# A Gentle Introduction to Homology, Cohomology, and Sheaf Cohomology

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# Chapter 1

# Introduction

One of the main problems, if the not, "the" problem, of topology is to understand when two spaces $X$ and $Y$ are similar or dissimilar. A related problem is to understand the connectivity structure of a space in terms of its holes and "higher-order" holes. Of course, one has to specify what "similar" means. Intuitively, two topological spaces $X$ and $Y$ are similar if there is a "good" bijection $f \colon X \to Y$ between them. More precisely, "good" means that $f$ is a continuous bijection whose inverse $f^{-1}$ is also continuous; in other words, $f$ is a *homeomorphism*. The notion of homeomorphism captures the notion proposed in the mid 1860's that $X$ can be deformed into $Y$ without tearing or overlapping. The problem then is to describe the equivalence classes of spaces under homeomorphism; it is a *classification problem*.

The classification problem for surfaces was investigated as early as the mid 1860's by Möbius and Jordan. These authors discovered that two (compact) surfaces are equivalent iff they have the same *genus* (the number of holes) and orientability type. Their "proof" could not be rigorous since they did not even have a precise definition of what a 2-manifold is! We have to wait until 1921 for a complete and rigorous proof of the classification theorem for compact surfaces; see Gallier and Xu [17] for a historical as well as technical account of this remarkable result.

What if $X$ and $Y$ do not have the nice structure of a surface or if they have higher-order dimension? In the words of Dieudonné, the problem is a "hopeless undertaking;" see Dieudonné's introduction [8].

The reaction to this fundamendal difficulty was the creation of algebraic and differential topology, whose major goal is to associate "invariant" objects to various types of spaces, so that homeomorphic spaces have "isomorphic" invariants. If two spaces $X$ and $Y$ happen to have some distinct invariant objects, then for sure they are not homeomorphic.

Poincaré was one of the major pioneers of this approach. At first these invariant objects were integers (Betti numbers and torsion numbers), but it was soon realized that much more information could be extracted from invariant algebraic structures such as groups, ring, and modules.

Three types of invariants can be assigned to a topological space:

(1)  Homotopy groups.

(2)  Homology groups.

(3)  Cohomology groups.

The above are listed in the chronological order of their discovery. It is interesting that the first homotopy group $\pi_1(X)$ of the space $X$, also called *fundamental group*, was invented by Poincaré (*Analysis Situs*, 1895), but homotopy basically did not evolve until the 1930s. One of the reasons is that the first homotopy group is generally nonabelian, which is harder to study.

On the other hand, homology and cohomology groups (or rings, or modules) are abelian, so results about commutative algebraic structures can be leveraged. This is true in particular if the ring $R$ is a PID, where the structure of the finitely generated $R$-modules is completely determined.

There are different kinds of homology groups. They usually correspond to some geometric intuition about decomposing a space into simple shapes such as triangles, tetrahedra, *ect,*. Cohomology is more abstract because it usually deal with functions on a space. However, we will see that it yields more information than homology precisely because certain kinds of operations on functions can be defined (cup and cap products).

As often in mathematics, some machinery that is created to solve a specific problem, here a problem in topology, unexpectedly finds fruitful applications to other parts of mathematics and becomes a major component of the arsenal of mathematical tools, in the present case *homological algebra* and *category theory*. In fact, category theory, invented by Mac Lane and Eilenberg, permeates algebraic topology and is really put to good use, rather than being a fancy attire that dresses up and obscures some simple theory, as it is used too often.

In view of the above discussion, it appears that algebraic topology might involve more algebra than topology. This is great if one is quite proficient in algebra, but not so good news for a novice who might be discouraged by the abstract and arcane nature of homological algebra. After all, what do the zig-zag lemma and the five lemma have to do with topology?

Unfortunately, it is true that a firm grasp of the basic concepts and results of homological algebra is essential to really understand what are the homology and the cohomology groups and what are their roles in topology.

One our goals is to attempt to demistify homological algebra. One should realize that the homology groups describe what man does in his home; in French, l'homme au logis. The cohomology groups describe what co-man does in his home; in French, le co-homme au logis, that is, la femme au logis. Obviously this is not politically correct, so cohomology should be renamed. The big question is: what is a better name for cohomology?

In the following sections we give a brief description of the topics that we are going to discuss in this book, and we try to provide motivations for the introduction of the concepts and tools involved. These sections introduce topics in the same order in which they are presented in the book. All historical references are taken from Dieudonné [8]. This is a remarkable account of the history of algebraic and differential topology from 1900 to the 1960's which contains a wealth of information.

## 1.1 Exact Sequences, Chain Complexes, Homology and Cohomology

There are various kinds of homology groups (simplicial, singular, cellular, *etc.*), but they all arise the same way, namely from a (possibly infinite) sequence called a *chain complex*

$$0 \xleftarrow{\;d_0\;} C_0 \xleftarrow{\;d_1\;} C_1 \xleftarrow{\phantom{xxxx}} \cdots \xleftarrow{\;d_{p-1}\;} C_{p-1} \xleftarrow{\;d_p\;} C_p \xleftarrow{\;d_{p+1}\;} C_{p+1} \xleftarrow{\phantom{xxxx}} \cdots \;,$$

in which the $C_p$ are vector spaces, or more generally abelian groups (typically freely generated), and the maps $d_p \colon C_p \to C_{p-1}$ are linear maps (homomorphisms of abelian groups) satisfying the condition

$$d_p \circ d_{p+1} = 0 \quad \text{for all } p \geq 0. \tag{$*_1$}$$

The elements of $C_p$ are called *p-chains* and the maps $d_p$ are called *boundary operators (or boundary maps)*. The intuition behind Condition ($*_1$) is that elements of the form $d_p(c) \in C_{p-1}$ with $c \in C_p$ are *boundaries*, and "a boundary has no boundary." For example, in $\mathbb{R}^2$, the points on the boundary of a closed unit disk form the unit circle, and the points on the unit circle have no boundary.

Since $d_p \circ d_{p+1} = 0$, we have $B_p(C) = \operatorname{Im} d_{p+1} \subseteq \operatorname{Ker} d_p = Z_p(C)$ so the quotient $Z_p(C)/B_p(C) = \operatorname{Ker} d_p/\operatorname{Im} d_{p+1}$ makes sense. The quotient module

$$H_p(C) = Z_p(C)/B_p(C) = \operatorname{Ker} d_p/\operatorname{Im} d_{p+1}$$

is the *p-th homology module* of the chain complex $C$. Elements of $Z_p$ are called *p-cycles* and elements of $B_p$ are called *p-boundaries*.

A condition stronger that Condition ($*_1$) is that

$$\operatorname{Im} d_{p+1} = \operatorname{Ker} d_p \quad \text{for all } p \geq 0. \tag{$**_1$}$$

A sequence satisfying Condition ($**_1$) is called an *exact sequence*. Thus, we can view the homology groups as a measure of the failure of a chain complex to be exact. Surprisingly, exact sequences show up in various areas of mathematics.

For example, given a topological space $X$, in singular homology the $C_p$'s are the abelian groups $C_p = S_p(X; \mathbb{Z})$ consisting of all (finite) linear combinations of the form $\sum n_i \sigma_i$, where

$n_i \in \mathbb{Z}$ and each $\sigma_i$, a *singular p-simplex*, is a continuous function $\sigma_i \colon \Delta^p \to X$ from the $p$-simplex $\Delta^p$ to the space $X$. A 0-simplex is a single point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a $p$-simplex is a higher-order generalization of a tetrahedron. A $p$-simplex $\Delta^p$ has $p+1$ *faces*, and the $i$th face is a $(p-1)$-singular simplex $\sigma \circ \phi_i^{p-1} \colon \Delta^{p-1} \to X$ defined in terms of a certain function $\phi_i^{p-1} \colon \Delta^{p-1} \to \Delta^p$; see Section 4.1. In the framework of singular homology, the boundary map $d_p$ is denoted by $\partial_p$, and for any singular $p$-simplex $\sigma$, $\partial\sigma$ is the singular $(p-1)$-chain given by

$$\partial\sigma = \sigma \circ \phi_0^{p-1} - \sigma \circ \phi_1^{p-1} + \cdots + (-1)^p \sigma \circ \phi_p^{p-1}.$$

A simple calculation confirms that $\partial_p \circ \partial_{p+1} = 0$. Consequently the free abelian groups $S_p(X; \mathbb{Z})$ together with the boundary maps $\partial_p$ form a chain complex denoted $S_*(X; \mathbb{Z})$ called the *simplicial chain complex* of $X$. Then the quotient module

$$H_p(X; \mathbb{Z}) = H_p(S_*(X; \mathbb{Z})) = \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1},$$

also denoted $H_p(X)$, is called the *p-th homology group* of $X$. Singular homology is discussed in Chapter 4, especially in Section 4.1.

Historically, singular homology did not come first. According to Dieudonné [8], singular homology emerged around 1925 in the work of Veblen, Alexander and Lefschetz (the "Princeton topologists," as Dieudonné calls them), and was defined rigorously and in complete generality by Eilenberg (1944). The definition of the homology modules $H_p(C)$ in terms of sequences of abelian groups $C_p$ and boundary homomorphisms $d_p \colon C_p \to C_{p-1}$ satisfying the condition $d_p \circ d_{p+1} = 0$ as quotients $\operatorname{Ker} d_p / \operatorname{Im} d_{p+1}$ seems to have been suggested to H. Hopf by Emmy Noether while Hopf was visiting Göttingen in 1925. Hopf used this definition in 1928, and independently so did Vietoris in 1926, and then Mayer in 1929.

The first occurrence of a chain complex is found in Poincaré's papers of 1900, although he did not use the formalism of modules and homomorphisms as we do now, but matrices instead. Poincaré introduced the homology of *simplicial complexes*, which are combinatorial triangulated objects objects made up of simplices. Given a simplicial complex $K$, we have free abelian groups $C_p(K)$ consisting of $\mathbb{Z}$-linear combinations of oriented $p$-simplices, and the boundary maps $\partial_p \colon C_p(K) \to C_{p-1}(K)$ are defined by

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i [\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p],$$

for any oriented $p$-simplex, $\sigma = [\alpha_0, \ldots, \alpha_p]$, where $[\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p]$ denotes the oriented $(p-1)$-simplex obtained by deleting vertex $\alpha_i$. Then we have a *simplicial chain complex* $(C_p(K), \partial_p)$ denoted $C_*(K)$, and the corresponding homology groups $H_p(C_*(K))$ are denoted $H_p(K)$ and called the *simplicial homology groups* of $K$. Simplicial homology is discussed in Chapter 5. We discussed singular homology first because it subsumes simplicial homology, as shown in Section 5.2.

A simplicial complex $K$ is a purely combinatorial object, thus it is not a space, but it has a *geometric realization* $K_g$, which is a (triangulated) topological space. This brings up the following question: if $K_1$ and $K_2$ are two simplicial complexes whose geometric realizations $(K_1)_g$ and $(K_2)_g$ are homeomorphic, are the simplicial homology groups $H_p(K_1)$ and $H_p(K_2)$ isomorphic?

Poincaré conjectured that the answer was "yes," and this conjecture was first proved by Alexander. The proof is nontrivial, and we present a version of it in Section 5.2.

The above considerations suggest that it would be useful to understand the relationship between the homology groups of two spaces $X$ and $Y$ related by a continuous map $f\colon X \to Y$. For this, we define mappings between chain complexes called chain maps.

Given two chain complexes $C$ and $C'$, a *chain map* $f\colon C \to C'$ is a family $f = (f_p)_{p \geq 0}$ of homomorphisms $f_p\colon C_p \to C'_p$ such that all the squares of the following diagram commute:

$$
\begin{array}{ccccccccccccc}
0 & \xleftarrow{d_0} & C_0 & \xleftarrow{d_1} & C_1 & \longleftarrow & \cdots & \xleftarrow{d_{p-1}} & C_{p-1} & \xleftarrow{d_p} & C_p & \xleftarrow{d_{p+1}} & C_{p+1} & \longleftarrow & \cdots \\
& & \downarrow{f_0} & & \downarrow{f_1} & & & & \downarrow{f_{p-1}} & & \downarrow{f_p} & & \downarrow{f_{p+1}} & & \\
0 & \xleftarrow{d'_0} & C'_0 & \xleftarrow{d'_1} & C'_1 & \longleftarrow & \cdots & \xleftarrow{d'_{p-1}} & C'_{p-1} & \xleftarrow{d'_p} & C'_p & \xleftarrow{d'_{p+1}} & C'_{p+1} & \longleftarrow & \cdots
\end{array} ,
$$

that is, $f_p \circ d_{p+1} = d'_{p+1} \circ f_{p+1}$, for all $p \geq 0$.

A chain map $f\colon C \to C'$ induces homomorphisms of homology

$$H_p(f)\colon H_p(C) \to H_p(C')$$

for all $p \geq 0$. Furthermore, given three chain complexes $C, C', C''$ and two chain maps $f\colon C \to C'$ and $g\colon C' \to C''$, we have

$$H_p(g \circ f) = H_p(g) \circ H_p(f) \quad \text{for all } p \geq 0$$

and

$$H_p(\mathrm{id}_C) = \mathrm{id}_{H_p(C)} \quad \text{for all } p \geq 0.$$

We say that the map $C \mapsto (H_p(C))_{p \geq 0}$ is *functorial* (to be more precise, it is a functor from the category of chain complexes and chain maps to the category of abelian groups and groups homomorphisms).

For example, in singular homology, a continuous function $f\colon X \to Y$ between two topological spaces $X$ and $Y$ induces a chain map $f_\sharp\colon S_*(X;\mathbb{Z}) \to S_*(Y;\mathbb{Z})$ between the two simplicial chain complexes $S_*(X;\mathbb{Z})$ and $S_*(Y;\mathbb{Z})$ associated with $X$ and $Y$, which in turn yield homology homomorphisms usually denoted $f_{*,p}\colon H_p(X;\mathbb{Z}) \to H_p(Y;\mathbb{Z})$. Thus the map $X \mapsto (H_p(X;\mathbb{Z}))_{p \geq 0}$ is a functor from the category of topological spaces and continuous maps to the category of abelian groups and groups homomorphisms. Functoriality implies

that if $f\colon X \to Y$ is a homeomorphism, then the maps $f_{*,p}\colon H_p(X;\mathbb{Z}) \to H_p(Y;\mathbb{Z})$ are *iso-morphisms*. Thus, the singular homology groups are topological invariants. This is one of the advantages of singular homology; topological invariance is basically obvious.

This is not the case for simplicial homology where it takes a fair amout of work to prove that if $K_1$ and $K_2$ are two simplicial complexes whose geometric realizations $(K_1)_g$ and $(K_2)_g$ are homeomorphic, then the simplicial homology groups $H_p(K_1)$ and $H_p(K_2)$ isomorphic.

One might wonder what happens if we reverse the arrows in a chain complex? Abstractly, this is how cohomology is obtained, although this point of view was not considered until at least 1935.

A *cochain complex* is a sequence

$$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \longrightarrow C^{p-1} \xrightarrow{d^{p-1}} C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} C^{p+2} \longrightarrow \cdots ,$$

in which the $C^p$ are abelian groups, and the maps $d^p\colon C^p \to C^{p+1}$ are homomorphisms of abelian groups satisfying the condition

$$d^{p+1} \circ d^p = 0 \quad \text{for all } p \geq 0 \tag{$*_2$}$$

The elements of $C^p$ are called *cochains* and the maps $d^p$ are called *coboundary maps*. This time, it is not clear how coboundary maps arise naturaly. Since $d^{p+1} \circ d^p = 0$, we have $B^p = \operatorname{Im} d^p \subseteq \operatorname{Ker} d^{p+1} = Z^{p+1}$, so the quotient $Z^p/B^p = \operatorname{Ker} d^{p+1}/\operatorname{Im} d^p$ makes sense and the quotient module

$$H^p(C) = Z^p/B^p = \operatorname{Ker} d^{p+1}/\operatorname{Im} d^p$$

is the *pth cohomology module* of the cochain complex $C$. Elements of $Z^p$ are called *p-cocycles* and elements of $B^p$ are called *p-coboundaries*.

There seems to be an unwritten convention that when dealing with homology we use subscripts, and when dealing with cohomology we use with superscripts. Also, the "dual" of any "notion" is the "co-notion."

As in the case of a chain complex, a condition stronger that Condition $(*_2)$ is that

$$\operatorname{Im} d_p = \operatorname{Ker} d_{p+1} \quad \text{for all } p \geq 0. \tag{$**_2$}$$

A sequence satisfying Condition $(**_2)$ is also called an *exact sequence*. Thus, we can view the cohomology groups as a measure of the failure of a cochain complex to be exact.

Given two cochain complexes $C$ and $C'$, a *(co)chain map* $f\colon C \to C'$ is a family $f = (f^p)_{p \geq 0}$ of homomorphisms $f^p\colon C^p \to C'^p$ such that all the squares of the following diagram commute:

$$
\begin{array}{ccccccccccc}
0 & \xrightarrow{d^{-1}} & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & \cdots \longrightarrow & C^{p-1} & \xrightarrow{d^{p-1}} & C^p & \xrightarrow{d^p} & C^{p+1} & \xrightarrow{d^{p+1}} & \cdots \\
& & \downarrow{f^0} & & \downarrow{f^1} & & & \downarrow{f^{p-1}} & & \downarrow{f^p} & & \downarrow{f^{p+1}} & & \\
0 & \xrightarrow{d'^{-1}} & C'^0 & \xrightarrow{d'^0} & C'^1 & \xrightarrow{d'^1} & \cdots \longrightarrow & C'^{p-1} & \xrightarrow{d'^{p-1}} & C'^p & \xrightarrow{d'^p} & C'^{p+1} & \xrightarrow{d'^{p+1}} & \cdots ,
\end{array}
$$

that is, $f^{p+1} \circ d^p = d'^p \circ f^p$ for all $p \geq 0$. A chain map $f \colon C \to C'$ induces homomorphisms of cohomology

$$H^p(f) \colon H^p(C) \to H^p(C')$$

for all $p \geq 0$. Furthermore, this assignment is functorial (more precisely, it is a functor from the category of cochain complexes and chain maps to the category of abelian groups and their homomorphisms).

At first glance cohomology appears to be very abstract so it is natural to look for explicit examples. A way to obtain a cochain complex is to apply the operator (functor) $\mathrm{Hom}_{\mathbb{Z}}(-, G)$ to a chain complex $C$, where $G$ is any abelian group. Given a fixed abelian group $A$, for any abelian group $B$ we denote by $\mathrm{Hom}_{\mathbb{Z}}(B, A)$ the abelian group of all homomorphisms from $B$ to $A$. Given any two abelian groups $B$ and $C$, for any homomorphism $f \colon B \to C$, the homomorphism $\mathrm{Hom}_{\mathbb{Z}}(f, A) \colon \mathrm{Hom}_{\mathbb{Z}}(C, A) \to \mathrm{Hom}_{\mathbb{Z}}(B, A)$ is defined by

$$\mathrm{Hom}_{\mathbb{Z}}(f, A)(\varphi) = \varphi \circ f \quad \text{for all } \varphi \in \mathrm{Hom}_{\mathbb{Z}}(C, A);$$

see the commutative diagram below:

$$
\begin{array}{ccc}
B & \xrightarrow{\ f\ } & C \\
& {\scriptstyle \mathrm{Hom}_{\mathbb{Z}}(f,A)(\varphi)} \searrow & \downarrow {\scriptstyle \varphi} \\
& & A.
\end{array}
$$

The map $\mathrm{Hom}_{\mathbb{Z}}(f, A)$ is also denoted by $\mathrm{Hom}_{\mathbb{Z}}(f, \mathrm{id}_A)$ or even $\mathrm{Hom}_{\mathbb{Z}}(f, \mathrm{id})$. Observe that the effect of $\mathrm{Hom}_{\mathbb{Z}}(f, \mathrm{id})$ on $\varphi$ is to precompose $\varphi$ with $f$.

If $f \colon B \to C$ and $g \colon C \to D$ are homomorphisms of abelian groups, a simple computation shows that

$$\mathrm{Hom}_R(g \circ f, \mathrm{id}) = \mathrm{Hom}_R(f, \mathrm{id}) \circ \mathrm{Hom}_R(g, \mathrm{id}).$$

Observe that $\mathrm{Hom}_{\mathbb{Z}}(f, \mathrm{id})$ and $\mathrm{Hom}_{\mathbb{Z}}(g, \mathrm{id})$ are composed in the reverse order of the composition of $f$ and $g$. It is also immediately verified that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_A, \mathrm{id}) = \mathrm{id}_{\mathrm{Hom}_{\mathbb{Z}}(A, G)}.$$

We say that $\mathrm{Hom}_{\mathbb{Z}}(-, \mathrm{id})$ is a *contravariant functor* (from the category of abelian groups and group homomorphisms to itself). Then given a chain complex

$$0 \xleftarrow{\ d_0\ } C_0 \xleftarrow{\ d_1\ } C_1 \xleftarrow{\quad} \cdots \xleftarrow{\ d_{p-1}\ } C_{p-1} \xleftarrow{\ d_p\ } C_p \xleftarrow{\ d_{p+1}\ } C_{p+1} \xleftarrow{\quad} \cdots ,$$

we can form the cochain complex

$$0 \xrightarrow{\ \mathrm{Hom}_{\mathbb{Z}}(d_0, \mathrm{id})\ } \mathrm{Hom}_{\mathbb{Z}}(C_0, G) \xrightarrow{\quad} \cdots \xrightarrow{\quad} \mathrm{Hom}_{\mathbb{Z}}(C_p, G) \xrightarrow{\ \mathrm{Hom}_{\mathbb{Z}}(d_{p+1}, \mathrm{id})\ } \mathrm{Hom}_{\mathbb{Z}}(C_{p+1}, G) \xrightarrow{\quad} \cdots$$

obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(-, G)$, and denoted $\mathrm{Hom}_{\mathbb{Z}}(C, G)$. The coboundary map $d^p$ is given by

$$d^p = \mathrm{Hom}_{\mathbb{Z}}(d_{p+1}, \mathrm{id}),$$

which means that for any $f \in \mathrm{Hom}_{\mathbb{Z}}(C_p, G)$, we have

$$d^p(f) = f \circ d_{p+1}.$$

Thus, for any $(p+1)$-chain $c \in C_{p+1}$ we have

$$(d^p(f))(c) = f(d_{p+1}(c)).$$

We obtain the cohomology groups $H^p(\mathrm{Hom}_{\mathbb{Z}}(C, G))$ associated with the cochain complex $\mathrm{Hom}_{\mathbb{Z}}(C, G)$. The cohomology groups $H^p(\mathrm{Hom}_{\mathbb{Z}}(C, G))$ are also denoted $H^p(C; G)$.

This process was applied to the simplicial chain complex $C_*(K)$ associated with a simplicial complex $K$ by Alexander and Kolmogoroff to obtain the *simplicial cochain complex* $\mathrm{Hom}_{\mathbb{Z}}(C_*(K); G)$ denoted $C^*(K; G)$ and the *simplicial cohomology groups* $H^p(K; G)$ of the simplicial complex $K$; see Section 5.4. Soon after, this process was applied to the singular chain complex $S_*(X; \mathbb{Z})$ of a space $X$ to obtain the *singular cochain complex* $\mathrm{Hom}_{\mathbb{Z}}(S_*(X; \mathbb{Z}); G)$ denoted $S^*(X; G)$ and the *singular cohomology groups* $H^p(X; G)$ of the space $X$; see Section 4.6.

Given a continuous map $f \colon X \to Y$, there is an induced chain map $f^\sharp \colon S^*(Y; G) \to S^*(X; G)$ between the singular cochain complexes $S^*(Y; G)$ and $S^*(X; G)$, and thus homomorphisms of cohomology $f^* \colon H^p(Y; G) \to H^p(X; G)$. Observe the reversal: $f$ is a map from $X$ to $Y$, but $f^*$ maps $H^p(Y; G)$ to $H^p(X; G)$. We say that the map $X \mapsto (H^p(X; G))_{p \geq 0}$ is a contravariant functor from the category of topological spaces and continuous maps to the category of abelian groups and their homomorphisms.

So far our homology groups have coefficients in $\mathbb{Z}$, but the process of forming a cochain complex $\mathrm{Hom}_{\mathbb{Z}}(C, G)$ from a chain complex $C$ allows the use of coefficients in any abelian group $G$, not just the integers. Actually, it is a trivial step to define chain complexes consisting of $R$-modules in any commutative ring $R$ with a multiplicative identity element 1, and such complexes yield homology modules $H_p(C; R)$ with coefficients in $R$. This process immediately applies to the singular homology groups $H_p(X; R)$ and to the simplicial homology groups $H_p(K; R)$. Also, given a chain complex $C$ where the $C_p$ are $R$-modules, for any $R$-module $G$ we can form the cochain complex $\mathrm{Hom}_R(C, G)$ and we obtain cohomology modules $H^p(C; G)$ with coefficients in any $R$-module $G$; see Section 4.6 and Section 12.5.

We can generalize homology with coefficients in a ring $R$ to modules with coefficients in a $R$-module $G$ by applying the operation (functor) $- \otimes_R G$ to a chain complex $C$ where the $C_p$'s are $R$-modules, to get the chain complex denoted $C \otimes_R G$. The homology groups of this complex are denoted $H_p(C, G)$. We will discuss this construction in Section 4.5 and Section 12.5.

If the ring $R$ is a PID, given a chain complex $C$ where the $C_p$ are $R$-modules, the homology groups $H_p(C; G)$ of the complex $C \otimes_R G$ are determined by the homology groups $H_{p-1}(C; R)$ and $H_p(C; R)$ *via* a formula called the *Universal Coefficient Theorem for Homology*; see Theorem 12.38. This formula involves a term $\mathrm{Tor}_1^R(H_{n-1}(C); G)$ that corresponds to the fact

that the operation $- \otimes_R G$ on linear maps generally does not preserve injectivity ($- \otimes_R G$ is not left-exact). These matters are discussed in Chapter 12.

Similarly, if the ring $R$ is a PID, given a chain complex $C$ where the $C_p$ are $R$-modules, the cohomology groups $H^p(C; G)$ of the complex $\mathrm{Hom}_R(C, G)$ are determined by the homology groups $H_{p-1}(C; R)$ and $H_p(C; R)$ *via* a formula called the *Universal Coefficient Theorem for Cohomology*; see Theorem 12.43. This formula involves a term $\mathrm{Ext}^1_R(H_{n-1}(C); G)$ that corresponds to the fact that if the linear map $f$ is injective, then $\mathrm{Hom}_R(f, \mathrm{id})$ is not necessarily surjective ($\mathrm{Hom}_R(-, G)$ is not right-exact). These matters are discussed in Chapter 12.

One of the advantages of singular homology (and cohomology) is that it is defined for *all* topological spaces, but one of its disadvantages is that in practice it is very hard to compute. On the other hand, simplicial homology (and cohomology) only applies to triangulable spaces (geometric realizations of simplicial complexes), but in principle it is computable (for finite complexes). One of the practical problems is that the triangulations involved may have a large number of simplices. J.H.C Whitehead invented a class of spaces called *CW complexes* that are more general than triangulable spaces and for which the computation of the singular homology groups is often more tractable. Unlike a simplicial complex, a CW complex is obtained by gluing spherical cells. CW complexes are discussed in Chapter 6.

There are at least four other ways of defining cohomology groups of a space $X$ by directly forming a cochain complex without using a chain complex and dualizing it by applying $\mathrm{Hom}_{\mathbb{Z}}(-, G)$:

(1) If $X$ is a smooth manifold, then there is the *de Rham complex* which uses the modules of smooth $p$-forms $\mathcal{A}^p(X)$ and the exterior derivatives $d^p \colon \mathcal{A}^p(X) \to \mathcal{A}^{p+1}(X)$. The corresponding cohomology groups are the *de Rham cohomology groups* $H^p_{\mathrm{dR}}(X)$. These are actually real vector spaces. De Rham cohomology is discussed in Chapter 3.

(2) If $X$ is any space and $\mathcal{U} = (U_i)_{i \in I}$ is any open cover of $X$, we can define the *Čech cohomology groups* $\check{H}^p(X, \mathcal{U})$ in a purely combinatorial fashion. Then we can define the notion of *refinement* of a cover and define the *Čech cohomology groups* $\check{H}^p(X, G)$ with values in an abelian group $G$ using a limiting process known as a *direct limit* (see Section 9.3, Definition 9.8). Čech cohomology is discussed in Chapter 10.

(3) If $X$ is any space, then there is the *Alexander–Spanier* cochain complex which yields the *Alexander–Spanier cohomology groups* $A^p_{\mathrm{A\text{-}S}}(X; G)$. Alexander–Spanier cohomology is discussed in Section 13.7 and in Chapter 14.

(4) Sheaf cohomology, based on derived functors and injective resolutions. This is the most general kind of cohomology of a space $X$, where cohomology groups $H^p(X, \mathcal{F})$ with values in a sheaf $\mathcal{F}$ on the space $X$ are defined. Intuitively, this means that the modules $\mathcal{F}(U)$ of "coefficients" in which these groups take values may vary with the open domain $U \subseteq X$. Sheaf cohomology is discussed in Chapter 13, and the algebraic machinery of derived functors is discussed in Chapter 12.

We will see that for topological manifolds, all these cohomology theories are equivalent; see Chapter 13. For paracompact spaces, Čech cohomology, Alexander–Spanier cohomology, and derived functor cohomology (for constant sheaves) are equivalent (see Chapter 13). In fact, Čech cohomology and Alexander–Spanier cohomology are equivalent for any space; see Chapter 14.

## 1.2   Relative Homology and Cohomology

In general, computing homology groups is quite difficult so it would be helpful if we had techniques that made this process easier. Relative homology and excision are two such tools that we discuss in this section.

Lefschetz (1928) introduced the *relative homology groups* $H_p(K, L; \mathbb{Z})$, where $K$ is a simplicial complex and $L$ is a subcomplex of $K$. The same idea immediately applies to singular homology and we can define the relative singular homology groups $H_p(X, A; R)$ where $A$ is a subspace of $X$. The intuition is that the module of $p$-chains of a relative chain complex consists of chains of $K$ modulo chains of $L$. For example, given a space $X$ and a subspace $A \subseteq X$, the *singular chain complex $S_*(X, A; R)$ of the pair* $(X, A)$ is the chain complex in which each $R$-module $S_p(X, A; R)$ is the quotient module

$$S_p(X, A; R) = S_p(X; R)/S_p(A; R).$$

It is easy to see that $S_p(X, A; R)$ is actually a free $R$-module; see Section 4.2.

Although this is not immediately apparent, the motivation is that the groups $H_p(A; R)$ and $H_p(X, A; R)$ are often "simpler" than the groups $H_p(X; R)$, and there is an exact sequence called the *long exact sequence of relative homology* that can often be used to come up with an inductive argument that allows the determination of $H_p(X; R)$ from $H_p(A; R)$ and $H_p(X, A; R)$. Indeed, we have the following exact sequence as shown in Section 4.2 (see Theorem 4.8):

$$\cdots \longrightarrow H_{p+2}(X, A; R) \underset{\partial_{*p+2}}{\longrightarrow}$$

$$\longrightarrow H_{p+1}(A; R) \xrightarrow{i_*} H_{p+1}(X; R) \xrightarrow{j_*} H_{p+1}(X, A; R) \underset{\partial_{*p+1}}{\longrightarrow}$$

$$\longrightarrow H_p(A; R) \xrightarrow{i_*} H_p(X; R) \xrightarrow{j_*} H_p(X, A; R) \underset{\partial_{*p}}{\longrightarrow}$$

$$\longrightarrow H_{p-1}(A; R) \longrightarrow \cdots$$

ending in

$$H_0(A; R) \longrightarrow H_0(X; R) \longrightarrow H_0(X, A; R) \longrightarrow 0.$$

Furthermore, if $(X, A)$ is a "good pair," then there is an isomorphism

$$H_p(X, A; R) \cong H_p(X/A, \{pt\}; R),$$

where pt stands for any point in $X$.

