there is no single viewing rectangle that reveals the true shape of the graph.



**FIGURE 8**

**EXAMPLE 5** Graph the function  $f(x) = \sin x + \frac{1}{100} \cos 100x$ .

**SOLUTION** Figure 8 shows the graph of  $f$  produced by a graphing calculator with viewing rectangle  $[-6.5, 6.5]$  by  $[-1.5, 1.5]$ . It looks much like the graph of  $y = \sin x$ , but perhaps with some bumps attached. If we zoom in to the viewing rectangle  $[-0.1, 0.1]$  by  $[-0.1, 0.1]$ , we can see much more clearly the shape of these bumps in Figure 9. The reason for this behavior is that the second term,  $\frac{1}{100} \cos 100x$ , is very small in comparison with the first term,  $\sin x$ . Thus we really need two graphs to see the true nature of this function.

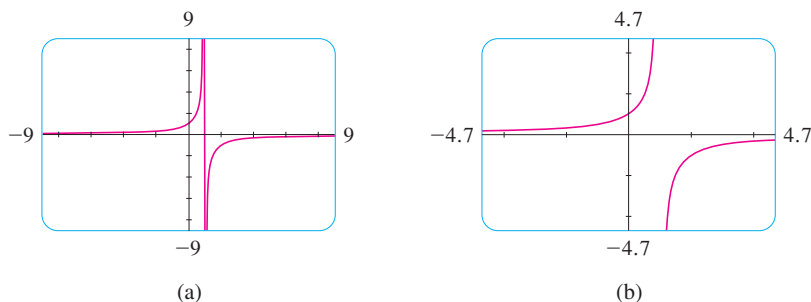


**FIGURE 9**

**EXAMPLE 6** Draw the graph of the function  $y = \frac{1}{1-x}$ .

**SOLUTION** Figure 10(a) shows the graph produced by a graphing calculator with viewing rectangle  $[-9, 9]$  by  $[-9, 9]$ . In connecting successive points on the graph, the calculator produced a steep line segment from the top to the bottom of the screen. That line segment is not truly part of the graph. Notice that the domain of the function  $y = 1/(1-x)$  is  $\{x \mid x \neq 1\}$ . We can eliminate the extraneous near-vertical line by experimenting with a change of scale. When we change to the smaller viewing rectangle  $[-4.7, 4.7]$  by  $[-4.7, 4.7]$  on this particular calculator, we obtain the much better graph in Figure 10(b).

Another way to avoid the extraneous line is to change the graphing mode on the calculator so that the dots are not connected.



**FIGURE 10**

**EXAMPLE 7** Graph the function  $y = \sqrt[3]{x}$ .

**SOLUTION** Some graphing devices display the graph shown in Figure 11, whereas others produce a graph like that in Figure 12. We know from Section 1.2 (Figure 13) that the graph in Figure 12 is correct, so what happened in Figure 11? The explanation is that some machines compute the cube root of  $x$  using a logarithm, which is not defined if  $x$  is negative, so only the right half of the graph is produced.

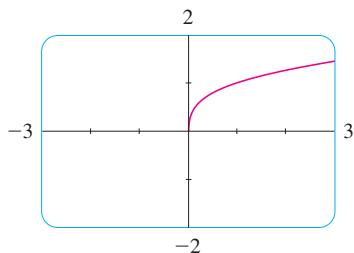


FIGURE 11

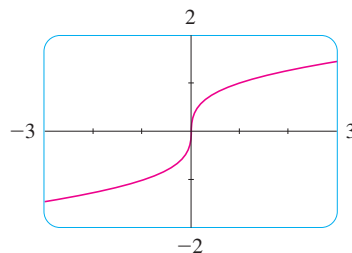


FIGURE 12

You can get the correct graph with Maple if you first type

```
with(RealDomain);
```

You should experiment with your own machine to see which of these two graphs is produced. If you get the graph in Figure 11, you can obtain the correct picture by graphing the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$$

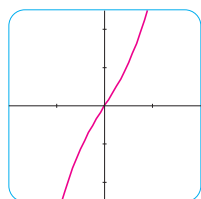
Notice that this function is equal to  $\sqrt[3]{x}$  (except when  $x = 0$ ).

To understand how the expression for a function relates to its graph, it's helpful to graph a **family of functions**, that is, a collection of functions whose equations are related. In the next example we graph members of a family of cubic polynomials.

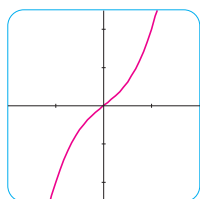
**V EXAMPLE 8** Graph the function  $y = x^3 + cx$  for various values of the number  $c$ . How does the graph change when  $c$  is changed?

**SOLUTION** Figure 13 shows the graphs of  $y = x^3 + cx$  for  $c = 2, 1, 0, -1,$  and  $-2$ . We see that, for positive values of  $c$ , the graph increases from left to right with no maximum or minimum points (peaks or valleys). When  $c = 0$ , the curve is flat at the origin. When  $c$  is negative, the curve has a maximum point and a minimum point. As  $c$  decreases, the maximum point becomes higher and the minimum point lower.

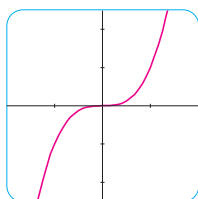
**TEC** In Visual G you can see an animation of Figure 13.



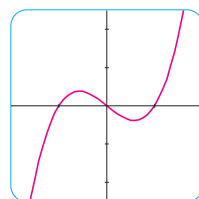
(a)  $y = x^3 + 2x$



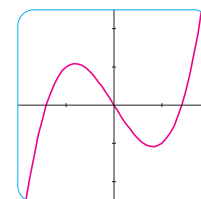
(b)  $y = x^3 + x$



(c)  $y = x^3$



(d)  $y = x^3 - x$



(e)  $y = x^3 - 2x$

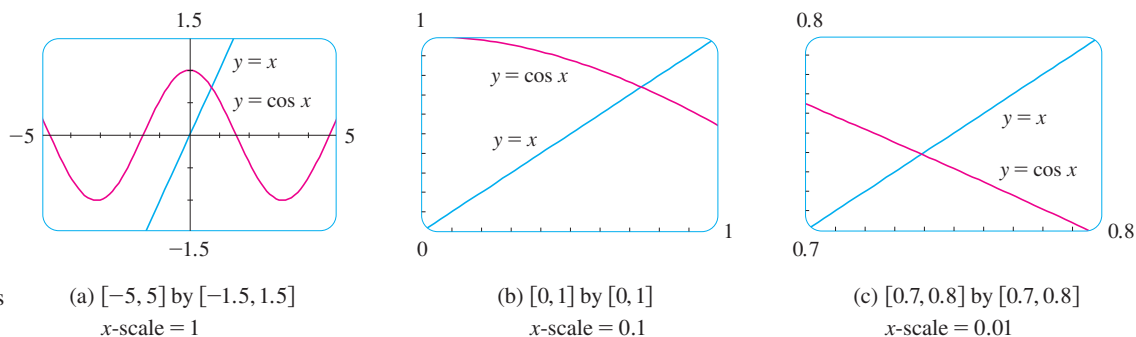
FIGURE 13

Several members of the family of functions  $y = x^3 + cx$ , all graphed in the viewing rectangle  $[-2, 2]$  by  $[-2.5, 2.5]$

**EXAMPLE 9** Find the solution of the equation  $\cos x = x$  correct to two decimal places.

**SOLUTION** The solutions of the equation  $\cos x = x$  are the  $x$ -coordinates of the points of intersection of the curves  $y = \cos x$  and  $y = x$ . From Figure 14(a) we see that there is

only one solution and it lies between 0 and 1. Zooming in to the viewing rectangle  $[0, 1]$  by  $[0, 1]$ , we see from Figure 14(b) that the root lies between 0.7 and 0.8. So we zoom in further to the viewing rectangle  $[0.7, 0.8]$  by  $[0.7, 0.8]$  in Figure 14(c). By moving the cursor to the intersection point of the two curves, or by inspection and the fact that the  $x$ -scale is 0.01, we see that the solution of the equation is about 0.74. (Many calculators have a built-in intersection feature.)

**FIGURE 14**

Locating the roots of  $\cos x = x$

(a)  $[-5, 5]$  by  $[-1.5, 1.5]$   
 $x$ -scale = 1

(b)  $[0, 1]$  by  $[0, 1]$   
 $x$ -scale = 0.1

(c)  $[0.7, 0.8]$  by  $[0.7, 0.8]$   
 $x$ -scale = 0.01

**G****Exercises**

- Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function  $f(x) = \sqrt{x^3 - 5x^2}$ .
  - $[-5, 5]$  by  $[-5, 5]$
  - $[0, 10]$  by  $[0, 2]$
  - $[0, 10]$  by  $[0, 10]$
- Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function  $f(x) = x^4 - 16x^2 + 20$ .
  - $[-3, 3]$  by  $[-3, 3]$
  - $[-10, 10]$  by  $[-10, 10]$
  - $[-50, 50]$  by  $[-50, 50]$
  - $[-5, 5]$  by  $[-50, 50]$
- 3–14 Determine an appropriate viewing rectangle for the given function and use it to draw the graph.
  - $f(x) = x^2 - 36x + 32$
  - $f(x) = x^3 + 15x^2 + 65x$
  - $f(x) = \sqrt{50 - 0.2x}$
  - $f(x) = \sqrt{15x - x^2}$
  - $f(x) = x^3 - 225x$
  - $f(x) = \frac{x}{x^2 + 100}$
  - $f(x) = \sin^2(1000x)$
  - $f(x) = \cos(0.001x)$
  - $f(x) = \sin \sqrt{x}$
  - $f(x) = \sec(20\pi x)$
  - $y = 10 \sin x + \sin 100x$
  - $y = x^2 + 0.02 \sin 50x$
- (a) Try to find an appropriate viewing rectangle for  $f(x) = (x - 10)^3 2^{-x}$ .  
(b) Do you need more than one window? Why?
- Graph the function  $f(x) = x^2 \sqrt{30 - x}$  in an appropriate viewing rectangle. Why does part of the graph appear to be missing?
- Graph the ellipse  $4x^2 + 2y^2 = 1$  by graphing the functions whose graphs are the upper and lower halves of the ellipse.
- Graph the hyperbola  $y^2 - 9x^2 = 1$  by graphing the functions whose graphs are the upper and lower branches of the hyperbola.
- 19–20 Do the graphs intersect in the given viewing rectangle? If they do, how many points of intersection are there?
  - $y = 3x^2 - 6x + 1$ ,  $y = 0.23x - 2.25$ ;  
 $[-1, 3]$  by  $[-2.5, 1.5]$
  - $y = 6 - 4x - x^2$ ,  $y = 3x + 18$ ;  $[-6, 2]$  by  $[-5, 20]$
- 21–23 Find all solutions of the equation correct to two decimal places.
  - $x^4 - x = 1$
  - $\sqrt{x} = x^3 - 1$
  - $\tan x = \sqrt{1 - x^2}$
- We saw in Example 9 that the equation  $\cos x = x$  has exactly one solution.
  - Use a graph to show that the equation  $\cos x = 0.3x$  has three solutions and find their values correct to two decimal places.



Graphing calculator or computer required

1. Homework Hints available at [stewartcalculus.com](http://stewartcalculus.com)

## A54 APPENDIX G GRAPHING CALCULATORS AND COMPUTERS

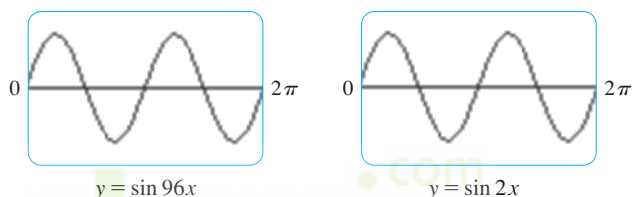
- (b) Find an approximate value of  $m$  such that the equation  $\cos x = mx$  has exactly two solutions.
25. Use graphs to determine which of the functions  $f(x) = 10x^2$  and  $g(x) = x^3/10$  is eventually larger (that is, larger when  $x$  is very large).
26. Use graphs to determine which of the functions  $f(x) = x^4 - 100x^3$  and  $g(x) = x^3$  is eventually larger.
27. For what values of  $x$  is it true that  $|\tan x - x| < 0.01$  and  $-\pi/2 < x < \pi/2$ ?
28. Graph the polynomials  $P(x) = 3x^5 - 5x^3 + 2x$  and  $Q(x) = 3x^5$  on the same screen, first using the viewing rectangle  $[-2, 2]$  by  $[-2, 2]$  and then changing to  $[-10, 10]$  by  $[-10,000, 10,000]$ . What do you observe from these graphs?
29. In this exercise we consider the family of root functions  $f(x) = \sqrt[n]{x}$ , where  $n$  is a positive integer.
- (a) Graph the functions  $y = \sqrt{x}$ ,  $y = \sqrt[4]{x}$ , and  $y = \sqrt[6]{x}$  on the same screen using the viewing rectangle  $[-1, 4]$  by  $[-1, 3]$ .
- (b) Graph the functions  $y = x$ ,  $y = \sqrt[3]{x}$ , and  $y = \sqrt[5]{x}$  on the same screen using the viewing rectangle  $[-3, 3]$  by  $[-2, 2]$ . (See Example 7.)
- (c) Graph the functions  $y = \sqrt{x}$ ,  $y = \sqrt[3]{x}$ ,  $y = \sqrt[4]{x}$ , and  $y = \sqrt[5]{x}$  on the same screen using the viewing rectangle  $[-1, 3]$  by  $[-1, 2]$ .
- (d) What conclusions can you make from these graphs?
30. In this exercise we consider the family of functions  $f(x) = 1/x^n$ , where  $n$  is a positive integer.
- (a) Graph the functions  $y = 1/x$  and  $y = 1/x^3$  on the same screen using the viewing rectangle  $[-3, 3]$  by  $[-3, 3]$ .
- (b) Graph the functions  $y = 1/x^2$  and  $y = 1/x^4$  on the same screen using the same viewing rectangle as in part (a).
- (c) Graph all of the functions in parts (a) and (b) on the same screen using the viewing rectangle  $[-1, 3]$  by  $[-1, 3]$ .
- (d) What conclusions can you make from these graphs?
31. Graph the function  $f(x) = x^4 + cx^2 + x$  for several values of  $c$ . How does the graph change when  $c$  changes?
32. Graph the function  $f(x) = \sqrt{1 + cx^2}$  for various values of  $c$ . Describe how changing the value of  $c$  affects the graph.
33. Graph the function  $y = x^n 2^{-x}$ ,  $x \geq 0$ , for  $n = 1, 2, 3, 4, 5$ , and 6. How does the graph change as  $n$  increases?

34. The curves with equations

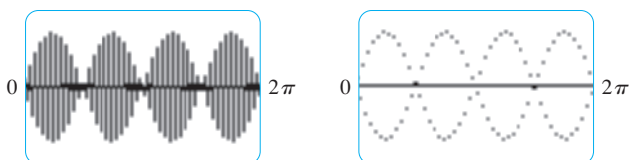
$$y = \frac{|x|}{\sqrt{c - x^2}}$$

are called **bullet-nose curves**. Graph some of these curves to see why. What happens as  $c$  increases?

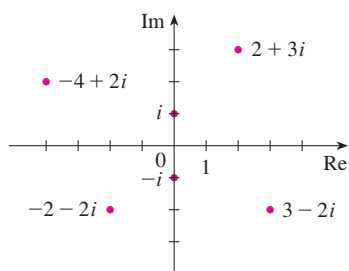
35. What happens to the graph of the equation  $y^2 = cx^3 + x^2$  as  $c$  varies?
36. This exercise explores the effect of the inner function  $g$  on a composite function  $y = f(g(x))$ .
- (a) Graph the function  $y = \sin(\sqrt{x})$  using the viewing rectangle  $[0, 400]$  by  $[-1.5, 1.5]$ . How does this graph differ from the graph of the sine function?
- (b) Graph the function  $y = \sin(x^2)$  using the viewing rectangle  $[-5, 5]$  by  $[-1.5, 1.5]$ . How does this graph differ from the graph of the sine function?
37. The figure shows the graphs of  $y = \sin 96x$  and  $y = \sin 2x$  as displayed by a TI-83 graphing calculator. The first graph is inaccurate. Explain why the two graphs appear identical. [Hint: The TI-83's graphing window is 95 pixels wide. What specific points does the calculator plot?]



38. The first graph in the figure is that of  $y = \sin 45x$  as displayed by a TI-83 graphing calculator. It is inaccurate and so, to help explain its appearance, we replot the curve in dot mode in the second graph. What two sine curves does the calculator appear to be plotting? Show that each point on the graph of  $y = \sin 45x$  that the TI-83 chooses to plot is in fact on one of these two curves. (The TI-83's graphing window is 95 pixels wide.)



## H Complex Numbers



**FIGURE 1**  
Complex numbers as points in the Argand plane

A **complex number** can be represented by an expression of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is a symbol with the property that  $i^2 = -1$ . The complex number  $a + bi$  can also be represented by the ordered pair  $(a, b)$  and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus the complex number  $i = 0 + 1 \cdot i$  is identified with the point  $(0, 1)$ .