The long exact sequence of relative homology is a corollary of one the staples of homology theory, the "zig-zag lemma." The zig-zag lemma says that for any short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of chain complexes $X, Y, Z$ there is a long exact sequence of cohomology

$$
\begin{array}{c}
\cdots \longrightarrow H^{p-1}(Z) \longrightarrow \\
\delta^{p-1} \\
H^p(X) \xrightarrow{f^*} H^p(Y) \xrightarrow{g^*} H^p(Z) \longrightarrow \\
\delta^p \\
H^{p+1}(X) \xrightarrow{f^*} H^{p+1}(Y) \xrightarrow{g^*} H^{p+1}(Z) \longrightarrow \\
\delta^{p+1} \\
H^{p+2}(X) \longrightarrow \cdots
\end{array}
$$

The zig-zag lemma is fully proved in Section 2.5; see Theorem 2.19. There is also a homology version of this theorem.

Another very important aspect of relative singular homology is that it satisfies the *excision axiom*, another useful tool to compute homology groups. This means that removing a subspace $Z \subseteq A \subseteq X$ which is clearly inside of $A$, in the sense that $Z$ is contained in the interior of $A$, does not change the relative homology group $H_p(X, A; R)$. More precisely, there is an isomorphism

$$H_p(X - Z, A - Z; R) \cong H_p(X, A; R);$$

see Section 4.3 (Theorem 4.12). A good illustration of the use of excision and of the long exact sequence of relative homology is the computation of the homology of the sphere $S^n$; see Section 4.4. Relative singular homology also satisfies another important property: the *homotopy axiom*, which says that if two spaces are homotopy equivalent, then their homology is isomorphic; see Theorem 4.7.

Following the procedure for obtaining cohomology from homology described in Section 1.1, by applying $\mathrm{Hom}_R(-, G)$ to the chain complex $S_*(X, A; R)$ we obtain the cochain complex $S^*(X, A; G) = \mathrm{Hom}_R(S_*(X, A; R), G)$, and thus the *singular relative cohomology groups* $H^p(X, A; G)$; see Section 4.7. In this case, we can think of the elements of $S^p(X, A; G)$ as linear maps (with values in $G$) on singular $p$-simplices in $X$ that vanish on singular $p$-simplices in $A$.

Fortunately, since each $S_p(X, A; R)$ is a free $R$-module, it can be shown that there is a long exact sequence of relative cohomology (see Theorem 4.33):

$$
\begin{aligned}
&\cdots \longrightarrow H^{p-1}(A; G) \longrightarrow \\
&\xrightarrow{\delta^*_{p-1}} H^p(X, A; G) \xrightarrow{(j^\top)^*} H^p(X; G) \xrightarrow{(i^\top)^*} H^p(A; G) \longrightarrow \\
&\xrightarrow{\delta^*_p} H^{p+1}(X, A; G) \xrightarrow{(j^\top)^*} H^{p+1}(X; G) \xrightarrow{(i^\top)^*} H^{p+1}(A; G) \longrightarrow \\
&\xrightarrow{\delta^*_{p+1}} H^{p+2}(X, A; G) \longrightarrow \cdots
\end{aligned}
$$

Relative singular cohomology also satisfies the excision axiom and the homotopy axioms (see Section 4.7).

## 1.3   Duality; Poincaré, Alexander, Lefschetz

Roughly speaking, duality is a kind of symmetry between the homology and the cohomology groups of a space. Historically, duality was formulated only for homology, but it was later found that more general formulations are obtained if both homology and cohomology are considered. We will discuss two duality theorems: Poincaré duality, and Alexander–Lefschetz duality. Original versions of these theorems were stated for homology and applied to special kinds of spaces. It took at least thirty years to obtain the versions that we will discuss.

The result that Poincaré considered as the climax of his work in algebraic topology was a *duality theorem* (even though the notion of duality was not very clear at the time). Since Poincaré was working with finite simplicial complexes, for him duality was a construction which, given a simplicial complex $K$ of dimension $n$, produced a "dual" complex $K^*$; see Munkres [38] (Chapter 8, Section 64). If done the right way, the matrices of the boundary maps $\partial \colon C_p(K) \to C_{p-1}(K)$ are transposes of the matrices of the boundary maps $\partial^* \colon C_{n-p+1}(K) \to C_{n-p}(K)$. As a consequence, the homology groups $H_p(K)$ and $H_{n-p}(K^*)$ are isomorphic. Note that this type of duality relates homology groups, not homology and cohomology groups as it usually does nowadays, for the good reason that cohomology did not exist until about 1935.

Around 1930, De Rham gave a version of Poincaré duality for smooth orientable, compact manifolds. If $M$ is a smooth, oriented, and compact $n$-manifolds, then there are isomorphisms

$$H^p_{\mathrm{dR}}(M) \cong (H^{n-p}_{\mathrm{dR}}(M))^*,$$

where $(H^{n-p}(M))^*$ is the dual of the vector space $H^{n-p}(M)$. This duality is actually induced by a nondegenerate pairing

$$\langle -, - \rangle \colon H^p_{\mathrm{dR}}(M) \times H^{n-p}_{\mathrm{dR}}(M) \to \mathbb{R}$$

given by integration, namely

$$\langle [\omega], [\eta] \rangle = \int_M \omega \wedge \eta,$$

where $\omega$ is a differential $p$-form and $\eta$ is a differential $(n-p)$-form. For details, see Chapter 3, Theorem 3.7. The proof uses several tools from the arsenal of homological algebra: the zig-zag lemma (in the form of Mayer–Vietoris sequences), the five lemma, and an induction on finite "good covers."

Around 1935, inspired by Pontrjagin's duality theorem and his introduction of the notion of nondegenerate pairing (see the end of this section), Alexander and Kolmogoroff independently started developing cohomology, and soon after this it was realized that because cohomology primarily deals with functions, it is possible to define various products. Among those, the *cup product* is particularly important because it induces a multiplication operation on what is called the *cohomology algebra* $H^*(X; R)$ of a space $X$, and the *cap product* yields a stronger version of Poincaré duality.

Recall that $S^*(X; R)$ is the $R$-module $\bigoplus_{p \geq 0} S^p(X; R)$, where the $S^p(X; R)$ are the singular cochain modules. For all $p, q \geq 0$, it possible to define a function

$$\smile : S^p(X; R) \times S^q(X; R) \to S^{p+q}(X; R),$$

called *cup product*. These functions induce a multiplication on $S^*(X; R)$ also called the cup product, which is bilinear, associative, and has an identity element. The cup product satisfies the following equation

$$\delta(c \smile d) = (\delta c) \smile d + (-1)^p c \smile (\delta d),$$

reminiscent of a property of the wedge product. This equation can be used to show that the cup product is a well defined on cohomology classes:

$$\smile : H^p(X; R) \times H^q(X; R) \to H^{p+q}(X; R).$$

These operations induce a multiplication operation on $H^*(X; R) = \bigoplus_{p \geq 0} H^p(X; R)$ which is bilinear and associative. Together with the cup product, $H^*(X; R)$ is called the *cohomology ring* of $X$. For details, see Section 4.8.

The cup product for simplicial cohomology was invented independently by Alexander and Kolmogoroff (in addition to simplicial cohomology) and presented at a conference held in Moscow in 1935. Alexander's original definition was not quite correct and he modified his definition following a suggestion of Čech (1936), independently found by Whitney (1938), who introduced the notation $\smile$. Eilenberg extended the definition of the cup product to singular cohomology (1944).

The significance of the cohomology ring is that two spaces $X$ and $Y$ may have isomorphic cohomology modules but nonisomorphic cohomology rings. Therefore, the cohomology ring is an invariant of a space $X$ that is finer than its cohomology.

Another product related to the cup product is the cap product. The *cap product* combines cohomology and homology classes, it is an operation

$$\frown\colon H^p(X;R) \times H_n(X;R) \to H_{n-p}(X;R);$$

see Section 7.2.

The cap product was introduced by Čech (1936) and independently by Whitney (1938), who introduced the notation $\frown$ and the name *cap product*. Again, Eilenberg generalized the cap product to singular homology and cohomology.

The cup product and the cap product are related by the following equation:

$$a(b \frown \sigma) = (a \smile b)(\sigma)$$

for all $a \in S^{n-p}(X;R)$, all $b \in S^p(X;R)$, and all $\sigma \in S_n(X;R)$, or equivalently using the bracket notation for evaluation as

$$\langle a, b \frown \sigma \rangle = \langle a \smile b, \sigma \rangle,$$

which shows that $\frown$ is the adjoint of $\smile$ with respect to the evaluation pairing $\langle -, - \rangle$.

The reason why the cap product is important is that it can be used to state a sharper version of Poincaré duality. First we need to talk about orientability.

If $M$ is a topological manifold of dimension $n$, it turns out that for every $x \in M$ the relative homology groups $H_p(M, M - \{x\}; \mathbb{Z})$ are either $(0)$ if $p \neq n$, or equal to $\mathbb{Z}$ if $p = n$. An *orientation* of $M$ is a choice of a generator $\mu_x \in H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ for each $x \in M$ which varies "'continuously" with $x$. A manifold that has an orientation is called *orientable*.

Technically, this means that for every $x \in M$, locally on some small open subset $U$ of $M$ containing $x$ there is some homology class $\mu_U \in H_n(M, M - U; \mathbb{Z})$ such that all the chosen $\mu_x \in H_n(M, M - \{x\}; \mathbb{Z})$ for all $x \in U$ are obtained as images of $\mu_U$. If such a $\mu_U$ can be found when $U = M$, we call it a *fundamental class* of $M$ and denote it by $\mu_M$; see Section 7.3. Readers familiar with differential geometry may think of the fundamental form as a discrete analog to the notion of volume form. The crucial result is that a compact manifold of dimension $n$ is orientable iff it has a unique fundamental class $\mu_M$; see Theorem 7.7.

The notion of orientability can be generalized to the notion of $R$-orientability. One of the advantages of this notion is that every manifold is $\mathbb{Z}/2\mathbb{Z}$ orientable. We can now state the Poincaré duality theorem in terms of the cap product.

If $M$ is compact and orientable, then there is a fundamental class $\mu_M$. In this case (if $0 \leq p \leq n$) we have a map

$$D_M\colon H^p(M;\mathbb{Z}) \to H_{n-p}(M;\mathbb{Z})$$

given by

$$D_M(\omega) = \omega \frown \mu_M.$$

*Poincaré duality* asserts that the map

$$D_M \colon \omega \mapsto \omega \frown \mu_M$$

is an isomorphism between $H^p(M; \mathbb{Z})$ and $H_{n-p}(M; \mathbb{Z})$; see Theorem 7.4.

Poincaré duality can be generalized to $R$-orientable manifolds for any commutative ring $R$, to coefficients in any $R$-module $G$, and to noncompact manifolds if we replace cohomology by cohomology with compact support (the modules $H_c^p(X; R)$); see Section 7.3, 7.4, and 7.5. If $R = \mathbb{Z}/2\mathbb{Z}$ Poincaré duality holds for all manifolds, orientable or not.

Another kind of duality was introduced by Alexander in 1922. Alexander considered a compact proper subset $A$ of the sphere $S^n$ ($n \geq 2$) which is a curvilinear cell complex ($A$ has some type of generalized triangulation). For the first time he defined the homology groups of the open subset $S^n - A$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ (so that he did not have to bother with signs), and he proved that for $p \leq n - 2$ there are isomorphisms

$$H_p(A; \mathbb{Z}/2\mathbb{Z}) \cong H_{n-p-1}(S^n - A; \mathbb{Z}/2\mathbb{Z}).$$

Since cohomology did not exist yet, the orginal version of Alexander duality was stated for homology.

Around 1928, Lefschetz started investigating homology with coefficients in $\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}$, or $\mathbb{Q}$, and defined relative homology. In his book published in 1930, using completely different methods from Alexander, Lefschetz proved a version of Alexander's duality in the case where $A$ is a subcomplex of $S^n$. Soon after he obtained a homological version of what we call the Lefschetz duality theorem in Section 14.3 (Theorem 14.9):

$$H^p(M, L; \mathbb{Z}) \cong H_{n-p}(M - L; \mathbb{Z}),$$

where $M$ and $L$ are complexes and $L$ is a subcomplex of $M$.

Both Alexander and Lefschetz duality can be generalized to the situation where in Alexander duality $A$ is an arbitrary closed subset of $S^n$, and in Lefschetz duality $L$ is any compact subset of $M$ and $M$ is orientable, but new kinds of cohomology need to be introduced: *Čech cohomology and Alexander–Spanier cohomology*, which turn out to be equivalent. This is a nontrivial theorem due to Dowker [10]. Then a duality theorem generalizing both Poincaré duality and Alexander–Lefschetz duality can be proved. These matters are discussed in Chapter 10, Section 13.7, and Chapter 14.

Proving the general version of Alexander–Lefschetz duality takes a significant amount of work because it requires defining relative versions of Čech cohomology and Alexander–Spanier cohomology, and to prove their equivalence as well as their equivalence to another definition in terms of direct limits of singular cohomology groups (see Definition 14.13 and Proposition 14.7).

Another mathematician who made important contributions, especially to duality theory, is Pontrjagin. In a paper published in 1931 Pontrjagin investigates the duality between

a closed subset $A$ of $\mathbb{R}^n$ homeomorphic to a simplicial complex and $\mathbb{R}^n - A$. Pontrjagin introduces for the first time the notion of a *nondegenerate pairing* $\varphi \colon U \times V \to G$ between two finitely abelian groups $U$ and $V$, where $G$ is another abelian group (he uses $G = \mathbb{Z}$ or $G = \mathbb{Z}/m\mathbb{Z}$). This is a bilinear map $\varphi \colon U \times V \to G$ such that if $\varphi(u, v) = 0$ for all $v \in V$ then $u = 0$, and if $\varphi(u, v) = 0$ for all $u \in U$ then $v = 0$. Pontrjagin proves that $U$ and $V$ are isomorphic for his choice of $G$, and applies the notion of nondegenerate pairing to Poincaré duality and to a version of Alexander duality for certain subsets of $\mathbb{R}^n$. Pontrjagin also introduces the important notion of *direct limit* (see Section 9.3, Definition 9.8) which, among other things, plays a crucial role in the definition of Čech cohomology and in the construction of a sheaf from a presheaf (see Chapter 11).

In another paper published in 1934, Pontrjagin states and proves his famous duality theory between discrete and compact abelian topological groups. In this situation, $U$ is a discrete group, $G = \mathbb{R}/\mathbb{Z}$, and $V = \widehat{U} = \mathrm{Hom}(U, \mathbb{R}/\mathbb{Z})$ (with the topology of simple convergence). Pontrjagin applies his duality theorem to a version of Alexander duality for compact subsets of $\mathbb{R}^n$ and for a version of Čech homology (cohomology had not been defined yet).

In the next section we introduce Čech Cohomology. It turns out that Čech cohomology accomodates very general types of coefficients, namely *presheaves and sheaves*. In Chapters 9 and 11 we introduce these notions that play a major role in many area of mathematics, especially algebraic geometry and algebraic topology.

One can say that from a historical point of view, all the notions we presented so far are discussed in the landmark book by Eilenberg and Steenrod [12] (1952). This is a beautiful book well worth reading, but it is not for the beginner. The next landmark book is Spanier's [47] (1966). It is easier to read than Eilenberg and Steenrod but still quite demanding.

The next era of algebraic topology begins with the introduction of the notion of sheaf by Jean Leray around 1946.

## 1.4   Presheaves, Sheaves, and Čech Cohomology

The machinery of sheaves is applicable to problems designated by the vague notion of "passage from local to global properties." When some mathematical object attached to a topological space $X$ can be "restricted" to any open subset $U$ of $X$, and that restriction is known for sufficiently small $U$, what can be said about that "global" object? For example, consider the continuous functions defined over $\mathbb{R}^2$ and their restrictions to open subsets of $\mathbb{R}^2$.

Problems of this type had arisen since the 1880's in complex analysis in several variables and had been studied by Poincaré, Cousin, and later H. Cartan and Oka. Beginning in 1942, Leray considered a similar problem in cohomology. Given a space $X$, when the cohomology $H^*(U; G) = \bigoplus_{p \geq 0} H^p(U; G)$ is known for sufficiently small $U$, what can be said about $H^*(X; G) = \bigoplus_{p \geq 0} H^p(X; G)$?

Leray devised some machinery in 1946 that was refined and generalized by H. Cartan, M. Lazard, A. Borel, Koszul, Serre, Godement, and others, to yield the notions of presheaves and sheaves.

Given a topological space $X$ and a class $\mathbf{C}$ of structures (a category), say sets, vector spaces, $R$-modules, groups, commutative rings, *etc.*, a *presheaf on $X$ with values in $\mathbf{C}$* consists of an assignment of some object $\mathcal{F}(U)$ in $\mathbf{C}$ to every open subset $U$ of $X$ and of a map $\mathcal{F}(i)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ of the class of structures in $\mathbf{C}$ to every inclusion $i\colon V \to U$ of open subsets $V \subseteq U \subseteq X$, such that

$$\mathcal{F}(i \circ j) = \mathcal{F}(j) \circ \mathcal{F}(i)$$
$$\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)},$$

for any two inclusions $i\colon V \to U$ and $j\colon W \to V$, with $W \subseteq V \subseteq U$.

Note that the order of composition is switched in $\mathcal{F}(i \circ j) = \mathcal{F}(j) \circ \mathcal{F}(i)$.

Intuitively, the map $\mathcal{F}(i)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ is a *restriction map* if we think of $\mathcal{F}(U)$ and $\mathcal{F}(V)$ as a sets of functions (which is often the case). For this reason, the map $\mathcal{F}(i)\colon \mathcal{F}(U) \to \mathcal{F}(V)$ is also denoted by $\rho_V^U\colon \mathcal{F}(U) \to \mathcal{F}(V)$, and the first equation of Definition 9.1 is expressed by

$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

Presheaves are typically used to keep track of local information assigned to a global object (the space $X$). It is usually desirable to use to consistent local information to recover some global information, but this requires a sharper notion, that of a sheaf.

The motivation for the extra condition that a sheaf should satisfy is this. Suppose we consider the presheaf of continuous functions on a topological space $X$. If $U$ is any open subset of $X$ and if $(U_i)_{i \in I}$ is an open cover of $U$, for any family $(f_i)_{i \in I}$ of continuous functions $f_i\colon U_i \to \mathbb{R}$, if $f_i$ and $f_j$ agree on every overlap $U_i \cap U_j$, then they $f_i$ patch to a unique continuous function $f\colon U \to \mathbb{R}$ whose restriction to $U_i$ is $f_i$.

Given a topological space $X$ and a class $\mathbf{C}$ of structures (a category), say sets, vector spaces, $R$-modules, groups, commutative rings, *etc.*, a *sheaf on $X$ with values in $\mathbf{C}$* is a presheaf $\mathcal{F}$ on $X$ such that for any open subset $U$ of $X$, for every open cover $(U_i)_{i \in I}$ of $U$ (that is, $U = \bigcup_{i \in I} U_i$ for some open subsets $U_i \subseteq U$ of $X$), the following conditions hold:

(G) (Gluing condition) For every family $(f_i)_{i \in I}$ with $f_i \in \mathcal{F}(U_i)$, if the $f_i$ are consistent, which means that
$$\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j) \quad \text{for all } i, j \in I,$$
then there is some $f \in \mathcal{F}(U)$ such that $\rho_{U_i}^U(f) = f_i$ for all $i \in I$.

(M) (Monopresheaf condition) For any two elements $f, g \in \mathcal{F}(U)$, if $f$ and $g$ agree on all the $U_i$, which means that

$$\rho_{U_i}^U(f) = \rho_{U_i}^U(g) \quad \text{for all } i \in I,$$

then $f = g$.

Many (but not all) objects defined on a manifold are sheaves: the smooth functions $C^\infty(U)$, the smooth differential $p$-forms $\mathcal{A}^p(U)$, the smooth vector fields $\mathfrak{X}(U)$, where $U$ is any open subset of $M$.

Given any commutative ring $R$ and a fixed $R$-module $G$, the *constant presheaf* $G_X$ is defined such that $G_X(U) = G$ for all nonempty open subsets $U$ of $X$, and $G_X(\emptyset) = (0)$. The *constant sheaf* $\widetilde{G}_X$ is the sheaf given by $\widetilde{G}_X(U) =$ the set of locally constant functions on $U$ (the functions $f\colon U \to G$ such that for every $x \in U$ there is some open subset $V$ of $U$ containing $x$ such that $f$ is constant on $V$), and $\widetilde{G}_X(\emptyset) = (0)$.

In general, a presheaf is not a sheaf. For example, the constant presheaf is not a sheaf. However, there is a procedure for converting a presheaf to a sheaf. We will return to this process in Section 1.5.

Čech cohomology with values in a presheaf of $R$-modules involves open covers of the topological space $X$.

Apparently, Čech himself did not introduce Čech cohomology, but he did introduce Čech homology using the notion of open cover (1932). Dowker, Eilenberg, and Steenrod introduced Čech cohomology in the early 1950's.

Given a topological space $X$, a family $\mathcal{U} = (U_j)_{j \in J}$ is an *open cover* of $X$ if the $U_j$ are open subsets of $X$ and if $X = \bigcup_{j \in J} U_j$. Given any finite sequence $I = (i_0, \ldots, i_p)$ of elements of some index set $J$ (where $p \geq 0$ and the $i_j$ are not necessarily distinct), we let

$$U_I = U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

Note that it may happen that $U_I = \emptyset$. We denote by $U_{i_0 \cdots \widehat{i_j} \cdots i_p}$ the intersection

$$U_{i_0 \cdots \widehat{i_j} \cdots i_p} = U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_p}$$

of the $p$ subsets obtained by omitting $U_{i_j}$ from $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ (the intersection of the $p+1$ subsets).

Now given a presheaf $\mathcal{F}$ of $R$-modules, the $R$-module of *Čech $p$-cochains* $C^p(\mathcal{U}, \mathcal{F})$ is the set of all functions $f$ with domain $J^{p+1}$ such that $f(i_0, \ldots, i_p) \in \mathcal{F}(U_{i_0 \cdots i_p})$; in other words,

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \ldots, i_p) \in J^{p+1}} \mathcal{F}(U_{i_0 \cdots i_p}),$$

the set of all $J^{p+1}$-indexed families $(f_{i_0, \ldots, i_p})_{(i_0, \ldots, i_p) \in J^{p+1}}$ with $f_{i_0, \ldots, i_p} \in \mathcal{F}(U_{i_0 \cdots i_p})$. Observe that the coefficients (the modules $\mathcal{F}(U_{i_0 \cdots i_p})$) can "vary" from open subset to open subset.

We have $p+1$ inclusion maps

$$\delta_j^p \colon U_{i_0 \cdots i_p} \longrightarrow U_{i_0 \cdots \widehat{i_j} \cdots i_p}, \quad 0 \leq j \leq p.$$

Each inclusion map $\delta_j^p \colon U_{i_0 \cdots i_p} \longrightarrow U_{i_0 \cdots \widehat{i_j} \cdots i_p}$ induces a map

$$\mathcal{F}(\delta_j^p) \colon \mathcal{F}(U_{i_0 \cdots \widehat{i_j} \cdots i_p}) \longrightarrow \mathcal{F}(U_{i_0 \cdots i_p})$$

which is none other that the restriction map $\rho^{U_{i_0 \cdots \widehat{i_j} \cdots i_p}}_{U_{i_0 \cdots i_p}}$ which, for the sake of notational simplicity, we also denote by $\rho^j_{i_0 \cdots i_p}$.

Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, the *coboundary maps* $\delta_{\mathcal{F}}^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ are given by

$$\delta_{\mathcal{F}}^p = \sum_{j=1}^{p+1} (-1)^j \mathcal{F}(\delta_j^{p+1}), \quad p \geq 0.$$

More explicitly, for any $p$-cochain $f \in C^p(\mathcal{U}, \mathcal{F})$, for any sequence $(i_0, \ldots, i_{p+1}) \in J^{p+2}$, we have

$$(\delta_{\mathcal{F}}^p f)_{i_0, \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho^j_{i_0 \cdots i_{p+1}} (f_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}}).$$

Unravelling the above definition for $p = 0$ we have

$$(\delta_{\mathcal{F}}^0 f)_{i,j} = \rho^0_{ij}(f_j) - \rho^1_{ij}(f_i),$$

and for $p = 1$ we have

$$(\delta_{\mathcal{F}}^1 f)_{i,j,k} = \rho^0_{ijk}(f_{j,k}) - \rho^1_{ijk}(f_{i,k}) + \rho^2_{ijk}(f_{i,j}).$$

It is easy to check that $\delta_{\mathcal{F}}^{p+1} \circ \delta_{\mathcal{F}}^p = 0$ for all $p \geq 0$, so we have a chain complex of cohomology

$$0 \xrightarrow{\delta_{\mathcal{F}}^{-1}} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_{\mathcal{F}}^0} C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots \xrightarrow{\delta_{\mathcal{F}}^{p-1}} C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_{\mathcal{F}}^p} C^{p+1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_{\mathcal{F}}^{p+1}} \cdots$$

and we can define the Čech cohomology groups as follows.

Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, the *Čech cohomology groups* $\check{H}^p(\mathcal{U}, \mathcal{F})$ *of the cover* $\mathcal{U}$ *with values in* $\mathcal{F}$ are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} \delta_{\mathcal{F}}^p / \operatorname{Im} \delta_{\mathcal{F}}^{p-1}, \quad p \geq 0.$$

The *classical Čech cohomology groups* $\check{H}^p(\mathcal{U}; G)$ *of the cover* $\mathcal{U}$ *with coefficients in the* $R$-*module* $G$ are the groups $\check{H}^p(\mathcal{U}, G_X)$, where $G_X$ is the constant sheaf on $X$ with values in $G$.

The next step is to define Čech cohomology groups that do not depend on the open cover $\mathcal{U}$. This is achieved by defining a notion of refinement on covers and by taking *direct*

*limits* (see Section 9.3, Definition 9.8). Čech had used such a method in defining his Čech homology groups, by introducing the notion of *inverse limit* (which, curiously, was missed by Pontrjagin whose introduced direct limits!).

Without going into details, given two covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ of a space $X$, we say that $\mathcal{V}$ *is a refinement of* $\mathcal{U}$, denoted $\mathcal{U} \prec \mathcal{V}$, if there is a function $\tau \colon J \to I$ such that

$$V_j \subseteq U_{\tau(j)} \quad \text{for all } j \in J.$$

Under this notion refinement, the open covers of $X$ form a directed preoder, and the family $(\check{H}^p(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ is what is called a direct mapping family so its direct limit

$$\varinjlim_{\mathcal{U}} \; \check{H}^p(\mathcal{U}, \mathcal{F})$$

makes sense. We define the *Čech cohomology groups* $\check{H}^p(X, \mathcal{F})$ *with values in* $\mathcal{F}$ by

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \; \check{H}^p(\mathcal{U}, \mathcal{F}).$$

The *classical Čech cohomology groups* $\check{H}^p(X; G)$ *with coefficients in the R-module G* are the groups $\check{H}^p(X, G_X)$ where $G_X$ is the constant presheaf with value $G$. All this is presented in Chapter 10.

A natural question to ask is how does the classical Čech cohomology of a space compare with other types of cohomology, in particular signular cohomology. In general, Čech cohomology can differ from singular cohomology, but for manifolds it agrees. Classical Čech cohomology also agrees with de Rham cohomology of the constant presheaf $\mathbb{R}_X$. These results are hard to prove; see Chapter 13.

## 1.5   Sheafification and Stalk Spaces

A *map (or morphism)* $\varphi \colon \mathcal{F} \to \mathcal{G}$ of presheaves (or sheaves) $\mathcal{F}$ and $\mathcal{G}$ on $X$ consists of a family of maps $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ of the class of structures in $\mathbf{C}$, for any open subset $U$ of $X$, such that

$$\varphi_V \circ (\rho_{\mathcal{F}})_V^U = (\rho_{\mathcal{G}})_V^U \circ \varphi_U$$

for every pair of open subsets $U, V$ such that $V \subseteq U \subseteq X$. Equivalently, the following diagrams commute for every pair of open subsets $U, V$ such that $V \subseteq U \subseteq X$

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\;\varphi_U\;} & \mathcal{G}(U) \\
{\scriptstyle (\rho_{\mathcal{F}})_V^U} \downarrow & & \downarrow {\scriptstyle (\rho_{\mathcal{G}})_V^U} \\
\mathcal{F}(V) & \xrightarrow[\;\varphi_V\;]{} & \mathcal{G}(V).
\end{array}
$$

The notion of kernel $\operatorname{Ker} \varphi$ and image $\operatorname{Im} \varphi$ of a presheaf or sheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$ is easily defined. The presheaf $\operatorname{Ker} \varphi$ is defined by $(\operatorname{Ker} \varphi)(U) = \operatorname{Ker} \varphi_U$, and the presheaf $\operatorname{Im} \varphi$ is

defined by $(\text{Im } \varphi)(U) = \text{Im } \varphi_U$. In the case or presheaves, they are also presheaves, but in the case of sheaves, the kernel $\text{Ker } \varphi$ is indeed a sheaf, but the image $\text{Im } \varphi$ is **not** a sheaf in general.

This failure of the image of a sheaf map to be a sheaf is a problem that causes significant technical complications. In particular, it is not clear what it means for a sheaf map to be surjective, and a "good" definition of the notion of an exact sequence of sheaves is also unclear.

Fortunately, there is a procedure for converting a presheaf $\mathcal{F}$ into a sheaf $\widetilde{\mathcal{F}}$ which is reasonably well-behaved. This procedure is called *sheafification*. There is a sheaf map $\eta \colon \mathcal{F} \to \widetilde{F}$ which is generally not injective.

The *sheafification* process is universal in the sense that given any presheaf $\mathcal{F}$ and any sheaf $\mathcal{G}$, for any presheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$, there is a unique sheaf map $\widehat{\varphi} \colon \widetilde{\mathcal{F}} \to \mathcal{G}$ such that

$$\varphi = \widehat{\varphi} \circ \eta_{\mathcal{F}}$$

as illustrated by the following commutative diagram



see Theorem 11.11.

The *sheafification* process involves constructing a topological space $\mathcal{SF}$ from the presheaf $\mathcal{F}$ that we call the *stalk space* of $\mathcal{F}$. Godement called it the *espace étalé*. The stalk space is the disjoint union of sets (modules) $\mathcal{F}_x$ called *stalks*. Each stalk $\mathcal{F}$ is the direct limit $\varinjlim(\mathcal{F}(U))_{U \ni x}$ of the family of modules $\mathcal{F}(U)$ for all "small" open sets $U$ containing $x$ (see Definition 11.1). There is a surjective map $p \colon \mathcal{SF} \to X$ which, under the topology given to $\mathcal{SF}$, is a local homeomorphism, which means that for every $y \in \mathcal{SF}$, there is some open subset $V$ of $\mathcal{SF}$ containing $y$ such that the restriction of $p$ to $V$ is a homeomorphism. The sheaf $\widetilde{\mathcal{F}}$ consists of the continuous sections of $p$, that is, the continuous functions $s \colon U \to \mathcal{SF}$ such that $p \circ s = \text{id}_U$, for any open subset $U$ of $X$. This construction is presented in detail in Section 11.1.

The construction of the pair $(\mathcal{SF}, p)$ from a presheaf $\mathcal{F}$ suggests another definition of a sheaf as a pair $(E, p)$, where $E$ is a topological space and $p \colon E \to X$ is a surjective local homeomorphism onto another space $X$. Such a pair $(E, p)$ is often called a *sheaf space*, but we prefer to call it a *stalk space*. This is the definition that was given by H. Cartan and M. Lazard around 1950. The sheaf $\Gamma E$ associated with the stalk space $(E, p)$ is defined as follows: for any open subset $U$ or $X$, the *sections* of $\Gamma E$ are the continuous sections $s \colon U \to E$, that is, the continuous functions such that $p \circ s = \text{id}$. We can also define a notion of map between two stalk spaces.

As this stage, given a topological space $X$ we have three categories:

(1) The category $\mathbf{Psh}(X)$ of presheaves and their morphisms.

(2) The category $\mathbf{Sh}(X)$ of sheaves and their morphisms.

(3) The category $\mathbf{StalkS}(X)$ of stalk spaces and their morphisms.

There is also a functor

$$S \colon \mathbf{PSh}(X) \to \mathbf{StalkS}(X)$$

from the category $\mathbf{PSh}(X)$ to the category $\mathbf{StalkS}(X)$ given by the construction of a stalk space $\mathcal{S}\mathcal{F}$ from a presheaf $\mathcal{F}$, and a functor

$$\Gamma \colon \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$$

from the category $\mathbf{StalkS}(X)$ to the category $\mathbf{Sh}(X)$, given by the sheaf $\Gamma E$ of continuous sections of $E$. Here, we are using the term functor in an informal way. A more precise definition is given in Section 1.7.

Note that every sheaf $\mathcal{F}$ is also a presheaf, and that every map $\varphi \colon \mathcal{F} \to \mathcal{G}$ of sheaves is also a map of presheaves. Therefore, we have an inclusion map

$$i \colon \mathbf{Sh}(X) \to \mathbf{PSh}(X),$$

which is a functor. As a consequence, $S$ restricts to an operation (functor)

$$S \colon \mathbf{Sh}(X) \to \mathbf{StalkS}(X).$$

There is also a map $\eta$ which maps a presheaf $\mathcal{F}$ to the sheaf $\Gamma S(\mathcal{F}) = \widetilde{\mathcal{F}}$. This map $\eta$ is a natural isomorphism between the functors id (the identity functor) and $\Gamma S$ from $\mathbf{Sh}(X)$ to itself.

We can also define a map $\epsilon$ which takes a stalk space $(E, p)$ and makes the stalk space $S\Gamma E$. The map $\epsilon$ is a natural isomorphism between the functors id (the identity functor) and $S\Gamma$ from $\mathbf{StalkS}(X)$ to itself.

Then we see that the two operations (functors)

$$S \colon \mathbf{Sh}(X) \to \mathbf{StalkS}(X) \quad \text{and} \quad \Gamma \colon \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$$

are almost mutual inverses, in the sense that there is a natural isomorphism $\eta$ between $\Gamma S$ and id and a natural isomorphism $\epsilon$ between $S\Gamma$ and id. In such a situation, we say that the classes (categories) $\mathbf{Sh}(X)$ and $\mathbf{StalkS}(X)$ are *equivalent*. The upshot is that it is basically a matter of taste (or convenience) whether we decide to work with sheaves or stalk spaces.

We also have the operator (functor)

$$\Gamma S \colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$$

which "sheafifies" a presheaf $\mathcal{F}$ into the sheaf $\widetilde{\mathcal{F}}$. Theorem 11.11 can be restated as saying that there is an isomorphism

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(\mathcal{F}, i(\mathcal{G})) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\widetilde{\mathcal{F}}, \mathcal{G}),$$

between the set (category) of maps between the presheaves $\mathcal{F}$ and $i(\mathcal{G})$ and the set (category) of maps between the sheaves $\widetilde{\mathcal{F}}$ and $\mathcal{G}$. In fact, such an isomorphism is natural, so in categorical terms, $i$ and $\widetilde{\ } = \Gamma S$ are *adjoint functors*.

All this is explained in Sections 11.2 and 11.3.

## 1.6 Cokernels and Images of Sheaf Maps

We still need to define the image of a sheaf map in such a way that the notion of exact sequence of sheaves makes sense. Recall that if $f \colon A \to B$ is a homomorphism of modules, the *cokernel* $\mathrm{Coker}\, f$ of $f$ is defined by $B/\mathrm{Im}\, f$. It is a measure of the surjectivity of $f$. We also have the projection map $\mathrm{coker}(f) \colon B \to \mathrm{Coker}\, f$, and observe that

$$\mathrm{Im}\, f = \mathrm{Ker}\, \mathrm{coker}(f).$$

The above suggests defining notions of cokernels of presheaf maps and sheaf maps. For a presheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$ this is easy, and we can define the *presheaf cokernel* $\mathrm{PCoker}(\varphi)$. It comes with a presheaf map $\mathrm{pcoker}(\varphi) \colon \mathcal{G} \to \mathrm{PCoker}(\varphi)$.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we define the *sheaf cokernel* $\mathrm{SCoker}(\varphi)$ as the sheafification of $\mathrm{PCoker}(\varphi)$. It also comes with a presheaf map $\mathrm{scoker}(\varphi) \colon \mathcal{G} \to \mathrm{SCoker}(\varphi)$.

Then it can be shown that if $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a sheaf map, $\mathrm{SCoker}(\varphi) = (0)$ iff the stalk maps $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ are surjective for all $x \in X$; see Proposition 11.18.

It follows that the "correct" definition for the image $\mathrm{SIm}\, \varphi$ of a sheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$ is

$$\mathrm{SIm}\, \varphi = \mathrm{Ker}\, \mathrm{scoker}(\varphi).$$

With this definition, a sequence of sheaves

$$\mathcal{F} \xrightarrow{\ \varphi\ } \mathcal{G} \xrightarrow{\ \psi\ } \mathcal{H}$$

is said to be exact if $\mathrm{SIm}\, \varphi = \mathrm{Ker}\, \psi$. Then it can be shown that

$$\mathcal{F} \xrightarrow{\ \varphi\ } \mathcal{G} \xrightarrow{\ \psi\ } \mathcal{H}$$

is an exact sequence of sheaves iff the sequence

$$\mathcal{F}_x \xrightarrow{\ \varphi_x\ } \mathcal{G}_x \xrightarrow{\ \psi_x\ } \mathcal{H}_x$$

is an exact sequence of $R$-modules (or rings) for all $x \in X$; see Proposition 11.23. This second characterization of exactness (for sheaves) is usually much more convenient than the first condition.

The definitions of cokernels and images of presheaves and sheaves as well as the notion of exact sequences of presheaves and sheaves are discussed in Sections 11.4 and 11.5.

## 1.7   Injective and Projective Resolutions; Derived Functors

In order to define, even informally, the concept of derived functor, we need to describe what are functors and exact functors.

Suppose we have two types of structures (categories) $\mathbf{C}$ and $\mathbf{D}$ (for concreteness, think of $\mathbf{C}$ as the class of $R$-modules over some commutative ring $R$ with an identity element 1 and of $\mathbf{D}$ as the class of abelian groups), and we have a transformation $T$ (a functor) which works as follows:

(i) Each object $A$ of $\mathbf{C}$ is mapped to some object $T(A)$ of $\mathbf{D}$.

(ii) Each map $A \xrightarrow{f} B$ between two objects $A$ and $B$ in $\mathbf{C}$ (of example, an $R$-linear map) is mapped to some map $T(A) \xrightarrow{T(f)} T(B)$ between the objects $T(A)$ and $T(B)$ in $\mathbf{D}$ (for example, a homomorphism of abelian groups) in such a way that the following properties hold:

(a) Given any two maps $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ between objects $A, B, C$ in $\mathbf{C}$ such that the composition $A \xrightarrow{g \circ f} C = A \xrightarrow{f} B \xrightarrow{g} C$ makes sense, the composition $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ makes sense in $\mathbf{D}$, and

$$T(g \circ f) = T(g) \circ T(f).$$

(b) If $A \xrightarrow{\mathrm{id}_A} A$ is the identity map of the object $A$ in $\mathbf{C}$, then $T(A) \xrightarrow{T(\mathrm{id}_A)} T(A)$ is the identity map of $T(A)$ in $\mathbf{D}$; that is,

$$T(\mathrm{id}_A) = \mathrm{id}_{T(A)}.$$

Whenever a transformation $T \colon \mathbf{C} \to \mathbf{D}$ satisfies the Properties (i), (ii) (a), (b), we call it a *(covariant) functor* from $\mathbf{C}$ to $\mathbf{D}$.

If $T \colon \mathbf{C} \to \mathbf{D}$ satisfies Properties (i), (b), and if Properties (ii) and (a) are replaced by the Properties (ii') and (a') below

(ii') Each map $A \xrightarrow{f} B$ between two objects $A$ and $B$ in $\mathbf{C}$ is mapped to some map $T(B) \xrightarrow{T(f)} T(A)$ between the objects $T(B)$ and $T(A)$ in $\mathbf{D}$ in such a way that the following properties hold:

(a') Given any two maps $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ between objects $A, B, C$ in $\mathbf{C}$ such that the composition $A \xrightarrow{g \circ f} C = A \xrightarrow{f} B \xrightarrow{g} C$ makes sense, the composition $T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A)$ makes sense in $\mathbf{D}$, and

$$T(g \circ f) = T(f) \circ T(g),$$

then $T$ is called a *contravariant functor* from **C** to **D**.

An example of a (covariant) functor is the functor $\text{Hom}(A, -)$ (for a fixed $R$-module $A$) from $R$-modules to $R$-modules which maps a module $B$ to the module $\text{Hom}(A, B)$ and a module homomorphism $f\colon B \to C$ to the module homomorphism $\text{Hom}(A, f)$ from $\text{Hom}(A, B)$ to $\text{Hom}(A, C)$ given by

$$\text{Hom}(A, f)(\varphi) = f \circ \varphi \quad \text{for all } \varphi \in \text{Hom}(A, B).$$

Another example is the functor $T$ from $R$-modules to $R$-modules such that $T(A) = A \otimes_R M$ for any $R$-module $A$, and $T(f) = f \otimes_R \text{id}_M$ for any $R$-linear map $f\colon A \to B$.

An example of a contravariant functor is the functor $\text{Hom}(-, A)$ (for a fixed $R$-module $A$) from $R$-modules to $R$-modules which maps a module $B$ to the module $\text{Hom}(B, A)$ and a module homomorphism $f\colon B \to C$ to the module homomorphism $\text{Hom}(f, A)$ from $\text{Hom}(C, A)$ to $\text{Hom}(B, A)$ given by

$$\text{Hom}(f, A)(\varphi) = \varphi \circ f \quad \text{for all } \varphi \in \text{Hom}(C, A).$$

Given a type of structures (category) **C** let us denote the set of all maps from an object $A$ to an object $B$ by $\text{Hom}_{\mathbf{C}}(A, B)$. For all the types of structures **C** that we will dealing with, each set $\text{Hom}_{\mathbf{C}}(A, B)$ has some additional structure; namely it is an abelian group.

Categories and functors were introduced by Eilenberg and Mac Lane, first in a paper published in 1942, and then in a more complete paper published in 1945.

Intuitively speaking an *abelian category* is a category in which the notion of kernel and cokernel of a map makes sense. Then we can define the notion of image of a map $f$ as the kernel of the cokernel of $f$, so the notion of exact sequence makes sense, as we did in Section 1.6. The categories of $R$-modules and the categories of sheaves (or presheaves) are abelian categories. For more details, see Section 11.5.

A sequence of $R$-modules and $R$-linear maps (more generally objects and maps between abelian categories)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \qquad (*)$$

is a *short exact sequence* if

(1) $f$ is injective.

(2) $\text{Im } f = \text{Ker } g$.

(3) $g$ is surjective.

According to Dieudonné [8], the notion of exact sequence first appeared in a paper of Hurewicz (1941), and then in a paper of Eilenberg and Steenrod and a paper of H. Cartan, both published in 1945. In 1947, Kelly and Picher generalized the notion of exact sequence

to chain complexes, and apparently introduced the terminology *exact sequence*. In their 1952 treatise [12], Eilenberg and Steenrod took the final step of allowing a chain complex to be indexed by $\mathbb{Z}$ (as we do in Section 2.3).

Given two types of structures (categories) $\mathbf{C}$ and $\mathbf{D}$ in each of which the concept of exactness is defined (abelian categories), given a (additive) functor $T\colon \mathbf{C} \to \mathbf{D}$, by applying $T$ to the short exact sequence $(*)$ we obtain the sequence

$$0 \longrightarrow T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \longrightarrow 0, \qquad\qquad (**)$$

which is a chain complex (since $T(g) \circ T(f) = 0$). Then the following question arises:

Is the sequence $(**)$ also exact?

In general, the answer is **no**, but weaker forms of presevation of exactness suggest themselves.

A functor $T\colon \mathbf{C} \to \mathbf{D}$, is said to be *exact* (resp. *left exact*, *right exact*) if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in $\mathbf{C}$, then the sequence

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow 0$$

is exact in $\mathbf{D}$ (left exact if the sequence

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C)$$

is exact, right exact if the sequence

$$T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow 0$$

is exact).

If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor, then $T$ is said to be *exact* (resp. *left exact*, *right exact*) if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in $\mathbf{C}$, then the sequence

$$0 \longrightarrow T(C) \longrightarrow T(B) \longrightarrow T(A) \longrightarrow 0$$

is exact in $\mathbf{D}$ (left exact if the sequence

$$0 \longrightarrow T(C) \longrightarrow T(B) \longrightarrow T(A)$$

is exact, right exact if the sequence

$$T(C) \longrightarrow T(B) \longrightarrow T(A) \longrightarrow 0$$

is exact).

For example, the functor $\mathrm{Hom}(-, A)$ is *left-exact* but not exact in general (see Section 2.1). Similarly, the functor $\mathrm{Hom}(A, -)$ is *left-exact* but not exact in general (see Section 2.2).

Modules for which the functor $\mathrm{Hom}(A, -)$ is exact play an important role. They are called *projective module*. Similarly, modules for which the functor $\mathrm{Hom}(-, A)$ is exact are called *injective modules*.

The functor $- \otimes_R M$ is *right-exact* but not exact in general (see Section 2.2). Modules $M$ for which the functor $- \otimes_R M$ is exact are called *flat*.

A good deal of homological algebra has to do with understanding how much a module fails to be projective, or injective (or flat).

Injective and projective modules also also characterized by extension properties.

(1) A module $P$ is projective iff for any surjective linear map $h \colon A \to B$ and any linear map $f \colon P \to B$, there is some linear map $\widehat{f} \colon P \to A$ lifting $f \colon P \to B$ in the sense that $f = h \circ \widehat{f}$, as in the following commutative diagram:

$$
\begin{array}{ccc}
 & & P \\
 & \overset{\widehat{f}}{\swarrow} & \big\downarrow {\scriptstyle f} \\
A & \underset{h}{\longrightarrow} & B \longrightarrow 0.
\end{array}
$$

(2) A module $I$ is injective if for any injective linear map $h \colon A \to B$ and any linear map $f \colon A \to I$, there is some linear map $\widehat{f} \colon B \to I$ extending $f \colon A \to I$ in the sense that $f = \widehat{f} \circ h$, as in the following commutative diagram:

$$
\begin{array}{ccc}
0 \longrightarrow & A & \overset{h}{\longrightarrow} B \\
 & {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle \widehat{f}} \\
 & I. &
\end{array}
$$

See Section 12.1.

Injective modules were introduced by Baer in 1940 and projective modules by Cartan and Eilenberg in the early 1950's. Every free module is projective. Injective modules are more elusive. If the ring $R$ is a PID an $R$-module $M$ is injective iff it is divisible (which means that for every nonzero $\lambda \in R$, the map given by $u \mapsto \lambda u$ for $u \in M$ is surjective).

One of the most useful properties of projective modules is that every module $M$ is the image of some projective (even free) module $P$, which means that there is a surjective homomorphism $\rho\colon P \to M$. Similarly, every module $M$ can be embedded in an injective module $I$, which means that there is an injective homomorphism $i\colon M \to I$. This second fact is harder to prove (see Baer's embedding theorem, Theorem 12.6).

The above properties can be used to construct inductively projective and injective resolutions of a module $M$, a process that turns out to be remarkably useful. Intuitively, projective resolutions measure how much a module deviates from being projective, and injective resolutions measure how much a module deviates from being injective,

Hopf introduced free resolutions in 1945. A few years later Cartan and Eilenberg defined projective and injective resolutions.

Given any $R$-module $A$, a *projective resolution* of $A$ is any exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} A \longrightarrow 0 \qquad (*_1)$$

in which every $P_n$ is a projective module. The exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0$$

obtained by truncating the projective resolution of $A$ after $P_0$ is denoted by $\mathbf{P}^A$, and the projective resolution $(*_1)$ is denoted by

$$\mathbf{P}^A \xrightarrow{p_0} A \longrightarrow 0.$$

Given any $R$-module $A$, an *injective resolution* of $A$ is any exact sequence

$$0 \longrightarrow A \xrightarrow{i_0} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^n \xrightarrow{d^n} I^{n+1} \longrightarrow \cdots \qquad (**_1)$$

in which every $I^n$ is an injective module. The exact sequence

$$I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^n \xrightarrow{d^n} I^{n+1} \longrightarrow \cdots$$

obtained by truncating the injective resolution of $A$ before $I^0$ is denoted by $\mathbf{I}_A$, and the injective resolution $(**_1)$ is denoted by

$$0 \longrightarrow A \xrightarrow{i_0} \mathbf{I}_A.$$

Now suppose that we have a functor $T\colon \mathbf{C} \to \mathbf{D}$, where $\mathbf{C}$ is the category of $R$-modules and $\mathbf{D}$ is the category of abelian groups. If we apply $T$ to $\mathbf{P}^A$ we obtain the chain complex

$$0 \longleftarrow T(P_0) \xleftarrow{T(d_1)} T(P_1) \xleftarrow{T(d_2)} \cdots \longleftarrow T(P_{n-1}) \xleftarrow{T(d_n)} T(P_n) \longleftarrow \cdots , \qquad \text{(Lp)}$$

denoted $T(\mathbf{P}^A)$. The above is no longer exact in general but it defines homology groups $H_p(T(\mathbf{P}^A))$.

Similarly If we apply $T$ to $\mathbf{I}_A$ we obtain the cochain complex

$$0 \longrightarrow T(I^0) \xrightarrow{T(d^0)} T(I^1) \xrightarrow{T(d^1)} \cdots \longrightarrow T(I^n) \xrightarrow{T(d^n)} T(I^{n+1}) \longrightarrow \cdots , \qquad \text{(Ri)}$$

denoted $T(\mathbf{I}_A)$. The above is no longer exact in general but it defines cohomology groups $H^p(T(\mathbf{I}_A))$.

The reason why projective resolutions are so special is that even though the homology groups $H_p(T(\mathbf{P}^A))$ appear to depend on the projective resolution $\mathbf{P}^A$, in fact they don't; *the groups $H_p(T(\mathbf{P}^A))$ only depend on $A$ and $T$*. This is proved in Theorem 12.22.

Similarly, the reason why injective resolutions are so special is that even though the cohomology groups $H^p(T(\mathbf{I}_A))$ appear to depend on the injective resolution $\mathbf{I}_A$, in fact they don't; *the groups $H^p(T(\mathbf{I}_A))$ only depend on $A$ and $T$*. This is proved in Theorem 12.21.

Proving the above facts takes some work; we make use of the *comparison theorems*; see Section 12.2, Theorem 12.12 and Theorem 12.15. In view of the above results, given a functor $T$ as above, Cartan and Eilenberg were led to define the *left derived functors $L_nT$* of $T$ by

$$L_nT(A) = H_n(T(\mathbf{P}^A)),$$

for any projective resolution $\mathbf{P}^A$ of $A$, and the *right derived functors $R^nT$* of $T$ by

$$R^nT(A) = H^n(T(\mathbf{I}_A)),$$

for any injective resolution $\mathbf{I}_A$ of $A$. The functors $L_nT$ and $R^nT$ can also be defined on maps. If $T$ is right-exact, then $L_0T$ is isomorphic to $T$ (as a functor), and if $T$ is left-exact, then $R^0T$ is isomorphic to $T$ (as a functor).

For example, the left derived functors of the right-exact functor $T_B(A) = A \otimes B$ (with $B$ fixed) are the "Tor" functors. We have $\text{Tor}_0^R(A, B) \cong A \otimes B$, and the functor $\text{Tor}_1^R(-, G)$ plays an important role in comparing the homology of a chain complex $C$ and the homology of the complex $C \otimes_R G$; see Section 12.5. Čech introduced the functor $\text{Tor}_1^R(-, G)$ in 1935 in terms of generators and relations. It is only after Whitney defined tensor products of arbitrary $\mathbb{Z}$-modules in 1938 that the defintion of Tor was expressed in the intrinsic form that we are now familar with.

There are also versions of left and right derived functors for contravariant functors. For example, the right derived functors of the contravariant left-exact functor $T_B(A) = \text{Hom}_R(A, B)$ (with $B$ fixed) are the "Ext" functors. We have $\text{Ext}_R^0(A, B) \cong \text{Hom}_R(A, B)$, and the functor $\text{Ext}_R^1(-, G)$ plays an important role in comparing the homology of a chain complex $C$ and the cohomology of the complex $\text{Hom}_R(C, G)$; see Section 12.5. The Ext functors were introduced in the context of algebraic topology by Eilenberg and Mac Lane (1942).

Everything we discussed so far is presented in Cartan and Eilenberg's groundbreaking book, Cartan–Eilenberg [7], published in 1956. It is in this book that the name *homological algebra* is introduced. MacLane [29] (1975) and Rotman [40] give more "gentle" presentations (see also Weibel [51] and Eisenbud [13]).

Derived functors can be defined for functors $T\colon \mathbf{C} \to \mathbf{D}$ where $\mathbf{C}$ or $\mathbf{D}$ is a more general category than the category of $R$-modules or the category of abelian groups. For example, in sheaf cohomology, the category $\mathbf{C}$ is the category of sheaves of rings. In general, it suffices that $\mathbf{C}$ and $\mathbf{D}$ are abelian categories.

We say that $\mathbf{C}$ *has enough projectives* if every object in $\mathbf{C}$ is the image of some projective object in $\mathbf{C}$, and that $\mathbf{C}$ *has enough injectives* if every object in $\mathbf{C}$ can be embedded (injectively) into some injective object in $\mathbf{C}$.

There are situations (for example, when dealing with sheaves) where it is useful to know that right derived functors can be computed by resolutions involving objects that are not necessarily injective, but $T$-acyclic, as defined below.

Given a left-exact functor $T\colon \mathbf{C} \to \mathbf{D}$, an object $J \in \mathbf{C}$ is *T-acyclic* if $R^n T(J) = (0)$ for all $n \geq 1$.

The following proposition shows that right derived functors can be computed using $T$-acyclic resolutions.

**Proposition**   Given an additive left-exact functor $T\colon \mathbf{C} \to \mathbf{D}$, for any $A \in \mathbf{C}$ suppose there is an exact sequence

$$0 \longrightarrow A \xrightarrow{\ \epsilon\ } J^0 \xrightarrow{\ d^0\ } J^1 \xrightarrow{\ d^1\ } J^2 \xrightarrow{\ d^2\ } \cdots \tag{$\dagger$}$$

in which every $J^n$ is $T$-acyclic (a right $T$-acyclic resolution $\mathbf{J}^A$). Then for every $n \geq 0$ we have a natural isomorphism between $R^n T(A)$ and $H^n(T(\mathbf{J}_A))$.

The above proposition is used several times in Chapter 13.

## 1.8   Universal $\delta$-Functors

The most important property of derived functors is that short exact sequences yield long exact sequences of homology or cohomology. This property was proved by Cartan and Eilenberg, but Grothendieck realized how crucial it was and this led him to the fundamental concept of *universal $\delta$-functor*. Since we will be using right derived functors much more than left derived functors we state the existence of the long exact sequences of cohomology for right derived functors.

**Theorem**   Assume the abelian category $\mathbf{C}$ has enough injectives, let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an exact sequence in $\mathbf{C}$, and let $T\colon \mathbf{C} \to \mathbf{D}$ be a left-exact (additive) functor.

(1) Then for every $n \geq 0$, there is a map

$$(R^n T)(A'') \xrightarrow{\delta^n} (R^{n+1}T)(A'),$$

and the sequence

$$
\begin{array}{l}
0 \longrightarrow T(A') \longrightarrow T(A) \longrightarrow T(A'') \longrightarrow \\
\qquad\qquad\qquad\quad \delta^0 \\
\longrightarrow (R^1 T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \\
\longrightarrow (R^n T)(A') \longrightarrow (R^n T)(A) \longrightarrow (R^n T)(A'') \longrightarrow \\
\qquad\qquad\qquad\quad \delta^n \\
\longrightarrow (R^{n+1}T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots
\end{array}
$$

is exact. This property is similar to the property of the zig-zag lemma from Section 1.2.

(2) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is another exact sequence in $\mathbf{C}$, and if there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0,
\end{array}
$$

then the induced diagram beginning with

$$
\begin{array}{ccccccc}
0 & \longrightarrow & T(A') & \longrightarrow & T(A) & \longrightarrow & T(A'') \xrightarrow{\delta_A^0} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T(B') & \longrightarrow & T(B) & \longrightarrow & T(B'') \xrightarrow[\delta_B^0]{}
\end{array}
$$

and continuing with

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & R^n T(A') & \longrightarrow & R^n T(A) & \longrightarrow & R^n T(A'') & \xrightarrow{\delta_A^n} & (R^{n+1}T)(A') & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & R^n T(B') & \longrightarrow & R^n T(B) & \longrightarrow & R^n T(B'') & \xrightarrow[\delta_B^n]{} & (R^{n+1}T)(B') & \longrightarrow \cdots
\end{array}
$$

is also commutative.

The proof of this result (Theorem12.25) is fairly involved and makes use of the Horseshoe lemma (Theorem 12.19).

The previous theorem suggests the definition of families of functors originally proposed by Cartan and Eilenberg [7] and then investigated by Grothendieck in his legendary "Tohoku" paper [21] (1957).

A $\delta$-*functors* consists of a countable family $T = (T^n)_{n \geq 0}$ of functors $T^n \colon \mathbf{C} \to \mathbf{D}$ that satisfy the two conditions of the previous theorem. There is a notion of map, also called morphism, between $\delta$-functors.

Given two $\delta$-functors $S = (S^n)_{n \geq 0}$ and $T = (T^n)_{n \geq 0}$ a *morphism* $\eta \colon S \to T$ between $S$ and $T$ is a family $\eta = (\eta^n)_{n \geq 0}$ of natural transformations $\eta^n \colon S^n \to T^n$ such that a certain diagram commutes.

Grothendieck also introduced the key notion of universal $\delta$-functor; see Grothendieck [21] (Chapter II, Section 2.2).

A $\delta$-functor $T = (T^n)_{n \geq 0}$ is *universal* if for every $\delta$-functor $S = (S^n)_{n \geq 0}$ and every natural transformation $\varphi \colon T^0 \to S^0$ there is a *unique* morphism $\eta \colon T \to S$ such that $\eta^0 = \varphi$; we say that $\eta$ *lifts* $\varphi$.

The reason why universal $\delta$-functors are important is the following kind of uniqueness property that shows that a universal $\delta$-functor *is completely determined by the component* $T^0$.

**Proposition**   Suppose $S = (S^n)_{n \geq 0}$ and $T = (T^n)_{n \geq 0}$ are both universal $\delta$-functors and there is an isomorphism $\varphi \colon S^0 \to T^0$ (a natural transformation $\varphi$ which is an isomorphism). Then there is a unique isomorphism $\eta \colon S \to T$ lifting $\varphi$.

One might wonder whether (universal) $\delta$-functors exist. Indeed there are plenty of them.

**Theorem**   Assume the abelian category $\mathbf{C}$ has enough injectives. For every additive left-exact functor $T \colon \mathbf{C} \to \mathbf{D}$, the family $(R^n T)_{n \geq 0}$ of right derived functors of $T$ is a $\delta$-functor. Furthermore $T$ is isomorphic to $R^0 T$.

In fact, the $\delta$-functors $(R^n T)_{n \geq 0}$ are universal.

Grothendieck came up with an ingenious sufficient condition for a $\delta$-functor to be universal: the notion of an *erasable* functor. Since Grothendieck's paper is written in French, this notion defined in Section 2.2 (page 141) of [21] is called *effaçable*, and many books and paper use it. Since the English translation of "effaçable" is "erasable," as advocated by Lang we will use the the English word.

A functor $T \colon \mathbf{C} \to \mathbf{D}$ is *erasable* (or *effaçable*) if for every object $A \in \mathbf{C}$ there is some object $M_A$ and an injection $u \colon A \to M_A$ such that $T(u) = 0$. In particular this will be the case if $T(M_A)$ is the zero object of $\mathbf{D}$.

Our favorite functors, namely the right derived functors $R^n T$, are erasable by injectives for all $n \geq 1$. The following result due to Grothendieck is crucial:

**Theorem**   Let $T = (T^n)_{n \geq 0}$ be a $\delta$-functor between two abelian categories $\mathbf{C}$ and $\mathbf{D}$. If $T^n$ is erasable for all $n \geq 1$, then $T$ is a universal $\delta$-functor.

Finally, by combining the previous results, we obtain the most important theorem about universal $\delta$-functors:

**Theorem** Assume the abelian category **C** has enough injectives. For every left-exact functor $T \colon \mathbf{C} \to \mathbf{D}$, the right derived functors $(R^n T)_{n \geq 0}$ form a universal $\delta$-functor such that $T$ is isomorphic to $R^0 T$. Conversely, every universal $\delta$-functor $T = (T^n)_{n \geq 0}$ is isomorphic to the right derived $\delta$-functor $(R^n T^0)_{n \geq 0}$.

After all, the mysterious universal $\delta$-functors are just the right derived functors of left-exact functors. As an example, the functors $\mathrm{Ext}^n_R(A, -)$ constitute a universal $\delta$-functor (for any fixed $R$-module $A$).

*The machinery of universal $\delta$-functors can be used to prove that different kinds of cohomology theories yield isomorphic groups.* If two cohomology theories $(H^n_S(-))_{n \geq 0}$ and $(H^n_T(-))_{n \geq 0}$ defined for objects in a category **C** (say, topological spaces) are given by universal $\delta$-functors $S$ and $T$ in the sense that the cohomology groups $H^n_S(A)$ and $H^n_T(A)$ are given by $H^n_S(A) = S^n(A)$ and $H^n_T(A) = T^n(A)$ for all objects $A \in \mathbf{C}$, and if $H^0_S(A)$ and $H^0_T(A)$ are isomorphic, then $H^n_S(A)$ and $H^n_T(A)$ are isomorphic for all $n \geq 0$. This technique willl be used in Chapter 13 to prove that sheaf cohomology and Čech cohomology are isomorphic for paracompact spaces.

In the next section, we will see how the machinery of right derived functors can be used to define sheaf cohomology (where the category **C** is the category of sheaves of $R$-modules, the category **D** is the catgeory of abelian groups, and $T$ is the "global section functor").

## 1.9 Sheaf Cohomology

Given a topological space $X$, we define the *global section functor* $\Gamma(X, -)$ such that for every sheaf of $R$-modules $\mathcal{F}$,
$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X).$$
This is a functor from the category $\mathbf{Sh}(X)$ of sheaves of $R$-modules over $X$ to the category of abelian groups.

A sheaf $\mathcal{I}$ is *injective* if for any injective sheaf map $h \colon \mathcal{F} \to \mathcal{G}$ and any sheaf map $f \colon \mathcal{F} \to \mathcal{I}$, there is some sheaf map $\widehat{f} \colon \mathcal{G} \to \mathcal{I}$ extending $f \colon \mathcal{F} \to \mathcal{I}$ in the sense that $f = \widehat{f} \circ h$, as in the following commutative diagram:

$$
\begin{array}{ccc}
0 \longrightarrow \mathcal{F} & \xrightarrow{\ h\ } & \mathcal{G} \\
\ \ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \widehat{f}} & \\
\mathcal{I}. & &
\end{array}
$$

This is the same diagram that we used to define injective modules in Section 1.7, but here, the category involved is the category of sheaves.

A nice feature of the category of sheaves of $R$-modules is that its has enough injectives.

**Proposition** For any sheaf $\mathcal{F}$ of $R$-modules, there is an injective sheaf $\mathcal{I}$ and an injective sheaf homomorphism $\varphi\colon \mathcal{F} \to \mathcal{I}$.

As in the case of modules, the fact that the category of sheaves has enough injectives implies that any sheaf has an injective resolution.

On the other hand, the category of sheaves does not have enough projectives. This is the reason why projective resolutions of sheaves are of little interest.

Another good property is that the global section functor is left-exact. Then as in the case of modules in Section 1.7, the cohomology groups induced by the right derived functors $R^p\Gamma(X, -)$ are well defined.

The *cohomology groups* of the sheaf $\mathcal{F}$ (or the *cohomology groups of $X$ with values in $\mathcal{F}$*), denoted by $H^p(X, \mathcal{F})$, are the groups $R^p\Gamma(X, -)(\mathcal{F})$ induced by the right derived functor $R^p\Gamma(X, -)$ (with $p \geq 0$).