The **real part** of the complex number  $a + bi$  is the real number  $a$  and the **imaginary part** is the real number  $b$ . Thus the real part of  $4 - 3i$  is 4 and the imaginary part is  $-3$ . Two complex numbers  $a + bi$  and  $c + di$  are **equal** if  $a = c$  and  $b = d$ , that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi) - (c + di) &= (a - c) + (b - d)i\end{aligned}$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di) \\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since  $i^2 = -1$ , this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

### EXAMPLE 1

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= (-1)(2 - 5i) + 3i(2 - 5i) \\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$

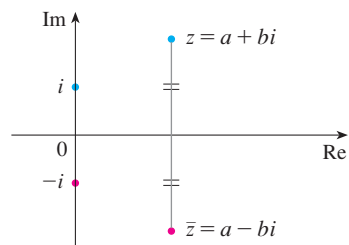
Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number  $z = a + bi$ , we define its **complex conjugate** to be  $\bar{z} = a - bi$ . To find the quotient of two complex numbers we multiply numerator and denominator by the complex conjugate of the denominator.

**EXAMPLE 2** Express the number  $\frac{-1 + 3i}{2 + 5i}$  in the form  $a + bi$ .

**SOLUTION** We multiply numerator and denominator by the complex conjugate of  $2 + 5i$ , namely  $2 - 5i$ , and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29}i$$

The geometric interpretation of the complex conjugate is shown in Figure 2:  $\bar{z}$  is the reflection of  $z$  in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.



**FIGURE 2**



## Properties of Conjugates

$$\overline{z + w} = \bar{z} + \bar{w} \quad \overline{zw} = \bar{z}\bar{w} \quad \overline{z^n} = \bar{z}^n$$

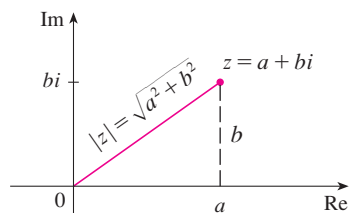


FIGURE 3

The **modulus**, or **absolute value**,  $|z|$  of a complex number  $z = a + bi$  is its distance from the origin. From Figure 3 we see that if  $z = a + bi$ , then

$$|z| = \sqrt{a^2 + b^2}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

and so

$$z\bar{z} = |z|^2$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Since  $i^2 = -1$ , we can think of  $i$  as a square root of  $-1$ . But notice that we also have  $(-i)^2 = i^2 = -1$  and so  $-i$  is also a square root of  $-1$ . We say that  $i$  is the **principal square root** of  $-1$  and write  $\sqrt{-1} = i$ . In general, if  $c$  is any positive number, we write

$$\sqrt{-c} = \sqrt{c} i$$

With this convention, the usual derivation and formula for the roots of the quadratic equation  $ax^2 + bx + c = 0$  are valid even when  $b^2 - 4ac < 0$ :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**EXAMPLE 3** Find the roots of the equation  $x^2 + x + 1 = 0$ .

**SOLUTION** Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3} i}{2}$$

We observe that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation  $ax^2 + bx + c = 0$  with real coefficients  $a$ ,  $b$ , and  $c$  are always complex conjugates. (If  $z$  is real,  $\bar{z} = z$ , so  $z$  is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.

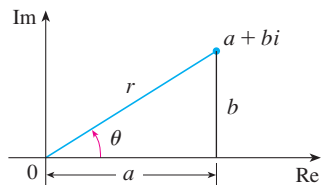


FIGURE 4

### Polar Form

We know that any complex number  $z = a + bi$  can be considered as a point  $(a, b)$  and that any such point can be represented by polar coordinates  $(r, \theta)$  with  $r \geq 0$ . In fact,

$$a = r \cos \theta \quad b = r \sin \theta$$

as in Figure 4. Therefore we have

$$z = a + bi = (r \cos \theta) + (r \sin \theta)i$$

Thus we can write any complex number  $z$  in the form

$$z = r(\cos \theta + i \sin \theta)$$

where  $r = |z| = \sqrt{a^2 + b^2}$  and  $\tan \theta = \frac{b}{a}$

The angle  $\theta$  is called the **argument** of  $z$  and we write  $\theta = \arg(z)$ . Note that  $\arg(z)$  is not unique; any two arguments of  $z$  differ by an integer multiple of  $2\pi$ .

**EXAMPLE 4** Write the following numbers in polar form.

(a)  $z = 1 + i$

(b)  $w = \sqrt{3} - i$

**SOLUTION**

(a) We have  $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$  and  $\tan \theta = 1$ , so we can take  $\theta = \pi/4$ . Therefore the polar form is

$$z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

(b) Here we have  $r = |w| = \sqrt{3 + 1} = 2$  and  $\tan \theta = -1/\sqrt{3}$ . Since  $w$  lies in the fourth quadrant, we take  $\theta = -\pi/6$  and

$$w = 2 \left[ \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right]$$

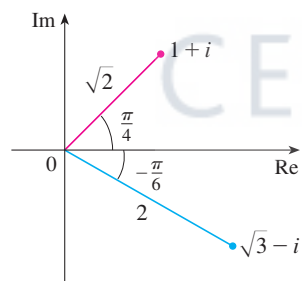


FIGURE 5

The numbers  $z$  and  $w$  are shown in Figure 5.

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

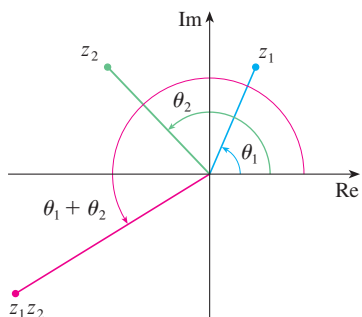


FIGURE 6

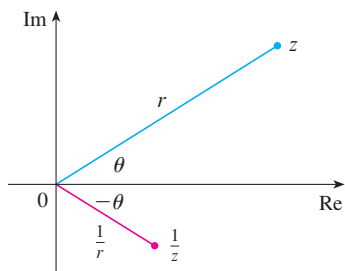


FIGURE 7

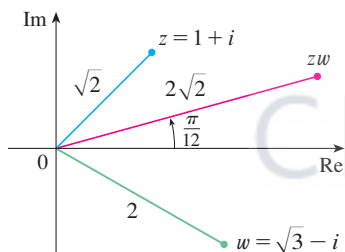


FIGURE 8

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments.* (See Figure 6.)

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments.*

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \quad z_2 \neq 0$$

In particular, taking  $z_1 = 1$  and  $z_2 = z$  (and therefore  $\theta_1 = 0$  and  $\theta_2 = \theta$ ), we have the following, which is illustrated in Figure 7.

If  $z = r(\cos \theta + i \sin \theta)$ , then  $\frac{1}{z} = \frac{1}{r}(\cos \theta - i \sin \theta)$ .

**EXAMPLE 5** Find the product of the complex numbers  $1 + i$  and  $\sqrt{3} - i$  in polar form.

**SOLUTION** From Example 4 we have

$$1 + i = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

and

$$\sqrt{3} - i = 2 \left[ \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right]$$

So, by Equation 1,

$$\begin{aligned} (1 + i)(\sqrt{3} - i) &= 2\sqrt{2} \left[ \cos \left( \frac{\pi}{4} - \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{4} - \frac{\pi}{6} \right) \right] \\ &= 2\sqrt{2} \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right) \end{aligned}$$

This is illustrated in Figure 8.

Repeated use of Formula 1 shows how to compute powers of a complex number. If

$$z = r(\cos \theta + i \sin \theta)$$

then

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$$

and

$$z^3 = zz^2 = r^3(\cos 3\theta + i \sin 3\theta)$$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

**2 De Moivre's Theorem** If  $z = r(\cos \theta + i \sin \theta)$  and  $n$  is a positive integer, then

$$z^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

This says that *to take the  $n$ th power of a complex number we take the  $n$ th power of the modulus and multiply the argument by  $n$ .*

**EXAMPLE 6** Find  $(\frac{1}{2} + \frac{1}{2}i)^{10}$ .

**SOLUTION** Since  $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$ , it follows from Example 4(a) that  $\frac{1}{2} + \frac{1}{2}i$  has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

So by De Moivre's Theorem,

$$\begin{aligned} \left( \frac{1}{2} + \frac{1}{2}i \right)^{10} &= \left( \frac{\sqrt{2}}{2} \right)^{10} \left( \cos \frac{10\pi}{4} + i \sin \frac{10\pi}{4} \right) \\ &= \frac{2^5}{2^{10}} \left( \cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2} \right) = \frac{1}{32} i \end{aligned}$$

De Moivre's Theorem can also be used to find the  $n$ th roots of complex numbers. An  $n$ th root of the complex number  $z$  is a complex number  $w$  such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

$$w = s(\cos \phi + i \sin \phi) \quad \text{and} \quad z = r(\cos \theta + i \sin \theta)$$

and using De Moivre's Theorem, we get

$$s^n(\cos n\phi + i \sin n\phi) = r(\cos \theta + i \sin \theta)$$

The equality of these two complex numbers shows that

$$s^n = r \quad \text{or} \quad s = r^{1/n}$$

and  $\cos n\phi = \cos \theta$  and  $\sin n\phi = \sin \theta$

From the fact that sine and cosine have period  $2\pi$  it follows that

$$n\phi = \theta + 2k\pi \quad \text{or} \quad \phi = \frac{\theta + 2k\pi}{n}$$

Thus  $w = r^{1/n} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right]$

Since this expression gives a different value of  $w$  for  $k = 0, 1, 2, \dots, n - 1$ , we have the following.

**3 Roots of a Complex Number** Let  $z = r(\cos \theta + i \sin \theta)$  and let  $n$  be a positive integer. Then  $z$  has the  $n$  distinct  $n$ th roots

$$w_k = r^{1/n} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right]$$

where  $k = 0, 1, 2, \dots, n - 1$ .

Notice that each of the  $n$ th roots of  $z$  has modulus  $|w_k| = r^{1/n}$ . Thus all the  $n$ th roots of  $z$  lie on the circle of radius  $r^{1/n}$  in the complex plane. Also, since the argument of each successive  $n$ th root exceeds the argument of the previous root by  $2\pi/n$ , we see that the  $n$ th roots of  $z$  are equally spaced on this circle.

**EXAMPLE 7** Find the six sixth roots of  $z = -8$  and graph these roots in the complex plane.

**SOLUTION** In trigonometric form,  $z = 8(\cos \pi + i \sin \pi)$ . Applying Equation 3 with  $n = 6$ , we get

$$w_k = 8^{1/6} \left( \cos \frac{\pi + 2k\pi}{6} + i \sin \frac{\pi + 2k\pi}{6} \right)$$

We get the six sixth roots of  $-8$  by taking  $k = 0, 1, 2, 3, 4, 5$  in this formula:

$$w_0 = 8^{1/6} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \sqrt{2} \left( \frac{\sqrt{3}}{2} + \frac{1}{2} i \right)$$

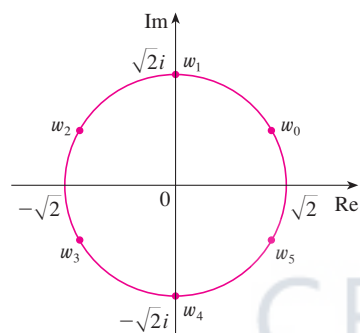
$$w_1 = 8^{1/6} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = \sqrt{2} i$$

$$w_2 = 8^{1/6} \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \sqrt{2} \left( -\frac{\sqrt{3}}{2} + \frac{1}{2} i \right)$$

$$w_3 = 8^{1/6} \left( \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) = \sqrt{2} \left( -\frac{\sqrt{3}}{2} - \frac{1}{2} i \right)$$

$$w_4 = 8^{1/6} \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = -\sqrt{2} i$$

$$w_5 = 8^{1/6} \left( \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right) = \sqrt{2} \left( \frac{\sqrt{3}}{2} - \frac{1}{2} i \right)$$



**FIGURE 9**  
The six sixth roots of  $z = -8$

All these points lie on the circle of radius  $\sqrt{2}$  as shown in Figure 9. ■

### Complex Exponentials

We also need to give a meaning to the expression  $e^z$  when  $z = x + iy$  is a complex number. The theory of infinite series as developed in Chapter 11 can be extended to the case where the terms are complex numbers. Using the Taylor series for  $e^x$  (11.10.11) as our guide, we define

$$\boxed{4} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

$$\boxed{5} \quad e^{z_1+z_2} = e^{z_1} e^{z_2}$$

If we put  $z = iy$ , where  $y$  is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \dots$$

$$\begin{aligned}
 \text{we get } e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\
 &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\
 &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\
 &= \cos y + i \sin y
 \end{aligned}$$

Here we have used the Taylor series for  $\cos y$  and  $\sin y$  (Equations 11.10.16 and 11.10.15). The result is a famous formula called **Euler's formula**:

6

$$e^{iy} = \cos y + i \sin y$$

Combining Euler's formula with Equation 5, we get

7

$$e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

**EXAMPLE 8** Evaluate: (a)  $e^{i\pi}$  (b)  $e^{-1+i\pi/2}$

**SOLUTION**

We could write the result of Example 8(a) as

$$e^{i\pi} + 1 = 0$$

This equation relates the five most famous numbers in all of mathematics: 0, 1,  $e$ ,  $i$ , and  $\pi$ .

(a) From Euler's equation [6] we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = \frac{1}{e} [0 + i(1)] = \frac{i}{e}$$

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos \theta + i \sin \theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

## H Exercises

**1–14** Evaluate the expression and write your answer in the form  $a + bi$ .

1.  $(5 - 6i) + (3 + 2i)$

2.  $(4 - \frac{1}{2}i) - (9 + \frac{5}{2}i)$

3.  $(2 + 5i)(4 - i)$

4.  $(1 - 2i)(8 - 3i)$

5.  $\overline{12 + 7i}$

6.  $\overline{2i(\frac{1}{2} - i)}$

7.  $\frac{1 + 4i}{3 + 2i}$

8.  $\frac{3 + 2i}{1 - 4i}$

9.  $\frac{1}{1 + i}$

10.  $\frac{3}{4 - 3i}$

11.  $i^3$

12.  $i^{100}$

13.  $\sqrt{-25}$

14.  $\sqrt{-3} \sqrt{-12}$

**15–17** Find the complex conjugate and the modulus of the number.

15.  $12 - 5i$

16.  $-1 + 2\sqrt{2}i$

17.  $-4i$

**18.** Prove the following properties of complex numbers.

(a)  $\overline{z + w} = \overline{z} + \overline{w}$  (b)  $\overline{zw} = \overline{z} \overline{w}$

(c)  $\overline{z^n} = \overline{z}^n$ , where  $n$  is a positive integer

[Hint: Write  $z = a + bi$ ,  $w = c + di$ .]

**A62 APPENDIX H** COMPLEX NUMBERS**19–24** Find all solutions of the equation.

**19.**  $4x^2 + 9 = 0$

**20.**  $x^4 = 1$

**21.**  $x^2 + 2x + 5 = 0$

**22.**  $2x^2 - 2x + 1 = 0$

**23.**  $z^2 + z + 2 = 0$

**24.**  $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

**25–28** Write the number in polar form with argument between 0 and  $2\pi$ .

**25.**  $-3 + 3i$

**26.**  $1 - \sqrt{3}i$

**27.**  $3 + 4i$

**28.**  $8i$

**29–32** Find polar forms for  $zw$ ,  $z/w$ , and  $1/z$  by first putting  $z$  and  $w$  into polar form.

**29.**  $z = \sqrt{3} + i$ ,  $w = 1 + \sqrt{3}i$

**30.**  $z = 4\sqrt{3} - 4i$ ,  $w = 8i$

**31.**  $z = 2\sqrt{3} - 2i$ ,  $w = -1 + i$

**32.**  $z = 4(\sqrt{3} + i)$ ,  $w = -3 - 3i$

**33–36** Find the indicated power using De Moivre's Theorem.

**33.**  $(1 + i)^{20}$

**34.**  $(1 - \sqrt{3}i)^5$

**35.**  $(2\sqrt{3} + 2i)^5$

**36.**  $(1 - i)^8$

**37–40** Find the indicated roots. Sketch the roots in the complex plane.**37.** The eighth roots of 1**38.** The fifth roots of 32**39.** The cube roots of  $i$ **40.** The cube roots of  $1 + i$ **41–46** Write the number in the form  $a + bi$ .

**41.**  $e^{i\pi/2}$

**42.**  $e^{2\pi i}$

**43.**  $e^{i\pi/3}$

**44.**  $e^{-i\pi}$

**45.**  $e^{2+i\pi}$

**46.**  $e^{\pi+i}$

**47.** Use De Moivre's Theorem with  $n = 3$  to express  $\cos 3\theta$  and  $\sin 3\theta$  in terms of  $\cos \theta$  and  $\sin \theta$ .**48.** Use Euler's formula to prove the following formulas for  $\cos x$  and  $\sin x$ :

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

**49.** If  $u(x) = f(x) + ig(x)$  is a complex-valued function of a real variable  $x$  and the real and imaginary parts  $f(x)$  and  $g(x)$  are differentiable functions of  $x$ , then the derivative of  $u$  is defined to be  $u'(x) = f'(x) + ig'(x)$ . Use this together with Equation 7 to prove that if  $F(x) = e^{rx}$ , then  $F'(x) = re^{rx}$  when  $r = a + bi$  is a complex number.**50.** (a) If  $u$  is a complex-valued function of a real variable, its indefinite integral  $\int u(x) dx$  is an antiderivative of  $u$ . Evaluate

$$\int e^{(1+i)x} dx$$

(b) By considering the real and imaginary parts of the integral in part (a), evaluate the real integrals

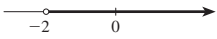
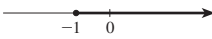
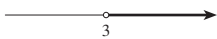

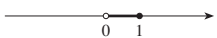
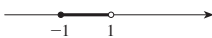
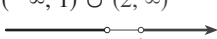


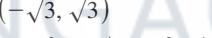

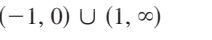
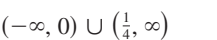
$$\int e^x \cos x dx \quad \text{and} \quad \int e^x \sin x dx$$

(c) Compare with the method used in Example 4 in Section 7.1.