To compute the sheaf cohomology groups $H^p(X, \mathcal{F})$, pick any resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \cdots$$

by injective sheaves $\mathcal{I}^n$, apply the global section functor $\Gamma(X, -)$ to obtain the complex of $R$-modules

$$0 \xrightarrow{\delta^{-1}} \mathcal{I}^0(X) \xrightarrow{\delta^0} \mathcal{I}^1(X) \xrightarrow{\delta^1} \mathcal{I}^2(X) \xrightarrow{\delta^2} \cdots,$$

and then

$$H^p(X, \mathcal{F}) = \operatorname{Ker} \delta^p / \operatorname{Im} \delta^{p-1}.$$

By Theorem 12.35 (stated in the previous section) the right derived functors $R^p\Gamma(X, -)$ constitute a universal $\delta$-functor, so all the properties of $\delta$-functors apply.

In principle, computing the cohomology groups $H^p(X, \mathcal{F})$ requires finding injective resolutions of sheaves. However injective sheaves are very big and hard to deal with. Fortunately, there is a class of sheaves known as *flasque* sheaves (due to Godement) which are $\Gamma(X, -)$-acyclic, and every sheaf has a resolution by flasque sheaves. Therefore, by Proposition 12.27 (stated in the previous section) the cohomology groups $H^p(X, \mathcal{F})$ can be computed using flasque resolutions.

Then we compare sheaf cohomology (defined by derived functors) to the other kinds of cohomology defined so far: de Rham, singular, Čech (for the constant sheaf $\widetilde{G}_X$).

If the space $X$ is paracompact, then it turns out that for any sheaf $\mathcal{F}$, the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ are isomorphic to the cohomology groups $H^p(X, \mathcal{F})$. Furthermore, if $\mathcal{F}$ is a presheaf, then the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ and $\check{H}^p(X, \widetilde{\mathcal{F}})$ are isomorphic, where $\widetilde{\mathcal{F}}$ is the sheafification of $\mathcal{F}$. Several other results (due to Leray and Henri Cartan) about the relationship between Čech cohomology and sheaf cohomology will be stated.

When $X$ is a topological manifold (thus paracompact), for every $R$-module $G$, we will show that the singular cohomology groups $H^p(X; G)$ are isomorphic to the cohomology

groups $H^p(X, \widetilde{G}_X)$ of the constant sheaf $\widetilde{G}_X$. Technically, we will need to define *soft* and *fine* sheaves.

We will also define Alexander–Spanier cohomology and prove that it is equivalent to sheaf cohomology (and Čech cohomology) for paracompact spaces and for the constant sheaf $\widetilde{G}_X$.

In summary, for manifolds, singular cohomology, Čech cohomology, Alexander–Spanier cohomology, and sheaf cohomology all agree (for the constant sheaf $\widetilde{G}_X$). For smooth manifolds, we can add de Rham cohomology to the above list of equivalent cohomology theories, for the constant sheaf $\widetilde{\mathbb{R}}_X$. All these results are presented in Chapter 13.

## 1.10   Suggestions On How to Use This Book

This book basically consists of two parts. The first part covers fairly basic material presented in the first eight chapters. The second part deals with more sophisticated material including sheaves, derived functors, and sheaf cohomology.

Chapter 3 on de Rham cohomology, Chapter 5 on simplicial homology and cohomology, and Chapter 6 on CW-complexes, are written in such a way that they are pretty much independent of each other and of the rest of book, and thus can be safely skipped. Readers who have never heard about differential forms can skip Chapter 3, although of course they will miss a nice facet of the global picture. Chapter 5 on simplicial homology and cohomology was included mostly for historical sake, and because they have a strong combinatorial and computational flavor. Chapter 6 on CW-complexes was included to show that there are tools for computing homology goups and to compensate for the lack of computational flavor of singular homology. However, CW-complexes can't really be understood without a good knowledge of singular homology.

Our feeling is that singular homology is simpler to define than the other homology theories, and since it is also more general, we decided to choose it as our first presentation of homology.

Our main goal is really to discuss cohomology, but except for de Rham cohomology, we feel that a two step process where we first present singular homology, and then singular cohomology as the result of applying the functor $\mathrm{Hom}(-, G)$, is less abrupt than discussing Čech cohomology (or Alexander–Spanier cohomology) first. If the reader prefers, he/she may to go directy to chapter 10.

In any case, we highy recommend first reading the first four sections of Chapter 2. Sections 2.5 and 2.6 can be skipped upon first reading. Next, either proceed with Chapter 3, or skip it, but read Chapter 4 entirely.

After this, we recommend reading Chapter 7 on Poincaré duality, since this is one of the jewels of algebraic topology.

The second part, starting with presheaves and sheaves in Chapter 9, relies on more algebra, especially Chapter 12 on derived functors. However, this is some of the most

beautiful material, so do not be discouraged if the going is tough. Skip proofs upon first reading and try to plow through as much as possible. Stop to take a break, and go back!

One of our goals is to fully prepare the reader to read books like Hartshorne [24] (Chapter III). Others have expressed the same goal, we hope to more successful.

We have borrowed some proofs of Steve Shatz from Shatz and Gallier [46], and many proofs in Chapter 12 are borrowed from Rotman [40]. Generally, we relied heavily on Bott and Tu [2], Bredon [4], Godement [18], Hatcher [25], Milnor and Stasheff [35], Munkres [38], Serre [44], Spanier [47], Tennison [48], and Warner [50]. These are wonderful books, and we hope that reading our book will prepare the reader to study them. We express our gratitude to these authors, and to all the others that have inspired us (including, of course, Dieudonné).

Since we made the decision not to include all proofs (this would have doubled if not tripled the size of the book!), we tried very hard to provide precise pointers to all omitted proofs. This may be irritating to the expert, but we believe that a reader with less knowledge will appreciate this. The reason for including a proof is that we feel that it presents a type of argument that the reader should be exposed to, but this often subjective and a reflection of our personal taste. When we omitted a proof, we tried to give an idea of what it would be, except when it was a really difficult proof. This should be an incentive for the reader to dig into these references.

# Chapter 2

# Homology and Cohomology

This chapter is an introduction to the crucial concepts and results of homological algebra needed to understand homology and cohomology in some depth. The two most fundamental concepts of homological algebra are:

(1) exact sequences.

(2) chain complexes.

Exact sequences are special kinds of chain complexes satisfying additional properties and the purpose of cohomology (and homology) is to "measure" the extend to which a chain complex fails to be an exact sequence. Remarkably, when this machinery is applied to topological spaces or manifolds, it yields some valuable topological information about these spaces.

In their simplest form chain complexes and exact sequences are built from vector spaces but a more powerful theory is obtained (at the cost of minor complications) if the vector spaces are replaced by $R$-modules, where $R$ is a commutative ring with a multiplicative identity element $1 \neq 0$. In particular, if $R = \mathbb{Z}$, then each space is just an abelian group. By a linear map we mean an $R$-linear map.

In Section 2.1 we introduce exact sequences and prove some of their most basic properties. Later in this chapter we prove two of their most important properties, namely the "zig-zag lemma" for cohomology, or long exact sequence of cohomology, and the "five lemma."

## 2.1   Exact Sequences

We begin with the notion of exact sequence.

**Definition 2.1.** A $\mathbb{Z}$-indexed sequence of $R$-modules and $R$-linear maps between them

$$\cdots \longrightarrow A_{p-1} \xrightarrow{f_{p-1}} A_p \xrightarrow{f_p} A_{p+1} \xrightarrow{f_{p+1}} A_{p+2} \longrightarrow \cdots$$

is *exact* if $\text{Im } f_p = \text{Ker } f_{p+1}$[1] for all $p \in \mathbb{Z}$. A sequence of $R$-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a *short exact sequence* if it is exact at $A, B, C$, which means that

1. $\text{Im } f = \text{Ker } g$.

2. $f$ is injective.

3. $g$ is surjective.

Observe that being exact at $A_{p+1}$, that is $\text{Im } f_p = \text{Ker } f_{p+1}$, implies that $f_{p+1} \circ f_p = 0$.

Given a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

since $g$ is surjective, $f$ is injective, and $\text{Im } f = \text{Ker } g$, by the first isomorphism theorem we have

$$C \cong B/\text{Ker } g = B/\text{Im } f \cong B/A.$$

Thus a short exact sequence amounts to a module $B$, a submodule $A$ of $B$, and the quotient module $C \cong B/A$.

The quotient module $B/\text{Im } f$ associated with the $R$-linear map $f \colon A \to B$ is a kind of "dual" of the submodule $\text{Ker } f$ which often comes up when dealing with exact sequences.

**Definition 2.2.** Given any $R$-linear map $f \colon A \to B$, the quotient module $B/\text{Im } f$ is called the *cokernel* of $f$ and is denoted by $\text{Coker } f$.

Observe that $\text{Coker } f = B/\text{Im } f \cong C = \text{Im } g$. Then given an exact sequence

$$\cdots \longrightarrow A_{p-2} \xrightarrow{f_{p-2}} A_{p-1} \xrightarrow{f_{p-1}} A_p \xrightarrow{f_p} A_{p+1} \xrightarrow{f_{p+1}} A_{p+2} \longrightarrow \cdots ,$$

we obtain short exact sequences as follows: if we focus on $A_p$, then there is a surjection $A_p \longrightarrow \text{Im } f_p$, and since $\text{Im } f_p = \text{Ker } f_{p+1}$ this is a surjection $A_p \longrightarrow \text{Ker } f_{p+1}$, and by the first isomorphism theorem and since $\text{Im } f_{p-1} = \text{Ker } f_p$, we have an isomorphism

$$A_p/\text{Im } f_{p-1} = A_p/\text{Ker } f_p \cong \text{Im } f_p = \text{Ker } f_{p+1}.$$

This means that we have the short exact sequence

$$0 \longrightarrow \text{Im } f_{p-1} \longrightarrow A_p \longrightarrow \text{Ker } f_{p+1} \longrightarrow 0.$$

---

[1]A good mnemonic for this equation is *ikea*; $i$ is the first letter in Im, and $k$ is the first letter in Ker.

By a previous remark Coker $f_{p-2} \cong \text{Im } f_{p-1}$, so we obtain the short exact sequence

$$0 \longrightarrow \text{Coker } f_{p-2} \longrightarrow A_p \longrightarrow \text{Ker } f_{p+1} \longrightarrow 0. \qquad (*_{\text{cok}})$$

Short exact sequences of this kind often come up in proofs (for example, the Universal Coefficient Theorems).

If we are dealing with vector spaces (that is, if $R$ is a field), then a standard result of linear algebra asserts that the isomorphism $A_p/\text{Ker } f_p \cong \text{Im } f_p$ yields the direct sum

$$A_p \cong \text{Ker } f_p \oplus \text{Im } f_p = \text{Im } f_{p-1} \oplus \text{Im } f_p.$$

As a consequence, if $A_{p-1}$ and $A_{p+1}$ are finite-dimensional, then so is $A_p$.

When the $R$-module $C$ is free, a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

has some special properties that play a crucial role when we dualize such a sequence.

**Definition 2.3.** A short exact sequence of $R$-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is said to *split* (or to be a *short split exact sequence*) if the submodule $f(A)$ is a direct summand in $B$, which means that $B$ is a direct sum $B = f(A) \oplus D$ for some submodule $D$ of $B$.

If a short exact sequence as in Definition 2.3 splits, since $\text{Im } f = \text{Ker } g$, $f$ is injective and $g$ is surjective, then the restriction of $g$ to $D$ is a bijection onto $C$ so there is an isomorphism $\theta \colon B \to A \oplus C$ defined so that the restriction of $\theta$ to $f(A)$ is equal to $f^{-1}$ and the restriction of $\theta$ to $D$ is equal to $g$.

**Proposition 2.1.** *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*be a short exact sequence of R-modules. The following properties are equivalent.*

*(1) The sequence splits.*

*(2) There is a linear map $p \colon B \to A$ such that $p \circ f = \text{id}_A$.*

*(3) There is a linear map $j \colon C \to B$ such that $g \circ j = \text{id}_C$.*

*Symbolically, we have the following diagram of linear maps:*

$$0 \longrightarrow A \underset{p}{\overset{f}{\rightleftarrows}} B \underset{j}{\overset{g}{\rightleftarrows}} C \longrightarrow 0.$$

*Proof.* It is easy to prove that (1) implies (2) and (3). Since $B = f(A) \oplus D$ for some submodule $D$ of $B$, if $\pi_1 \colon A \oplus D \to A$ is the first projection and $f^{-1} \oplus \mathrm{id}_D \colon f(A) \oplus D \to A \oplus D$ be the isomorphism induced by $f^{-1}$, then let $p = \pi_1 \circ (f^{-1} \oplus \mathrm{id}_D)$. It is clear that $p \circ f = \pi_1 \circ (f^{-1} \oplus \mathrm{id}_D) \circ f = \mathrm{id}_A$. Define $j \colon C \to D$ as the inverse of the restriction of $g$ to $D$ (which is bijective, as we said earlier). Obviously $g \circ j = \mathrm{id}_C$.

If (2) holds, let us prove that

$$B = f(A) \oplus \operatorname{Ker} p.$$

For any $b \in B$, we can write $b = f(p(b)) + (b - f(p(b)))$. Obviously $f(p(b)) \in f(A)$, and since $p \circ f = \mathrm{id}_A$ we have

$$p(b - f(p(b))) = p(b) - p(f(p(b))) = p(b) - (p \circ f)(p(b)) = p(b) - p(b) = 0,$$

so $(b - f(p(b))) \in \operatorname{Ker} p$, which shows that $B = f(A) + \operatorname{Ker} p$. If $b \in f(A) \cap \operatorname{Ker} p$, then $b = f(a)$ for some $a \in A$, so $0 = p(b) = p(f(a)) = a$, and thus $b = f(0) = 0$. We conclude that $B = f(A) \oplus \operatorname{Ker} p$, as claimed.

If (3) holds, let us prove that

$$B = f(A) \oplus \operatorname{Im} j.$$

Since $\operatorname{Im} f = \operatorname{Ker} g$, this is equivalent to

$$B = \operatorname{Ker} g \oplus \operatorname{Im} j.$$

For any $b \in B$, we can write $b = (b - j(g(b))) + j(g(b))$. Clearly $j(g(b)) \in \operatorname{Im} j$, and since $g \circ j = \mathrm{id}_C$ we have

$$g(b - j(g(b)) = g(b) - g(j(g(b))) = g(b) - (g \circ j)(g(b)) = g(b) - g(b) = 0,$$

so $(b - j(g(b))) \in \operatorname{Ker} g$. If $b \in \operatorname{Ker} g \cap \operatorname{Im} j$, then $b = j(c)$ for some $c \in C$, and so $0 = g(b) = g(j(c)) = c$, thus $b = j(c) = j(0) = 0$. We conclude that $B = \operatorname{Ker} g \oplus \operatorname{Im} j$.    $\square$

**Corollary 2.2.** *Let*

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

*be a short exact sequence of R-modules. If $C$ is free then the exact sequence splits.*

*Proof.* Pick a basis $(e_i)_{i \in I}$ in $C$. Define the linear map $j \colon C \to B$ by choosing any vector $b_i \in B$ such that $g(b_i) = e_i$ (since $g$ is surjective, this is possible) and setting $j(e_i) = b_i$. Then

$$(g \circ j)(e_i) = g(b_i) = e_i.$$

so $g \circ j = \mathrm{id}_C$, and by Proposition 2.2 the sequence splits since (3) implies (1).    $\square$

The following example is an exact sequence of abelian groups ($\mathbb{Z}$-modules) that does not split

$$0 \longrightarrow m\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0,$$

where $i$ is the inclusion map and $\pi$ is the projection map such that $\pi(n) = n \bmod m$, the residue of $n$ modulo $m$ (with $m \geq 1$). Indeed, any surjective homomorphism $p$ from $\mathbb{Z}$ to $m\mathbb{Z}$ would have to map 1 to $m$, but then $p \circ i \neq \text{id}$.

Some of the fundamental and heavily used results about exact sequences include the "zig-zag lemma" and the "five lemma." We will encounter these lemma later on. The following (apparently unnamed) result is also used a lot.

**Proposition 2.3.** *Consider any diagram*

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C
\end{array}
$$

*in which the left and right squares commute and $\alpha, \beta, \gamma$ are isomorphisms. If the top row is exact, then the bottom row is also exact.*

*Proof.* The commutativity of the left and right squares implies that

$$\gamma \circ g \circ f = g' \circ f' \circ \alpha.$$

Since the top row is exact, $g \circ f = 0$, so $g' \circ f' \circ \alpha = 0$, and since $\alpha$ is an ismorphism, $g' \circ f' = 0$. It follows that $\text{Im } f' \subseteq \text{Ker } g'$.

Conversely assume that $b' \in \text{Ker } g'$. Since $\beta$ is an isomorphism there is some $b \in B$ such that $\beta(b) = b'$, and since $g'(b') = 0$ we have

$$(g' \circ \beta)(b) = 0.$$

Since the right square commutes $g' \circ \beta = \gamma \circ g$, so

$$(\gamma \circ g)(b) = 0.$$

Since $\gamma$ is an isomorphism, $g(b) = 0$. Since the top row is exact, $\text{Im } f \subseteq \text{Ker } g$, so there is some $a \in A$ such that $f(a) = b$, which implies that

$$(\beta \circ f)(a) = \beta(b) = b'.$$

Since the left square commutes $\beta \circ f = f' \circ \alpha$, and we deduce that

$$f'(\alpha(a)) = b',$$

which proves that $\text{Ker } g' \subseteq \text{Im } f'$. Therefore, $\text{Im } f' \subseteq \text{Ker } g'$, as claimed. $\qquad\square$

A common way to define cohomology is to apply duality to homology so we review duality in $R$-modules to make sure that we are on firm grounds.

**Definition 2.4.** Given an $R$-module $A$, the $R$-module $\mathrm{Hom}(A, R)$ of all $R$-linear maps from $A$ to $R$ (also called *R-linear forms*) is called the *dual* of $A$. Given any two $R$-modules $A$ and $B$, for any $R$-linear map $f\colon A \to B$, the $R$-linear map $f^\top\colon \mathrm{Hom}(B, R) \to \mathrm{Hom}(A, R)$ defined by

$$f^\top(\varphi) = \varphi \circ f \quad \text{for all } \varphi \in \mathrm{Hom}(B, R)$$

is called the *dual linear map* of $f$; see the commutative diagram below:

$$
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & B \\
& {}_{f^\top(\varphi)}\searrow & \downarrow{\varphi} \\
& & R.
\end{array}
$$

The dual linear map $f^\top$ is also denoted by $\mathrm{Hom}(f, R)$ (or $\mathrm{Hom}(f, \mathrm{id}_R)$).

If $f\colon A \to B$ and $g\colon B \to C$ are linear maps of $R$-modules, a simple computation shows that

$$(g \circ f)^\top = f^\top \circ g^\top.$$

Note the reversal in the order of composition of $f^\top$ and $g^\top$. It is also immediately verified that

$$\mathrm{id}_A^\top = \mathrm{id}_{\mathrm{Hom}(A,R)}.$$

Here are some basic properties of the behavior of duality applied to exact sequences.

**Proposition 2.4.** *Let $g\colon B \to C$ be a linear map between $R$-modules.*

(a) *If $g$ is an isomorphism then so is $g^\top$.*

(b) *If $g$ is the zero map then so is $g^\top$.*

(c) *If the sequence*

$$B \overset{g}{\longrightarrow} C \longrightarrow 0$$

*is exact, then the sequence*

$$0 \longrightarrow \mathrm{Hom}(C, R) \overset{g^\top}{\longrightarrow} \mathrm{Hom}(B, R)$$

*is also exact.*

*Proof.* Properties (a) and (b) are immediate and left as an exercise.

Asume that the sequence $B \overset{g}{\longrightarrow} C \longrightarrow 0$ is exact which means that $g$ is surjective. Let $\psi \in \mathrm{Hom}(C, R)$ and assume that $g^\top(\psi) = 0$, which means that $\psi \circ g = 0$, that is, $\psi(g(b)) = 0$ for all $b \in B$. Since $g$ is surjective, we have $\psi(c) = 0$ for all $c \in C$, that is, $\psi = 0$ and $g^\top$ is injective. $\qquad\square$

**Proposition 2.5.** *If the following sequence of $R$-modules*

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*is exact, then the sequence*

$$0 \longrightarrow \mathrm{Hom}(C, R) \xrightarrow{g^\top} \mathrm{Hom}(B, R) \xrightarrow{f^\top} \mathrm{Hom}(A, R)$$

*is also exact. Furthermore, if*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*is a split short exact sequence, then*

$$0 \longrightarrow \mathrm{Hom}(C, R) \xrightarrow{g^\top} \mathrm{Hom}(B, R) \xrightarrow{f^\top} \mathrm{Hom}(A, R) \longrightarrow 0$$

*is also a split short exact sequence.*

*Proof.* Since $g$ is surjective, by Proposition 2.4(c), $g^\top$ is injective. Since $\mathrm{Im}\, f = \ker g$, we have $g \circ f = 0$, so $f^\top \circ g^\top = 0$, which shows that $\mathrm{Im}\, g^\top \subseteq \mathrm{Ker}\, f^\top$. Conversely, we prove that if $f^\top(\psi) = 0$ for some $\psi \in \mathrm{Hom}(B, R)$, then $\psi = g^\top(\varphi)$ for some $\varphi \in \mathrm{Hom}(C, R)$.

Since $f^\top(\psi) = \psi \circ f$, if $f^\top(\psi) = 0$ then $\psi$ vanishes on $f(A)$. Thus $\psi$ induces a linear map $\psi' \colon B/f(A) \to R$ such that $\psi = \psi' \circ \pi$ where $\pi \colon B \to B/f(A)$ is the canonical projection. The exactness of the sequence implies that $g$ induces an isomorphism $g' \colon B/f(A) \to C$, and we have the following commutative diagram:



if we let $\varphi = \psi' \circ (g')^{-1}$, then we have a linear form $\varphi \in \mathrm{Hom}(C, R)$, and

$$g^\top(\varphi) = \varphi \circ g = \psi' \circ (g')^{-1} \circ g = \psi,$$

as desired. Therefore, the dual sequence is exact at $\mathrm{Hom}(B, R)$.

If our short exact sequence is split, then there is a map $p \colon B \to A$ such that $p \circ f = \mathrm{id}_A$, so we get $f^\top \circ p^\top = \mathrm{id}_{\mathrm{Hom}(A,R)}$, which shows that $f^\top$ is surjective, and $p^\top \colon \mathrm{Hom}(A, R) \to \mathrm{Hom}(B, R)$ splits the dual sequence. $\qquad\square$

If $f \colon A \to B$ is injective, then $f^\top \colon \mathrm{Hom}(B, R) \to \mathrm{Hom}(A, R)$ is not necessarily surjective. For example, we have the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where $\times 2(n) = 2n$, but the map $(\times 2)^\top$ is not surjective. This is because for any $\varphi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ we have $(\times 2)^\top(\varphi) = \varphi \circ \times 2$ and this function maps $\mathbb{Z}$ into $2\mathbb{Z}$. Thus the image of $(\times 2)^\top$ is not all of $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$.

Combining Corollary 2.2 and Proposition 2.5 we get the following result.

**Proposition 2.6.** *If*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*is a short exact sequence and if $C$ is a free $R$-module, then*

$$0 \longrightarrow \mathrm{Hom}(C, R) \xrightarrow{g^\top} \mathrm{Hom}(B, R) \xrightarrow{f^\top} \mathrm{Hom}(A, R) \longrightarrow 0$$

*is a split short exact sequence.*

The proposition below will be needed in the proof of the Universal Coefficient Theorem for cohomology (Theorem 12.43).

Let $M$ and $G$ be $R$-modules, and let $B \subseteq Z \subseteq M$ be some submodules of $M$. Define $B^0$ and $Z^0$ by

$$B^0 = \{\varphi \in \mathrm{Hom}(M, G) \mid \varphi(a) = 0 \quad \text{for all } b \in B\}$$
$$Z^0 = \{\varphi \in \mathrm{Hom}(M, G) \mid \varphi(z) = 0 \quad \text{for all } z \in Z\}.$$

**Proposition 2.7.** *For any $R$-modules $M, G$, and $B \subseteq Z \subseteq M$, if $M = Z \oplus Z'$ for some submodule $Z'$ of $M$, then then we have an isomorphism*

$$\mathrm{Hom}(Z/B, G) \cong B^0/Z^0.$$

*Proof.* Define a map $\eta \colon B^0 \to \mathrm{Hom}(Z/B, G)$ as follows: For any $\varphi \in B^0$, that is any $\varphi \in \mathrm{Hom}(M, G)$ such that $\varphi$ vanishes on $B$, let $\eta(\varphi) \in \mathrm{Hom}(Z/B, G)$ be the linear map defined such that

$$\eta(\varphi)(\alpha) = \varphi(z) \quad \text{for any } z \in \alpha \in Z/B.$$

For any other $z' \in \alpha$ we have $z' = z + b$ for some $b \in B$, and then

$$\varphi(z + b) = \varphi(z) + \varphi(b) = \varphi(z)$$

since $\varphi$ vanishes on $B$. Therefore any map $\varphi \in B^0$ is constant on the each equivalence class in $Z/B$, and $\eta(\varphi)$ is well defined. The map $\eta$ is surjective because if $f$ is any linear map in $\mathrm{Hom}(Z/B, G)$, we can define the linear map $\varphi_0 \colon Z \to G$ by

$$\varphi_0(z) = f([z]) \quad \text{for all } [z] \in Z.$$

Since $f \in \mathrm{Hom}(Z/B, G)$, we have $\varphi_0(b) = f([b]) = 0$ for all $b \in B$. Since $M = Z \oplus Z'$, we can extend $\varphi_0$ to a linear map $\varphi \colon M \to G$, for example by setting $\varphi \equiv 0$ on $Z'$, and then $\varphi$ is a map in $\mathrm{Hom}(M, G)$ vanishing on $B$, and by definition $\eta(\varphi) = f$, since

$$\eta(\varphi)([z]) = \varphi(z) = \varphi_0(z) = f([z]) \quad \text{for all } [z] \in Z/B.$$

Finally, for any $\varphi \in B^0$, since $\varphi$ is constant on any equivalence class in $Z/B$, we have $\eta(\varphi) = 0$ iff $\eta(\varphi)([z]) = 0$ for all $[z] \in Z/B$ iff $\varphi(z) = 0$ for all $z \in Z$, iff $\varphi \in Z^0$. Therefore $\mathrm{Ker}\, \eta = Z^0$, and consequently by the First Isomorphism Theorem,

$$B^0/Z^0 \cong \mathrm{Hom}(Z/B, G),$$

as claimed. $\square$

We will also need the next proposition. Let $M$ and $G$ be $R$-modules, and let $B$ be a submodule of $M$. As above, let

$$B^0 = \{f \in \mathrm{Hom}(M, G) \mid f|B \equiv 0\},$$

the set of $R$-linear maps $f \colon M \to G$ that vanish on $B$.

**Proposition 2.8.** *Let $M$ and $G$ be $R$-modules, and let $B$ be a submodule of $M$. There is an isomorphism*

$$\kappa \colon B^0 \to \mathrm{Hom}(M/B, G)$$

*defined by*

$$(\kappa(f))([u]) = f(u) \quad \text{for all } [u] \in M/B.$$

*Proof.* We need to check that the definition of $\kappa(f)$ does not depend on the representative $u \in M$ chosen in the equivalence class $[u] \in M/B$. Indeed, if $v = u + b$ some $b \in B$, we have

$$f(v) = f(u + b) = f(u) + f(b) = f(u),$$

since $f(b) = 0$ for all $b \in B$. The formula $\kappa(f)([u]) = f(u)$ makes it obvious that $\kappa(f)$ is linear since $f$ is linear. The mapping $\kappa$ is injective. This is because if $\kappa(f_1) = \kappa(f_2)$, then

$$\kappa(f_1)([u]) = \kappa(f_2)([u])$$

for all $u \in M$, and because $\kappa(f_1)([u]) = f_1(u)$ and $\kappa(f_2)([u]) = f_2(u)$, we get $f_1(u) = f_2(u)$ for all $u \in M$, that is, $f_1 = f_2$. The mapping $\kappa$ is surjective because given any linear map $\varphi \in \mathrm{Hom}(M/B, G)$, if we define $f$ by

$$f(u) = \varphi([u])$$

for all $u \in M$, then $f$ is linear, vanishes on $B$, and clearly, $\kappa(f) = \varphi$. Therefore, we have the isomorphism $\kappa \colon B^0 \to \mathrm{Hom}(M/B, G)$, as claimed. $\square$

**Remark:** Proposition 2.8 is actually the special case of Proposition 2.7 where $Z = (0)$, since in this case $Z^0 = M$ and $Z' = M$. We feel that it is still instructive to give a direct proof of Proposition 2.8.

If we look carefully at the proofs of Propositions 2.4–2.6, we see that they go through with the ring $R$ replaced by any fixed $R$-module $A$. This suggests looking at more general versions of Hom.

## 2.2   The Functors $\mathrm{Hom}(-, A)$, $\mathrm{Hom}(A, -)$, and $- \otimes A$

In this section we consider several operators $T$ on $R$-modules that map a module $A$ to another module $T(A)$, and a module homomorphism $f \colon A \to B$ to a module homomorphism $T(f) \colon T(A) \to T(B)$, or to a homomorphism $T(f) \colon T(B) \to T(A)$ (note the reversal). Given any two module homomorphism $f \colon A \to B$ and $g \colon B \to C$, if $T$ does not reverse the direction of maps then $T(g \circ f) = T(g) \circ T(f)$, else $T(g \circ f) = T(f) \circ T(g)$. We also have $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for all $A$. Such operators are called *functors* (*covariant* in the first case, *contravariant* if it reverses the direction of maps).

We begin with the $\mathrm{Hom}_R(-, A)$-functor, which reverses the direction of the maps.

**Definition 2.5.** Given a fixed $R$-module $A$, for any $R$-module $B$ we denote by $\mathrm{Hom}_R(B, A)$ the $R$-module of all $R$-linear maps from $B$ to $A$. Given any two $R$-modules $B$ and $C$, for any $R$-linear map $f \colon B \to C$, the $R$-linear map $\mathrm{Hom}_R(f, A) \colon \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(B, A)$ is defined by

$$\mathrm{Hom}_R(f, A)(\varphi) = \varphi \circ f \quad \text{for all } \varphi \in \mathrm{Hom}_R(C, A);$$

see the commutative diagram below:

$$
\begin{array}{ccc}
B & \xrightarrow{\ f\ } & C \\
 & \searrow{\scriptstyle \mathrm{Hom}_R(f,A)(\varphi)} & \downarrow{\scriptstyle \varphi} \\
 & & A.
\end{array}
$$

Observe that $\mathrm{Hom}_R(f, A)(\varphi)$ is $\varphi$ composed with $f$, that is its result is to *pull back* along $f$ any map $\varphi$ from $C$ to $A$ to a map from $B$ to $A$.[2] The map $\mathrm{Hom}_R(f, A)$ is also denoted by $\mathrm{Hom}_R(f, \mathrm{id}_A)$, or for short $\mathrm{Hom}_R(f, \mathrm{id})$. Some authors denote $\mathrm{Hom}_R(f, A)$ by $f^*$.

If $f \colon B \to C$ and $g \colon C \to D$ are linear maps of $R$-modules, a simple computation shows that

$$\mathrm{Hom}_R(g \circ f, A) = \mathrm{Hom}_R(f, A) \circ \mathrm{Hom}_R(g, A).$$

Observe that $\mathrm{Hom}_R(f, A)$ and $\mathrm{Hom}_R(g, A)$ are composed in the reverse order of the composition of $f$ and $g$. It is also immediately verified that

$$\mathrm{Hom}_R(\mathrm{id}_A, A) = \mathrm{id}_{\mathrm{Hom}_R(A,A)}.$$

---

[2] A trick to remember that $\mathrm{Hom}_R(f, A)$ composes $\varphi$ on the left of $f$ is that $f$ is the leftmost argument in $\mathrm{Hom}_R(f, A)$.

Thus, $\mathrm{Hom}_R(-, A)$ is a (contravariant) functor. To simplify notation, we usually omit the subscript $R$ in $\mathrm{Hom}_R(-, A)$ unless confusion arises.

**Proposition 2.9.** *Let $A$ be any fixed $R$-module and let $g\colon B \to C$ be a linear map between $R$-modules.*

(a) *If $g$ is an isomorphism then so is $\mathrm{Hom}(g, A)$.*

(b) *If $g$ is the zero map then so is $\mathrm{Hom}(g, A)$.*

(c) *If the sequence*

$$B \xrightarrow{\ g\ } C \longrightarrow 0$$

*is exact, then the sequence*

$$0 \longrightarrow \mathrm{Hom}(C, A) \xrightarrow{\ \mathrm{Hom}(g, A)\ } \mathrm{Hom}(B, A)$$

*is also exact.*

The proof of Proposition 2.9 is identical to the proof of Proposition 2.4.

**Proposition 2.10.** *Let $M$ be any fixed $R$-module. If the following sequence of $R$-modules*

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

*is exact, then the sequence*

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{\ \mathrm{Hom}(g, M)\ } \mathrm{Hom}(B, M) \xrightarrow{\ \mathrm{Hom}(f, M)\ } \mathrm{Hom}(A, M)$$

*is also exact. Furthermore, if*

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

*is a split short exact sequence, then*

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{\ \mathrm{Hom}(g, M)\ } \mathrm{Hom}(B, M) \xrightarrow{\ \mathrm{Hom}(f, M)\ } \mathrm{Hom}(A, M) \longrightarrow 0$$

*is also a split short exact sequence.*

The proof of Proposition 2.10 is identical to the proof of Proposition 2.5. We say that $\mathrm{Hom}(-, M)$ is a *left-exact* functor.

**Remark:** It can be shown that the sequence

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

is exact iff the sequence

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{\ \mathrm{Hom}(g, M)\ } \mathrm{Hom}(B, M) \xrightarrow{\ \mathrm{Hom}(f, M)\ } \mathrm{Hom}(A, M)$$

is exact for all $R$-modules $M$; see Dummit and Foote [11] (Chapter 10, Theorem 33).

**Proposition 2.11.** *Let $M$ be any fixed $R$-module. If*

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

*is a short exact sequence and if $C$ is a free $R$-module, then*

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{\ \mathrm{Hom}(g,M)\ } \mathrm{Hom}(B, M) \xrightarrow{\ \mathrm{Hom}(f,M)\ } \mathrm{Hom}(A, M) \longrightarrow 0$$

*is a split short exact sequence.*

There is also a version of the Hom-functor, $\mathrm{Hom}_R(A, -)$, in which the first slot is held fixed.

**Definition 2.6.** Given a fixed $R$-module $A$, for any $R$-module $B$ we denote by $\mathrm{Hom}_R(A, B)$ the $R$-module of all $R$-linear maps from $A$ to $B$. Given any two $R$-modules $B$ and $C$, for any $R$-linear map $f\colon B \to C$, the $R$-linear map $\mathrm{Hom}_R(A, f)\colon \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, C)$ is defined by

$$\mathrm{Hom}_R(A, f)(\varphi) = f \circ \varphi \quad \text{for all } \varphi \in \mathrm{Hom}_R(A, B);$$

see the commutative diagram below:



Observe that $\mathrm{Hom}_R(A, f)(\varphi)$ is $f$ composed with $\varphi$, that is its result is to *push forward* along $f$ any map $\varphi$ from $A$ to $B$ to a map from $A$ to $C$.[3] The map $\mathrm{Hom}_R(A, f)$ is also denoted by $\mathrm{Hom}_R(\mathrm{id}_A, f)$, or for short $\mathrm{Hom}_R(\mathrm{id}, f)$. Some authors denote $\mathrm{Hom}_R(A, f)$ by $f_*$.

If $f\colon B \to C$ and $g\colon C \to D$ are linear maps of $R$-modules, a simple computation shows that

$$\mathrm{Hom}_R(A, g \circ f) = \mathrm{Hom}_R(A, g) \circ \mathrm{Hom}_R(A, f).$$

It is also immediately verified that

$$\mathrm{Hom}_R(\mathrm{id}_A, A) = \mathrm{id}_{\mathrm{Hom}_R(A,A)}.$$

Thus, $\mathrm{Hom}_R(A, -)$ is a (covariant) functor.

The $\mathrm{Hom}_R(A, -)$-functor has properties analogous to those of the $\mathrm{Hom}_R(-, A)$-functor, except that sequences are not reversed. Again, to simplify notation, we usually omit the subscript $R$ in $\mathrm{Hom}_R(A, -)$ unless confusion arises.

---

[3]A trick to remember that $\mathrm{Hom}_R(A, f)$ composes $\varphi$ on the right of $f$ is that $f$ is the rightmost argument in $\mathrm{Hom}_R(A, f)$.

**Proposition 2.12.** *Let $M$ be any fixed $R$-module. If the following sequence of $R$-modules*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

*is exact, then the sequence*

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M, B) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}(M, C)$$

*is also exact. Furthermore, if*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*is a split short exact sequence, then*

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M, B) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}(M, C) \longrightarrow 0$$

*is also a split short exact sequence.*

The proof of Proposition 2.12 is left as an exercise. We say that $\mathrm{Hom}(M, -)$ is a *left-exact* functor.

If $f \colon A \to B$ is surjective, then $\mathrm{Hom}(C, f) \colon \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$ is not necessarily surjective. For example, we have the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where $\times 2(n) = 2n$, but if $C = \mathbb{Z}/2\mathbb{Z}$, the map

$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \pi) \colon \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

is not surjective. This is because any map $\varphi \colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must map 1 to 0. In $\mathbb{Z}/2\mathbb{Z}$ we have $1 + 1 = 0$, so $\varphi(1 + 1) = \varphi(0) = 0$, but if $\varphi(1) \neq 0$, then $\varphi(1 + 1) = \varphi(1) + \varphi(1) = 2\varphi(1) \neq 0$ in $\mathbb{Z}$, a contradiction. Therefore, $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = (0)$, and yet $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ contains the identity map.

**Remark:** It can be shown that the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact iff the sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M, B) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}(M, C)$$

is exact for all $R$-modules $M$. See Dummit and Foote [11] (Chapter 10, Theorem 28).

**Proposition 2.13.** *Let $M$ be any fixed $R$-module. If*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*is a short exact sequence and if $C$ is a free $R$-module, then*

$$0 \longrightarrow \operatorname{Hom}(M, A) \xrightarrow{\operatorname{Hom}(M,f)} \operatorname{Hom}(M, B) \xrightarrow{\operatorname{Hom}(M,g)} \operatorname{Hom}(M, C) \longrightarrow 0$$

*is a split short exact sequence.*

A more complete discussion of the functor $\operatorname{Hom}(-, A)$ is found in Munkres [38] (Chapter 5, §41), and a thorough presentation in MacLane [29], Cartan–Eilenberg [7], Rotman [40], and Weibel [51].

Another operation on modules that plays a crucial role is the tensor product. Let $M$ be a fixed $R$-module. For any $R$-module $A$, we have the $R$-module $A \otimes_R M$, and for any $R$-linear map $f \colon B \to C$ we have the $R$-linear map $f \otimes_R \operatorname{id}_M \colon B \otimes_R M \to C \otimes_R M$. To simplify notation, unless confusion arises, we will drop the subscript $R$ on $\otimes_R$.

If $f \colon B \to C$ and $g \colon C \to D$ are linear maps of $R$-modules, a simple computation shows that

$$(g \otimes \operatorname{id}_M) \circ (f \otimes \operatorname{id}_M) = (g \circ f) \otimes \operatorname{id}_M.$$

It is also immediately verified that

$$\operatorname{id}_M \otimes \operatorname{id}_M = \operatorname{id}_{M \otimes M}.$$

Thus, $- \otimes M$ is a (covariant) functor. Similarly we have the functor $M \otimes -$ obtained by holding the first slot fixed. This functor has the same properties as $- \otimes M$ so we will not consider it any further.

We would like to understand the behavior of the functor $- \otimes M$ with respect to exact sequences.

A crucial fact is that if $f \colon B \to C$ is injective, then $f \otimes \operatorname{id}_M$ may not be injective. For example, if we let $R = \mathbb{Z}$, then the inclusion map $i \colon \mathbb{Z} \to \mathbb{Q}$ is injective, but if $M = \mathbb{Z}/2\mathbb{Z}$, then

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = (0),$$

since we can write

$$a \otimes b = (a/2) \otimes (2b) = (a/2) \otimes 0 = 0.$$

Thus, $i \otimes \operatorname{id}_M \colon \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} = i \otimes \operatorname{id}_M \colon \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to (0)$, which is not injective. Thus, $- \otimes M$ is not left-exact. However, it is right-exact, as we now show.

**Proposition 2.14.** *Let $f \colon A \to B$ and $f' \colon A' \to B'$ be two $R$-linear maps. If $f$ and $f'$ are surjective then*

$$f \otimes f' \colon A \otimes A' \to B \otimes B'$$

*is surjective, and its kernel $\operatorname{Ker}(f \otimes f')$ is spanned by all tensors of the form $a \otimes a'$ for which either $a \in \operatorname{Ker} f$ or $a' \in \operatorname{Ker} f'$.*

*Proof.* Let $H$ be the submodule of $A \otimes A'$ spanned by all tensors of the form $a \otimes a'$ for which either $a \in \mathrm{Ker}\, f$ or $a' \in \mathrm{Ker}\, f'$. Obviously, $f \otimes f'$ vanishes on $H$, so it factors through a $R$-linear map

$$\Phi \colon (A \otimes A')/H \to B \otimes B'$$

as illustrated in the following diagram:

$$A \otimes A' \xrightarrow{\ \pi\ } (A \otimes A')/H$$

$$f \otimes f' \searrow \quad \downarrow \Phi$$

$$B \otimes B'.$$

We prove that $\Phi$ is an isomorphism by defining an inverse $\Psi$ for $\Phi$. We begin by defining a function

$$\psi \colon B \times B' \to (A \otimes A')/H$$

by setting

$$\psi(b, b') = \overline{a_1 \otimes a_1'}$$

for all $b \in B$ and all $b' \in B'$, where $a_1 \in A$ is any element such that $f(a_1) = b$ and $a_1' \in A'$ is any element such that $f'(a_1') = b'$, which exist since $f$ and $f'$ are surjective. We need to check that $\psi$ does not depend on the choice of $a_1 \in f^{-1}(b)$ and $a_1' \in (f')^{-1}(b')$. If $f(a_2) = b$ and $f'(a_2') = b'$, with $a_2 \in A$ and $a_2' \in A'$, since we can write

$$a_1 \otimes a_1' - a_2 \otimes a_2' = (a_1 - a_2) \otimes a_1' + a_2 \otimes (a_1' - a_2'),$$

and since $f(a_1 - a_2) = f(a_1) - f(a_2) = b - b = 0$, and $f'(a_1' - a_2') = f'(a_1') - f'(a_2') = b' - b' = 0$, we see that $a_1 \otimes a_1' - a_2 \otimes a_2' \in H$, thus

$$\overline{a_1 \otimes a_1'} = \overline{a_2 \otimes a_2'}$$

an $\psi$ is well defined. We check immediately that $\psi$ is $R$-bilinear, so $\psi$ induces a $R$-linear map

$$\Psi \colon B \otimes B' \to (A \otimes A')/H.$$

It remains to check that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identity maps, which is easily verified on generators. $\qquad \square$

We can now show that $- \otimes M$ is right-exact.

**Proposition 2.15.** *Suppose the sequence*

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

*is exact. Then the sequence*

$$A \otimes M \xrightarrow{\ f \otimes \mathrm{id}_M\ } B \otimes M \xrightarrow{\ g \otimes \mathrm{id}_M\ } C \otimes M \longrightarrow 0$$

*is exact. If $f$ is injective and the first sequence splits, then $f \otimes \mathrm{id}_M$ is injective and the second sequence splits.*

*Proof.* Since the first sequence is exact, $g$ is surjective and Proposition 2.14 implies that $g \otimes \mathrm{id}_M$ is surjective, and that its kernel $H$ is the submodule of $B \otimes M$ spanned by all elements of the form $b \otimes z$ with $b \in \mathrm{Ker}\, g$ and $z \in M$. On the other hand the image $D$ of $f \otimes \mathrm{id}_M$ is the submodule spanned by all elements of the form $f(a) \otimes z$, with $a \in A$ and $z \in M$. Since $\mathrm{Im}\, f = \mathrm{Ker}\, g$, we have $H = D$; that is, $\mathrm{Im}\, (f \otimes \mathrm{id}_M) = \mathrm{Ker}\, (g \otimes \mathrm{id}_M)$.

Suppose that $f$ is injective and the first sequence splits. Let $p \colon B \to A$ be a $R$-linear map such that $p \circ f = \mathrm{id}_A$. Then

$$(p \otimes \mathrm{id}_M) \circ (f \circ \mathrm{id}_M) = (p \circ f) \otimes (\mathrm{id}_M \circ \mathrm{id}_M) = \mathrm{id}_A \otimes \mathrm{id}_M = \mathrm{id}_{A \otimes M},$$

so $f \otimes \mathrm{id}_M$ is injective and $p \otimes \mathrm{id}_M$ splits the second sequence. $\qquad\square$

Proposition 2.15 says that the functor $- \otimes M$ is *right-exact*. A more complete discussion of the functor $- \otimes M$ is found in Munkres [38] (Chapter 6, §50), and a thorough presentation in MacLane [29], Cartan–Eilenberg [7], Rotman [40], and Weibel [51].

## 2.3    Abstract Chain Complexes and Their Cohomology

The notion of a chain complex is obtained from the notion of an exact sequence by relaxing the requirement $\mathrm{Im}\, f_p = \mathrm{Ker}\, f_{p+1}$ to $f_{p+1} \circ f_p = 0$.

**Definition 2.7.** A *(differential) complex* (or *chain complex*) is a $\mathbb{Z}$-graded $R$-module

$$C = \bigoplus_{p \in \mathbb{Z}} C^p,$$

together with a $R$-linear map

$$d \colon C \to C$$

such that $dC^p \subseteq C^{p+1}$ and $d \circ d = 0$. We denote the restriction of $d$ to $C^p$ by $d^p \colon C^p \to C^{p+1}$.

Given a complex $(C, d)$, we define the $\mathbb{Z}$-graded $R$-modules

$$B^*(C) = \mathrm{Im}\, d, \qquad Z^*(C) = \mathrm{Ker}\, d.$$

Since $d \circ d = 0$, we have

$$B^*(C) \subseteq Z^*(C) \subseteq C$$

so the quotient spaces $Z^p(C)/B^p(C)$ make sense and we can define cohomology.

**Definition 2.8.** Given a differential complex $(C, d)$ of $R$-modules, we define the *cohomology space* $H^*(C)$ by

$$H^*(C) = \bigoplus_{p \in \mathbb{Z}} H^p(C),$$

where the $p$th cohomology group ($R$-module) $H^p(C)$ is the quotient space

$$H^p(C) = (\operatorname{Ker} d \cap C^p)/(\operatorname{Im} d \cap C^p) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1} = Z^p(C)/B^p(C).$$

Elements of $C^p$ are called *p-cochains* or *cochains*, elements of $Z^p(C)$ are called *p-cocycles* or *cocycles*, and elements of $B^p(C)$ are called *p-coboundaries* or *coboundaries*. Given a cocycle $a \in Z^p(C)$, its *cohomology class* $a + \operatorname{Im} d^{p-1}$ is denoted by $[a]$. A complex $C$ is said to be *acyclic* if its cohomology is trivial, that is $H^p(C) = (0)$ for all $p$, which means that $C$ is an exact sequence.

We often drop the complex $C$ when writing $Z^p(C)$, $B^p(C)$ of $H^p(C)$.

Typically, when dealing with cohomology we consider chain complexes such that $C^p = (0)$ for all $p < 0$:

$$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \longrightarrow C^{p-1} \xrightarrow{d^{p-1}} C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} C^{p+2} \longrightarrow \cdots$$

We can deal with homology by assuming that $C^p = (0)$ for all $p > 0$. In this case, we have a chain complex of the form

$$\cdots \longrightarrow C^{-(p+1)} \xrightarrow{d^{-(p+1)}} C^{-p} \xrightarrow{d^{-p}} C^{-(p-1)} \xrightarrow{d^{-(p-1)}} \cdots \longrightarrow C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} 0.$$

It is customary to use positive indices and to convert the above diagram to the diagram shown below in which every negative upper index $-p$ is replaced by the positive lower index $p$

$$\cdots \longrightarrow C_{p+1} \xrightarrow{d_{p+1}} C_p \xrightarrow{d_p} C_{p-1} \xrightarrow{d_{p-1}} \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0.$$

An equivalent diagram is obtained by also reversing the direction of the arrows:

$$0 \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \xleftarrow{d_{p-1}} C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \longleftarrow \cdots .$$

Which diagram is preferred is a matter of taste.[4] We also denote the space $H^{-p}(C)$ (where $p \geq 0$) by $H_p(C)$ and call it the $p$th *homology space*. Note that

$$H_p(C) = \operatorname{Ker} d_p / \operatorname{Im} d_{p+1},$$

and if we write $Z_p(C) = \operatorname{Ker} d_p$ and $B_p(C) = \operatorname{Im} d_{p+1}$, we also have

$$H_p(C) = Z_p(C)/B_p(C),$$

elements of $C_p$ are called *chains*, elements of $Z_p(C)$ are called *cycles*, and elements of $B_p(C)$ are called *boundaries*. Singular homology defined in Section 4.6 is such an example.

---

[4]Notice that applying $\operatorname{Hom}(-, R)$ to the second diagram reverses all the arrows so that a complex of cohomology is obtained. For this reason, we have a slight preference for the second diagram.

**Remark:** When dealing with cohomology, it is customary to use superscripts for denoting the cochains groups $C^p$, the cohomology groups $H^p(C)$, the coboundary maps $d^p$, *etc.*, and to write complexes with the arrows going from left to right so that the superscripts increase. However, when dealing with homology, it is customary to use subscripts for denoting the chains groups $C_p$, the homology groups $H_p(C)$, the boundary maps $d_p$, *etc.*, and to write homology complexes with decreasing indices and arrows going from left to right, or complexes with increasing indices and arrows going from right to left. In homology, the boundary maps $d_p \colon C_p \to C_{p-1}$ are usually denoted by $\partial_p$, and in cohomology the coboundary maps $d^p \colon C^p \to C^{p+1}$ are usually denoted by $\delta^p$.

## 2.4   Chain Maps and Chain Homotopies

We know that homomorphisms between $R$-modules play a very important role in the theory of $R$-modules. There are two notions of maps between chain complexes that also play an important role in homology and cohomology theory.

**Definition 2.9.** Given two complexes $(C, d_C)$ and $(D, d_D)$, a *chain map* $f \colon C \to D$ is a family $f = (f^p)$ of $R$-linear maps $f^p \colon C^p \to D^p$ such that

$$d_D \circ f^p = f^{p+1} \circ d_C \quad \text{for all } p \in \mathbb{Z},$$

equivalently all the squares in the following diagram commute:

$$
\begin{array}{ccccccccc}
\cdots \xrightarrow{\;d_C\;} & C^{p-1} & \xrightarrow{\;d_C\;} & C^p & \xrightarrow{\;d_C\;} & C^{p+1} & \xrightarrow{\;d_C\;} & C^{p+2} & \xrightarrow{\;d_C\;} \cdots \\
& \downarrow{\scriptstyle f^{p-1}} & & \downarrow{\scriptstyle f^p} & & \downarrow{\scriptstyle f^{p+1}} & & \downarrow{\scriptstyle f^{p+2}} & \\
\cdots \xrightarrow{\;d_D\;} & D^{p-1} & \xrightarrow{\;d_D\;} & D^p & \xrightarrow{\;d_D\;} & D^{p+1} & \xrightarrow{\;d_D\;} & D^{p+2} & \xrightarrow{\;d_D\;} \cdots
\end{array}
$$

A chain map of complexes $f \colon C \to D$ induces a map $f^* \colon H^*(C) \to H^*(D)$ between the cohomology spaces $H^*(C)$ and $H^*(D)$, which means that each map $f^p \colon C^p \to D^p$ induces a homomorphism $(f^p)^* \colon H^p(C) \to H^p(D)$.

**Proposition 2.16.** *Given a chain map of complexes $f \colon C \to D$, for every $p \in \mathbb{Z}$, the function $(f^p)^* \colon H^p(C) \to H^p(D)$ defined such that*

$$(f^p)^*([a]) = [f^p(a)] \quad \text{for all } a \in Z^p(C)$$

*is a homomorphism. Therefore, $f \colon C \to D$ induces a homomorphism $f^* \colon H^*(C) \to H^*(D)$.*

*Proof.* First, we show that if $[a]$ is a cohomology class in $H^p(C)$ with $a \in Z^p(C)$ ($a$ is a cocycle), then $f^p(a) \in Z^p(D)$; that is, $f^p(a)$ is a cocycle. Since $a \in Z^p(C)$ we have $d_C(a) = 0$, and since by the commutativity of the squares of the diagram of Definition 2.9

$$d_D \circ f^p = f^{p+1} \circ d_C,$$

we get
$$d_D \circ f^p(a) = f^{p+1} \circ d_C(a) = 0,$$
which shows that $f^p(a) \in Z^p(D)$, that is $f^p(a)$ is a cocycle.

Next we show that $[f^p(a)]$ does not depend on the choice of $a$ in the equivalence class $[a]$. If $[b] = [a]$ with $a, b \in Z^p(C)$, then $a - b \in B^p(C)$, which means that $a - b = d_C(x)$ for some $x \in C^{p-1}$. We have
$$d_D \circ f^{p-1} = f^p \circ d_C,$$
which implies that
$$f^p(a - b) = f^p \circ d_C(x) = d_D \circ f^{p-1}(x),$$
and since $f^p$ is linear we get $f^p(a) - f^p(b) = d_D \circ f^{p-1}(x)$, that is, $f^p(a) - f^p(b) \in \operatorname{Im} d_D$, which means that $[f^p(a)] = [f^p(b)]$. Thus, $(f^p)^*([a]) = [f^p(a)]$ is well defined.

The fact that $(f^p)^*$ is a homomorphism is standard and follows immediately from the definition of $(f^p)^*$. $\qquad\square$

There are situations, for instance when defining Čech cohomology groups, where we have different maps $f \colon C \to D$ and $g \colon C \to D$ betweeen two complexes $C$ and $D$ and yet we would like the induced maps $f^* \colon H^*(C) \to H^*(D)$ and $g^* \colon H^*(C) \to H^*(D)$ to be identical, that is, $f^* = g^*$. A sufficient condition is the existence of a certain kind of map between $C$ and $D$ called a chain homotopy.

**Definition 2.10.** Given two chain maps $f \colon C \to D$ and $g \colon C \to D$, a *chain homotopy* between $f$ and $g$ is a family $s = (s^p)_{p \in \mathbb{Z}}$ of $R$-linear maps $s^p \colon C^p \to D^{p-1}$ such that
$$d_D \circ s^p + s^{p+1} \circ d_C = f^p - g^p \quad \text{for all } p \in \mathbb{Z}.$$

As a diagram, a chain homotopy is given by a family of slanted arrows as below, where we write $h = f - g$:



The following proposition clarifies this somewhat mysterious definition.

**Proposition 2.17.** *Given two chain maps $f \colon C \to D$ and $g \colon C \to D$ between two complexes $C$ and $D$, if $s$ is a chain homotopy between $f$ and $g$, then $f^* = g^*$.*

*Proof.* If $[a]$ is a cohomology class in $H^p(C)$, where $a$ is a cocycle in $Z^p(C)$, that is $a \in C^p$ and $d_C(a) = 0$, we have
$$((f^p)^* - (g^p)^*)([a]) = [f^p(a) - g^p(a)] = [d_D \circ s^p(a) + s^{p+1} \circ d_C(a)],$$

and since $a$ is a cocycle $d_C(a) = 0$ so

$$((f^p)^* - (g^p)^*)([a]) = [d_D \circ s^p(a)] = 0,$$

since $d_D \circ s^p(a)$ is a coboundary in $B^p(D)$.                    □

## 2.5   The Long Exact Sequence of Cohomology or Zig-Zag Lemma

The following result is the first part of one of the most important results of (co)homology theory.

**Proposition 2.18.** *Any short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*of complexes $X, Y, Z$ yields a cohomology sequence*

$$H^p(X) \xrightarrow{f^*} H^p(Y) \xrightarrow{g^*} H^p(Z)$$

*which is exact for every $p$, which means that $\mathrm{Im}\, f^* = \mathrm{Ker}\, g^*$ for all $p$.*

*Proof.* Consider the following diagram where the rows are exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X^{p-1} & \xrightarrow{f^{p-1}} & Y^{p-1} & \xrightarrow{g^{p-1}} & Z^{p-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_Y} & & \downarrow{\scriptstyle d_Z} & & \\
0 & \longrightarrow & X^p & \xrightarrow{f^p} & Y^p & \xrightarrow{g^p} & Z^p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_Y} & & \downarrow{\scriptstyle d_Z} & & \\
0 & \longrightarrow & X^{p+1} & \xrightarrow{f^{p+1}} & Y^{p+1} & \xrightarrow{g^{p+1}} & Z^{p+1} & \longrightarrow & 0.
\end{array}
$$

Since we have a short exact sequence, $f^p$ is injective, $g^p$ is surjective, and $\mathrm{Im}\, f^p = \mathrm{Ker}\, g^p$ for all $p$. Consequently $g^p \circ f^p = 0$, and for for every cohomology class $[a] \in H^p(X)$, we have

$$g^* \circ f^*([a]) = g^*([f^p(a)]) = [g^p(f^p(a))] = 0,$$

which implies that $\mathrm{Im}\, f^* \subseteq \mathrm{Ker}\, g^*$. To prove the inclusion in the opposite direction, we need to prove that for every $[b] \in H^p(Y)$ such that $g^*([b]) = 0$ (where $b \in Y^p$ is a cocycle) there is some $[a] \in H^p(X)$ such that $f^*([a]) = [b]$.

If $g^*([b]) = [g^p(b)] = 0$ then $g^p(b)$ must be a coboundary, which means that $g^p(b) = d_Z(c)$ for some $c \in Z^{p-1}$. Since $g^{p-1}$ is surjective, there is some $b_1 \in Y^{p-1}$ such that $c = g^{p-1}(b_1)$. Now $g$ being a chain map the top right square commutes, that is

$$d_Z \circ g^{p-1} = g^p \circ d_Y,$$

so

$$g^p(b) = d_Z(c) = d_Z(g^{p-1}(b_1)) = g^p(d_Y(b_1)),$$

which implies that

$$g^p(b - d_Y(b_1)) = 0.$$

By exactness of the short exact sequence, $\operatorname{Im} f^p = \operatorname{Ker} g^p$ for all $p$, and there is some $a \in X^p$ such that

$$f^p(a) = b - d_Y(b_1).$$

If we can show that $a$ is a cocycle, then

$$f^*([a]) = [f^p(a)] = [b - d_Y(b_1)] = [b],$$

proving that $f^*([a]) = [b]$, as desired.

Thus, we need to prove that $d_X(a) = 0$. Since $f^{p+1}$ is injective, it suffices to show that $f^{p+1}(d_X(a)) = 0$. But $f$ is a chain map so the left lower square commutes, that is

$$d_Y \circ f^p = f^{p+1} \circ d_X,$$

and we have

$$f^{p+1}(d_X(a)) = d_Y(f^p(a)) = d_Y(b - d_Y(b_1)) = d_Y(b) - d_Y \circ d_Y(b) = 0$$

since $b$ is a cocycle, so $d_Y(b) = 0$ and $d_Y \circ d_Y = 0$ since $Y$ is a differential complex.    □

In general, a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of complexes does not yield an exact sequence

$$0 \longrightarrow H^p(X) \xrightarrow{f^*} H^p(Y) \xrightarrow{g^*} H^p(Z) \longrightarrow 0$$

for all (or any) $p$. However, one of the most important results in homological algebra is that a short exact sequence of complexes yields a so-called *long exact sequence* of cohomology groups.

This result is often called the "zig-zag lemma" for cohomology; see Munkres [38] (Chapter 3, Section 24). The proof involves a lot of "diagram chasing." It is not particularly hard, but a bit tedious and not particularly illuminating. Still, this is a very important result so we provide a complete and detailed proof.

**Theorem 2.19.** *(Long exact sequence of cohomology, or zig-zag lemma for cohomology) For any short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*of complexes $X, Y, Z$, there are homomorphisms $\delta^p \colon H^p(Z) \to H^{p+1}(X)$ such that we obtain a long exact sequence of cohomology of the following form:*



*(for all $p$).*

*Proof.* The main step is the construction of the homomorphisms $\delta^p \colon H^p(Z) \to H^{p+1}(X)$. We suggest that upon first reading the reader looks at the construction of $\delta^p$ and then skips the proofs of the various facts that need to be established.

Consider the following diagram where the rows are exact:



To define $\delta^p([c])$ where $[c] \in H^p(Z)$ is a cohomology class ($c \in Z^p$ is a cocycle, that is $d_Z(c) = 0$), pick any $b \in Y^p$ such that $g^p(b) = c$, push $b$ down to $Y^{p+1}$ by applying $d_Y$ obtaining $d_Y(b)$, and then pull $d_Y(b)$ back to $X^{p+1}$ by applying $(f^{p+1})^{-1}$, obtaining $a = (f^{p+1})^{-1}(d_Y(b))$. Then, set

$$\delta^p([c]) = [a].$$

Schematically, starting with an element $c \in Z^p$, we follow the path from right to left in the diagram below.

$$
\begin{array}{ccc}
Y^p & \xrightarrow{\ g^p\ } & Z^p \\
\big\downarrow{\scriptstyle d_Y} & b \longmapsfrom c = g^p(b) & \\
X^{p+1} \xrightarrow{\ f^{p+1}\ } Y^{p+1} & & \big\downarrow{\scriptstyle d_Z} \\
a \longmapsfrom d_Y(b) \xrightarrow{\ g^{p+1}\ } 0 &
\end{array}
$$

In order to ensure that $\delta^p$ is well defined, we must check five facts:

(a) For any $c \in Z^p$ such that $d_Z(c) = 0$ and any $b \in Y^p$, if $g^p(b) = c$, then $d_Y(b) \in \operatorname{Im} f^{p+1}$. This guarantees that $a = (f^{p+1})^{-1}(d_Y(b))$ is well-defined since $f^{p+1}$ is injective.

(b) The element $a \in X^{p+1}$ is a cocycle; more precisely, if $f^{p+1}(a) = d_Y(b)$ for some $b \in Y^p$, then $d_X(a) = 0$.

(c) The homology class $[a]$ does not depend on the choice of $b$ in $(g^p)^{-1}(c)$; that is, for all $b_1, b_2 \in Y^p$ and all $a_1, a_2 \in X^{p+1}$, if $g^p(b_1) = g^p(b_2) = c$ and $f^{p+1}(a_1) = d_Y(b_1)$, $f^{p+1}(a_2) = d_Y(b_2)$, then $[a_1] = [a_2]$.

(d) The map $\delta^p$ is a linear map.

(e) The homology class $[a]$ does not depend on the choice of the cocycle $c$ in the cohomology class $[c]$. Since $\delta^p$ is linear, it suffices to show that if $c$ is a coboundary in $Z^p$, then for any $b$ such that $g^p(b) = c$ and any $a \in X^{p+1}$ such that $f^{p+1}(a) = d_Y(b)$, then $[a] = 0$.

Recall that since $f$ and $g$ are chain maps, the top, middle, and bottom left and right squares commute.

(a) Since $\operatorname{Im} f^{p+1} = \operatorname{Ker} g^{p+1}$, it suffices to show that $g^{p+1}(d_Y(b)) = 0$. However, since the middle right square commutes and $d_Z(c) = 0$ ($c$ is a cocycle),

$$
g^{p+1}(d_Y(b)) = d_Z(g^p(b)) = d_Z(c) = 0,
$$

as desired.

(b) Since $f^{p+2}$ is injective, $d_X(a) = 0$ iff $f^{p+2} \circ d_X(a) = 0$, and since the lower left square commutes

$$
f^{p+2} \circ d_X(a) = d_Y \circ f^{p+1}(a) = d_Y \circ d_Y(b) = 0,
$$

so $d_X(a) = 0$, as claimed.