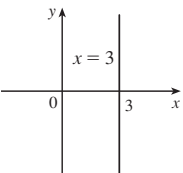
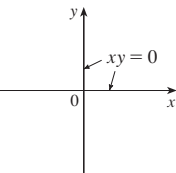
This page contains answers for this chapter only

**APPENDIXES**

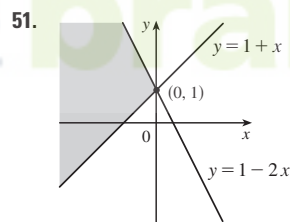
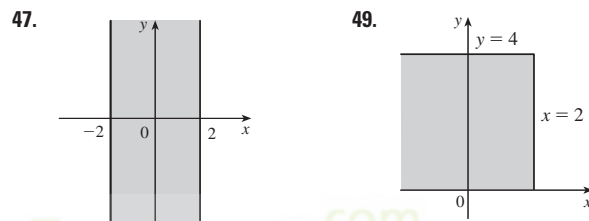
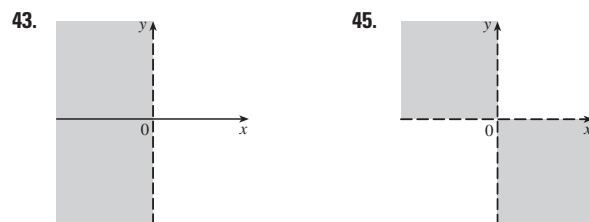
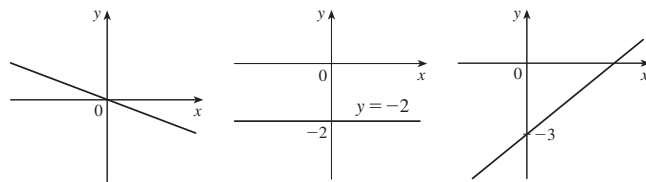
**EXERCISES A ■ PAGE A9**

1. 18    3.  $\pi$     5.  $5 - \sqrt{5}$     7.  $2 - x$
9.  $|x + 1| = \begin{cases} x + 1 & \text{for } x \geq -1 \\ -x - 1 & \text{for } x < -1 \end{cases}$     11.  $x^2 + 1$
13.  $(-2, \infty)$     15.  $[-1, \infty)$
- 
- 
17.  $(3, \infty)$     19.  $(2, 6)$
- 
- 
21.  $(0, 1]$     23.  $[-1, \frac{1}{2})$
- 
- 
25.  $(-\infty, 1) \cup (2, \infty)$     27.  $[-1, \frac{1}{2})$
- 
- 
29.  $(-\infty, \infty)$     31.  $(-\sqrt{3}, \sqrt{3})$
- 
- 
33.  $(-\infty, 1]$     35.  $(-1, 0) \cup (1, \infty)$
- 
- 
37.  $(-\infty, 0) \cup (\frac{1}{4}, \infty)$
- 
39.  $10 \leq C \leq 35$     41. (a)  $T = 20 - 10h, 0 \leq h \leq 12$   
 (b)  $-30^\circ\text{C} \leq T \leq 20^\circ\text{C}$     43.  $\pm \frac{3}{2}$     45.  $2, -\frac{4}{3}$
47.  $(-3, 3)$     49.  $(3, 5)$     51.  $(-\infty, -7] \cup [-3, \infty)$
53.  $[1.3, 1.7]$     55.  $[-4, -1] \cup [1, 4]$
57.  $x \geq (a + b)c/(ab)$     59.  $x > (c - b)/a$

**EXERCISES B ■ PAGE A15**

1. 5    3.  $\sqrt{74}$     5.  $2\sqrt{37}$     7. 2    9.  $-\frac{9}{2}$
17. 
19. 

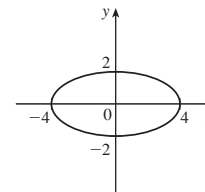
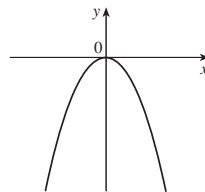
21.  $y = 6x - 15$     23.  $2x - 3y + 19 = 0$
25.  $5x + y = 11$     27.  $y = 3x - 2$     29.  $y = 3x - 3$
31.  $y = 5$     33.  $x + 2y + 11 = 0$     35.  $5x - 2y + 1 = 0$
37.  $m = -\frac{1}{3}, b = 0$     39.  $m = 0, b = -2$     41.  $m = \frac{3}{4}, b = -3$



53.  $(0, -4)$     55. (a)  $(4, 9)$     (b)  $(3.5, -3)$     57.  $(1, -2)$
59.  $y = x - 3$     61. (b)  $4x - 3y - 24 = 0$

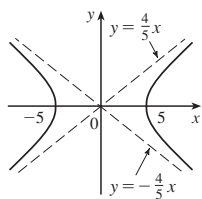
**EXERCISES C ■ PAGE A23**

1.  $(x - 3)^2 + (y + 1)^2 = 25$     3.  $x^2 + y^2 = 65$
5.  $(2, -5), 4$     7.  $(-\frac{1}{2}, 0), \frac{1}{2}$     9.  $(\frac{1}{4}, -\frac{1}{4}), \sqrt{10}/4$
11. Parabola    13. Ellipse

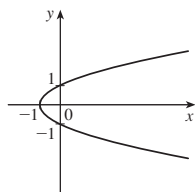




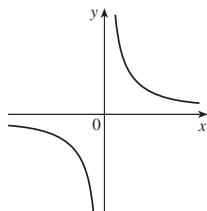
**15. Hyperbola**



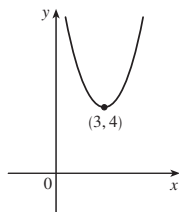
**19. Parabola**



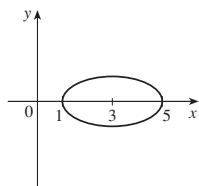
**23. Hyperbola**



**27. Parabola**

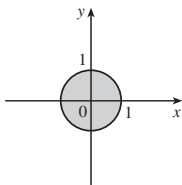


**31. Ellipse**

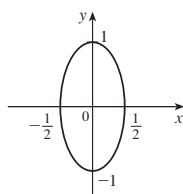


**35.**  $y = x^2 - 2x$

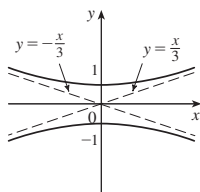
**37.**



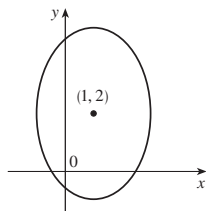
**17. Ellipse**



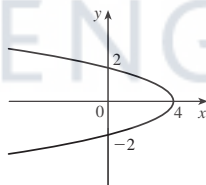
**21. Hyperbola**



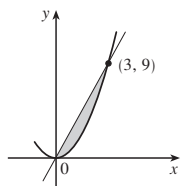
**25. Ellipse**



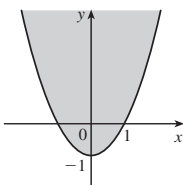
**29. Parabola**



**33.**



**39.**

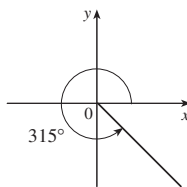


**EXERCISES D ■ PAGE A32**

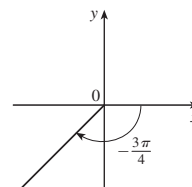
**1.**  $7\pi/6$     **3.**  $\pi/20$     **5.**  $5\pi$     **7.**  $720^\circ$     **9.**  $75^\circ$

**11.**  $-67.5^\circ$     **13.**  $3\pi$  cm    **15.**  $\frac{2}{3}$  rad =  $(120/\pi)^\circ$

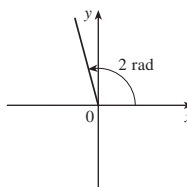
**17.**



**19.**



**21.**



**23.**  $\sin(3\pi/4) = 1/\sqrt{2}$ ,  $\cos(3\pi/4) = -1/\sqrt{2}$ ,  $\tan(3\pi/4) = -1$ ,  $\csc(3\pi/4) = \sqrt{2}$ ,  $\sec(3\pi/4) = -\sqrt{2}$ ,  $\cot(3\pi/4) = -1$

**25.**  $\sin(9\pi/2) = 1$ ,  $\cos(9\pi/2) = 0$ ,  $\csc(9\pi/2) = 1$ ,  $\cot(9\pi/2) = 0$ ,  $\tan(9\pi/2)$  and  $\sec(9\pi/2)$  undefined

**27.**  $\sin(5\pi/6) = \frac{1}{2}$ ,  $\cos(5\pi/6) = -\sqrt{3}/2$ ,  $\tan(5\pi/6) = -1/\sqrt{3}$ ,  $\csc(5\pi/6) = 2$ ,  $\sec(5\pi/6) = -2/\sqrt{3}$ ,  $\cot(5\pi/6) = -\sqrt{3}$

**29.**  $\cos \theta = \frac{4}{5}$ ,  $\tan \theta = \frac{3}{4}$ ,  $\csc \theta = \frac{5}{3}$ ,  $\sec \theta = \frac{5}{4}$ ,  $\cot \theta = \frac{4}{3}$

**31.**  $\sin \phi = \sqrt{5}/3$ ,  $\cos \phi = -\frac{2}{3}$ ,  $\tan \phi = -\sqrt{5}/2$ ,  $\csc \phi = 3/\sqrt{5}$ ,  $\cot \phi = -2/\sqrt{5}$

**33.**  $\sin \beta = -1/\sqrt{10}$ ,  $\cos \beta = -3/\sqrt{10}$ ,  $\tan \beta = \frac{1}{3}$ ,  $\csc \beta = -\sqrt{10}$ ,  $\sec \beta = -\sqrt{10}/3$

**35.** 5.73576 cm    **37.** 24.62147 cm    **59.**  $\frac{1}{15}(4 + 6\sqrt{2})$

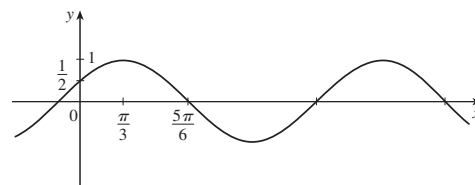
**61.**  $\frac{1}{15}(3 + 8\sqrt{2})$     **63.**  $\frac{24}{25}$     **65.**  $\pi/3, 5\pi/3$

**67.**  $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$     **69.**  $\pi/6, \pi/2, 5\pi/6, 3\pi/2$

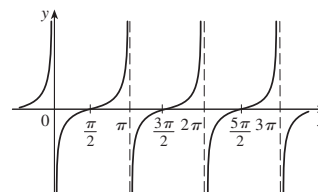
**71.**  $0, \pi, 2\pi$     **73.**  $0 \leq x \leq \pi/6$  and  $5\pi/6 \leq x \leq 2\pi$

**75.**  $0 \leq x < \pi/4$ ,  $3\pi/4 < x < 5\pi/4$ ,  $7\pi/4 < x \leq 2\pi$

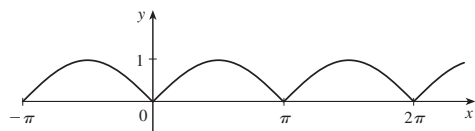
**77.**



**79.**



**81.**



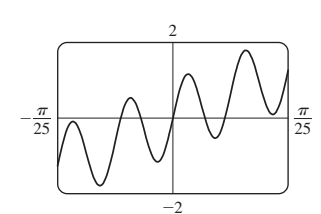
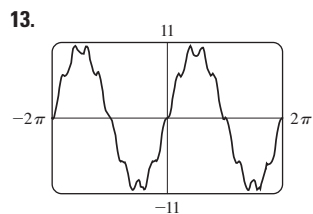
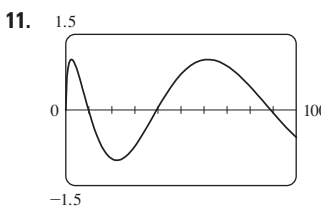
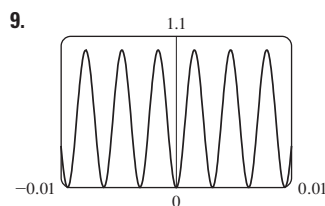
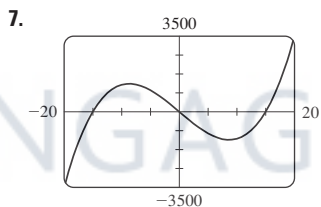
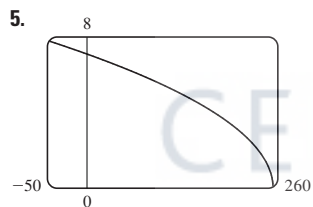
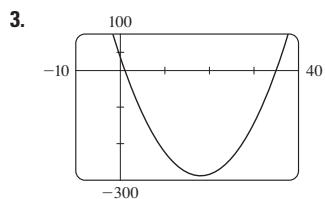
**89.** 14.34457 cm<sup>2</sup>

**EXERCISES E ■ PAGE A38**

1.  $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$     3.  $3^4 + 3^5 + 3^6$   
 5.  $-1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$     7.  $1^{10} + 2^{10} + 3^{10} + \dots + n^{10}$   
 9.  $1 - 1 + 1 - 1 + \dots + (-1)^{n-1}$     11.  $\sum_{i=1}^{10} i$   
 13.  $\sum_{i=1}^{19} \frac{i}{i+1}$     15.  $\sum_{i=1}^n 2i$     17.  $\sum_{i=0}^5 2^i$     19.  $\sum_{i=1}^n x^i$   
 21. 80    23. 3276    25. 0    27. 61    29.  $n(n+1)$   
 31.  $n(n^2 + 6n + 17)/3$     33.  $n(n^2 + 6n + 11)/3$   
 35.  $n(n^3 + 2n^2 - n - 10)/4$   
 41. (a)  $n^4$     (b)  $5^{100} - 1$     (c)  $\frac{97}{300}$     (d)  $a_n - a_0$   
 43.  $\frac{1}{3}$     45. 14    49.  $2^{n+1} + n^2 + n - 2$

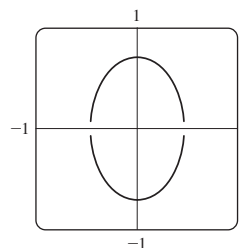
**EXERCISES G ■ PAGE A53**

1. (c)



15. (b) Yes; two are needed

17.

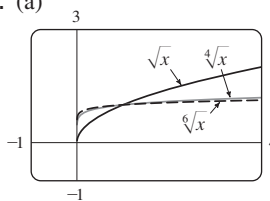


19. No    21. -0.72, 1.22

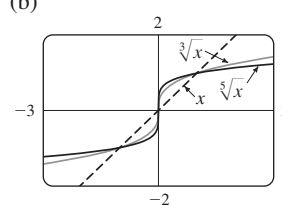
23. 0.65    25.  $g$

27.  $-0.31 < x < 0.31$

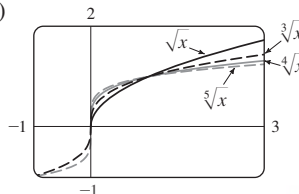
29. (a)



(b)

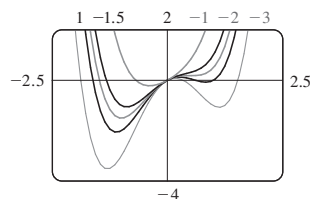


(c)



(d) Graphs of even roots are similar to  $\sqrt{x}$ , graphs of odd roots are similar to  $\sqrt[3]{x}$ . As  $n$  increases, the graph of  $y = \sqrt[n]{x}$  becomes steeper near 0 and flatter for  $x > 1$ .

31.



If  $c < -1.5$ , the graph has three humps: two minimum points and a maximum point. These humps get flatter as  $c$  increases until at  $c = -1.5$  two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as  $c$  increases.

33. The hump gets larger and moves to the right.

35. If  $c < 0$ , the loop is to the right of the origin; if  $c > 0$ , the loop is to the left. The closer  $c$  is to 0, the larger the loop.