(c) Assume that $g^p(b_1) = g^p(b_2) = c$. Then $g^p(b_1 - b_2) = 0$, and since $\operatorname{Im} f^p = \operatorname{Ker} g^p$, there is some $\widetilde{a} \in X^p$ such that $b_1 - b_2 = f^p(\widetilde{a})$. Using the fact that the middle left square commutes we have

$$
\begin{aligned}
f^{p+1}(a_1 - a_2) &= f^{p+1}(a_1) - f^{p+1}(a_2) \\
&= d_Y(b_1) - d_Y(b_2) = d_Y(b_1 - b_2) \\
&= d_Y(f^p(\widetilde{a})) = f^{p+1}(d_X(\widetilde{a})),
\end{aligned}
$$

and the injectivity of $f^{p+1}$ yields $a_1 - a_2 = d_X(\widetilde{a})$, which implies that $[a_1] = [a_2]$.

(d) The fact that $\delta^p$ is linear is an immediate consequence of the fact that all the maps involved in its definition are linear.

(e) Let $c \in Z^p$ be a coboundary, which means that $c = d_Z(\widetilde{c})$ for some $\widetilde{c} \in Z^{p-1}$. Since $g^{p-1}$ is surjective, there is some $b_1 \in Y^{p-1}$ such that $g^{p-1}(b_1) = \widetilde{c}$, and since the top right sqare commutes $d_Z \circ g^{p-1} = g^p \circ d_Y$, and we get

$$
c = d_Z(\widetilde{c}) = d_Z(g^{p-1}(b_1)) = g^p(d_Y(b_1)).
$$

By (c), to compute the cohomology class $[a]$ such that $\delta^p([c]) = [a]$ we can pick any $b \in Y^p$ such that $g^p(b) = c$, and since $c = g^p(d_Y(b_1))$ we can pick $b = d_Y(b_1)$ and then we obtain

$$
d_Y(b) = d_Y \circ d_Y(b_1) = 0.
$$

Since $f^{p+1}$ is injective, if $a \in X^{p+1}$ is the unique element such that $f^{p+1}(a) = d_Y(b) = 0$, then $a = 0$, and thus $[a] = 0$.

It remains to prove that

$$
\operatorname{Im}(g^p)^* = \operatorname{Ker} \delta^p \quad \text{and} \quad \operatorname{Im} \delta^p = \operatorname{Ker}(f^{p+1})^*.
$$

For any cohomology class $[b] \in H^p(Y)$ for some $b \in Y^p$ such that $d_Y(b) = 0$ ($b$ is a cocycle), since $(g^p)^*([b]) = [g^p(b)]$, if we write $c = g^p(b)$ then $c$ is a cocycle in $Z^p$, and by definition of $\delta^p$ we have

$$
\delta^p((g^p)^*([b])) = \delta^p([c]) = [(f^{p+1})^{-1}(d_Y(b))] = [(f^{p+1})^{-1}(0)] = 0.
$$

Thus, $\operatorname{Im}(g^p)^* \subseteq \operatorname{Ker} \delta^p$.

Conversely, assume that $\delta^p([c]) = 0$, for some $c \in Z^p$ such that $d_Z(c) = 0$. By definition of $\delta^p$, we have $\delta^p([c]) = [a]$ where $a \in X^{p+1}$ is given by $f^{p+1}(a) = d_Y(b)$ for any $b \in Y^p$ such that $g^p(b) = c$, and since $[a] = 0$ the element $a$ must be a coboundary, which means that $a = d_X(a_1)$ for some $a_1 \in X^p$. Then, by commutativity of the left middle square we have

$$
d_Y(b) = f^{p+1}(a) = f^{p+1}(d_X(a_1)) = d_Y(f^p(a_1)),
$$

so $d_Y(b - f^p(a_1)) = 0$, that is $b - f^p(a_1)$ is a cycle in $Y^p$. Since $\operatorname{Im} f^p = \operatorname{Ker} g^p$ we have $g^p \circ f^p = 0$, which implies that

$$c = g^p(b) = g^p(b - f^p(a)).$$

It follows that $(g^p)^*([b - f^p(a)]) = [c]$, proving that $\operatorname{Ker} \delta^p \subseteq \operatorname{Im} (g^p)^*$.

For any $[c] \in H^p(Z)$, since $\delta^p([c]) = [a]$ where $f^{p+1}(a) = d_Y(b)$ for any $b \in Y^p$ such that $g^p(b) = c$, as $d_Y(b)$ is a coboundary we have

$$(f^{p+1})^*(\delta^p([c])) = (f^{p+1})^*([a]) = [f^{p+1}(a)] = [d_Y(b)] = 0,$$

and thus $\operatorname{Im} \delta^p \subseteq \operatorname{Ker} (f^{p+1})^*$.

Conversely, assume that $(f^{p+1})^*([a]) = 0$, for some $a \in X^{p+1}$ with $d_X(a) = 0$, which means that $f^{p+1}(a) = d_Y(b)$ for some $b \in Y^p$. Since $\operatorname{Im} f^{p+1} = \operatorname{Ker} g^{p+1}$ we have $g^{p+1} \circ f^{p+1} = 0$, so by commutativity of the middle right square

$$d_Z(g^p(b)) = g^{p+1}(d_Y(b)) = g^{p+1}(f^{p+1}(a)) = 0,$$

which means that $g^p(b)$ is a cocycle in $Z^p$, and since $f^{p+1}(a) = d_Y(b)$ by definition of $\delta^p$

$$\delta^p([g^p(b)]) = [a],$$

showing that $\operatorname{Ker} (f^{p+1})^* \subseteq \operatorname{Im} \delta^p$. $\qquad\square$

The maps $\delta^p \colon H^p(Z) \to H^{p+1}(X)$ are called *connecting homomorphisms*. The kind of argument used to prove Theorem 2.19 is known as *diagram chasing*.

**Remark:** The construction of the connecting homomorphisms $\delta^p \colon H^p(Z) \to H^{p+1}(X)$ is often obtained as a corollary of the *snake lemma*. This is the approach followed in the classical texts by MacLane [29] and Cartan–Eilenberg [7]. These books assume that the reader already has a fair amount of background in algebraic topology and the proofs are often rather terse or left to reader as "easy exercises" in diagram chasing. Bott and Tu [2] refer to MacLane for help but as we just said MacLane leaves many details as exercises to the reader. More recent texts such as Munkres [38], Rotman [41], Madsen and Tornehave [31], Tu [49] and Hatcher [25] show more compassion for the reader and provide much more details. Still, except for Hatcher and Munkres who give all the steps of the proof (for homology, and sometimes quickly) certain steps are left as "trivial" exercises (for example, step (e)). At the risk of annoying readers who have some familiarity with homological algebra we decided to provide all gory details of the proof so that readers who are novice in this area have a place to fall back if they get stuck, even if these proofs are not particularly illuminating (and rather tedious).

The assignment of a long exact sequence of cohomology to a short exact sequences of complexes is "natural" in the sense that it also applies to morphisms of short exact sequences of complexes.

**Definition 2.11.** Given two short exact sequences of complexes

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X' \xrightarrow{\ f'\ } Y' \xrightarrow{\ g'\ } Z' \longrightarrow 0,$$

a *morphism* between these two exact sequences is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & X' & \xrightarrow{\ f'\ } & Y' & \xrightarrow{\ g'\ } & Z' & \longrightarrow & 0,
\end{array}
$$

where $\alpha, \beta, \gamma$ are chain maps.

The following proposition gives a precise meaning to the naturality of the assignment of a long exact sequence of cohomology to a short exact sequences of complexes.

**Proposition 2.20.** *For any morphism of exact sequences of chain complexes*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & X' & \xrightarrow{\ f'\ } & Y' & \xrightarrow{\ g'\ } & Z' & \longrightarrow & 0,
\end{array}
$$

*the following diagram of cohomology commutes:*

$$
\begin{array}{ccccccccc}
\longrightarrow & H^p(X) & \xrightarrow{\ f^*\ } & H^p(Y) & \xrightarrow{\ g^*\ } & H^p(Z) & \xrightarrow{\ \delta^p\ } & H^{p+1}(X) & \longrightarrow \\
& \downarrow{\scriptstyle\alpha^*} & & \downarrow{\scriptstyle\beta^*} & & \downarrow{\scriptstyle\gamma^*} & & \downarrow{\scriptstyle\alpha^*} & \\
\longrightarrow & H^p(X') & \xrightarrow{(f')^*} & H^p(Y') & \xrightarrow{(g')^*} & H^p(Z') & \xrightarrow{(\delta')^p} & H^{p+1}(X') & \longrightarrow
\end{array}
$$

*Proof.* A proof of Proposition 2.20 for homology can be found in Munkres [38] (Chapter 3, Section 24, Theorem 24.2) and Hatcher [25] (Chapter 2, Section 2.1). The proof is a "diagram chasing" argument which can be modified to apply to cohomology as we now show. The first two squares commute because they already commute at the cochain level by definition of a morphism so we only have to prove that the third square commutes.

Recall how $\delta^p(\xi)$ is defined where $\xi = [c] \in H^p(Z)$ is represented by a cocycle $c \in Z^p$: pick any $b \in Y^p$ such that $g^p(b) = c$, push $b$ down to $Y^{p+1}$ by applying $d_Y$ obtaining $d_Y(b)$, and then pull $d_Y(b)$ back to $X^{p+1}$ by applying $(f^{p+1})^{-1}$, obtaining $a = (f^{p+1})^{-1}(d_Y(b))$; set

$\delta^p([c]) = [a]$. Schematically,

$$
\begin{array}{ccc}
Y^p & \xrightarrow{\;g^p\;} & Z^p \\
\downarrow{\scriptstyle d_Y} & b \longleftarrow\!\!| \; c = g^p(b) & \\
X^{p+1} \xrightarrow{\;f^{p+1}\;} Y^{p+1} & & \downarrow{\scriptstyle d_Z} \\
a \longleftarrow\!\!| \; d_Y(b) & \xrightarrow{\;g^{p+1}\;} & 0
\end{array}
$$

Since $a \in X^{p+1}$ is a cocycle and $\alpha$ is a chain map $\alpha(a) \in X'^{p+1}$ is a cocycle. Similarly $\gamma(c) \in Z'^{p+1}$ is a cocycle, and by definition $\gamma^*([c]) = [\gamma(c)]$. We claim that

$$(\delta')^p([\gamma(c)]) = [\alpha(a)].$$

Since $c = g^p(b)$ we have $\gamma(c) = \gamma \circ g^p(b)$ and since the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;g\;} & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & X' & \xrightarrow{\;f'\;} & Y' & \xrightarrow{\;g'\;} & Z' & \longrightarrow & 0
\end{array}
\qquad (*)
$$

commutes, we have $\gamma(c) = \gamma \circ g^p(b) = g'^p \circ \beta(b)$. Consider the following diagram:

$$
\begin{array}{ccc}
Y'^p & \xrightarrow{\;g'^p\;} & Z'^p \\
\downarrow{\scriptstyle d_{Y'}} & \beta(b) \longleftarrow\!\!| \; \gamma(c) = g'^p(\beta(b)) & \\
X'^{p+1} \xrightarrow{\;f'^{p+1}\;} Y'^{p+1} & & \downarrow{\scriptstyle d_{Z'}} \\
\alpha(a) \longleftarrow\!\!| \; d_{Y'}(\beta(b)) & \xrightarrow{\;g'^{p+1}\;} & 0.
\end{array}
$$

By commutativity of the diagram $(*)$, the fact that $\beta$ is a chain map, and since $f^{p+1}(a) = d_Y(b)$, we have

$$f'^{p+1}(\alpha(a)) = \beta(f^{p+1}(a)) = \beta(d_Y(b)) = d_{Y'}(\beta(b)),$$

which shows that $(\delta')^p([\gamma(c)]) = [\alpha(a)]$. But $\delta^p([c]) = [a]$, so we get

$$(\delta')^p(\gamma^*([c])) = (\delta')^p([\gamma(c)]) = [\alpha(a)] = \alpha^*([a]) = \alpha^*(\delta^p([c])),$$

namely

$$(\delta')^p \circ \gamma^* = \alpha^* \circ \delta^p,$$

as claimed. $\qquad\square$

Given two complexes $(X, d_X)$ and $(Y, d_Y)$, the complex $X \oplus Y$ consists of the modules $X^p \oplus Y^p$ and of the maps

$$X^p \oplus Y^p \xrightarrow{\ d_X^p \oplus d_Y^p\ } X^{p+1} \oplus Y^{p+1}$$

defined such that $(d_X^p \oplus d_Y^p)(x + y) = d_X^p(x) + d_Y^p(y)$, for all $x \in X^p$ and all $y \in Y^p$. It is immediately verified that $(d_X^{p+1} \oplus d_Y^{p+1}) \circ (d_X^P \oplus d_Y^p) = 0$. The following proposition is easy to prove.

**Proposition 2.21.** *For any two complexes $(X, d_X)$ and $(Y, d_Y)$, we have isomorphisms*

$$H^p(X \oplus Y) \cong H^p(X) \oplus H^p(Y)$$

*for all $p$.*

*Sketch of proof.* It is easy to check that

$$\operatorname{Ker} d_{X \oplus Y}^p \cong \operatorname{Ker} d_X^p \oplus \operatorname{Ker} d_Y^p$$
$$\operatorname{Im} d_{X \oplus Y}^p \cong \operatorname{Im} d_X^p \oplus \operatorname{Im} d_Y^p,$$

from which the results follows.                                                          $\square$

In the next chapter we discuss an example of a long exact sequence of cohomology arising from two open subsets $U_1, U_2$ of a manifold $M$ that involves the cohomology space $H^p(U_1 \cup U_2)$ and the cohomology spaces $H^{p-1}(U_1 \cap U_2)$, $H^p(U_1)$ and $H^p(U_2)$. This long exact sequence is known as the *Mayer–Vietoris sequence*. If $U$ is covered by a finite family $(U_i)_{i=1}^r$ of open sets and if this family is a "good cover," then by an inductive argument involving the *Mayer–Vietoris sequence* it is possible to prove that the cohomology spaces $H^p(U)$ are finite-dimensional.

Any decent introduction to homological algebra must dicsuss the "five lemma" (due to Steenrod). Together with the zig-zag lemma, this one of its most useful results.

## 2.6   The Five Lemma

As a warm up, let us consider the "short five lemma," from MacLane [29] (Chapter I, Section 3, Lemma 3.1).

**Proposition 2.22.** *(Short Five Lemma) Consider the following diagram in which the rows are exact and all the squares commute.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow[\ f'\ ]{} & B' & \xrightarrow[\ g'\ ]{} & C' & \longrightarrow & 0
\end{array}
$$

*If $\alpha$ and $\gamma$ are isomorphisms, then $\beta$ is also an isomorphism.*

*Proof.* First we prove that $\beta$ is injective. Assume that $\beta(b) = 0$ for some $b \in B$. Then $g'(\beta(b)) = 0$, and since the right square commutes, $0 = g'(\beta(b)) = \gamma(g(b))$. Since $\gamma$ is injective (it is an isomorphism), $\gamma(g(b)) = 0$ implies that

$$g(b) = 0.$$

Since the top row is exact and $b \in \mathrm{Ker}\, g = \mathrm{Im}\, f$, there is some $a \in A$ such that

$$f(a) = b. \tag{$*_1$}$$

Here is a summary of the situation so far:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & a \in A & \xrightarrow{\ f\ } & b \in B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \alpha(a) \in A' & \underset{f'}{\longrightarrow} & B' & \underset{g'}{\longrightarrow} & C' & \longrightarrow & 0
\end{array}
$$

Since the left square commutes, using $(*_1)$ we have

$$f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0.$$

Since the bottom row is exact, $f'$ is injective so $\alpha(a) = 0$, and since $\alpha$ is injective (it is an isomorphism), $a = 0$. But then by $(*_1)$ we have $b = f(a) = 0$, which shows that $\beta$ is injective.

   We now prove that $\beta$ is surjective. Pick any $b' \in B'$. Since $\gamma$ is surjective (it is an isomorphism), there is some $c \in C$ such that

$$\gamma(c) = g'(b'). \tag{$*_2$}$$

Since the top row is exact, $g$ is surjective so there is some $b \in B$ such that

$$g(b) = c. \tag{$*_3$}$$

Since the right square commutes, by $(*_2)$ and $(*_3)$ we have

$$g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = g'(b'),$$

which implies $g'(\beta(b) - b') = 0$. Since the bottom row is exact and $\beta(b) - b' \in \mathrm{Ker}\, g' = \mathrm{Im}\, f'$ there is some $a \in A'$ such that

$$f'(a') = \beta(b) - b'. \tag{$*_4$}$$

Since $\alpha$ is surjective (it is an isomorphism), there is some $a \in A$ such that

$$\alpha(a) = a'. \tag{$*_5$}$$

Here is a summary of the situation so far:

$$0 \longrightarrow a \in A \xrightarrow{\quad f \quad} b \in B \xrightarrow{\quad g \quad} c \in C \longrightarrow 0$$

$$\downarrow \alpha \qquad\qquad \downarrow \beta \qquad\qquad \downarrow \gamma$$

$$0 \longrightarrow a' \in A' \xrightarrow[f']{} \beta(b) - b' \in B' \xrightarrow[g']{} g'(b') \in C' \longrightarrow 0$$

Since the left square commutes, using $(*_4)$ and $(*_5)$ we obtain

$$\beta(f(a)) = f'(\alpha(a)) = f'(a') = \beta(b) - b',$$

which implies that $b' = \beta(b - f(a))$, showing that $\beta$ is surjective.        □

Observe that the proof shows that if $\alpha$ and $\gamma$ are injective, then $\beta$ is injective, and if $\alpha$ and $\gamma$ are surjective, then $\beta$ is surjective.

**Proposition 2.23.** *(Five Lemma) Consider the following diagram in which the rows are exact and all the squares commute.*

$$\longrightarrow A \xrightarrow{\quad f_1 \quad} B \xrightarrow{\quad f_2 \quad} C \xrightarrow{\quad f_3 \quad} D \xrightarrow{\quad f_4 \quad} E \longrightarrow$$

$$\downarrow \alpha_1 \qquad \downarrow \alpha_2 \qquad \downarrow \alpha_3 \qquad \downarrow \alpha_4 \qquad \downarrow \alpha_5$$

$$\longrightarrow A' \xrightarrow[f'_1]{} B' \xrightarrow[f'_2]{} C' \xrightarrow[f'_3]{} D' \xrightarrow[f'_4]{} E' \longrightarrow$$

*If $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are isomorphisms, then $\alpha_3$ is also an isomorphism.*

*Proof.* The proof of Proposition 2.23 can be found in any book on homological algebra, for example MacLane [29], Cartan–Eilenberg [7], and Rotman [41], but the reader may be put off by the fact that half of the proof is left to the reader (at least, Rotman proves the surjectivity part, which is slightly harder, and MacLane gives a complete proof of the short five lemma). The five lemma is fully proved in Spanier [47] and Hatcher [25]. Because it is a "fun" proof by diagram-chasing we present the proof in Spanier [47] (Chapter 4, Section 5, Lemma 11).

First, we prove that $\alpha_3$ is injective. Assume that $\alpha_3(c) = 0$ for some $c \in C$. Then $f'_3 \circ \alpha_3(c) = 0$, and by commutativity of the third square, $\alpha_4 \circ f_3(c) = 0$. Since $\alpha_4$ is injective (it is an isomorphism),

$$f_3(c) = 0.$$

Since the top row is exact and $c \in \operatorname{Ker} f_3 = \operatorname{Im} f_2$, there is some $b \in B$ such that

$$f_2(b) = c.$$

Since the second square commutes,

$$f'_2 \circ \alpha_2(b) = \alpha_3 \circ f_2(b) = \alpha_3(c) = 0,$$

and since the bottom is exact and $\alpha_2(b) \in \operatorname{Ker} f_2' = \operatorname{Im} f_1'$, there is some $a' \in A'$ such that

$$f_1'(a') = \alpha_2(b). \tag{$*_1$}$$

Since $\alpha_1$ is surjective (it is an isomorphism) there is some $a \in A$ such that

$$\alpha_1(a) = a'.$$

Here is a summary of the situation so far:

$$
\begin{array}{ccccccccc}
\longrightarrow & a \in A & \xrightarrow{f_1} & b \in B & \xrightarrow{f_2} & c \in C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E & \longrightarrow \\
& \downarrow{\alpha_1} & & \downarrow{\alpha_2} & & \downarrow{\alpha_3} & & \downarrow{\alpha_4} & & \downarrow{\alpha_5} & \\
\longrightarrow & a' \in A' & \xrightarrow{f_1'} & \alpha_2(b) \in B' & \xrightarrow{f_2'} & C' & \xrightarrow{f_3'} & D' & \xrightarrow{f_4'} & E' & \longrightarrow
\end{array}
$$

By the commutativity of the first square and $(*_1)$,

$$\alpha_2 \circ f_1(a) = f_1' \circ \alpha_1(a) = f_1'(a') = \alpha_2(b),$$

and since $\alpha_2$ is injective (it is an isomorphism), $b = f_1(a)$. Since the top row is exact $f_2 \circ f_1 = 0$, so

$$c = f_2(b) = f_2 \circ f_1(a) = 0,$$

proving that $\alpha_3$ is injective.

Next we prove that $\alpha_3$ is surjective. Pick $c' \in C'$. Since $\alpha_4$ is surjective (it is an isomorphism) there is some $d \in D$ such that

$$\alpha_4(d) = f_3'(c'). \tag{$*_2$}$$

Since the bottom row is exact $f_4' \circ f_3' = 0$ and since the fourth square commutes we have

$$0 = f_4' \circ f_3'(c') = f_4' \circ \alpha_4(d) = \alpha_5 \circ f_4(d).$$

Since $\alpha_5$ is injective (it is an isomorphism),

$$f_4(d) = 0,$$

and since the top row is exact and $d \in \operatorname{Ker} f_4 = \operatorname{Im} f_3$, there is some $c \in C$ such that

$$f_3(c) = d. \tag{$*_3$}$$

Since the third square commutes, using $(*_3)$ and $(*_2)$ we have

$$f_3' \circ \alpha_3(c) = \alpha_4 \circ f_3(c) = \alpha_4(d) = f_3'(c'),$$

so $f_3'(\alpha_3(c) - c') = 0$. Since the bottom row is exact and $\alpha_3(c) - c' \in \text{Ker } f_3' = \text{Im } f_2'$, there is some $b' \in B'$ such that

$$f_2'(b') = \alpha_3(c) - c'. \tag{$*_4$}$$

Since $\alpha_2$ is surjective (it is an isomorphism) there is some $b \in B$ such that

$$\alpha_2(b) = b'. \tag{$*_5$}$$

Here is a summary of the situation so far:

$$
\begin{array}{ccccccccc}
\longrightarrow & A & \xrightarrow{\ f_1\ } & b \in B & \xrightarrow{\ f_2\ } & c \in C & \xrightarrow{\ f_3\ } & d \in D & \xrightarrow{\ f_4\ } & E & \longrightarrow \\
 & \downarrow{\alpha_1} & & \downarrow{\alpha_2} & & \downarrow{\alpha_3} & & \downarrow{\alpha_4} & & \downarrow{\alpha_5} & \\
\longrightarrow & A' & \xrightarrow[\ f_1'\ ]{} & b' \in B' & \xrightarrow[\ f_2'\ ]{} & \alpha_3(c) - c' \in C' & \xrightarrow[\ f_3'\ ]{} & f_3'(c') \in D' & \xrightarrow[\ f_4'\ ]{} & E' & \longrightarrow
\end{array}
$$

Then using $(*_4)$ and $(*_5)$ and the fact that the second square commutes we have

$$\alpha_3(f_2(b)) = f_2'(\alpha_2(b)) = f_2'(b') = \alpha_3(c) - c',$$

which implies that $c' = \alpha_3(c - f_2(b))$, showing that $\alpha_3$ is surjective.   $\square$

**Remark:** The hypotheses of the five lemma can be weakened. One can check that the proof goes through if $\alpha_2$ and $\alpha_4$ are isomorphisms, $\alpha_1$ is surjective, and $\alpha_5$ is injective.

# Chapter 3

# de Rham Cohomology

## 3.1 Review of de Rham Cohomology

Let $M$ be a smooth manifold. The de Rham cohomology is based on differential forms. If $\mathcal{A}^p(M)$ denotes the real vector space of smooth $p$-forms on $M$, then we know that there is a mapping $d^p \colon \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ called *exterior differentiation*, and $d^p$ satisfies the crucial property

$$d^{p+1} \circ d^p = 0 \quad \text{for all } p \geq 0.$$

Recall that $\mathcal{A}^0(M) = C^\infty(M)$, the space of all smooth (real-valued) functions on $M$.

**Definition 3.1.** The sequence of vector spaces and linear maps between them satisfying $d^{p+1} \circ d^p = 0$ given by

$$\mathcal{A}^0(M) \xrightarrow{d^0} \mathcal{A}^1(M) \xrightarrow{d^1} \mathcal{A}^2(M) \xrightarrow{d^2} \cdots \xrightarrow{d^{p-1}} \mathcal{A}^p(M) \xrightarrow{d^p} \mathcal{A}^{p+1}(M) \xrightarrow{d^{p+1}} \cdots$$

is called a *differential complex*.

We can package together the vector spaces $\mathcal{A}^p(M)$ as the direct sum $\mathcal{A}^*(M)$ given by

$$\mathcal{A}^*(M) = \bigoplus_{p \geq 0} \mathcal{A}^p(M)$$

called the *de Rham complex* of $M$, and the family of maps $(d^p)$ as the map

$$d \colon \mathcal{A}^*(M) \to \mathcal{A}^*(M),$$

where $d$ on the $p$th summand $\mathcal{A}^p(M)$ is equal to $d^p$, so that

$$d \circ d = 0.$$

Furthermore, we know that $d$ is an *anti-derivation*, which means that

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^p \omega \wedge d\tau, \quad \omega \in \mathcal{A}^p(M), \tau \in \mathcal{A}^q(M).$$

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For example, if $M = \mathbb{R}^3$, then

$$d^0 \colon \mathcal{A}^0(M) \to \mathcal{A}^1(M)$$

correspond to grad,

$$d^1 \colon \mathcal{A}^1(M) \to \mathcal{A}^2(M)$$

corresponds to curl, and

$$d^2 \colon \mathcal{A}^2(M) \to \mathcal{A}^3(M)$$

corresponds to div.

In fact, $\mathcal{A}^*(U)$ is defined for every open subset $U$ of $M$, and $\mathcal{A}^*$ is a sheaf of differential complexes.

**Definition 3.2.** A form $\omega \in \mathcal{A}^p(M)$ is *closed* if

$$d\omega = 0,$$

*exact* if

$$\omega = d\tau \quad \text{for some } \tau \in \mathcal{A}^{p-1}(M).$$

Let $Z^p(M)$ denote the subspace of $\mathcal{A}^p(M)$ consisting of closed $p$-forms, $B^p(M)$ denote the subspace of $\mathcal{A}^p(M)$ consisting of exact $p$-forms, with $B^0(M) = (0)$ (the trivial vector space), and let

$$Z^*(M) = \bigoplus_{p \geq 0} Z^p(M), \quad B^*(M) = \bigoplus_{p \geq 0} B^p(M).$$

Since $d \circ d = 0$, we have $B^p(M) \subseteq Z^p(M)$ for all $p \geq 0$ but the converse is generally false.

**Definition 3.3.** The *de Rham cohomology* of a smooth manifold $M$ is the real vector space $H^*_{\mathrm{dR}}(M)$ given by the direct sum

$$H^*_{\mathrm{dR}}(M) = \bigoplus_{p \geq 0} H^p_{\mathrm{dR}}(M),$$

where the cohomology group (actually, real vector space) $H^p_{\mathrm{dR}}(M)$ is the quotient vector space

$$H^p_{\mathrm{dR}}(M) = Z^p(M)/B^p(M).$$

Thus, the cohomology group (vector space) $H^*_{\mathrm{dR}}(M)$ gives some measure of the failure of closed forms to be exact.

**Definition 3.4.** A *gradation* of a vector space $V$ is family $(V_p)$ of subspaces $V_p \subseteq V$ such that

$$V = \bigoplus_{p \geq 0} V_p.$$

In this case, we say that $V$ is a *graded vector space*.

Note that by definition $H_{\mathrm{dR}}^*(M)$ is a graded vector space.

Exterior multiplication in $\mathcal{A}^*(M)$ induces a ring structure on the vector space $H_{\mathrm{dR}}^*(M)$. First, it is clear by definition that

$$B^*(M) \subseteq Z^*(M) \subseteq \mathcal{A}^*(M).$$

We claim that $Z^*(M)$ is a subring of $\mathcal{A}^*(M)$ and that $B^*(M)$ is an ideal in $Z^*(M)$.

*Proof.* Assume that $d\omega = 0$ and $d\tau = 0$ for some $\omega \in Z^p(M)$ and some $\tau \in Z^q(M)$. Then since $d$ is an anti-derivation, we have

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^p \omega \wedge d\tau = 0 \wedge \tau + (-1)^p \omega \wedge 0 = 0,$$

which shows that $\omega \wedge \tau \in Z^*(M)$. Therefore, $Z^*(M)$ is a subring of $\mathcal{A}^*(M)$.

Next, assume that $\omega \in Z^p(M)$ and $\tau \in B^q(M)$, so that $d\omega = 0$ and $\tau = d\alpha$ for some $\alpha \in \mathcal{A}^{q-1}(M)$. Then, we have

$$d(\omega \wedge (-1)^p \alpha) = d\omega \wedge (-1)^p \alpha + (-1)^p \omega \wedge (-1)^p d\alpha = 0 \wedge (-1)^p \alpha + \omega \wedge \tau = \omega \wedge \tau,$$

which shows that $\omega \wedge \tau \in B^*(M)$, so $B^*(M)$ is an ideal in $Z^*(M)$ $\qquad \square$

Since $B^*(M)$ is an ideal in $Z^*(M)$, the quotient ring $Z^*(M)/B^*(M)$ is well-defined, and $H_{\mathrm{dR}}^*(M) = Z^*(M)/B^*(M)$ is a ring under the multiplication induced by $\wedge$. Therefore, $H_{\mathrm{dR}}^*(M)$ is an $\mathbb{R}$-algebra.

A variant of de Rham cohomology is *de Rham cohomology with compact support*, where we consider the vector space $\mathcal{A}_c^*(M)$ of differential forms with compact support. As before, we have the subspaces $B_c^*(M) \subseteq Z_c^*(M)$, and we let

$$H_{\mathrm{dR},c}^*(M) = Z_c^*(M)/B_c^*(M).$$

The *Poincaré Lemma's* are the following results:

**Proposition 3.1.** *The following facts hold:*

$$H_{\mathrm{dR}}^p(\mathbb{R}^n) = \begin{cases} 0 & \text{unless } p \neq 0 \\ \mathbb{R} & \text{if } p = 0, \end{cases}$$

*and*

$$H_{\mathrm{dR},c}^p(\mathbb{R}^n) = \begin{cases} 0 & \text{unless } p \neq 0 \\ \mathbb{R} & \text{if } p = n. \end{cases}$$

*Theses facts also hold if $\mathbb{R}^n$ is replaced by any nonempty convex subset of $\mathbb{R}^n$ (or even a star-shaped subset of $\mathbb{R}^n$).*

## 3.2   The Mayer–Vietoris Argument

Let $M$ be a smooth manifold and assume that $M = U_1 \cup U_2$ for two open subsets $U_1$ and $U_2$ of $M$. The inclusion maps $i_k \colon U_k \to M$ and $j_k \colon U_1 \cap U_2 \to U_k$ for $k = 1, 2$ induce a pullback map $f \colon \mathcal{A}^*(M) \to \mathcal{A}^*(U_1) \oplus \mathcal{A}^*(U_2)$ given by $f = (i_1^*, i_2^*)$ and a pullback map $g \colon \mathcal{A}^*(U_1) \oplus \mathcal{A}^*(U_2) \to \mathcal{A}^*(U_1 \cap U_2)$ given by $g = j_1^* - j_2^*$. We have the following short exact sequence.