**EXERCISES H ■ PAGE A61**

1.  $8 - 4i$     3.  $13 + 18i$     5.  $12 - 7i$     7.  $\frac{11}{13} + \frac{10}{13}i$   
 9.  $\frac{1}{2} - \frac{1}{2}i$     11.  $-i$     13.  $5i$     15.  $12 + 5i, 13$

**A134 APPENDIX I ANSWERS TO ODD-NUMBERED EXERCISES**

17.  $4i, 4$     19.  $\pm \frac{3}{2}i$     21.  $-1 \pm 2i$

23.  $-\frac{1}{2} \pm (\sqrt{7}/2)i$     25.  $3\sqrt{2} [\cos(3\pi/4) + i \sin(3\pi/4)]$

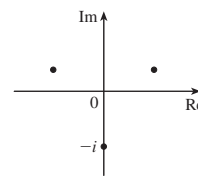
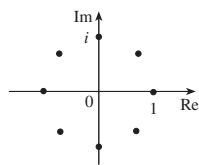
27.  $5\{\cos[\tan^{-1}(\frac{4}{3})] + i \sin[\tan^{-1}(\frac{4}{3})]\}$

29.  $4[\cos(\pi/2) + i \sin(\pi/2)], \cos(-\pi/6) + i \sin(-\pi/6),$   
 $\frac{1}{2}[\cos(-\pi/6) + i \sin(-\pi/6)]$

31.  $4\sqrt{2} [\cos(7\pi/12) + i \sin(7\pi/12)],$   
 $(2\sqrt{2})[\cos(13\pi/12) + i \sin(13\pi/12)], \frac{1}{4}[\cos(\pi/6) + i \sin(\pi/6)]$

33.  $-1024$     35.  $-512\sqrt{3} + 512i$

37.  $\pm 1, \pm i, (1/\sqrt{2})(\pm 1 \pm i)$     39.  $\pm(\sqrt{3}/2) + \frac{1}{2}i, -i$



41.  $i$     43.  $\frac{1}{2} + (\sqrt{3}/2)i$     45.  $-e^2$

47.  $\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta,$

$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$

CENGAGE **brain**.com

# Index

RP denotes Reference Page numbers.

Numbers followed by asterisks denote pages in Sections 6.2\*, 6.3\*, and 6.4\*.

- Abel, Niels, 160  
 absolute maximum and minimum values, 198, 970, 975  
 absolute value, 16, A6, A54  
 absolute value function, 16  
 absolutely convergent series, 756  
 acceleration as a rate of change, 121, 164  
 acceleration of a particle, 887  
   components of, 890  
   as a vector, 887  
 Achilles and the tortoise, 5  
 adaptive numerical integration, 539  
 addition formulas for sine and cosine, A29  
 addition of vectors, 816, 819  
 Airy, Sir George, 770  
 Airy function, 770  
 algebraic function, 30  
 alternating harmonic series, 753, 756  
 alternating series, 751  
 Alternating Series Estimation Theorem, 754  
 Alternating Series Test, 751  
 analytic geometry, A10  
 angle(s), A24  
   between curves, 195  
   of deviation, 207  
   negative or positive, A25  
   between planes, 845  
   standard position, A25  
   between vectors, 825, 826  
 angular momentum, 895  
 angular speed, 888  
 antiderivative, 269  
 antidifferentiation formulas, 270  
 aphelion, 707  
 apolune, 701  
 approach path of an aircraft, 156  
 approximate integration, 530  
 approximating cylinder, 354  
 approximating surface, 570  
 approximation  
   by differentials, 185  
   to  $e$ , 396  
   linear, 183, 941, 945  
   linear, to a tangent plane, 941  
   by the Midpoint Rule, 302, 531  
   by Newton's method, 263  
   by an  $n$ th-degree Taylor polynomial, 189  
   quadratic, 189  
   by Riemann sums, 296  
   by Simpson's Rule, 535, 537  
   tangent line, 183  
   by Taylor polynomials, 792  
   by Taylor's Inequality, 780, 793  
   by the Trapezoidal Rule, 532  
 Archimedes, 329  
 Archimedes' Principle, 381, 1158  
 arc curvature, 877  
 arc length, 562, 878  
   of a parametric curve, 672  
   of a polar curve, 691  
   of a space curve, 877, 878  
 arc length contest, 569  
 arc length formula, 563  
 arc length function, 565  
 arcsine function, 453  
 area, 2, 284  
   of a circle, 504  
   under a curve, 284, 289, 295  
   between curves, 344, 345  
   of an ellipse, 503  
   by exhaustion, 2, 64  
   by Green's Theorem, 1111  
   enclosed by a parametric curve, 671  
   in polar coordinates, 678, 689  
   of a sector of a circle, 689  
   surface, 674, 1038, 1128, 1130  
   of a surface of a revolution, 569, 575  
 area function, 309  
 area problem, 2, 284  
 argument of a complex number, A55  
 arithmetic-geometric mean, 726  
 arrow diagram, 11  
 astroid, 162, 669  
 asymptote(s), 238  
   in graphing, 238  
   horizontal, 225, 238  
   of a hyperbola, 698, A20  
   slant, 238, 241  
   vertical, 57, 238  
 asymptotic curve, 244  
 autonomous differential equation, 613  
 auxiliary equation, 1167  
   complex roots of, 1169  
   real roots of, 1168  
 average cost function, 259  
 average rate of change, 108, 164, 886  
 average speed of molecules, 552  
 average value of a function, 373, 374, 594, 1003, 1051  
 average velocity, 4, 47, 106, 164  
 axes, coordinate, 810, A11  
 axes of an ellipse, A19  
 axis of a parabola, 694  
 bacterial growth, 629, 634  
 Barrow, Isaac, 3, 64, 114, 310, 329  
 base of a cylinder, 352  
 base of a logarithm, 404, 441\*  
   change of, 406, 442\*  
 baseball and calculus, 376  
 basis vectors, 820  
 Bernoulli, James, 618, 645  
 Bernoulli, John, 470, 480, 618, 664, 778  
 Bernoulli differential equation, 645  
 Bessel, Friedrich, 766  
 Bessel function, 163, 766, 770  
 Bézier, Pierre, 677  
 Bézier curves, 663, 677  
 binomial coefficients, 784  
 binomial series, 784  
   discovery by Newton, 791  
 binomial theorem, 128, RP1  
 binormal vector, 882  
 blackbody radiation, 801  
 blood flow, 170, 261, 588  
 boundary curve, 1146  
 boundary-value problem, 1171  
 bounded sequence, 721  
 bounded set, 975  
 Boyle's Law, 174  
 brachistochrone problem, 664  
 Brahe, Tycho, 891  
 branches of a hyperbola, 698, A20  
 Buffon's needle problem, 602  
 bullet-nose curve, 154

**A136 INDEX**

- $C^1$  transformation, 1064  
 cable (hanging), 463  
 calculator, graphing, 244, 662, 685, A46. *See*  
     also computer algebra system  
 calculus, 8  
     differential, 3  
     integral, 2, 3  
     invention of, 8, 329  
 cancellation equations  
     for inverse functions, 386  
     for inverse trigonometric functions, 386,  
     454  
     for logarithms, 404, 441\*  
 cans, minimizing manufacturing cost of, 262  
 Cantor, Georg, 737  
 Cantor set, 737  
 capital formation, 591  
 cardiac output, 589  
 cardioid, 162, 682  
 carrying capacity, 176, 217, 605, 631  
 Cartesian coordinate system, A11  
 Cartesian plane, A11  
 Cassini, Giovanni, 689  
 CAS. *See* computer algebra system  
 catenary, 463  
 Cauchy, Augustin-Louis, 76, 1008  
 Cauchy-Schwarz Inequality, 831  
 Cauchy's Mean Value Theorem, 476  
 Cavalieri, 537  
 Cavalieri's Principle, 362  
 center of gravity. *See* center of mass  
 center of mass, 578, 579, 1028, 1089  
     of a lamina, 1029  
     of a plate, 581  
     of a solid, 1047  
     of a surface, 1136  
     of a wire, 1089  
 centripetal force, 899  
 centroid of a plane region, 580  
 centroid of a solid, 1047  
 Chain Rule, 148, 151  
     for several variables, 948, 950, 951  
 change of base, formula for, 407, 442\*  
 change of variable(s)  
     in a double integral, 1023, 1065, 1068  
     in integration, 330  
     in a triple integral, 1053, 1058, 1070  
 characteristic equation, 1167  
 charge, electric, 167, 1027, 1028, 1047, 1184  
 charge density, 1028, 1047  
 chemical reaction, 167  
 circle, area of, 504  
 circle, equation of, A16  
 circle of curvature, 883  
 circular cylinder, 352  
 circular paraboloid, 856  
 circulation of a vector field, 1150  
 cissoid of Diocles, 668, 687  
 Clairaut, Alexis, 931  
 Clairaut's Theorem, 931  
 clipping planes, 850  
 closed curve, 1101  
 closed interval, A3  
 Closed Interval Method, 202  
     for a function of two variables, 976  
 closed set, 975  
 closed surface, 1140  
 Cobb, Charles, 903  
 Cobb-Douglas production function, 904,  
     934, 987  
 cochleoid, 710  
 coefficient(s)  
     binomial, 784  
     of friction, 147, 205  
     of inequality, 351  
     of a polynomial, 27  
     of a power series, 765  
     of static friction, 861  
 combinations of functions, 39  
 comets, orbits of, 708  
 common ratio, 729  
 comparison properties of the integral, 305  
 comparison test for improper integrals, 549  
 Comparison Test for series, 746  
 Comparison Theorem for integrals, 549  
 complementary equation, 1173  
 Completeness Axiom, 722  
 complex conjugate, A53  
 complex exponentials, A58  
 complex number(s), A53  
     addition and subtraction of, A53  
     argument of, A55  
     division of, A53, A56  
     equality of, A53  
     imaginary part of, A53  
     modulus of, A54  
     multiplication of, A53, A56  
     polar form, A55  
     powers of, A57  
     principal square root of, A54  
     real part of, A3  
     roots of, A58  
 component function, 864, 1081  
 components of acceleration, 890  
 components of a vector, 817, 828  
 composition of functions, 40, 148  
     continuity of, 89, 922  
     derivative of, 150  
 compound interest, 450, 479  
 compressibility, 168  
 computer algebra system, 53, 526, 662  
     for integration, 526, 775  
     pitfalls of using, 53  
 computer algebra system, graphing with, A46  
     a curve, 244  
     function of two variables, 906  
     level curves, 910  
     parametric equations, 662  
     parametric surface, 1126  
     partial derivatives, 931  
     polar curve, 685  
     sequence, 719  
     space curve, 867  
     vector field, 1082  
 concavity, 216  
 Concavity Test, 217, A43  
 concentration, 168  
 conchoid, 665, 687  
 conditionally convergent series, 757  
 conductivity (of a substance), 1144  
 cone, 694, 854  
     parametrization of, 1126  
 conic section, 694, 702  
     directrix, 694, 702  
     eccentricity, 702  
     focus, 694, 696, 702  
     polar equation, 704  
     shifted, 699, A21  
     vertex (vertices), 694  
 conjugates, properties of, A54  
 connected region, 1101  
 conservation of energy, 1105  
 conservative vector field, 1085, 1106  
 constant force, 368, 829  
 constant function, 126  
 Constant Multiple Law of limits, 62  
 Constant Multiple Rule, 128  
 constraint, 981, 985  
 consumer surplus, 587, 588  
 continued fraction expansion, 726  
 continuity  
     of a function, 81, 865  
     of a function of three variables, 922  
     of a function of two variables, 920  
     on an interval, 84  
     from the left or right, 83  
     continuous compounding of  
         interest, 450, 479  
 continuous random variable, 592  
 contour curves, 907  
 contour map, 907, 933  
 convergence  
     absolute, 756  
     conditional, 757  
     of an improper integral, 544, 547  
     interval of, 767  
     radius of, 767  
     of a sequence, 716  
     of a series, 729  
 convergent improper integral, 544, 547  
 convergent sequence, 716  
 convergent series, 729  
     properties of, 733  
 conversion, cylindrical to rectangular  
     coordinates, 1052  
 cooling tower, hyperbolic, 856  
 coordinate axes, 810, A11  
 coordinate planes, 810  
 coordinate system, A2  
     Cartesian, A11

- cylindrical, 1052
- polar, 678
- retangular, A11
- spherical, 1057
- three-dimensional rectangular, 810
- coplanar vectors, 837
- Coriolis acceleration, 898
- Cornu's spiral, 676
- cosine function, A26
  - derivative of, 143
  - graph of, 31, A31
  - power series for, 782, 74
- cost function, 171, 255
- critical number, 201
- critical point(s), 970, 980
- critically damped vibration, 1182
- cross product, 832
  - direction of, 834
  - geometric characterization of, 835
  - magnitude of, 835
  - properties of, 836
- cross-section, 352
  - of a surface, 851
- cubic function, 27
- curl of a vector field, 1115
- current, 167
- curvature, 677, 879
- curve(s)
  - asymptotic, 244
  - Bézier, 663, 677
  - boundary, 1146
  - bullet-nose, 154
  - cissock of Diocles, 687
  - closed, 1101
  - Cornu's spiral, 676
  - demand, 587
  - devil's, 162
  - dog saddle, 915
  - epicycloid, 669
  - equipotential, 914
  - grid, 1124
  - helix, 865
  - length of, 562, 877
  - level, 907
  - monkey saddle, 915
  - orientation of, 1092, 1108
  - orthogonal, 163
  - ovals of Cassini, 689
  - parametric, 660, 865
  - piecewise-smooth, 1088
  - polar, 680
  - serpentine, 137
  - simple, 1102
  - smooth, 562
  - space, 864, 865
  - strophoid, 693, 711
  - swallowtail catastrophe, 668
  - toroidal spiral, 867
  - trochoid, 667
  - twisted cubic, 867
  - witch of Maria Agnesi, 667
- curve fitting, 25
- curve-sketching procedure, 237
- cusps, 665
- cycloid, 663
- cylinder, 352, 851
  - parabolic, 851
  - parametrization of, 1126
- cylindrical coordinate system, 1052
  - conversion equations for, 1052
  - triple integrals in, 1053
- cylindrical coordinates, 1054
- cylindrical shell, 363
- damped vibration, 1181
- damping constant, 1181
- decay, law of natural, 446
- decay, radioactive, 448
- decreasing function, 19
- decreasing sequence, 720
- definite integral, 296, 998
  - properties of, 303
  - Substitution Rule for, 333
  - of a vector function, 875
- definite integration
  - by parts, 512, 514, 515
  - by substitution, 333
- degree of a polynomial, 27
- del ( $\nabla$ ), 960
- delta ( $\Delta$ ) notation, 108
- demand curve, 256, 587
- demand function, 255, 587
- De Moivre, Abraham, A56
- De Moivre's Theorem, A56
- density
  - of a lamina, 1027
  - linear, 166, 167, 324
  - liquid, 577
  - mass vs. weight, 577
  - of a solid, 1047
- dependent variable, 10, 902, 950
- derivative(s), 104, 107, 114, 188
  - of a composite function, 148
  - of a constant function, 126
  - directional, 957, 958, 961
  - domain of, 114
  - of exponential functions, 395, 397
  - as a function, 114
  - higher, 120
  - higher partial, 930
  - of hyperbolic functions, 464
  - of an integral, 311
  - of an inverse function, 410, 411
  - of inverse trigonometric functions, 453
  - left-hand, 126
  - of logarithmic functions, 411, 421\*, 424\*
  - normal, 1122
  - notation, 117
  - notation for partial, 927
  - partial, 926
- of a polynomial, 126
- of a power function, 127, 133
- of a power series, 772
- of a product, 130
- of a quotient, 132
- as a rate of change, 104
- right-hand, 126
- second, 120, 874
- second directional, 968
- second partial, 930
- as the slope of a tangent, 104, 107
- third, 121
- of trigonometric functions, 140, 144
- of a vector function, 871
- Descartes, René, A11
- descent of aircraft, determining start of, 156
- determinant, 832
- devil's curve, 162
- Difference Law of limits, 62
- difference quotient, 12
- Difference Rule, 129
- differentiable function, 117, 942
- differential, 185, 943, 945
- differential calculus, 3
- differential equation, 139, 270, 271, 446, 603, 604, 606
  - autonomous, 613
  - Bernoulli, 645
  - family of solutions, 604, 607
  - first-order, 606
  - general solution of, 607
  - homogeneous, 1166
  - linear, 640
  - linearly independent solutions, 1167
  - logistic, 631, 727
  - nonhomogeneous, 1166, 1173
  - order of, 606
  - partial, 932
  - second-order, 606, 1166
  - separable, 618
  - solution of, 606
- differentiation, 117
  - formulas for, 136, RP5
  - formulas for vector functions, 874
  - implicit, 157, 158, 929, 952
  - logarithmic, 416, 427\*
  - partial, 924, 929, 930
  - of a power series, 772
  - term-by-term, 772
  - of a vector function, 874
- differentiation operator, 117
- Direct Substitution Property, 64
- directed line segment, 815
- direction field, 609, 610
- direction numbers, 842
- directional derivative, 957, 958, 961
  - maximum value of, 962
  - of a temperature function, 957, 958
  - second, 958
- directrix, 694, 702

**A138 INDEX**

- discontinuity, 82, 83  
 discontinuous function, 82  
 discontinuous integrand, 547  
 disk method for approximating volume, 354  
 dispersion, 207  
 displacement, 106, 325  
 displacement vector, 815, 829  
 distance  
   between lines, 847  
   between planes, 847  
   between point and line in space, 839  
   between point and plane, 839  
   between points in a plane, A11  
   between points in space, 812  
   between real numbers, A7  
 distance formula, A12  
   in three dimensions, 812  
 distance problem, 291  
 divergence  
   of an improper integral, 544, 547  
   of an infinite series, 729  
   of a sequence, 716  
   of a vector field, 1118  
 Divergence, Test for, 733  
 Divergence Theorem, 1153  
 divergent improper integral, 544, 547  
 divergent sequence, 716  
 divergent series, 729  
 division of power series, 787  
 DNA, helical shape of, 866  
 dog saddle, 915  
 domain of a function, 10, 902  
 domain sketching, 902  
 Doppler effect, 956  
 dot product, 824  
   in component form, 824  
   properties of, 825  
 double-angle formulas, A29  
 double integral, 998, 1000  
   change of variable in, 1065, 1068  
   over general regions, 1012, 1013  
   Midpoint Rule for, 1002  
   in polar coordinates, 1021, 1022, 1023  
   properties of, 1005, 1017  
   over rectangles, 998  
 double Riemann sum, 1001  
 Douglas, Paul, 903  
 Dumpster design, minimizing cost of, 980  
 dye dilution method, 589  
  
 $e$  (the number), 396, 423\*  
   as a limit, 417, 443\*  
   as a sum of an infinite series, 781  
 eccentricity, 702  
 electric charge, 1027, 1028, 1047  
 electric circuit, 617, 620, 643  
   analysis of, 1184  
 electric current to a flash bulb, 46  
 electric field (force per unit charge), 1084  
 electric flux, 1143  
  
 electric force, 1084  
 elementary function, integrability of, 522  
 element of a set, A3  
 ellipse, 165, 696, 702, A19  
   area, 503  
   directrix, 702  
   eccentricity, 702  
   foci, 696, 702  
   major axis, 696, 707  
   minor axis, 696  
   polar equation, 704, 707  
   reflection property, 697  
   rotated, 163  
   vertices, 696  
 ellipsoid, 852, 854  
 elliptic paraboloid, 852, 854  
 empirical model, 25  
 end behavior of a function, 236  
 endpoint extreme values, 199  
 energy  
   conservation of, 1105  
   kinetic, 377, 1105  
   potential, 1105  
 epicycloid, 669  
 epitrochoid, 676  
 equation(s)  
   cancellation, 386  
   of a circle, A17  
   differential (*see* differential equation)  
   of an ellipse, 696, 704, A19  
   of a graph, A16, A17  
   heat conduction, 937  
   of a hyperbola, 697, 698, 699, 704, A20  
   Laplace's, 932, 1119  
   of a line, A12, A13, A14, A16  
   of a line in space, 840, 841  
   of a line through two points, 842  
   linear, 844, A14  
   logistic difference, 727  
   logistic differential, 605, 639  
   Lotka-Volterra, 647  
    $n$ th-degree, 160  
   of a parabola, 694, 704, A18  
   parametric, 660, 841, 865, 1123  
   of a plane, 843  
   of a plane through three points, 845  
   point-slope, A12  
   polar, 680, 704  
   predator-prey, 646, 647  
   second-degree, A16  
   slope-intercept, A13  
   of a space curve, 865  
   of a sphere, 813  
   symmetric, 842  
   two-intercept form, A16  
   van der Waals, 938  
   vector, 840  
   wave, 932  
 equilateral hyperbola, A21  
 equilibrium point, 648  
  
 equilibrium solution, 605, 647  
 equipotential curves, 914  
 equivalent vectors, 816  
 error  
   in approximate integration, 532, 533  
   percentage, 187  
   relative, 187  
   in Taylor approximation, 793  
 error bounds, 534, 538  
 error estimate  
   for alternating series, 754  
   for the Midpoint Rule, 532, 533  
   for Simpson's Rule, 538  
   for the Trapezoidal Rule, 532, 533  
 error function, 403  
 escape velocity, 552  
 estimate of the sum of a series, 742, 749,  
   754, 759  
 Euclid, 64  
 Eudoxus, 2, 64, 329  
 Euler, Leonhard, 55, 613, 739, 745, 781  
 Euler's formula, A59  
 Euler's Method, 613, 614  
 even function, 17, 238  
 expected values, 1035  
 exponential decay, 446  
 exponential function(s), 32, 391, 397, RP4  
   with base  $a$ , 391, 415, 437\*  
   derivative, of 395, 415, 432\*, 438\*  
   graphs of, 393, 398, 439\*  
   integration of, 301, 331, 400, 786, 787  
   limits of, 394, 399, 431\*  
   power series for, 779  
   properties of, 394, 431\*  
 exponential graph, 398  
 exponential growth, 398, 634  
 exponents, laws of, 394, 431\*, 437\*  
 extrapolation, 26  
 extreme value, 198  
 Extreme Value Theorem, 199, 975  
  
 family  
   of epicycloids and hypocycloids, 668  
   of exponential functions, 393  
   of functions, 247, 249, A50  
   of parametric curves, 664  
   of solutions, 604, 607  
 fat circles, 161, 568  
 Fermat, Pierre, 3, 114, 200, 329, A11  
 Fermat's Principle, 260  
 Fermat's Theorem, 200  
 Fibonacci, 715, 726  
 Fibonacci sequence, 715, 726  
 field  
   conservative, 1085  
   electric, 1084  
   force, 1084  
   gradient, 966, 1084  
   gravitational, 1084  
   incompressible, 1119

- irrotational, 1118
- scalar, 1081
- vector, 1080, 1081
- velocity, 1080, 1083
- First Derivative Test, 215
  - for Absolute Extreme Values, 253
- first octant, 810
- first-order linear differential equation, 606, 640
- first-order optics, 798
- fixed point of a function, 104, 213
- flash bulb, current to, 46
- flow lines, 1086
- fluid flow, 1083, 1119, 1142
- flux, 588, 589, 1141, 1143
- flux integral, 1141
- FM synthesis, 247
- foci, 696
- focus, 694, 702
  - of a conic section, 702
  - of an ellipse, 696, 702
  - of a hyperbola, 697
  - of a parabola, 694
- folium of Descartes, 157, 711
- force, 368
  - centripetal, 899
  - constant, 368, 829
  - exerted by fluid, 576, 577
  - resultant, 821
  - torque, 837
- force field, 1080, 1084
- forced vibrations, 1183
- Fourier, Joseph, 173
- Fourier series, finite, 502
- four-leaved rose, 682
- fractions (partial), 508, 509
- Frenet-Serret formulas, 886
- Fresnel, Augustin, 313
- Fresnel function, 313
- frustum, 361, 362
- Fubini, Guido, 1008
- Fubini's Theorem, 1008, 1041
- function(s), 10, 902
  - absolute value, 16
  - Airy, 770
  - algebraic, 30
  - arc length, 565, 877
  - arcsine, 453
  - area, 309
  - arrow diagram of, 11
  - average cost, 259
  - average value of, 374, 594, 1003, 1051
  - Bessel, 163, 766, 770
  - Cobb-Douglas production, 904, 934, 987
  - combinations of, 39
  - component, 864, 1081
  - composite, 40, 148, 922
  - constant, 126
  - continuity of, 82, 920, 922
  - continuous, 865
  - cost, 171, 255
  - cubic, 27
  - decreasing, 19
  - demand, 255, 587
  - derivative of, 107
  - differentiability of, 117, 942
  - discontinuous, 82
  - domain of, 10, 902
  - elementary, 522
  - error, 403, 436\*
  - even, 17, 238
  - exponential, 32, 391, 397, RP4
  - extreme values of, 198
  - family of, 247, 349, A50
  - fixed point of, 104, 213
  - Fresnel, 313
  - Gompertz, 636, 639
  - gradient of, 960, 962
  - graph of, 11, 904
  - greatest integer, 68
  - harmonic, 932
  - Heaviside, 44, 54
  - homogeneous, 956
  - hyperbolic, 462
  - implicit, 157
  - increasing, 19
  - integrable, 1000
  - inverse, 385
  - inverse cosine, 455
  - inverse hyperbolic, 464, 466
  - inverse sine, 453
  - inverse tangent, 456
  - inverse trigonometric, 453
  - joint density, 1032, 1047
  - limit of, 50, 72, 917, 922
  - linear, 23, 905
  - logarithmic, 32, 404, 441\*
  - machine diagram of, 11
  - marginal cost, 109, 171, 255, 324
  - marginal profit, 256
  - marginal revenue, 256
  - maximum and minimum values of, 198, 970
  - natural exponential, 397, 399, 429\*, 431\*
  - natural logarithmic, 405, 421\*
  - nondifferentiable, 119
  - of  $n$  variables, 911
  - odd, 18, 238
  - one-to-one, 384
  - periodic, 238
  - piecewise defined, 16
  - polynomial, 27, 921
  - position, 106
  - potential, 1085
  - power, 28, 133
  - probability density, 592, 1032
  - profit, 256
  - quadratic, 27
  - ramp, 44
  - range of, 10, 902
  - rational, 30, 508, 921
  - reciprocal, 30
  - reflected, 36
  - representation as a power series, 770
  - representations of, 10, 12
  - revenue, 256
  - root, 29
  - of several variables, 902, 910
  - shifted, 36
  - sine integral, 319
  - smooth, 562
  - step, 17
  - stretched, 36
  - tabular, 13
  - of three variables, 910
  - transformation of, 36
  - translation of, 36
  - trigonometric, 31, A26
  - of two variables, 902
  - value of, 10, 11
  - vector, 826
- Fundamental Theorem of Calculus, 310, 312, 317
  - higher-dimensional versions, 1159
  - for line integrals, 1099
  - for vector functions, 875
- $G$  (gravitational constant), 174, 373
- Gabriel's horn, 574
- Galileo, 664, 671, 694
- Galois, Evariste, 160
- Gause, G. F., 634
- Gauss, Karl Friedrich, 1153, A35
- Gaussian optics, 798
- Gauss's Law, 1143
- Gauss's Theorem, 1153
- geometric series, 729
- geometry of a tetrahedron, 840
- Gibbs, Joseph Willard, 821
- Gini, Corrado, 351
- Gini coefficient, 351
- Gini index, 351
- global maximum and minimum, 198
- Gompertz function, 636, 639
- gradient, 960, 962
- gradient vector, 960, 962
  - interpretations of, 1066
- gradient vector field, 1066, 1084
- graph(s)
  - of an equation, A16, A17
  - of equations in three dimensions, 811
  - of exponential functions, 393, 398, RP4
  - of a function, 11
  - of a function of two variables, 904
  - of logarithmic functions, 404, 407
  - of a parametric curve, 660
  - of a parametric surface, 1136
  - polar, 680, 685
  - of power functions, 29, RP3
  - of a sequence, 719
  - of trigonometric functions, 31, A30, RP2



**A140 INDEX**

- graphing calculator, 244, 662, 685, A46  
graphing device. *See* computer algebra system  
gravitation law, 174, 373  
gravitational acceleration, 368  
gravitational field, 1084  
great circle, 1063  
greatest integer function, 68  
Green, George, 1109, 1152  
Green's identities, 1122  
Green's Theorem, 1108, 1152  
    vector forms, 1120  
Gregory, James, 149, 499, 537,  
    774, 778  
Gregory's series, 774  
grid curves, 1124  
growth, law of natural, 446, 630  
growth rate, 168, 324  
    relative, 446, 630
- half-angle formulas, A29  
half-life, 448  
half-space, 911  
hare-lynx system, 650  
harmonic function, 932  
harmonic series, 732, 741  
    alternating, 753  
heat conduction equation, 937  
heat conductivity, 1144  
heat flow, 1143  
heat index, 924  
Heaviside, Oliver, 54  
Heaviside function, 44, 54  
Hecht, Eugene, 185, 188, 797  
helix, 865  
hidden line rendering, 850  
higher derivatives, 120  
higher partial derivatives, 930  
homogeneous differential equation, 1166  
homogeneous function, 956  
Hooke's Law, 369, 1180  
horizontal asymptote, 225, 238  
horizontal line, equation of, A13  
Horizontal Line Test, 384  
horizontal plane, 811  
Hubble Space Telescope, 203  
Huygens, Christiaan, 664  
hydrostatic pressure and force, 576, 577  
hydro-turbine optimization, 990  
hyperbola, 162, 697, 702, A20  
    asymptotes, 698, A20  
    branches, 698, A20  
    directrix, 702  
    eccentricity, 702  
    equation, 698, 699, 704, A20  
    equilateral, A21  
    foci, 697, 702  
    polar equation, 704  
    reflection property, 702  
    vertices, 698
- hyperbolic function(s), 462  
    derivatives of, 464  
    inverse, 466  
hyperbolic identities, 463  
hyperbolic paraboloid, 853, 854  
hyperbolic substitution, 505, 506  
hyperboloid, 854  
hypersphere, 1051  
hypocycloid, 668
- $i$  (imaginary number), A53  
 $\mathbf{i}$  (standard basis vector), 820  
I/D Test, 214  
ideal gas law, 176, 938  
image of a point, 1065  
image of a region, 1065  
implicit differentiation, 157, 158, 929, 952  
implicit function, 157, 158  
Implicit Function Theorem, 953, 954  
improper integral, 543  
    convergence or divergence of, 544, 547  
impulse of a force, 376  
incompressible velocity field, 1119  
increasing function, 19  
increasing sequence, 720  
Increasing/Decreasing Test, 214  
increment, 108, 945  
indefinite integral(s), 321  
    table of, 322  
independence of path, 1100  
independent random variable, 1034  
independent variable, 10, 902, 950  
indeterminate difference, 474  
indeterminate forms of limits, 469  
indeterminate power, 474  
indeterminate product, 473  
index of summation, A34  
inequalities, rules for, A4  
inertia (moment of), 1030, 1047, 1098  
infinite discontinuity, 83  
infinite interval, 543, 544  
infinite limit, 56, 79, 230  
infinite sequence. *See* sequence  
infinite series. *See* series  
inflection point, 218  
initial condition, 607  
initial point  
    of a parametric curve, 661  
    of a vector, 815, 1170  
initial-value problem, 607  
inner product, 824  
instantaneous rate of change, 48, 108, 164  
instantaneous rate of growth, 169  
instantaneous rate of reaction, 168  
instantaneous velocity, 48, 106, 164  
integer, A2  
integrable function, 1000  
integral(s)  
    approximations to, 302  
    change of variables in, 407, 1023, 1064,  
        1068, 1070  
    comparison properties of, 305  
    conversion to cylindrical coordinates, 1053  
    conversion to polar coordinates, 1022  
    conversion to spherical coordinates, 1058  
    definite, 295, 998  
    derivative of, 312  
    double (*see* double integral)  
    evaluating, 298  
    improper, 543  
    indefinite, 321  
    iterated, 1006, 1007  
    line (*see* line integral)  
    patterns in, 529  
    properties of, 303  
    surface, 1134, 1141  
    of symmetric functions, 334  
    table of, 511, 519, 524, RP6–10  
    triple, 1041, 1042  
    units for, 326  
integral calculus, 2, 3  
Integral Test, 740  
integrand, 296  
    discontinuous, 547  
integration, 296  
    approximate, 530  
    by computer algebra system, 526  
    of exponential functions, 301, 400  
    formulas, 511, 519, RP6–10  
    indefinite, 321  
    limits of, 296  
    numerical, 530  
    partial, 1007  
    by partial fractions, 508  
    by parts, 512, 513, 514  
    of a power series, 772  
    of rational functions, 508  
    by a rationalizing substitution, 516  
    reversing order of, 1009, 1017  
    over a solid, 1054  
    substitution in, 330  
    tables, use of, 524  
    term-by-term, 772  
    of a vector function, 871  
intercepts, 238, A19  
interest compounded continuously, 450, 479  
Intermediate Value Theorem, 89  
intermediate variable, 950  
interpolation, 26  
intersection  
    of planes, 845  
    of polar graphs, area of, 690  
    of sets, A3  
    of three cylinders, 1056  
interval, A3  
interval of convergence, 767  
inverse cosine function, 455  
inverse function(s), 384, 385