**Proposition 3.2.** *For any smooth manifold $M$, if $M = U_1 \cup U_2$ for any two open subsets $U_1$ and $U_2$, then we have the short exact sequence*

$$0 \longrightarrow \mathcal{A}^*(M) \xrightarrow{\ f\ } \mathcal{A}^*(U_1) \oplus \mathcal{A}^*(U_2) \xrightarrow{\ g\ } \mathcal{A}^*(U_1 \cap U_2) \longrightarrow 0.$$

*Proof.* The proof is not really difficult. It involves the use of a partition of unity. For details, see Bott and Tu [2] (Chapter 1, Proposition 2.3) or Madsen and Tornehave [31] (Chapter 5, Theorem 5.1).                                                                    $\square$

The short exact sequence given by Proposition 3.2 is called the *Mayer–Vietoris sequence*.

If we apply Theorem 2.19 to the Mayer–Vietoris sequence we obtain the long Mayer–Vietoris cohomology sequence shown below:

$$
\begin{array}{c}
\cdots \xrightarrow{\hspace{3cm}} H_{\mathrm{dR}}^{p-1}(U_1 \cap U_2) \quad \xrightarrow{\ \delta^{p-1}\ } \\[1em]
H_{\mathrm{dR}}^{p}(M) \xrightarrow{\ f^*\ } H_{\mathrm{dR}}^{p}(U_1) \oplus H_{\mathrm{dR}}^{p}(U_2) \xrightarrow{\ g^*\ } H_{\mathrm{dR}}^{p}(U_1 \cap U_2) \quad \xrightarrow{\ \delta^{p}\ } \\[1em]
H_{\mathrm{dR}}^{p+1}(M) \xrightarrow{\ f^*\ } H_{\mathrm{dR}}^{p+1}(U_1) \oplus H_{\mathrm{dR}}^{p+1}(U_2) \xrightarrow{\ g^*\ } H_{\mathrm{dR}}^{p+1}(U_1 \cap U_2) \quad \xrightarrow{\ \delta^{p+1}\ } \\[1em]
H_{\mathrm{dR}}^{p+2}(M) \xrightarrow{\hspace{3cm}} \cdots
\end{array}
$$

(for all $p$).

This long exact sequence implies that

$$H_{\mathrm{dR}}^{p}(M) \cong \operatorname{Im} \delta^{p-1} \oplus \operatorname{Im} f^*.$$

It follows that if the spaces $H_{\mathrm{dR}}^{p-1}(U_1 \cap U_2)$, $H_{\mathrm{dR}}^{P}(U_1)$ and $H_{\mathrm{dR}}^{P}(U_2)$ are finite-dimensional, then so is $H_{\mathrm{dR}}^{p}(M)$. This suggests an inductive argument on the number of open subsets in a finite cover of $M$. For this argument to succeed, such covers must have some special properties about intersections of these opens subsets; Bott and Tu call them *good covers*.
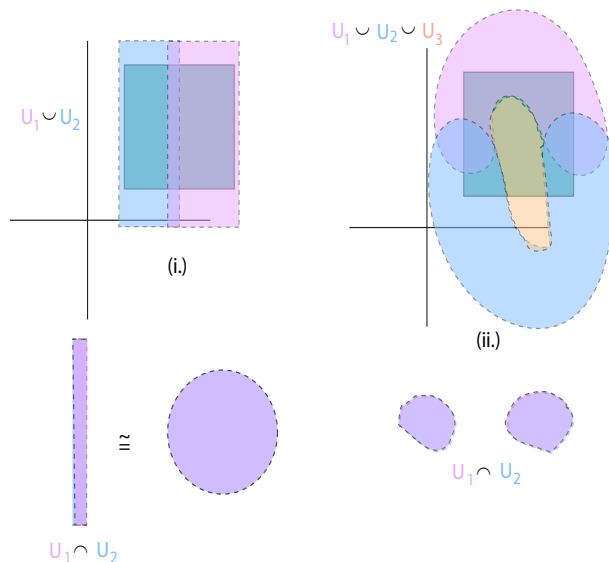
Figure 3.1: The manifold $M$ is a an open unit square of $\mathbb{R}^2$. Figure $(i.)$ is a good cover of $M$ while Figure $(ii.)$ is not a good cover of $M$ since $U_1 \cap U_2$ is isomorphic to the disjoint union of two open disks.

**Definition 3.5.** Given a smooth manifold $M$ of dimension $n$, an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of $M$ is called a *good cover* if all finite nonempty intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. A manifold which has a finite good cover is said to be of *finite type*. See Figure 3.1.

Fortunately, every smooth manifold has a good cover.

**Theorem 3.3.** *Every smooth manifold $M$ has a good cover. If $M$ is a compact manifold, then $M$ has a finite good cover.*

*Proof Sketch.* The proof of Theorem 3.3 makes use of some differential geometry. First, using a partitition of unity argument we can prove that every manifold has a Riemannian metric.

The second step uses the fact that in a Riemannian manifold, evey point $p$ has a geodesically convex neighborhood $U$, which means that any two points $p_1, p_2 \in U$ can be joined by a geodesic that stays inside $U$. Now, any intersection of geodesically convex neighborhoods is still geodesically convex, and a geodesically convex neighborhood is diffeomorphic to $\mathbb{R}^n$, so any open cover consisting of geodesically convex open subsets is a good cover.         $\square$

The above argument can be easily adapted to prove that every open cover of a manifold can be refined to a good open cover.

We can now prove that the de Rham cohomoloy spaces of a manifold endowed with a finite good cover are finite-dimensional. To simply notation, we write $H^p$ instead of $H^p_{\mathrm{dR}}$.

**Theorem 3.4.** *If a manifold $M$ has a finite good cover, then the cohomology vector spaces $H^p(M)$ are finite-dimensional for all $p \geq 0$.*

*Proof.* We proceed by induction on the number of open sets in a good cover $(V_1, \ldots, V_p)$. If $p = 1$, then $V_1$ itself is diffeomorphic to $\mathbb{R}^n$, and by the Poincaré lemma (Proposition 3.1) the cohomology spaces are either $(0)$ or $\mathbb{R}^n$. Thus, the base case holds.

For the induction step, assume that the cohomology of a manifold having a good cover with at most $p$ open sets is finite-dimensional, and let $\mathcal{U} = (V_1, \ldots, V_{p+1})$ be a good cover with $p + 1$ open subsets. The open subset $(V_1 \cup \cdots \cup V_p) \cap V_{p+1}$ has a good cover with $p$ open subsets, namely $(V_1 \cap V_{p+1}, \ldots, V_p \cap V_{p+1})$. See Figures 3.2 and 3.3 By the induction hypothesis, the vector spaces $H^p(V_1 \cup \cdots \cup V_p)$, $H^p(V_p)$ and $H^p((V_1 \cup \cdots \cup V_p) \cap V_{p+1})$ are finite-dimensional for all $p$, so by the consequence of the long Mayer–Vietoris cohomology sequence stated just before definition 3.5, with $M = V_1 \cup \cdots \cup V_{p+1}$, $U_1 = V_1 \cup \cdots \cup V_p$, and $U_2 = V_{p+1}$, we conclude that the vector spaces $H^p(U_1 \cup \cdots \cup V_{p+1})$ are finite-dimensional for all $p$, which concludes the induction step.  $\square$
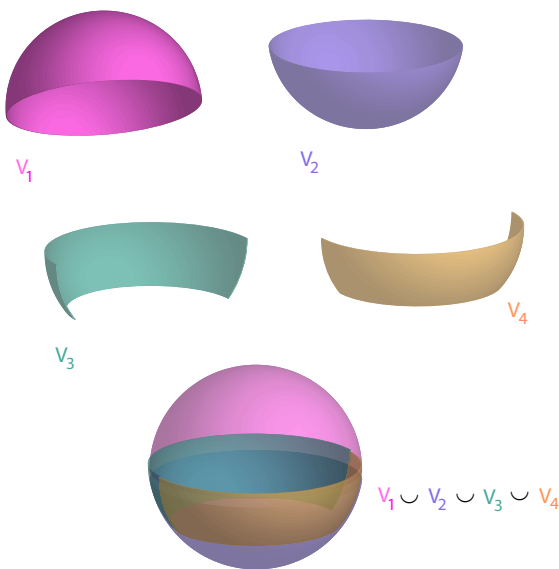


Figure 3.2: A good cover of $S^2$ consisting of four open sets. Note $V_1 \cap V_2 = V_3 \cap V_4 = \emptyset$.

As a special case of Theorem 3.4, we see that the cohomology of any compact manifold is finite-dimensional.

A similar result holds de Rham cohomology with compact support, but we have to be a little careful because in general, the pullback of a form with compact support by a smooth map may *not* have compact support. Fortunately, the Mayer–Vietoris sequence only needs inclusion maps between open sets.
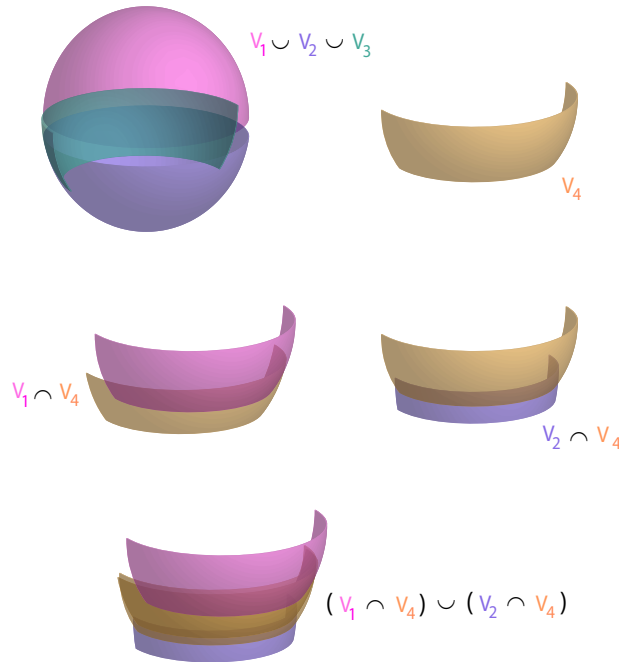
Figure 3.3: The inductive good cover construction applied to $V_1 \cup V_2 \cup V_3 \cup V_4$, a good cover of $S^2$.

Given any two open subsets $U, V$ of $M$, if $U \subseteq V$ and $i\colon U \to V$ is the inlusion map, there is an induced map $i_*\colon \mathcal{A}_c^p(U) \to \mathcal{A}_c^p(V)$ defined such that

$$
\begin{aligned}
(i_*(\omega))(p) &= \omega(p) && \text{if } p \in U \\
(i_*(\omega))(p) &= 0 && \text{if } p \in V - \operatorname{supp} \omega.
\end{aligned}
$$

We say that $\omega$ has been extended to $V$ by zero. Notice that unlike the definition of the pullback $f^*\omega$ of a form $\omega \in \mathcal{A}^p(V)$ by a smooth map $f\colon U \to V$ where $f^*\omega \in \mathcal{A}^p(U)$, the map $i_*$ pushes a form $\omega \in \mathcal{A}_c^p(U)$ *forward* to a form $i_*\omega \in \mathcal{A}_c^p(V)$. If $i\colon U \to V$ and $j\colon V \to W$ are two inclusions, then $(j \circ i)_* = j_* \circ i_*$, with no reversal of the order of $i_*$ and $j_*$.

Let $M$ be a smooth manifold and assume that $M = U_1 \cup U_2$ for two open subsets $U_1$ and $U_2$ of $M$. The inclusion maps $i_k\colon U_k \to M$ and $j_k\colon U_1 \cap U_2 \to U_k$ for $k = 1, 2$ induce a map $s\colon \mathcal{A}_c^*(U_1) \oplus \mathcal{A}_c^*(U_2) \to \mathcal{A}_c^*(M)$ given by $s(\omega_1, \omega_2) = (i_1)_*(\omega_1) + (i_2)_*(\omega_2)$ and a map $j\colon \mathcal{A}_c^*(U_1 \cap U_2) \to \mathcal{A}_c^*(U_1) \oplus \mathcal{A}_c^*(U_2)$ given by $j(\omega) = ((j_1)_*(\omega), -(j_2)_*(\omega))$. We have the following short exact sequence called the *Mayer–Vietoris sequence for cohomoloy with compact support*.

**Proposition 3.5.** *For any smooth manifold $M$, if $M = U_1 \cup U_2$ for any two open subsets*

$U_1$ and $U_2$, then we have the short exact sequence

$$0 \longrightarrow \mathcal{A}_c^*(U_1 \cap U_2) \xrightarrow{\ j\ } \mathcal{A}_c^*(U_1) \oplus \mathcal{A}_c^*(U_2) \xrightarrow{\ s\ } \mathcal{A}_c^*(M) \longrightarrow 0.$$

For a proof of Proposition 3.5, see Bott and Tu [2] (Chapter 1, Proposition 2.7). Observe that compared to the Mayer–Vietoris sequence of Proposition 3.2, the direction of the arrows is reversed.

If we apply Theorem 2.19 to the Mayer–Vietoris sequence of Proposition 3.5 we obtain the long Mayer–Vietoris sequence for cohomology with compact support shown below:

$$\cdots \longrightarrow H_{\mathrm{dR,c}}^{p-1}(M) \longrightarrow$$
$$\xrightarrow{\ \delta_c^{p-1}\ }$$
$$\longrightarrow H_{\mathrm{dR,c}}^{p}(U_1 \cap U_2) \xrightarrow{\ j^*\ } H_{\mathrm{dR,c}}^{p}(U_1) \oplus H_{\mathrm{dR,c}}^{p}(U_2) \xrightarrow{\ s^*\ } H_{\mathrm{dR,c}}^{p}(M) \longrightarrow$$
$$\xrightarrow{\ \delta_c^{p}\ }$$
$$\longrightarrow H_{\mathrm{dR,c}}^{p+1}(U_1 \cap U_2) \xrightarrow{\ j^*\ } H_{\mathrm{dR,c}}^{p+1}(U_1) \oplus H_{\mathrm{dR,c}}^{p+1}(U_2) \xrightarrow{\ s^*\ } H_{\mathrm{dR,c}}^{p+1}(M) \longrightarrow$$
$$\xrightarrow{\ \delta_c^{p+1}\ }$$
$$\longrightarrow H_{\mathrm{dR,c}}^{p+2}(U_1 \cap U_2) \longrightarrow \cdots$$

(for all $p$). Then, using the above sequence and the Poincaré lemma, using basically the same proof as in Theorem 3.4 we obtain the following result.

**Theorem 3.6.** *If a manifold $M$ has a finite good cover, then the vector spaces $H_{\mathrm{dR,c}}^{p}(M)$ of cohomology with compact support are finite-dimensional for all $p \geq 0$.*

The long exact sequences of cohomology induced by Proposition 3.2 and Proposition 3.5 can be combined to prove a version of Poincaré duality. Following Bott and Tu [2] we give a brief presentation of this result.

## 3.3   Poincaré Duality on an Orientable Manifold

Let $M$ be a smooth orientable manifold without boundary of dimension $n$. In this section, to simplify notation we write $H^p(M)$ for $H_{\mathrm{dR}}(M)$ and $H_c^p(M)$ for $H_{\mathrm{dR,c}}(M)$. For any form $\omega \in \mathcal{A}^p(M)$ and any form with compact support $\eta \in \mathcal{A}_c^{n-p}(M)$, the support of the $n$-form $\omega \wedge \eta$ is contained in both supports of $\omega$ an $\eta$, so $\omega \wedge \eta$ also has compact support and $\int_M \omega \wedge \eta$ makes sense. Since $B^*(M)$ is an ideal in $Z^*(M)$ and by Stokes' Theorem $\int_M d\omega = 0$, we have a well-defined map

$$\langle -, - \rangle \colon H^p(M) \times H_c^{n-p}(M) \longrightarrow \mathbb{R}$$

defined by

$$\langle [\omega], [\eta] \rangle = \int_M \omega \wedge \eta,$$

for any closed form $\omega \in \mathcal{A}^p(M)$ and any closed form with compact support $\eta \in \mathcal{A}_c^{n-p}(M)$. The above map is clearly bilinear so it is a pairing. Recall that if the vector spaces $H^p(M)$ and $H_c^{n-p}(M)$ are finite-dimensional (which is the case if $M$ has a finite good cover) and if the pairing is nondegenerate, then it induces a natural isomorphism between $H^p(M)$ and the dual space $(H_c^{n-p}(M))^*$ of $H_c^{n-p}(M)$.

**Theorem 3.7.** *(Poincaré duality) Let $M$ be a smooth oriented $n$-dimensional manifold. If $M$ has a finite good cover, then the map*

$$\langle -, - \rangle \colon H^p(M) \times H_c^{n-p}(M) \longrightarrow \mathbb{R}$$

*is a nondegenerate pairing. This implies that we have isomorphisms*

$$H^p(M) \cong (H_c^{n-p}(M))^*$$

*for all $p$ with $0 \leq p \leq n$. In particular, if $M$ is compact then*

$$H^p(M) \cong (H^{n-p}(M))^*$$

*for all $p$ with $0 \leq p \leq n$.*

The proof of Theorem 3.7 uses induction on the size of a finite good cover for $M$. For the induction step, the long exact sequences of cohomology induced by Proposition 3.2 and Proposition 3.5 are combined in a clever way, and the five lemma (Proposition 2.23) is used. Proofs of Theorem 3.7 are given in Bott and Tu [2] (Chapter 1, pages 44-46), and in more details in Madsen and Tornehave [31] (Chapter 13).

The first step of the proof is to dualize the second long exact sequence of cohomology. It turns out that this yields an exact sequence, and for this we need the following proposition. This is actually a special case of Proposition 2.6, but it does not hurt to give a direct proof.

**Proposition 3.8.** *Let $A, B, C$ be three vector spaces and let $\varphi \colon A \to B$ and $\psi \colon B \to C$ be two linear maps such that the sequence*

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

*is exact at $B$. Then the sequence*

$$C^* \xrightarrow{\psi^\top} B^* \xrightarrow{\varphi^\top} A^*$$

*is exact at $B^*$.*

*Proof.* Recall that $\varphi^\top \colon B^* \to A^*$ is the linear map defined such that $\varphi^\top(f) = f \circ \varphi$ for every linear form $f \in B^*$ and similarly $\psi^\top \colon C^* \to B^*$ is given by $\psi^\top(g) = g \circ \psi$ for every linear form $g \in C^*$. The fact that the first sequence is exact at $B$ means that $\operatorname{Im} \varphi = \operatorname{Ker} \psi$, which implies $\psi \circ \varphi = 0$, thus $\varphi^\top \circ \psi^\top = 0$, so $\operatorname{Im} \psi^\top \subseteq \operatorname{Ker} \varphi^\top$. Conversely, we need to prove that $\operatorname{Ker} \varphi^\top \subseteq \operatorname{Im} \psi^\top$.

Pick any $f \in \operatorname{Ker} \varphi^\top$, which means that $\varphi^\top(f) = 0$, that is $f \circ \varphi = 0$. Consequently $\operatorname{Im} \varphi \subseteq \operatorname{Ker} f$, and since $\operatorname{Im} \varphi = \operatorname{Ker} \psi$ we have

$$\operatorname{Ker} \psi \subseteq \operatorname{Ker} f.$$

We are going to construct a linear form $g \in C^*$ such that $f = g \circ \psi = \psi^\top(g)$. Observe that it suffices to construct such a linear form defined on $\operatorname{Im} \psi$, because such a linear form can then be extended to the whole of $C$.

Pick any basis $(v_i)_{i \in I}$ in $\operatorname{Im} \psi$, and let $(u_i)_{i \in I}$ be any family of vectors in $B$ such that $\psi(u_i) = v_i$ for all $i \in I$. Then, by a familiar argument $(u_i)_{i \in I}$ is linearly independent and it spans a subspace $D$ of $B$ such that

$$B = \operatorname{Ker} \psi \oplus D.$$

Define $g \colon C \to K$ such that
$$g(v_i) = f(u_i), \quad i \in I.$$

We claim that $f = g \circ \psi$.

Indeed, $f(u_i) = g(v_i) = (g \circ \psi)(u_i)$ for all $i \in I$, and if $w \in \operatorname{Ker} \psi$, since $\operatorname{Ker} \psi \subseteq \operatorname{Ker} f$, we have

$$f(w) = 0 = (g \circ \psi)(w) = 0.$$

Therefore, $f = g \circ \psi = \psi^\top(g)$, which shows that $f \in \operatorname{Im} \psi^\top$, as desired.   $\square$

By applying Proposition 3.8 to the second long exact sequence of cohomology (of compact support), we obtain the following long exact sequence:

$$\cdots \xrightarrow{\;(\delta_c^{p+1})^\top\;} H_c^{p+2}(U_1 \cap U_2)^* \longrightarrow$$

$$\longrightarrow H_c^{p+1}(M)^* \xrightarrow{(s^*)^\top} H_c^{p+1}(U_1)^* \oplus H_c^{p+1}(U_2)^* \xrightarrow[(\delta_c^p)^\top]{(j^*)^\top} H_c^{p+1}(U_1 \cap U_2)^* \longrightarrow$$

$$\longrightarrow H_c^p(M)^* \xrightarrow{(s^*)^\top} H_c^p(U_1)^* \oplus H_c^p(U_2)^* \xrightarrow[(\delta_c^{p-1})^\top]{(j^*)^\top} H_c^p(U_1 \cap U_2)^* \longrightarrow$$

$$\longrightarrow H_c^{p-1}(M)^* \longrightarrow \cdots$$

(for all $p$).

Let us denote by $\theta_M^p \colon H^p(M) \to (H_c^{n-p}(M))^*$ the isomorphism given by Theorem 3.7. The following propositions are shown in Bott and Tu [2] (Chapter 1, Lemma 5.6), and in Madsen and Tornehave [31] (Chapter 13, Lemma 13.6 and Lemma 13.7).

**Proposition 3.9.** *For any two open subsets $U$ and $V$ of a manifold $M$, if $U \subseteq V$ and $i \colon U \to V$ is the inclusion map, then the following diagrams commute for all $p$:*

$$
\begin{array}{ccc}
H^p(V) & \xrightarrow{\ i^*\ } & H^p(U) \\
\theta_V^p \downarrow & & \downarrow \theta_U^p \\
H_c^{n-p}(V)^* & \xrightarrow[\ i_*^\top\ ]{} & H_c^{n-p}(U)^*.
\end{array}
$$

**Proposition 3.10.** *For any two open subsets $U_1$ and $U_2$ of a manifold $M$, if $U = U_1 \cup U_2$ then the following diagrams commute for all $p$:*

$$
\begin{array}{ccc}
H^p(U_1 \cap U_2) & \xrightarrow{\ \delta^p\ } & H^{p+1}(U) \\
\theta_{U_1 \cap U_2}^p \downarrow & & \downarrow \theta_U^{p+1} \\
H_c^{n-p}(U_1 \cap U_2)^* & \xrightarrow[(-1)^{p+1}(\delta_c^{n-p-1})^\top]{} & H_c^{n-p-1}(U)^*.
\end{array}
$$

Using Proposition 3.9 and Proposition 3.10, we obtain a diagram in which the top and bottom rows are exact and every square commutes. Here is a fragment of this diagram in which we have omitted the labels of the horizontal arrows to unclutter this diagram:

$$
\begin{array}{ccccc}
\longrightarrow & H^{p-1}(U_1) \oplus H^{p-1}(U_2) & \longrightarrow & H^{p-1}(U_1 \cap U_2) & \longrightarrow \\
& \downarrow{\scriptstyle \theta_{U_1}^{p-1} \oplus \theta_{U_2}^{p-1}} & & \downarrow{\scriptstyle \theta_{U_1 \cap U_2}^{p-1}} & \\
\longrightarrow & H_c^{n-p+1}(U_1)^* \oplus H_c^{n-p+1}(U_2)^* & \longrightarrow & H_c^{n-p+1}(U_1 \cap U_2)^* & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccccc}
\longrightarrow & H^p(U) & \longrightarrow & H^p(U_1) \oplus H^p(U_2) & \longrightarrow & H^p(U_1 \cap U_2) & \longrightarrow \\
& \downarrow{\scriptstyle \theta_U^p} & & \downarrow{\scriptstyle \theta_{U_1}^p \oplus \theta_{U_2}^p} & & \downarrow{\scriptstyle \theta_{U_1 \cap U_2}^p} & \\
\longrightarrow & H_c^{n-p}(U)^* & \longrightarrow & H_c^{n-p}(U_1)^* \oplus H_c^{n-p}(U_2)^* & \longrightarrow & H_c^{n-p}(U_1 \cap U_2)^* & \longrightarrow
\end{array}
$$

Now, here is the crucial step of the proof. Suppose we can prove that the maps $\theta_{U_1}^p, \theta_{U_2}^p$ and $\theta_{U_1 \cap U_2}^p$ are isomorphisms for all $p$. Then, by the five lemma (Proposition 2.23), we can conclude that the maps $\theta_U^p$ are also isomorphisms.

We can now give the main part of the proof of Theorem 3.7 using induction on the size of a finite good cover.

*Proof sketch of Theorem 3.7.* Let $\mathcal{U} = (V_1, \ldots, V_p)$ be a good cover for the orientable manifold $M$. We proceed by induction on $p$. If $p = 1$, then $M = V_1$ is diffeomorphic to $\mathbb{R}^n$ and by the Poincaré lemma (Proposition 3.1) we have

$$H_{\mathrm{dR}}^p(\mathbb{R}^n) = \begin{cases} 0 & \text{unless } p \neq 0 \\ \mathbb{R} & \text{if } p = 0, \end{cases}$$

and

$$H_{\mathrm{dR},c}^p(\mathbb{R}^n) = \begin{cases} 0 & \text{unless } p \neq 0 \\ \mathbb{R} & \text{if } p = n, \end{cases}$$

so we have the desired isomorphisms.

Assume inductively that Poincaré duality holds for any orientable manifold having a good cover with at most $p$ open subsets, and let $(V_1, \ldots, V_{p+1})$ be a cover with $p + 1$ open subsets. Observe that $(V_1 \cup \cdots \cup V_p) \cap V_{p+1}$ has a good cover with $p$ open subsets, namely $(V_1 \cap V_{p+1}, \ldots, V_p \cap V_{p+1})$. By the induction hypothesis applied to $U_1 = V_1 \cup \cdots \cup V_p$, $U_2 = V_{p+1}$, and $U = M = V_1 \cup \cdots \cup V_{p+1}$, the maps $\theta_{U_1}^p, \theta_{U_2}^p$ and $\theta_{U_1 \cap U_2}^p$ in the diagram shown just after Proposition 3.10 are isomorphisms for all $p$, so by the five lemma (Proposition 2.23) we can conclude that the maps $\theta_U^p$ are also isomorphisms, establishing the induction step. $\qquad\square$

As a corollary of Poincaré duality, if $M$ is an orientable and connected manifold, then $H^0(M) \cong \mathbb{R}$, and so $H_c^n(M) \cong \mathbb{R}$. In particular, if $M$ is compact then $H^n(M) \cong \mathbb{R}$.

**Remark:** As explained in Bott and Tu [2], the assumption that the good cover is finite is not necessary. Then, the statement of Poincaré duality is that if $M$ is any orientable manifold of dimension $n$, then there are isomorphisms

$$H^p(M) \cong (H_c^{n-p}(M))^*$$

for all $p$ with $0 \leq p \leq n$, even if $H^p(M)$ is infinite dimensional. However, the statement obtained by taking duals, namely

$$H_c^p(M) \cong (H^{n-p}(M))^*,$$

is generally false.

In Chapter 1 of their book, Bott and Tu derive more consequences of the Mayer–Vietoris method. The interested reader is referred to Bott and Tu [2].

The de Rham cohomology is a very effective tool to deal with manifolds but one of the drawbacks of using real coefficients is that torsion phenomena are overlooked. There are other cohomology theories of finer grain that use coefficients in rings such as $\mathbb{Z}$. One of the simplest uses singular chains, and we discuss it in the next chapter.

# Chapter 4

# Singular Homology and Cohomology

## 4.1 Singular Homology

In this section we only assume that our space $X$ is a Hausdorff topological space, and we consider continuous maps between such spaces. Singular homology (and cohomology) arises from chain complexes built from singular chains (and cochains). Singular chains are defined in terms of certain convex figures generalizing line segments, triangles and tetrahedra called standard $n$-simplices. We adopt the definition from Milnor and Stasheff [35].

**Definition 4.1.** For any integer $n \geq 0$, the *standard $n$-simplex* $\Delta^n$ is the convex subset of $\mathbb{R}^{n+1}$ consisting of the set of points

$$\Delta^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + t_1 + \cdots + t_n = 1,\ t_i \geq 0\}.$$

The $n+1$ points corresponding to the canonical basis vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ are called the *vertices* of the simplex $\Delta^n$.

The simplex $\Delta^n$ is the convex hull of the $n+1$ points $(e_1, \ldots, e_{n+1})$ since we can write

$$\Delta^n = \{t_0 e_1 + t_1 e_2 + \cdots + t_n e_{n+1} \mid t_0 + t_1 + \cdots + t_n = 1,\ t_i \geq 0\}.$$

Thus, $\Delta^n$ is a subset of $\mathbb{R}^{n+1}$. In particular, when $n = 0$, the 0-simplex $\Delta^0$ consists of the single points $t_0 = 1$ on $\mathbb{R}$. Some simplices are illustrated in Figure 4.1.

**Remark:** Other authors such as Bott and Tu [2] and Warner [50] define the $n$-simplex $\Delta^n$ as a convex subset of $\mathbb{R}^n$. In their definition, if we denote the point corresponding to the origin of $\mathbb{R}^n$ as $e_0$, then

$$\Delta^n = \{t_0 e_0 + t_1 e_1 + \cdots + t_n e_n \mid t_0 + t_1 + \cdots + t_n = 1,\ t_i \geq 0\}.$$
$$= \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_1 + \cdots + t_n \leq 1,\ t_i \geq 0\}.$$

Some of these simplices are illustrated in Figure 4.2.

These points of view are equivalent but one should be careful that the notion of face of a singular simplex (see below) is defined slightly differently.
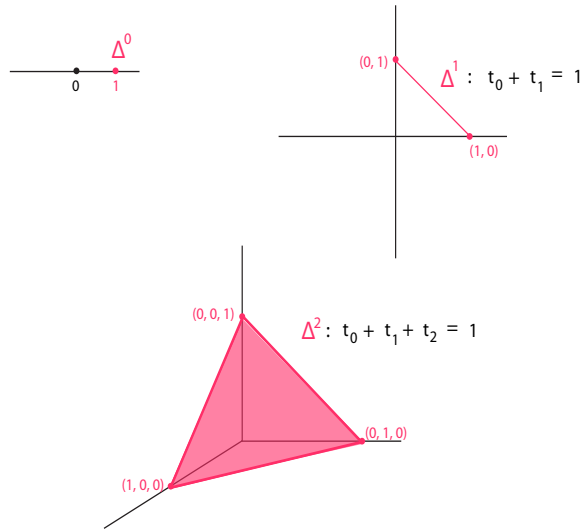
Figure 4.1: The simplices $\Delta^0, \Delta^1, \Delta^2$.

**Definition 4.2.** Given a topological space $X$, a *singular p-simplex* is any continuous map $\sigma\colon \Delta^p \to X$ (with $p \geq 0$). If $p \geq 1$, the *ith face (map)* of the singular $p$-simplex $\sigma$ is the $(p-1)$-singular simplex

$$\sigma \circ \phi_i^{p-1}\colon \Delta^{p-1} \to X, \quad 0 \leq i \leq p,$$

where $\phi_i^{p-1}\colon \Delta^{p-1} \to \Delta^p$ is the map given by

$$\phi_0^{p-1}(t_1, \ldots, t_p) = (0, t_1, \ldots, t_p)$$
$$\phi_i^{p-1}(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_p) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_p), \quad 1 \leq i \leq p-1$$
$$\phi_p^{p-1}(t_0, \ldots, t_{p-1}) = (t_0, \ldots, t_{p-1}, 0).$$

Some singular 1-simplices and singular 2-simplices are illustrated in Figure 4.3.