- inverse sine function, 453  
 inverse square laws, 35  
 inverse tangent function, 456  
 inverse transformation, 1065  
 inverse trigonometric functions, 455  
 irrational number, A2  
 irrotational vector field, 1118  
 isothermal, 907, 914  
 isothermal compressibility, 168  
 iterated integral, 1006, 1007
- j** (standard basis vector), 820  
 Jacobi, Carl, 1067  
 Jacobian of a transformation, 1067, 1070  
 jerk, 122  
 joint density function, 1032, 1047  
 joule, 368  
 jump discontinuity, 83
- k** (standard basis vector), 820  
 kampyle of Eudoxus, 162  
 Kepler, Johannes, 706, 891  
 Kepler's Laws, 706, 891, 892, 896  
 kinetic energy, 377, 1105  
 Kirchhoff's Laws, 611, 1184  
 Kondo, Shigeru, 781
- Lagrange, Joseph-Louis, 209, 210, 982  
 Lagrange multiplier, 981, 982  
 lamina, 580, 1027, 1029  
 Laplace, Pierre, 932, 1119  
 Laplace operator, 1119  
 Laplace's equation, 932, 1119  
 lattice point, 196  
 law of conservation of angular momentum, 895  
 Law of Conservation of Energy, 1106  
 law of cosines, A33  
 law of gravitation, 373  
 law of laminar flow, 170, 588  
 law of natural growth or decay, 446  
 laws of exponents, 394, 431\*, 437\*  
 laws of logarithms, 405, 422\*  
 learning curve, 609  
 least squares method, 26, 979  
 least upper bound, 722  
 left-hand derivative, 126  
 left-hand limit, 55, 76  
 Leibniz, Gottfried Wilhelm, 3, 117, 310, 329, 618, 791  
 Leibniz notation, 117  
 lemniscate, 162  
 length
  - of a curve, 562
  - of a line segment, A7, A12
  - of a parametric curve, 672
  - of a polar curve, 691
  - of a space curve, 877
  - of a vector, 818
- level curve(s), 907, 910  
 level surface, 911
  - tangent plane to, 964
- l'Hospital, Marquis de, 470, 480  
 l'Hospital's Rule, 470, 480
  - origins of, 480
- libration point, 268  
 limaçon, 686  
 limit(s), 2, 50
  - calculating, 62
  - $e$  (the number) as, 417, 443\*
  - of exponential functions, 394, 399, 431\*
  - of a function, 50, 73
  - of a function of three variables, 922
  - of a function of two variables, 917
  - infinite, 56, 79, 230
  - at infinity, 223, 224, 230
  - of integration, 296
  - left-hand, 55, 76
  - of logarithmic functions, 405, 407
  - one-sided, 55, 76
  - precise definitions, 72, 76, 79, 231
  - properties of, 62
  - right-hand, 55, 76
  - of a sequence, 5, 286, 716
  - involving sine and cosine functions, 140, 141, 143
  - of a trigonometric function, 142
  - of a vector function, 864
- Limit Comparison Test, 748  
 Limit Laws, 62, A39
  - for functions of two variables, 920
  - for sequences, 717
- linear approximation, 183, 941, 945  
 linear combination, 1166  
 linear density, 166, 167, 324  
 linear differential equation, 640, 1166  
 linear equation, A14
  - of a plane, 844
- linear function, 23, 905  
 linearity of an integral, 1005  
 linearization, 183, 941  
 linearly independent solutions, 1167  
 linear model, 23  
 linear regression, 26  
 line(s) in the plane, 45, A12
  - equation of, A12, A13, A14
  - equation of, through two points, 842
  - horizontal, A13
  - normal, 135
  - parallel, A14
  - perpendicular, A14
  - secant, 45, 46
  - slope of, A12
  - tangent, 44, 46, 104
- line(s) in space
  - normal, 965
  - parametric equations of, 841
  - skew, 843
  - symmetric equations of, 842
- tangent, 872
  - vector equation of, 840, 841
- line integral, 1087
  - Fundamental Theorem for, 1099
  - for a plane curve, 1087
  - with respect to arc length, 1090
  - for a space curve, 1092
  - work defined as, 1094
  - of vector fields, 1094, 1095
- liquid force, 576, 577  
 Lissajous figure, 662, 668  
 lithotripsy, 697  
 local maximum and minimum values, 198, 970  
 logarithm(s), 32, 404
  - laws of, 405, 422\*
  - natural, 405, 422\*
  - notation for, 405, 442\*
- logarithmic differentiation, 416, 427\*  
 logarithmic function(s), 32, 404, 421\*, 441\*
  - with base  $a$ , 404
  - derivatives of, 411, 424\*, 441\*
  - graphs of, 404, 407
  - limits of, 405, 407
  - properties of, 404, 422\*
- logistic difference equation, 727  
 logistic differential equation, 605, 631  
 logistic model, 605, 630  
 logistic sequence, 727  
 LORAN system, 701  
 Lorenz curve, 351  
 Lotka-Volterra equations, 647
- machine diagram of a function, 11  
 Maclaurin, Colin, 745  
 Maclaurin series, 777, 778
  - table of, 785
- magnitude of a vector, 818  
 major axis of ellipse, 696  
 marginal cost function, 109, 171, 255, 324  
 marginal productivity, 934  
 marginal profit function, 256  
 marginal propensity to consume or save, 736  
 marginal revenue function, 256  
 mass
  - of a lamina, 1027
  - of a solid, 1047
  - of a surface, 1136
  - of a wire, 1089
- mass, center of. *See* center of mass  
 mathematical induction, 99, 101, 723
  - principle of, 99, 101, A36
- mathematical model. *See* model(s),  
     mathematical  
 maximum and minimum values, 198, 970  
 mean life of an atom, 552  
 mean of a probability density function, 594  
 Mean Value Theorem, 208, 209
  - for double integrals, 1076
  - for integrals, 374

**A142 INDEX**

- mean waiting time, 594  
 median of a probability density function, 596  
 method of cylindrical shells, 363  
 method of exhaustion, 2, 64  
 method of Lagrange multipliers, 981, 982, 985  
 method of least squares, 26, 979  
 method of undetermined coefficients, 1173, 1177  
 midpoint formula, A16  
 Midpoint Rule, 302, 532  
   for double integrals, 1002  
   error in using, 532  
   for triple integrals, 1049  
 minor axis of ellipse, 696  
 mixing problems, 622  
 Möbius, August, 1139  
 Möbius strip, 1133, 1139  
 model(s), mathematical, 13, 23  
   Cobb-Douglas, for production costs, 904, 934, 987  
   comparison of natural growth vs. logistic, 634  
   of electric current, 611  
   empirical, 25  
   exponential, 32, 394, 395, 446, 447  
   Gompertz function, 636, 639  
   linear, 23  
   logarithmic, 32  
   polynomial, 28  
   for population growth, 394, 604, 636  
   power function, 28  
   predator-prey, 646  
   rational function, 30  
   seasonal-growth, 639  
   trigonometric, 31, 32  
   for vibration of membrane, 766  
   von Bertalanffy, 655  
 modeling  
   with differential equations, 604  
   motion of a spring, 606  
   population growth, 394, 395, 446, 447, 604, 630, 636, 654  
 modulus, A58  
 moment  
   about an axis, 579, 1029  
   of inertia, 1030, 1047, 1098  
   of a lamina, 580, 1029  
   of a mass, 579  
   about a plane, 1047  
   polar, 1031  
   second, 1030  
   of a solid, 1047  
   of a system of particles, 579  
 momentum of an object, 376  
 monkey saddle, 915  
 monotonic sequence, 720  
 Monotonic Sequence Theorem, 722  
 motion of a projectile, 888  
 motion in space, 886  
   motion of a spring, force affecting  
     damping, 1181  
     resonance, 1184  
     restoring, 1180  
 movie theater seating, 461  
 multiple integrals. *See* double integral;  
   triple integral(s)  
 multiplication of power series, 787  
 multiplier (Lagrange), 981, 982, 985  
 multiplier effect, 736  
 natural exponential function, 397, 399, 429\*  
   derivative of, 397  
   graph of, 398  
   power series for, 778  
   properties of, 399, 431\*  
 natural growth law, 446, 630  
 natural logarithm function, 405, 421\*  
   derivative of, 397, 407  
   limits of, 407  
   properties of, 404, 422\*  
 $n$ -dimensional vector, 819  
 negative angle, A25  
 net area, 297  
 Net Change Theorem, 324  
 net investment flow, 591  
 newton (unit of force), 368  
 Newton, Sir Isaac, 3, 8, 64, 114, 117, 310, 329, 791, 892, 896  
 Newton's Law of Cooling, 449, 609  
 Newton's Law of Gravitation, 174, 373, 892, 1083  
 Newton's method, 263  
 Newton's Second Law of Motion, 368, 377, 88, 892, 1180  
 Nicomedes, 665  
 nondifferentiable function, 119  
 nonhomogeneous differential equation, 1166, 1173  
 nonparallel planes, 845  
 normal component of acceleration, 890, 891  
 normal derivative, 1122  
 normal distribution, 596  
 normal line, 135, 965  
 normal plane, 883  
 normal vector, 844, 882  
 $n$ th-degree equation, finding roots of, 160  
 $n$ th-degree Taylor polynomial, 189, 779  
 number  
   complex, A53  
   integer, A2  
   irrational, A2  
   rational, A2  
   real, A2  
 numerical integration, 530  
 $O$  (origin), 810  
 octant, 810  
 odd function, 18, 238  
 one-sided limits, 55, 76  
 one-to-one function, 384  
 one-to-one transformation, 1065  
 open interval, A3  
 open region, 1101  
 optics  
   first-order, 798  
   Gaussian, 798  
   third-order, 798  
 optimization problems, 198, 250  
 orbit of a planet, 892  
 order of a differential equation, 606  
 order of integration, reversed, 1009, 1017  
 ordered pair, A10  
 ordered triple, 810  
 Oresme, Nicole, 732  
 orientation of a curve, 1092, 1108  
 orientation of a surface, 1139  
 oriented surface, 1139  
 origin, 810, A2, A10  
 orthogonal curves, 163  
 orthogonal projection, 831  
 orthogonal surfaces, 969  
 orthogonal trajectory, 163, 621  
 orthogonal vectors, 826  
 osculating circle, 883  
 osculating plane, 883  
 Ostrogradsky, Mikhail, 1153  
 ovals of Cassini, 689  
 overdamped vibration, 1182  
 Pappus, Theorem of, 583  
 Pappus of Alexandria, 583  
 parabola, 694, 702, A18  
   axis, 694  
   directrix, 694  
   equation, 694, 695  
   focus, 694, 702  
   polar equation, 704  
   reflection property, 196  
   vertex, 694  
 parabolic cylinder, 851  
 paraboloid, 852, 856  
 paradoxes of Zeno, 5  
 parallel lines, A14  
 parallel planes, 845  
 parallel vectors, 817  
 parallelepiped, 352  
   volume of, 837  
 Parallelogram Law, 816, 831  
 parameter, 660, 841, 865  
 parametric curve, 660, 865  
   arc length of, 672  
   area under, 671  
   slope of tangent line to, 669  
 parametric equations, 660, 841, 865  
   of a line in space, 841  
   of a space curve, 865  
   of a surface, 1123  
   of a trajectory, 889

- parametric surface, 1123
  - graph of, 1136
  - surface area of, 1128, 1129
  - surface integral over, 1135
  - tangent plane to, 1127
- parametrization of a space curve, 878
  - with respect to arc length, 879
  - smooth, 879
- paraxial rays, 185
- partial derivative(s), 926
  - of a function of more than three variables, 929
  - interpretations of, 927
  - notations for, 927
  - as a rate of change, 926
  - rules for finding, 927
  - second, 930
  - as slopes of tangent lines, 927
- partial differential equation, 932
- partial fractions, 508, 509
- partial integration, 512, 513, 514, 10007
- partial sum of a series, 728
- particle, motion of, 886
- parts, integration by, 512, 513, 514
- pascal (unit of pressure), 577
- path, 1100
- patterns in integrals, 529
- pendulum, approximating the period
  - of, 185, 188
- percentage error, 187
- perihelion, 707
- perilune, 701
- period, 238
- periodic function, 238
- perpendicular lines, A14
- perpendicular vectors, 826
- phase plane, 648
- phase portrait, 648
- phase trajectory, 648
- piecewise defined function, 16
- piecewise-smooth curve, 1088
- Planck's Law, 801
- plane region of type I, 1013
- plane region of type II, 1014
- plane(s)
  - angle between, 845
  - coordinate, 810
  - equation(s) of, 840, 843, 844
  - equation of, through three points, 845
  - horizontal, 811
  - line of intersection, 845
  - normal, 883
  - osculating, 883
  - parallel, 845
  - tangent to a surface, 939, 964, 1127
  - vertical, 902
- planetary motion, 891
  - laws of, 706
- planimeter, 1111
- point of inflection, 218
- point(s) in space
  - coordinates of, 810
  - distance between, 812
  - projection of, 811
- point-slope equation of a line, A12
- Poiseuille, Jean-Louis-Marie, 170
- Poiseuille's Laws, 188, 261, 589
- polar axis, 678
- polar coordinate system, 678
  - area in, 641
  - conic sections in, 702
  - conversion of double integral to, 1021, 1022
  - conversion equations for Cartesian coordinates, 379, 680
- polar curve, 680
  - arc length of, 691
  - graph of, 680
  - symmetry in, 683
  - tangent line to, 683
- polar equation, graph of, 680
- polar equation of a conic, 704
- polar form of a complex number, A55
- polar graph, 680
- polar moment of inertia, 1031
- polar rectangle, 1021
- polar region, area of, 689
- pole, 678
- polynomial, 27
- polynomial function, 27
  - of two variables, 921
- population growth, 446, 629
  - of bacteria, 629, 634
  - of insects, 518
  - models, 604
  - world, 394, 395, 446, 447
- position function, 106
- position vector, 818
- positive angle, A25
- positive orientation
  - of a boundary curve, 1146
  - of a closed curve, 1188
  - of a surface, 1140
- potential, 556
- potential energy, 1105
- potential function, 1085
- pound (unit of force), 368
- power, 1110
- power consumption, approximation of, 326
- power function(s), 28, 133
  - derivative of, 126
- Power Law of limits, 63
- Power Rule, 127, 134, 151, 416
- power series, 765
  - coefficients of, 765
  - for cosine and sine, 782
  - differentiation of, 772
  - division of, 787
  - for exponential function, 782
  - integration of, 772
- interval of convergence, 767
  - multiplication of, 787
  - radius of convergence, 767
  - representations of functions as, 771
- predator-prey model, 176, 646, 647
- pressure exerted by a fluid, 576, 577
- prime notation, 107, 129
- principal square root of a complex number, A54
- principal unit normal vector, 882
- principle of mathematical induction, 98, A36
- principle of superposition, 1175
- probability, 592, 1032
- probability density function, 592, 1032
- problem-solving principles, 97
  - uses of, 279, 330, 341, 485
- producer surplus, 590
- product
  - cross, 832 (*see also* cross product)
  - dot, 824 (*see also* dot product)
  - scalar, 824
  - scalar triple, 836
  - triple, 836
- product formulas, A29
- Product Law of limits, 62
- Product Rule, 130
- profit function, 256
- projectile, path of, 668, 888
- projection, 811, 828
  - orthogonal, 831
- $p$ -series, 741
- quadrant, A11
- quadratic approximation, 189, 980
- quadratic function, 27
- quadric surface(s), 851
  - cone, 854
  - cylinder, 851
  - ellipsoid, 854
  - hyperboloid, 854
  - paraboloid, 852, 853, 854
  - table of graphs, 854
- quaternion, 821
- Quotient Law of limits, 62
- Quotient Rule, 132
- radian measure, 140 A24
- radiation from stars, 801
- radioactive decay, 448
- radiocarbon dating, 452
- radius of convergence, 767
- radius of gyration, 1032
- rainbow, formation and location of, 206
- rainbow angle, 207
- ramp function, 44
- range of a function, 10, 902
- rate of change
  - average, 107, 164
  - derivative as, 108
  - instantaneous, 48, 108, 164

**A144 INDEX**

- rate of growth, 169, 324  
rate of reaction, 110, 168, 324  
rates, related, 176  
rational function, 30, 509, 921  
    continuity of, 85  
    integration of, 508  
rational number, A2  
rationalizing substitution for integration, 516  
Ratio Test, 758  
Rayleigh-Jeans Law, 801  
real line, A3  
real number, A2  
rearrangement of a series, 761  
reciprocal function, 30  
rectangular coordinate system, 811, A11  
    conversion to cylindrical  
        coordinates, 1052  
    conversion to spherical coordinates, 1057  
rectilinear motion, 272  
recursion relation, 1189  
reduction formula, 491  
reflecting a function, 36  
reflection property  
    of conics, 196  
    of an ellipse, 697  
    of a hyperbola, 702  
    of a parabola, 196  
region  
    connected, 1101  
    under a graph, 284, 289  
    open, 1101  
    plane, of type I or II, 1013, 1014  
    simple plane, 1109  
    simple solid, 1153  
    simply-connected, 1102  
    solid (of type 1, 2, or 3), 1042, 1043, 1044  
    between two graphs, 344  
regression, linear, 26  
related rates, 176  
relative error, 187  
relative growth rate, 446, 630  
relative maximum or minimum, 198  
remainder estimates  
    for the Alternating Series, 754  
    for the Integral Test, 742  
remainder of the Taylor series, 779  
removable discontinuity, 83  
representation(s) of a function, 10, 12, 13  
    as a power series, 770  
resonance, 1184  
restoring force, 1180  
resultant force, 821  
revenue function, 256  
reversing order of integration, 1009, 1017  
revolution, solid of, 357  
revolution, surface of, 569  
Riemann, Georg Bernhard, 296  
Riemann sum(s), 296  
    for multiple integrals, 1001, 1041  
right circular cylinder, 352  
right-hand derivative, 126  
right-hand limit, 55, 76  
right-hand rule, 810, 834  
Roberval, Gilles de, 316, 671  
rocket science, 988  
Rolle, Michel, 208  
roller coaster, design of, 140  
roller derby, 1063  
Rolle's Theorem, 208  
root function, 29  
Root Law of limits, 64  
Root Test, 760  
roots of a complex number, A57  
roots of an  $n$ th-degree equation, 160  
rubber membrane, vibration of, 766  
ruling of a surface, 851  
rumors, rate of spread, 172  
saddle point, 971  
sample point, 289, 296, 999  
satellite dish, parabolic, 856  
scalar, 817  
scalar equation of a plane, 844  
scalar field, 1081  
scalar multiple of a vector, 817  
scalar product, 824  
scalar projection, 828  
scalar triple product, 836  
    geometric characterization of, 837  
scatter plot, 13  
seasonal-growth model, 639  
secant function, A26  
    derivative of, 144  
    graph of, A31  
secant line, 3, 45, 46, 48  
secant vector, 872  
second derivative, 120, 874  
    of a vector function, 874  
Second Derivative Test, 218  
Second Derivatives Test, 971  
second directional derivative, 968  
second moment of inertia, 1030  
second-order differential equation, 606  
    solutions of, 1166, 1171  
second partial derivative, 930  
sector of a circle, area of, 689  
separable differential equation, 618  
sequence, 5, 14  
    bounded, 721  
    convergent, 716  
    decreasing, 720  
    divergent, 716  
    Fibonacci, 715  
    graph of, 719  
    increasing, 720  
    limit of, 5, 286, 716  
    logistic, 727  
    monotonic, 720  
    of partial sums, 728  
    term of, 714  
series, 6, 728  
    absolutely convergent, 756  
    alternating, 751  
    alternating harmonic, 753, 756, 757  
    binomial, 784  
    coefficients of, 765  
    conditionally convergent, 757  
    convergent, 729  
    divergent, 729  
    geometric, 729  
    Gregory's, 774  
    harmonic, 732, 741  
    infinite, 728  
    Maclaurin, 777, 778  
     $p$ -, 741  
    partial sum of, 728  
    power, 765  
    rearrangement of, 761  
    strategy for testing, 763  
    sum of, 6, 729  
    Taylor, 777, 778  
    term of, 728  
    trigonometric, 765  
series solution of a differential  
    equation, 1188  
set, bounded or closed, 975  
set notation, A3  
serpentine, 137  
shell method for approximating volume, 363  
shift of a function, 36  
shifted conics, 699, A21  
shock absorber, 1181  
Sierpinski carpet, 737  
sigma notation, 290, A34  
simple curve, 1102  
simple harmonic motion, 155  
simple plane region, 1109  
simple solid region, 1153  
simply-connected region, 1102  
Simpson, Thomas, 536, 537, 996  
Simpson's Rule, 535, 537  
    error bounds for, 538  
sine function, A26  
    derivative of, 143, 144  
    graph of, 31, A31  
    power series for, 782  
sine integral function, 319  
sink, 1157  
skew lines, 843  
slant asymptote, 238, 241  
slope, A12  
    of a curve, 104  
slope field, 610  
slope-intercept equation of a line, A13  
smooth curve, 562, 879  
smooth function, 562  
smooth parametrization, 879  
smooth surface, 1128  
Snell's Law, 260  
snowflake curve, 806

- solid, 352  
     volume of, 352, 353, 360, 364, 1042, 1043  
 solid angle, 1163  
 solid of revolution, 357  
     rotated on a slant, 575  
     volume of, 360, 364, 575  
 solid region, 1153  
 solution curve, 610  
 solution of a differential equation, 606  
 solution of predator-prey equations, 647  
 source, 1157  
 space, three-dimensional, 810  
 space curve, 864, 865, 866, 867  
     arc length of, 877  
 speed of a particle, 109, 886  
 sphere  
     equation of, 813  
     flux across, 1141  
     parametrization of, 1125  
     surface area of, 1129  
 spherical coordinate system, 1057  
     conversion equations for, 1057  
     triple integrals in, 1058  
 spherical wedge, 1058  
 spherical zones, 601  
 spring constant, 369, 606, 1180  
 Squeeze Theorem, 68, A42  
     for sequences, 718  
 standard basis vectors, 820  
 standard deviation, 596  
 standard position of an angle, A25  
 stationary points, 970  
 steady state solution, 1186  
 stellar stereography, 552  
 step function, 17  
 Stokes, Sir George, 1147, 1152  
 Stokes' Theorem, 1146  
 strategy  
     for integration, 518, 519  
     for optimization problems, 250, 251  
     for problem solving, 97  
     for related rates, 179  
     for testing series, 763  
     for trigonometric integrals, 497, 498  
 streamlines, 1086  
 stretching of a function, 36  
 strophoid, 693, 711  
 Substitution Rule, 330, 331  
     for definite integrals, 333  
 subtraction formulas for sine and cosine, A29  
 sum, 289  
     of a geometric series, 730  
     of an infinite series, 729  
     of partial fractions, 509  
     Riemann, 296  
     telescoping, 732  
     of vectors, 816  
 Sum Law of limits, 62  
 Sum Rule, 129  
 summation notation, A34  
 supply function, 590  
 surface(s)  
     closed, 1140  
     graph of, 1136  
     level, 911  
     oriented, 1139  
     parametric, 1123  
     positive orientation of, 1140  
     quadric, 851  
     smooth, 1128  
 surface area, 571  
     of a parametric surface, 674, 1128, 1129  
     of a sphere, 1129  
     of a surface  $z = f(x, y)$ , 1037, 1038, 1130  
 surface integral, 1134  
     over a parametric surface, 1135  
     of a vector field, 1140  
 surface of revolution, 569  
     parametric representation of, 1127  
     surface area of, 571  
 swallowtail catastrophe curve, 668  
 symmetric equations of a line, 842  
 symmetric functions, integrals of, 334  
 symmetry, 17, 238, 334  
     in polar graphs, 683  
 symmetry principle, 580  
  
 $T$  and  $T^{-1}$  transformations, 1064, 1065  
 table of differentiation formulas, 136, RP5  
 tables of integrals, 519, RP6–10  
     use of, 524  
 tabular function, 13  
 tangent function, A26  
     derivative of, 144  
     graph of, 32, A31  
 tangent line(s), 104  
     to a curve, 3, 44, 104  
     early methods of finding, 114  
     to a parametric curve, 669, 670  
     to a polar curve, 683  
     to a space curve, 873  
     vertical, 120  
 tangent line approximation, 183  
 tangent plane  
     to a level surface, 939, 964  
     to a parametric surface, 1127  
     to a surface  $F(x, y, z) = k$ , 940, 964  
     to a surface  $z = f(x, y)$ , 939  
 tangent plane approximation, 941  
 tangent problem, 2, 3, 44, 104  
 tangent vector, 872  
 tangential component of acceleration, 890  
 tautochrone problem, 664  
 Taylor, Brook, 778  
 Taylor polynomial, 189, 779, 980  
     applications of, 792  
 Taylor series, 777, 778  
 Taylor's Inequality, 780  
  
 techniques of integration, summary, 519  
 telescoping sum, 732  
 temperature-humidity index, 912, 924  
 term of a sequence, 714  
 term of a series, 728  
 term-by-term differentiation and integration, 772  
 terminal point of a parametric curve, 661  
 terminal point of a vector, 815  
 terminal velocity, 626  
 Test for Divergence, 733  
 tests for convergence and divergence  
     of series  
         Alternating Series Test, 751  
         Comparison Test, 746  
         Integral Test, 740  
         Limit Comparison Test, 748  
         Ratio Test, 758  
         Root Test, 760  
         summary of tests, 763  
 tetrahedron, 840  
 third derivative, 121  
 third-order optics, 798  
 Thomson, William (Lord Kelvin), 1109, 1147, 1152  
 three-dimensional coordinate systems, 810, 811  
 TNB frame, 882  
 toroidal spiral, 867  
 torque, 895  
 Torricelli, Evangelista, 671  
 Torricelli's Law, 174  
 torsion of a space curve, 885  
 torus, 362, 1134  
 total differential, 944  
 total electric charge, 1029, 1047  
 total fertility rate, 191  
 trace of a surface, 851  
 trajectory, parametric equations for, 889  
 transfer curve, 899  
 transformation, 1064  
     of a function, 36  
     inverse, 1065  
     Jacobian of, 1067, 1070  
     one-to-one, 1065  
 translation of a function, 36  
 Trapezoidal Rule, 532  
     error in, 532  
 tree diagram, 932  
 trefoil knot, 867  
 Triangle Inequality, 78, A8  
     for vectors, 831  
 Triangle Law, 816  
 trigonometric functions, 31, A26  
     derivatives of, 140, 144  
     graphs of, 31, 32, A30, A31  
     integrals of, 321, 519  
     inverse, 453  
     limits involving, 141, 143  
     trigonometric identities, A28

## A146 INDEX

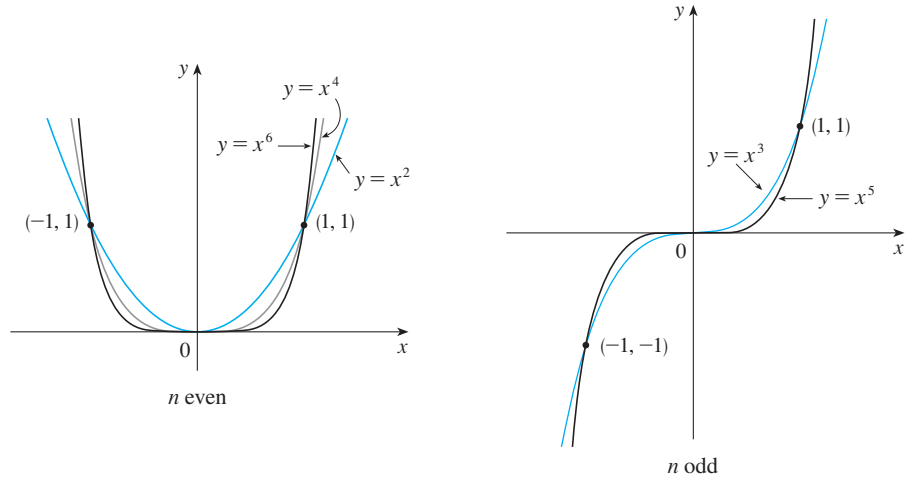
- trigonometric integrals, 495  
 strategy for evaluating, 497, 498
- trigonometric series, 765
- trigonometric substitutions, 502  
 table of, 502
- triple integral(s), 1041, 1042  
 applications of, 1046  
 in cylindrical coordinates, 1053  
 over a general bounded region, 1042  
 Midpoint Rule for, 1049  
 in spherical coordinates, 1058, 1059
- triple product, 836
- triple Riemann sum, 1041
- trochoid, 667
- Tschirnhausen cubic, 162, 350
- twisted cubic, 867
- type I or type II plane region, 1013, 1014
- type 1, 2, or 3 solid region, 1042, 1043, 1044
- ultraviolet catastrophe, 801
- underdamped vibration, 1182
- undetermined coefficients, method of, 1173, 1177
- uniform circular motion, 888
- union of sets, A3
- unit normal vector, 882
- unit tangent vector, 872
- unit vector, 821
- value of a function, 10
- van der Waals equation, 163, 938
- variable(s)  
 change of, 330  
 continuous random, 592  
 dependent, 10, 902, 950  
 independent, 10, 902, 950  
 independent random, 1034  
 intermediate, 950
- variables, change of. *See* change of variable(s)
- variation of parameters, method of, 1177, 1178
- vascular branching, 261
- vector(s), 815  
 acceleration, 887  
 addition of, 816, 818  
 algebraic, 818, 819  
 angle between, 825  
 basis, 820  
 binormal, 882  
 combining speed, 823  
 components of, 828  
 coplanar, 837  
 cross product of, 832  
 difference, 818  
 displacement, 829  
 dot product, 825  
 equality of, 816  
 force, 1083  
 geometric representation of, 818  
 gradient, 960, 962  
**i**, **j**, and **k**, 820  
 length of, 818  
 magnitude of, 818  
 multiplication of, 817, 819  
 $n$ -dimensional, 819  
 normal, 844  
 orthogonal, 826  
 parallel, 817  
 perpendicular, 826  
 position, 818  
 properties of, 819  
 representation of, 818  
 scalar multiple of, 817  
 standard basis, 820  
 tangent, 872  
 three-dimensional, 818  
 triple product, 837  
 two-dimensional, 818  
 unit, 821  
 unit normal, 882  
 unit tangent, 872  
 velocity, 886  
 zero, 816
- vector equation  
 of a line, 840, 841  
 of a plane, 844
- vector field, 1080, 1081  
 conservative, 1085  
 curl of, 1115  
 divergence of, 1118  
 electric flux of, 1143  
 flux of, 1141  
 force, 1080, 1084  
 gradient, 1084  
 gravitational, 1084  
 incompressible, 1119  
 irrotational, 1118  
 line integral of, 1094, 1095  
 potential function, 1104  
 surface integral of, 1141  
 velocity, 1080, 1083
- vector function, 864  
 continuity of, 865  
 derivative of, 871  
 integration of, 875  
 limit of, 864
- vector product, 832  
 properties of, 836
- vector projection, 828
- vector triple product, 837
- vector-valued function. *See* vector function
- continuous, 865  
 limit of, 864
- velocity, 3, 47, 106, 164, 324  
 average, 4, 47, 106, 164  
 instantaneous, 48, 106, 164
- velocity field, 1083  
 airflow, 1080  
 ocean currents, 1080  
 wind patterns, 1080
- velocity gradient, 170
- velocity problem, 47, 105
- velocity vector, 886
- velocity vector field, 1080
- Verhulst, Pierre-François, 605
- vertex of a parabola, 694
- vertical asymptote, 57, 238
- vertical line, A13
- Vertical Line Test, 15
- vertical tangent line, 120
- vertical translation of a graph, 36
- vertices of an ellipse, 696
- vertices of a hyperbola, 698
- vibration of a rubber membrane, 766
- vibration of a spring, 1180
- vibrations, 1180, 1181, 1183
- viewing rectangle, A46
- visual representations of a function, 10, 12
- volume, 353  
 by cross-sections, 352, 353, 589  
 by cylindrical shells, 363  
 by disks, 354, 357  
 by double integrals, 998  
 of a hypersphere, 1051  
 by polar coordinates, 1024  
 of a solid, 354, 1000  
 of a solid of revolution, 357, 575  
 of a solid on a slant, 575  
 by triple integrals, 1046  
 by washers, 256, 357
- Volterra, Vito, 647
- von Bertalanffy model, 655
- Wallis, John, 3
- Wallis product, 494
- washer method, 356
- wave equation, 932
- Weierstrass, Karl, 517
- weight (force), 368
- wind-chill index, 903
- wind patterns in San Francisco Bay  
 area, 1080
- witch of Maria Agnesi, 137, 667
- work (force), 368, 369  
 defined as a line integral, 1094
- Wren, Sir Christopher, 674
- $x$ -axis, 810, A10
- $x$ -coordinate, 810, A10
- $x$ -intercept, A13, A19
- $X$ -mean, 1035
- $y$ -axis, 810, A10
- $y$ -coordinate, 810, A10
- $y$ -intercept, A13, A19
- $Y$ -mean, 1035
- $z$ -axis, 810
- $z$ -coordinate, 7810
- Zeno, 5
- Zeno's paradoxes, 5
- zero vectors, 816

Cut here and keep for reference

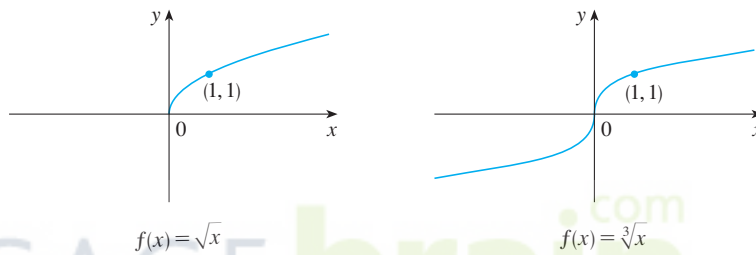
## SPECIAL FUNCTIONS

### Power Functions $f(x) = x^a$

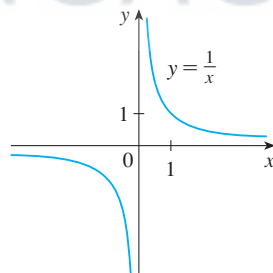
(i)  $f(x) = x^n$ ,  $n$  a positive integer



(ii)  $f(x) = x^{1/n} = \sqrt[n]{x}$ ,  $n$  a positive integer



(iii)  $f(x) = x^{-1} = \frac{1}{x}$

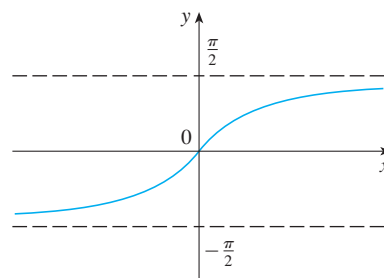


### Inverse Trigonometric Functions

$$\arcsin x = \sin^{-1}x = y \iff \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\arccos x = \cos^{-1}x = y \iff \cos y = x \text{ and } 0 \leq y \leq \pi$$

$$\arctan x = \tan^{-1}x = y \iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$



$$\lim_{x \rightarrow -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

$$\lim_{x \rightarrow \infty} \tan^{-1}x = \frac{\pi}{2}$$



## SPECIAL FUNCTIONS

### Exponential and Logarithmic Functions

$$\log_a x = y \iff a^y = x$$

$$\ln x = \log_e x, \text{ where } \ln e = 1$$

$$\ln x = y \iff e^y = x$$

#### Cancellation Equations

$$\log_a(a^x) = x \quad a^{\log_a x} = x$$

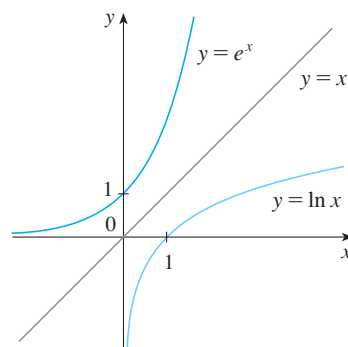
$$\ln(e^x) = x \quad e^{\ln x} = x$$

#### Laws of Logarithms

$$1. \log_a(xy) = \log_a x + \log_a y$$

$$2. \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$3. \log_a(x^r) = r \log_a x$$

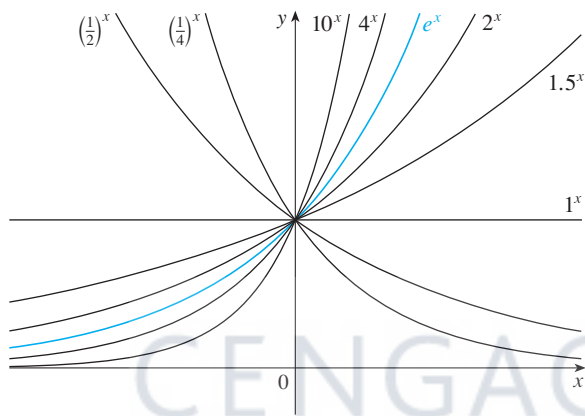


$$\lim_{x \rightarrow -\infty} e^x = 0$$

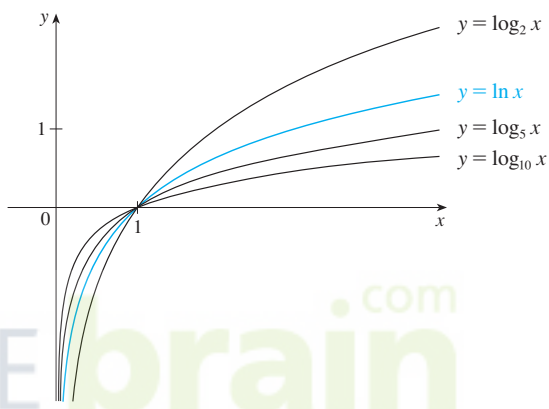
$$\lim_{x \rightarrow \infty} e^x = \infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

$$\lim_{x \rightarrow \infty} \ln x = \infty$$



Exponential functions



Logarithmic functions

### Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

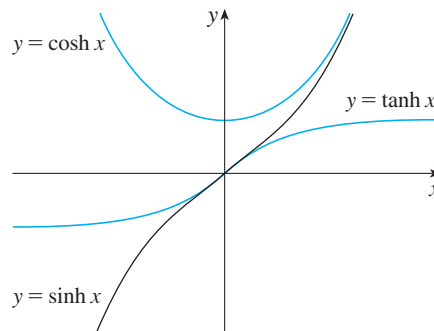
$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$



### Inverse Hyperbolic Functions

$$y = \sinh^{-1} x \iff \sinh y = x$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$y = \cosh^{-1} x \iff \cosh y = x \text{ and } y \geq 0$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$y = \tanh^{-1} x \iff \tanh y = x$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

## DIFFERENTIATION RULES

### General Formulas

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}[cf(x)] = cf'(x)$
- $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
- $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
- $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$  (Product Rule)
- $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$  (Quotient Rule)
- $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$  (Chain Rule)
- $\frac{d}{dx}(x^n) = nx^{n-1}$  (Power Rule)

### Exponential and Logarithmic Functions

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx} \ln|x| = \frac{1}{x}$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

### Trigonometric Functions

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

### Inverse Trigonometric Functions

- $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

### Hyperbolic Functions

- $\frac{d}{dx}(\sinh x) = \cosh x$
- $\frac{d}{dx}(\cosh x) = \sinh x$
- $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
- $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
- $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
- $\frac{d}{dx}(\operatorname{coth} x) = -\operatorname{csch}^2 x$

### Inverse Hyperbolic Functions

- $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$
- $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$
- $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$
- $\frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$
- $\frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$
- $\frac{d}{dx}(\operatorname{coth}^{-1}x) = \frac{1}{1-x^2}$

## TABLE OF INTEGRALS

## Basic Forms

1.  $\int u \, dv = uv - \int v \, du$
2.  $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
3.  $\int \frac{du}{u} = \ln |u| + C$
4.  $\int e^u \, du = e^u + C$
5.  $\int a^u \, du = \frac{a^u}{\ln a} + C$
6.  $\int \sin u \, du = -\cos u + C$
7.  $\int \cos u \, du = \sin u + C$
8.  $\int \sec^2 u \, du = \tan u + C$
9.  $\int \csc^2 u \, du = -\cot u + C$
10.  $\int \sec u \tan u \, du = \sec u + C$
11.  $\int \csc u \cot u \, du = -\csc u + C$
12.  $\int \tan u \, du = \ln |\sec u| + C$
13.  $\int \cot u \, du = \ln |\sin u| + C$
14.  $\int \sec u \, du = \ln |\sec u + \tan u| + C$
15.  $\int \csc u \, du = \ln |\csc u - \cot u| + C$
16.  $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C, \quad a > 0$
17.  $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$
18.  $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$
19.  $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u+a}{u-a} \right| + C$
20.  $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C$

Forms Involving  $\sqrt{a^2 + u^2}, \quad a > 0$ 

21.  $\int \sqrt{a^2 + u^2} \, du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$
22.  $\int u^2 \sqrt{a^2 + u^2} \, du = \frac{u}{8} (a^2 + 2u^2) \sqrt{a^2 + u^2} - \frac{a^4}{8} \ln(u + \sqrt{a^2 + u^2}) + C$
23.  $\int \frac{\sqrt{a^2 + u^2}}{u} \, du = \sqrt{a^2 + u^2} - a \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + C$
24.  $\int \frac{\sqrt{a^2 + u^2}}{u^2} \, du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln(u + \sqrt{a^2 + u^2}) + C$
25.  $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln(u + \sqrt{a^2 + u^2}) + C$
26.  $\int \frac{u^2 \, du}{\sqrt{a^2 + u^2}} = \frac{u}{2} \sqrt{a^2 + u^2} - \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$
27.  $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln \left| \frac{\sqrt{a^2 + u^2} + a}{u} \right| + C$
28.  $\int \frac{du}{u^2 \sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$
29.  $\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 + u^2}} + C$

## TABLE OF INTEGRALS

### Forms Involving $\sqrt{a^2 - u^2}$ , $a > 0$

$$30. \int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

$$31. \int u^2 \sqrt{a^2 - u^2} du = \frac{u}{8} (2u^2 - a^2) \sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C$$

$$32. \int \frac{\sqrt{a^2 - u^2}}{u} du = \sqrt{a^2 - u^2} - a \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$$

$$33. \int \frac{\sqrt{a^2 - u^2}}{u^2} du = -\frac{1}{u} \sqrt{a^2 - u^2} - \sin^{-1} \frac{u}{a} + C$$

$$34. \int \frac{u^2 du}{\sqrt{a^2 - u^2}} = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

$$35. \int \frac{du}{u \sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$$

$$36. \int \frac{du}{u^2 \sqrt{a^2 - u^2}} = -\frac{1}{a^2 u} \sqrt{a^2 - u^2} + C$$

$$37. \int (a^2 - u^2)^{3/2} du = -\frac{u}{8} (2u^2 - 5a^2) \sqrt{a^2 - u^2} + \frac{3a^4}{8} \sin^{-1} \frac{u}{a} + C$$

$$38. \int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 - u^2}} + C$$

### Forms Involving $\sqrt{u^2 - a^2}$ , $a > 0$

$$39. \int \sqrt{u^2 - a^2} du = \frac{u}{2} \sqrt{u^2 - a^2} - \frac{a^2}{2} \ln |u + \sqrt{u^2 - a^2}| + C$$

$$40. \int u^2 \sqrt{u^2 - a^2} du = \frac{u}{8} (2u^2 - a^2) \sqrt{u^2 - a^2} - \frac{a^4}{8} \ln |u + \sqrt{u^2 - a^2}| + C$$

$$41. \int \frac{\sqrt{u^2 - a^2}}{u} du = \sqrt{u^2 - a^2} - a \cos^{-1} \frac{a}{|u|} + C$$

$$42. \int \frac{\sqrt{u^2 - a^2}}{u^2} du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln |u + \sqrt{u^2 - a^2}| + C$$

$$43. \int \frac{du}{\sqrt{u^2 - a^2}} = \ln |u + \sqrt{u^2 - a^2}| + C$$

$$44. \int \frac{u^2 du}{\sqrt{u^2 - a^2}} = \frac{u}{2} \sqrt{u^2 - a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 - a^2}| + C$$

$$45. \int \frac{du}{u^2 \sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$$

$$46. \int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2 \sqrt{u^2 - a^2}} + C$$

CENGAGE **brain**.com

## TABLE OF INTEGRALS

Forms Involving  $a + bu$ 

$$47. \int \frac{u \, du}{a + bu} = \frac{1}{b^2} (a + bu - a \ln |a + bu|) + C$$

$$48. \int \frac{u^2 \, du}{a + bu} = \frac{1}{2b^3} [(a + bu)^2 - 4a(a + bu) + 2a^2 \ln |a + bu|] + C$$

$$49. \int \frac{du}{u(a + bu)} = \frac{1}{a} \ln \left| \frac{u}{a + bu} \right| + C$$

$$50. \int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln \left| \frac{a + bu}{u} \right| + C$$

$$51. \int \frac{u \, du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2} \ln |a + bu| + C$$

$$52. \int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2} \ln \left| \frac{a + bu}{u} \right| + C$$

$$53. \int \frac{u^2 \, du}{(a + bu)^2} = \frac{1}{b^3} \left( a + bu - \frac{a^2}{a + bu} - 2a \ln |a + bu| \right) + C$$

$$54. \int u\sqrt{a + bu} \, du = \frac{2}{15b^2} (3bu - 2a)(a + bu)^{3/2} + C$$

$$55. \int \frac{u \, du}{\sqrt{a + bu}} = \frac{2}{3b^2} (bu - 2a)\sqrt{a + bu} + C$$

$$56. \int \frac{u^2 \, du}{\sqrt{a + bu}} = \frac{2}{15b^3} (8a^2 + 3b^2u^2 - 4abu)\sqrt{a + bu} + C$$

$$57. \int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}} \right| + C, \quad \text{if } a > 0$$

$$= \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a + bu}{-a}} + C, \quad \text{if } a < 0$$

$$58. \int \frac{\sqrt{a + bu}}{u} \, du = 2\sqrt{a + bu} + a \int \frac{du}{u\sqrt{a + bu}}$$

$$59. \int \frac{\sqrt{a + bu}}{u^2} \, du = -\frac{\sqrt{a + bu}}{u} + \frac{b}{2} \int \frac{du}{u\sqrt{a + bu}}$$

$$60. \int u^n \sqrt{a + bu} \, du = \frac{2}{b(2n + 3)} \left[ u^n (a + bu)^{3/2} - na \int u^{n-1} \sqrt{a + bu} \, du \right]$$

$$61. \int \frac{u^n \, du}{\sqrt{a + bu}} = \frac{2u^n \sqrt{a + bu}}{b(2n + 1)} - \frac{2na}{b(2n + 1)} \int \frac{u^{n-1} \, du}{\sqrt{a + bu}}$$

$$62. \int \frac{du}{u^n \sqrt{a + bu}} = -\frac{\sqrt{a + bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{du}{u^{n-1} \sqrt{a + bu}}$$

## TABLE OF INTEGRALS

### Trigonometric Forms

$$63. \int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4} \sin 2u + C$$

$$64. \int \cos^2 u \, du = \frac{1}{2}u + \frac{1}{4} \sin 2u + C$$

$$65. \int \tan^2 u \, du = \tan u - u + C$$

$$66. \int \cot^2 u \, du = -\cot u - u + C$$

$$67. \int \sin^3 u \, du = -\frac{1}{3}(2 + \sin^2 u) \cos u + C$$

$$68. \int \cos^3 u \, du = \frac{1}{3}(2 + \cos^2 u) \sin u + C$$

$$69. \int \tan^3 u \, du = \frac{1}{2} \tan^2 u + \ln |\cos u| + C$$

$$70. \int \cot^3 u \, du = -\frac{1}{2} \cot^2 u - \ln |\sin u| + C$$

$$71. \int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| + C$$

$$72. \int \csc^3 u \, du = -\frac{1}{2} \csc u \cot u + \frac{1}{2} \ln |\csc u - \cot u| + C$$

$$73. \int \sin^n u \, du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u \, du$$

$$74. \int \cos^n u \, du = \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u \, du$$

$$75. \int \tan^n u \, du = \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u \, du$$

$$76. \int \cot^n u \, du = \frac{-1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u \, du$$

$$77. \int \sec^n u \, du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u \, du$$

$$78. \int \csc^n u \, du = \frac{-1}{n-1} \cot u \csc^{n-2} u + \frac{n-2}{n-1} \int \csc^{n-2} u \, du$$

$$79. \int \sin au \sin bu \, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$$

$$80. \int \cos au \cos bu \, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$$

$$81. \int \sin au \cos bu \, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$$

$$82. \int u \sin u \, du = \sin u - u \cos u + C$$

$$83. \int u \cos u \, du = \cos u + u \sin u + C$$

$$84. \int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$$

$$85. \int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$$

$$86. \int \sin^n u \cos^m u \, du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} u \cos^m u \, du$$

$$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m} \int \sin^n u \cos^{m-2} u \, du$$

### Inverse Trigonometric Forms

$$87. \int \sin^{-1} u \, du = u \sin^{-1} u + \sqrt{1-u^2} + C$$

$$88. \int \cos^{-1} u \, du = u \cos^{-1} u - \sqrt{1-u^2} + C$$

$$89. \int \tan^{-1} u \, du = u \tan^{-1} u - \frac{1}{2} \ln(1+u^2) + C$$

$$90. \int u \sin^{-1} u \, du = \frac{2u^2-1}{4} \sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$$

$$91. \int u \cos^{-1} u \, du = \frac{2u^2-1}{4} \cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$$

$$92. \int u \tan^{-1} u \, du = \frac{u^2+1}{2} \tan^{-1} u - \frac{u}{2} + C$$

$$93. \int u^n \sin^{-1} u \, du = \frac{1}{n+1} \left[ u^{n+1} \sin^{-1} u - \int \frac{u^{n+1} du}{\sqrt{1-u^2}} \right], \quad n \neq -1$$

$$94. \int u^n \cos^{-1} u \, du = \frac{1}{n+1} \left[ u^{n+1} \cos^{-1} u + \int \frac{u^{n+1} du}{\sqrt{1-u^2}} \right], \quad n \neq -1$$

$$95. \int u^n \tan^{-1} u \, du = \frac{1}{n+1} \left[ u^{n+1} \tan^{-1} u - \int \frac{u^{n+1} du}{1+u^2} \right], \quad n \neq -1$$

## TABLE OF INTEGRALS

## Exponential and Logarithmic Forms

$$96. \int u e^{au} du = \frac{1}{a^2} (au - 1)e^{au} + C$$

$$97. \int u^n e^{au} du = \frac{1}{a} u^n e^{au} - \frac{n}{a} \int u^{n-1} e^{au} du$$

$$98. \int e^{au} \sin bu du = \frac{e^{au}}{a^2 + b^2} (a \sin bu - b \cos bu) + C$$

$$99. \int e^{au} \cos bu du = \frac{e^{au}}{a^2 + b^2} (a \cos bu + b \sin bu) + C$$

$$100. \int \ln u du = u \ln u - u + C$$

$$101. \int u^n \ln u du = \frac{u^{n+1}}{(n+1)^2} [(n+1) \ln u - 1] + C$$

$$102. \int \frac{1}{u \ln u} du = \ln |\ln u| + C$$

## Hyperbolic Forms

$$103. \int \sinh u du = \cosh u + C$$

$$104. \int \cosh u du = \sinh u + C$$

$$105. \int \tanh u du = \ln \cosh u + C$$

$$106. \int \coth u du = \ln |\sinh u| + C$$

$$107. \int \operatorname{sech} u du = \tan^{-1} |\sinh u| + C$$

$$108. \int \operatorname{csch} u du = \ln \left| \tanh \frac{1}{2} u \right| + C$$

$$109. \int \operatorname{sech}^2 u du = \tanh u + C$$

$$110. \int \operatorname{csch}^2 u du = -\operatorname{coth} u + C$$

$$111. \int \operatorname{sech} u \tanh u du = -\operatorname{sech} u + C$$

$$112. \int \operatorname{csch} u \coth u du = -\operatorname{csch} u + C$$

Forms Involving  $\sqrt{2au - u^2}$ ,  $a > 0$ 

$$113. \int \sqrt{2au - u^2} du = \frac{u-a}{2} \sqrt{2au - u^2} + \frac{a^2}{2} \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$114. \int u \sqrt{2au - u^2} du = \frac{2u^2 - au - 3a^2}{6} \sqrt{2au - u^2} + \frac{a^3}{2} \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$115. \int \frac{\sqrt{2au - u^2}}{u} du = \sqrt{2au - u^2} + a \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$116. \int \frac{\sqrt{2au - u^2}}{u^2} du = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$117. \int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$118. \int \frac{u du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$119. \int \frac{u^2 du}{\sqrt{2au - u^2}} = -\frac{(u+3a)}{2} \sqrt{2au - u^2} + \frac{3a^2}{2} \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$120. \int \frac{du}{u \sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$$