Note that a singular $p$-simplex $\sigma$ has $p+1$ faces. The *ith* face $\sigma \circ \phi_i^{p-1}$ is sometimes denoted by $\sigma^i$. For example, if $p = 1$, since there is only one variable on $\mathbb{R}^1$ and $\Delta^0 = \{1\}$, the maps $\phi_0^0, \phi_1^0\colon \Delta^0 \to \Delta^1$ are given by

$$\phi_0^0(1) = (0, 1), \quad \phi_1^0(1) = (1, 0).$$

For $p = 2$, the maps $\phi_0^1, \phi_1^1, \phi_2^1\colon \Delta^1 \to \Delta^2$ are given by

$$\phi_0^1(t_1, t_2) = (0, t_1, t_2), \quad \phi_1^1(t_0, t_2) = (t_0, 0, t_2), \quad \phi_2^1(t_0, t_1) = (t_0, t_1, 0).$$

There does not seem to be any standard notation for the set of all singular $p$-simplicies on $X$. We propose the notation $S_{\Delta^p}(X)$.
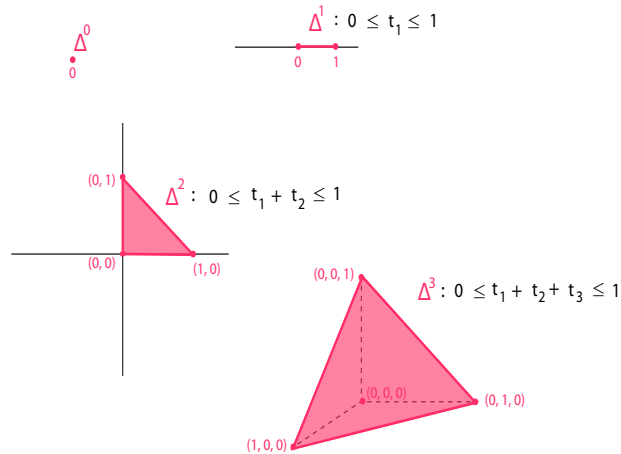
Figure 4.2: Some simplices according to the second definition.

**Remark:** In Definition 4.2 we may replace $X$ by any open subset $U$ of $X$, in which case a continuous map $\sigma \colon \Delta^p \to U$ is called a *singular p-simplex in U*. If $X$ is a smooth manifold, following Warner [50], we define a *differentiable singular p-simplex* in $U$ to be a singular $p$-simplex $\sigma$ which can be extended to a smooth map of some open subset of $\mathbb{R}^{n+1}$ containing $\Delta^p$ into $U$.

We now come to the crucial definition of singular $p$-chains. In the framework of singular homology (and cohomology) we have the extra degree of freedom of choosing the coefficients. The set of coefficients will be a commutative ring with unit denoted by $R$. Better results are obtained if we assume that $R$ is a PID. In most cases, we may assume that $R = \mathbb{Z}$.

**Definition 4.3.** Given a topological space $X$ and a commutative ring $R$, a *singular p-chain with coefficients in R* is any formal linear combination $\alpha = \sum_{i=1}^m \lambda_i \sigma_i$ of singular $p$-simplices $\sigma_i$ with coefficients $\lambda_i \in R$. The *singular chain group* $S_p(X; R)$ is the free $R$-module consisting of all singular $p$-chains; it is generated by the set $S_{\Delta^p}(X)$ of singular $p$-simplices. We set $S_p(X; R) = (0)$ for $p < 0$. If $p \geq 1$, given any singular $p$-simplex $\sigma$, its *boundary* $\partial \sigma$ is the singular $(p-1)$-chain given by

$$\partial \sigma = \sigma \circ \phi_0^{p-1} - \sigma \circ \phi_1^{p-1} + \cdots + (-1)^p \sigma \circ \phi_p^{p-1}.$$

Extending the map $\partial$ to $S_p(X; R)$ by linearity, we obtain the *boundary homomorphism*

$$\partial \colon S_p(X; R) \to S_{p-1}(X; R).$$

When we want to be very precise, we write $\partial_p \colon S_p(X; R) \to S_{p-1}(X; R)$. We define $S_*(X; R)$ as the direct sum

$$S_*(X; R) = \bigoplus_{p \geq 0} S_p(X; R).$$
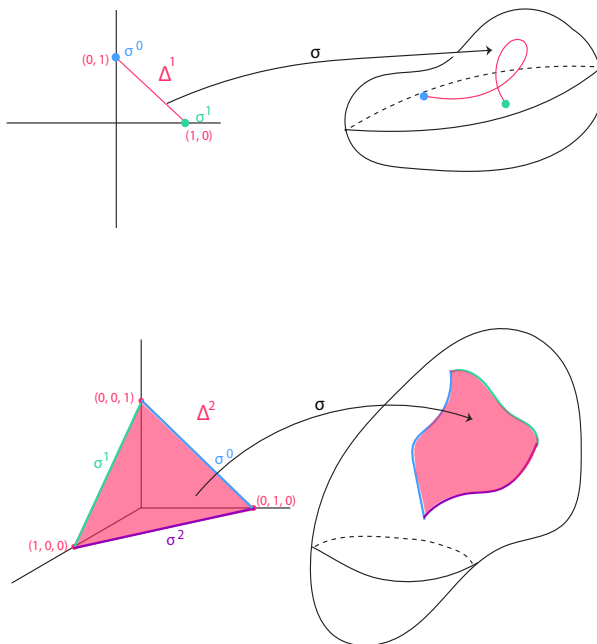
Figure 4.3: Some singular simplices.

Then, the boundary maps $\partial_p$ yield the boundary map $\partial\colon S_*(X;R) \to S_*(X;R)$. For example, the boundary of a singular 1-simplex $\sigma$ is $\sigma(0,1) - \sigma(1,0)$. The boundary of a singular 2-simplex $\sigma$ is

$$\sigma^0 - \sigma^1 + \sigma^2,$$

where $\sigma^0, \sigma^1, \sigma^2$ are the faces of $\sigma$, in this case, three curves in $X$. For example, $\sigma^0$ is the curve given by the map

$$(t_1, t_2) \mapsto \sigma(0, t_1, t_2)$$

from $\Delta^1$ to $X$, where $t_1 + t_2 = 1$ and $t_1, t_2 \geq 0$.

The following result is easy to check

**Proposition 4.1.** *Given a topological space $X$ and a commutative ring $R$, the boundary map $\partial\colon S_*(X;R) \to S_*(X;R)$ satisfies the equation*

$$\partial \circ \partial = 0.$$

We can put together the maps $\partial_p\colon S_p(X;R) \to S_{p-1}(X;R)$ to obtain the following chain complex of homology

$$0 \xleftarrow{\partial_0} S_0(X;R) \xleftarrow{\partial_1} S_1(X;R) \xleftarrow{\hspace{1cm}} \cdots \xleftarrow{\partial_{p-1}} S_{p-1}(X;R) \xleftarrow{\partial_p} S_p(X;R) \xleftarrow{\partial_{p+1}} \cdots$$

in which the direction of the arrows is from right to left. Note that if we replace every nonnegative index $p$ by $-p$ in $\partial_p, S_p(X; R)$ *etc.*, then we obtain a chain complex as defined in Section 2.3 and we now have all the ingredients to define homology groups. We have the familiar spaces $Z_p(X; R) = \operatorname{Ker} \partial_p$ of *singular p-cycles*, and $B_p(X; R) = \operatorname{Im} \partial_{p+1}$ of *singular p-boundaries*. By Proposition 4.1, $B_p(X; R)$ is a submodule of $Z_p(X; R)$ so we obtain homology spaces:

**Definition 4.4.** Given a topological space $X$ and a commutative ring $R$, for any $p \geq 0$ the *singular homology module* $H_p(X; R)$ is defined by

$$H_p(X; R) = \ker \partial_p / \operatorname{Im} \partial_{p+1} = Z_p(X; R)/B_p(X; R).$$

We set $H_p(X; R) = (0)$ for $p < 0$ and define $H_*(X; R)$ as the direct sum

$$H_*(X; R) = \bigcup_{p \geq 0} H_p(X; R)$$

and call it the *singular homology of $X$ with coefficients in $R$.*

The spaces $H_p(X; R)$ are $R$-modules but following common practice we often refer to them as groups.

A singular 0-chain is a linear combination $\sum_{i=1}^{m} \lambda_i P_i$ of points $P_i \in X$. Because the boundary of a singular 1-simplex is the difference of two points, if $X$ is path-connected, it is easy to see that a singular 0-chain is the boundary of a singular 1-chain iff $\sum_{i=1}^{m} \lambda_i = 0$. Thus, $X$ is path connected iff

$$H_0(X; R) = R.$$

More generally, we have the following proposition.

**Proposition 4.2.** *Given any topological space $X$, for any commutative ring $R$ with an identity element, $H_0(X; R)$ is a free $R$-module. If $(X_\alpha)_{\alpha \in I}$ is the collection of path components of $X$ and if $\sigma_\alpha$ is a singular $0$-simplex whose image is in $X_\alpha$, then the homology classes $[\sigma_\alpha]$ form a basis of $H_0(X; R)$.*

Proposition 4.2 is proved in Munkres [38] (Chapter 4, Section 29, Theorem 29.2). In particular, if $X$ has $m$ path-connected components, then $H_0(X; R) \cong \underbrace{R \oplus \cdots \oplus R}_{m}$.

We leave it as an exercise (or look at Bott and Tu [2], Chapter III, §15) to show that the homology groups of $\mathbb{R}^n$ are given by

$$H_p(\mathbb{R}^n; R) = \begin{cases} (0) & \text{if } p \geq 1 \\ R & \text{if } p = 0. \end{cases}$$

The same result holds if $\mathbb{R}^n$ is replaced by any nonempty convex subset of $\mathbb{R}^n$, or a space consisting of a single point.

The homology groups (with coefficients in $\mathbb{Z}$) of the compact surfaces can be completely determined. Some of them, such as the projective plane, have $\mathbb{Z}/2\mathbb{Z}$ as a homology group.

If $X$ and $Y$ are two topological spaces and if $f\colon X \to Y$ is a continuous function between them, then we have induced homomorphisms $H_p(f)\colon H_p(X; R) \to H_p(Y; R)$ between the homology groups of $X$ and the homology groups of $Y$. We say that homology is functorial.

**Proposition 4.3.** *If $X$ and $Y$ are two topological spaces and if $f\colon X \to Y$ is a continuous function between them, then there are homomorphisms $H_p(f)\colon H_p(X; R) \to H_p(Y; R)$ for all $p \geq 0$.*

*Proof.* To prove the proposition we show that there is a chain map between the chain complexes associated with $X$ and $Y$ and apply Proposition 2.16. Given any singular $p$-simplex $\sigma\colon \Delta^p \to X$ we obtain a singular $p$-simplex $f\sigma\colon \Delta^p \to Y$ obtained by composing with $f$, namely $f\sigma = f \circ \sigma$. Since $S_p(X; R)$ is freely generated by $S_{\Delta^p}(X; R)$, the map $\sigma \mapsto f\sigma$ from $S_{\Delta^p}(X; R)$ to $S_p(Y; R)$ extends uniquely to a homomorphism $S_p(f)\colon S_p(X; R) \to S_p(Y; R)$. It is immediately verified that the following diagrams are commutative

$$
\begin{array}{ccc}
S_{p+1}(X; R) & \xrightarrow{\partial^X_{p+1}} & S_p(X; R) \\
{\scriptstyle S_{p+1}(f)}\downarrow & & \downarrow{\scriptstyle S_p(f)} \\
S_{p+1}(Y; R) & \xrightarrow[\partial^Y_p]{} & S_p(Y; R),
\end{array}
$$

which means that the maps $S_p(f)\colon S_p(X; R) \to S_p(Y; R)$ form a chain map $S(f)$. By Proposition 2.16, we obtain homomorphisms $S_p(f)^*\colon H_p(X; R) \to H_p(Y; R)$ for all $p$, which we denote by $H_p(f)$. $\square$

Following the convention that in homology subscripts are used to denote objects, the map $S_p(f)\colon S_p(X; R) \to S_p(Y; R)$ is also denoted $f_{\sharp,p}\colon S_p(X; R) \to S_p(Y; R)$, and the map $H_p(f)\colon H_p(X; R) \to H_p(Y; R)$ is also denoted $f_{*p}\colon H_p(X; R) \to H_p(Y; R)$ (or simply $f_*\colon H_p(X; R) \to H_p(Y; R)$).

Proposition 4.3 implies that if two spaces $X$ and $Y$ are homeomorphic, then $X$ and $Y$ have isomorphic homology. This gives us a way of showing that some spaces are not homeomorphic: if for some $p$ the homology groups $H_p(X; R)$ and $H_p(Y; R)$ are not isomorphic, then $X$ and $Y$ are not homeomorphic.

Actually, it turns out that the homology groups of two homotopy equivalent spaces are isomorphic. Intuitively, two continuous maps $f, g\colon X \to Y$ are homotopic is $f$ can be continuously deformed into $g$, which means that there is a one-parameter family $F(-, t)$ of continuous maps $F(-, t)\colon X \to Y$ varying continuously in $t \in [0, 1]$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Here is the formal definition.

**Definition 4.5.** Two continuous maps $f, g \colon X \to Y$ (where $X$ and $Y$ are topological spaces) are *homotopic* if there is a continuous function $F \colon X \times [0, 1] \to Y$ (called a *homotopy with fixed ends*) such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x) \quad \text{for all } x \in X.$$

We write $f \simeq g$. See Figure 4.4.
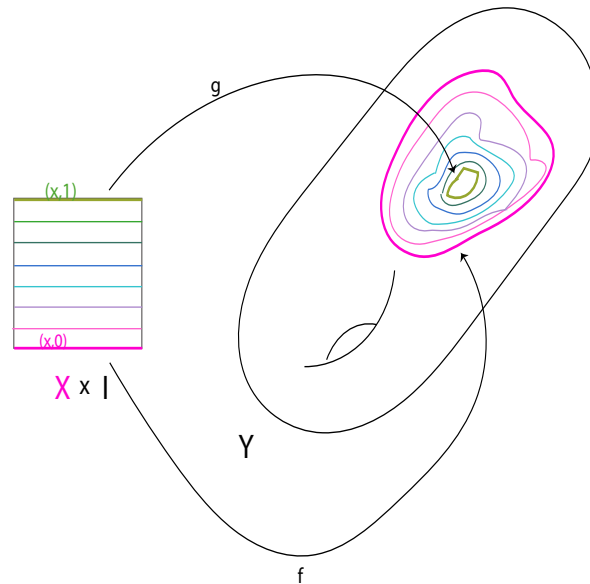


Figure 4.4: The homotopy $F$ between $X \times I$ and $Y$, where $X = [0, 1]$ and $Y$ is the torus.

A space $X$ is said to be *contractible* if the identity map $\mathrm{id}_X \colon X \to X$ is homotopic to a constant function with domain $X$. For example, any convex subset of $\mathbb{R}^n$ is contractible. Intuitively, a contractible space can be continuously deformed to a single point, so it is topologically trivial. In particular, it cannot contain holes. An example of a contractible set is shown in Figure 4.5.

A *deformation retraction* of a space $X$ onto a subspace $A$ is a homotopy $F \colon X \times [0, 1] \to X$ such that $F(x, 0) = x$ for all $x \in X$, $F(x, t) = x$ for all $x \in A$ and all $t \in (0, 1]$, and $F(X, 1) = A$. In this case, $A$ is called a *deformation retract* of $X$. An example of deformation retract is shown in Figure 4.6.

Topologically, homeomorphic spaces should be considered equivalent. From the point of view of homotopy, experience has shown that the more liberal notion of homotopy equivalence is the right notion of equivalence.
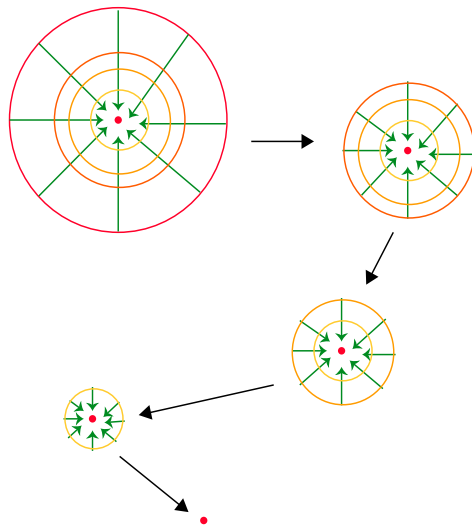
Figure 4.5: A contractible set

**Definition 4.6.** Two topological spaces $X$ and $Y$ are *homotopy equivalent* if there are continuous functions $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X, \quad f \circ g \simeq \mathrm{id}_Y.$$

We write $X \simeq Y$. See Figure 4.7.

A great deal of homotopy theory has to do with developing tools to decide when two spaces are homotopy equivalent. It turns out that homotopy equivalent spaces have isomorphic homology. In this sense homology theory is cruder than homotopy theory. However, homotopy groups are generally more complicated and harder to compute than homology groups. For one thing, homotopy groups are generally nonabelian, whereas homology groups are abelian,

**Proposition 4.4.** *Given any two continuous maps $f, g\colon X \to Y$ (where $X$ and $Y$ are topological spaces), if $f$ and $g$ are homotopic then the chain maps $S(f), S(g)\colon S_*(X; R) \to S_*(Y; R)$ are chain homotopic (see Definition 2.10).*

Proofs of Proposition 4.4 can be found in MacLane [29] (Chapter II, Theorem 8.2) and Hatcher [25] (Chapter 2, Theorem 2.10). The idea is to reduce to proof to the case where the space $Y$ is the cylinder $X \times [0, 1]$. In this case we have the two continuous maps $b, t\colon X \to X \times [0, 1]$ given by $b(x) = (x, 0)$ and $t(x) = (x, 1)$, which are clearly homotopic. Then one shows that a chain homotopy can be constructed between the chain maps $S(t)$ and $S(b)$.

As a corollary of Proposition 4.4, we obtain the following important result.
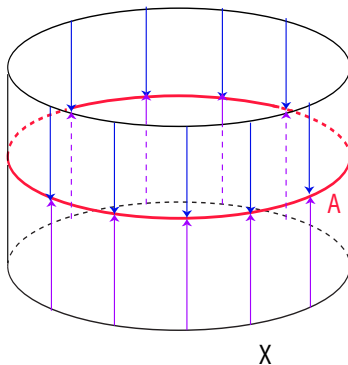
Figure 4.6: A deformation retract of the cylinder $X$ onto its median circle $A$

**Proposition 4.5.** *Given any two continuous maps $f, g \colon X \to Y$ (where $X$ and $Y$ are topological spaces), if $f$ and $g$ are homotopic and $H_p(f), H_p(g) \colon H_p(X; R) \to H_p(Y; R)$ are the induced homomorphisms, then $H_p(f) = H_p(g)$ for all $p \geq 0$. As a consequence, if $X$ and $Y$ are homotopy equivalent then the homology groups $H_p(X; R)$ and $H_p(Y; R)$ are isomorphic for all $p \geq 0$,*

*Proof.* By Proposition 4.4 there is a chain homotopy between $S(f) \colon S_*(X; R) \to S_*(Y; R)$ and $S(g) \colon S_*(X; R) \to S_*(Y; R)$, and by Proposition 2.17 the induced homomorphisms $H_p(f), H_p(g) \colon H_p(X; R) \to H_p(Y; R)$ are identical. If $f \colon X \to Y$ and $g \colon Y \to X$ are two maps making $X$ and $Y$ chain homotopic, we have $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, so by the first part of the proposition

$$H_p(g \circ f) = H_p(g) \circ H_p(f) = H_p(\mathrm{id}_X) = \mathrm{id}_{H_p(X;R)}$$

and

$$H_p(f \circ g) = H_p(f) \circ H_p(g) = H_p(\mathrm{id}_Y) = \mathrm{id}_{H_p(Y;R)},$$

which shows that the maps $H_p(f) \colon H_p(X; R) \to H_p(Y; R)$ are isomorphisms with inverses $H_p(g)$. $\square$

## 4.2 Relative Singular Homology Groups

A more flexible theory is obtained if we consider homology groups $H_p(X, A)$ associated with pairs of spaces $(A, X)$, where $A$ is a subspace of $X$ (assuming for simplicity that $R = \mathbb{Z}$, that is, integer coefficients). The quotient space $X/A$ is obtained from $X$ by identifying $A$ with a single point. Then, if $(X, A)$ is a "good pair," which means that $A$ is a nonempty closed subspace that is a deformation retract of some neighborhood in $X$ (for example if $X$ is a cell complex and $A$ is a nonempty subcomplex), it turns out that

$$H_p(X, A) \cong H_p(X/A, \{\mathrm{pt}\}),$$
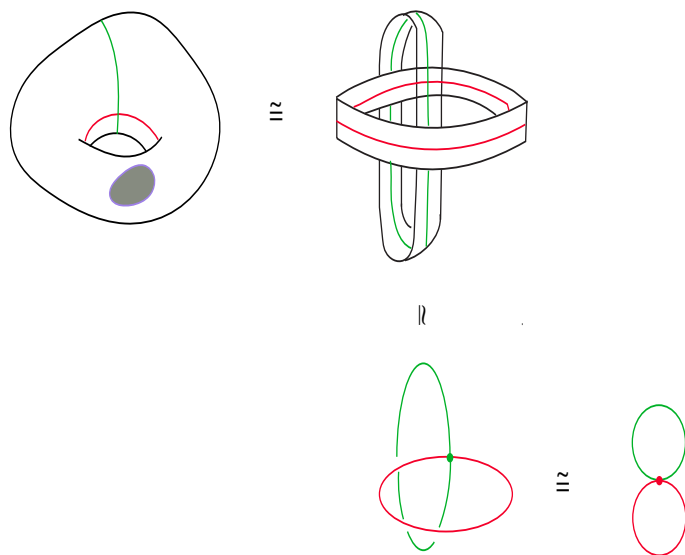
Figure 4.7: The punctured torus is homotopically equivalent to the figure eight.

where pt stands for any point in $X$. (see Hatcher [25], Proposition 2.22).

It can also be shown that the homology groups $H_p(X, \{pt\})$ are equal to the reduced homology groups of $X$, which are usually denoted by $\widetilde{H}_p(X)$, or more precisely by $\widetilde{H}_p(X; \mathbb{Z})$ (see Hatcher [25], Proposition 2.22).

**Definition 4.7.** Given a nonempty space $X$, the *reduced homology groups*

$$\widetilde{H}_0(X; R) = \operatorname{Ker} \epsilon / \operatorname{Im} \partial_1$$
$$\widetilde{H}_p(X; R) = \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1}, \quad p > 0$$

are defined by the *augmented chain complex*

$$0 \longleftarrow R \xleftarrow{\epsilon} S_0(X; R) \xleftarrow{\partial_1} S_1(X; R) \longleftarrow \cdots \xleftarrow{\partial_{p-1}} S_{p-1}(X; R) \xleftarrow{\partial_p} S_p(X; R) \xleftarrow{\partial_{p+1}} \cdots ,$$

where $\epsilon \colon S_0(X; R) \to R$ is the unique $R$-linear map such that $\epsilon(\sigma) = 1$ for every singular 0-simplex $\sigma \colon \Delta^0 \to X$ in $S_{\Delta^0}(X)$, given by

$$\epsilon\left(\sum_i \lambda_i \sigma_i\right) = \sum_i \lambda_i.$$

It is immediate to see that $\epsilon \circ \partial_1 = 0$, so $\operatorname{Im} \partial_1 \subseteq \operatorname{Ker} \epsilon$. By definition $H_0(X; R) = S_0(X; R)/\operatorname{Im} \partial_1$. The module $S_0(X; R)$ is a free $R$-module isomorphic to the direct sum $\bigoplus_{\sigma \in S_{\Delta^0}(X)} R$ with one copy of $R$ for every $\sigma \in S_{\Delta^0}(X)$, so by choosing one of the copies of

$R$ we can define an injective $R$-linear map $s\colon R \to S_0(X;R)$ such that $\epsilon \circ s = \mathrm{id}$, and we obtain the following short split exact sequence:

$$0 \longrightarrow \mathrm{Ker}\, \epsilon \longrightarrow S_0(X;R) \underset{s}{\overset{\epsilon}{\rightleftarrows}} R \longrightarrow 0.$$

Thus

$$S_0(X;R) \cong \mathrm{Ker}\, \epsilon \oplus R,$$

and since $\mathrm{Im}\, \partial_1 \subseteq \mathrm{Ker}\, \epsilon$, we get

$$S_0(X;R)/\mathrm{Im}\, \partial_1 \cong (\mathrm{Ker}\, \epsilon/\mathrm{Im}\, \partial_1) \oplus R,$$

which yields

$$H_0(X;R) = \widetilde{H}_0(X;R) \oplus R$$
$$H_p(X;R) = \widetilde{H}_p(X;R), \quad p > 0.$$

In the special case where $R = \mathbb{Z}$,

$$H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$$
$$H_p(X) = \widetilde{H}_p(X), \quad p > 0.$$

One of the reasons for introducing the reduced homology groups is that

$$\widetilde{H}_0(\{\mathrm{pt}\};R) = (0),$$

whereas

$$H_0(\{\mathrm{pt}\};R) = R.$$

On the other hand

$$H_p(\{\mathrm{pt}\};R) = \widetilde{H}_p(\{\mathrm{pt}\};R) = (0), \quad \text{if } p > 0.$$

Since $A$ is a subspace of $X$, each singular simplex $\sigma\colon \Delta^p \to A$ yields a singular simplex $\sigma\colon \Delta^p \to X$ by composing $\sigma$ with the the inclusion map from $A$ to $X$, so the singular complex $S_*(A;R)$ is a subcomplex of the singular complex $S_*(X;R)$. Let $S_p(X,A;R)$ be the quotient module

$$S_p(X,A;R) = S_p(X;R)/S_p(A;R)$$

and let $S_*(X,A;R)$ be the corresponding graded module (the direct sum of the $S_p(X,A;R)$).

The boundary map $\partial_{X,p}\colon S_p(X;R) \to S_{p-1}(X;R)$ of the original complex $S_*(X;R)$ restricts to the boundary map $\partial_{A,p}\colon S_p(A;R) \to S_{p-1}(A;R)$ of the complex $S_*(A;R)$ so the quotient map $\partial_p\colon S_p(X,A;R) \to S_{p-1}(X,A;R)$ induced by $\partial_{X,p}$ and given by

$$\partial_p(\sigma + S_p(A;R)) = \partial_{X,p}(\sigma) + S_{p-1}(A;R)$$

for every singular $p$-simplex $\sigma$ is a boundary map for the chain complex $S_*(X, A; R)$. The chain complex $S_*(X, A; R)$

$$0 \xleftarrow{\partial_0} S_0(X, A; R) \xleftarrow{\partial_1} S_1(X, A; R) \longleftarrow \cdots \xleftarrow{\partial_{p-1}} S_{p-1}(X, A; R) \xleftarrow{\partial_p} S_p(X, A; R) \xleftarrow{\partial_{p+1}} \cdots$$

is called the *singular chain complex* of the pair $(X, A)$.

**Definition 4.8.** Given a pair $(X, A)$ where $A$ is a subspace of $X$, the *singular relative homology groups* $H_p(X, A; R)$ of $(X, A)$ are defined by

$$H_p(X, A; R) = H_p(S_*(X; R)/S_*(A; R)),$$

the singular homology groups of the chain complex $S_*(X, A; R)$. For short, we often drop the word "singular" in singular relative homology group.

Observe that the quotient module $S_p(X, A; R) = S_p(X; R)/S_p(A; R)$ is a free module. Indeed, the family of cosets of the form $\sigma + S_p(A; R)$ where the image of the singular $p$-simplex $\sigma$ does not lie in $A$ forms a basis of $S_p(X, A; R)$.

The relative homology group $H_p(X, A; R)$ is also expressed as the quotient

$$H_p(X, A; R) = Z_p(X, A; R)/B_p(X, A; R),$$

where $Z_p(X, A; R)$ is the group of *relative $p$-cycles*, namely those chains $c \in S_p(X; R)$ such that $\partial_p c \in S_{p-1}(A; R)$, and $B_p(X, A; R)$ is the group of *relative $p$-boundaries*, where $c \in B_p(X, A; R)$ iff $c = \partial_{p+1}\beta + \gamma$ with $\beta \in S_{p+1}(X; R)$ and $\gamma \in S_p(A; R)$. An illustration of the notion of relative cycle is shown in Figure 4.8 and of a relative boundary in Figure 4.9.

A single space $X$ may be regarded as the pair $(X, \emptyset)$, and so $H_p(X, \emptyset; R) = H_p(X; R)$.

**Definition 4.9.** Given two pairs $(X, A)$ and $(Y, B)$ with $A \subseteq X$ and $B \subseteq Y$, a map $f\colon (X, A) \to (Y, B)$ is a continuous function $f\colon X \to Y$ such that $f(A) \subseteq B$. A *homotopy* $F$ between two maps $f, g\colon (X, A) \to (Y, B)$ is a homotopy between $f$ and $g$ such that $F(A \times [0, 1]) \subseteq B$; we write $f \simeq g$. Two pairs $(X, A)$ and $(Y, B)$ are *homotopy equivalent* if there exist maps $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, A)$ such that $g \circ f \simeq (\mathrm{id}_X, \mathrm{id}_A)$ and $f \circ g \simeq (\mathrm{id}_Y, \mathrm{id}_B)$.

Proposition 4.3 is easily generalized to pairs of spaces.

**Proposition 4.6.** *If $(X, A)$ and $(Y, A)$ are pairs of spaces and if $f\colon (X, A) \to (Y, B)$ is a continuous map between them, then there are homomorphisms $H_p(f)\colon H_p(X, A; R) \to H_p(Y, B; R)$ for all $p \geq 0$.*
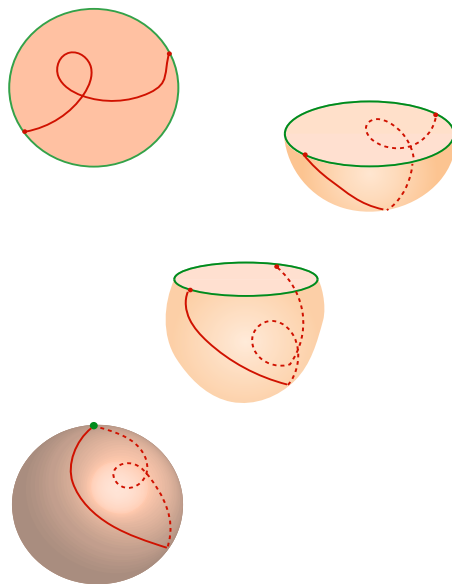
Figure 4.8: Let $X$ be the closed unit disk and $A$ its circular boundary. Let $p = 1$. The red curve is a relative cycle since its boundary is in $A$. We show the effect of collapsing $A$ a point, namely transforming $X$ into a unit sphere.
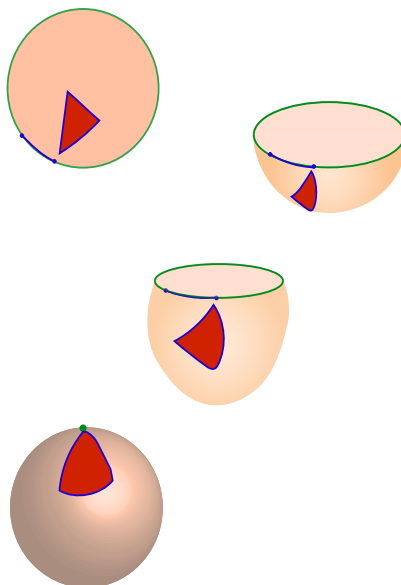


Figure 4.9: Let $X$ be the closed unit disk and $A$ its circular boundary. Let $p = 1$. The burnt orange triangle and the blue arc form a relative boundary.

*Proof sketch.* Given any singular $p$-simplex $\sigma\colon \Delta^p \to X$ by composition with $f$ we obtain the singular $p$-simplex $f\sigma\colon \Delta^p \to Y$, and since $S_p(X;R)$ is freely generated by $S_{\Delta^p}(X;R)$ we get a homomorphism $S_p(f)\colon S_p(X;R) \to S_p(Y;R)$. Consider the composite map $\varphi\colon S_p(X;R) \to S_p(Y;R)/S_p(B;R)$ given by

$$S_p(X;R) \xrightarrow{S_p(f)} S_p(Y;R) \xrightarrow{\pi_{Y,B}} S_p(Y;R)/S_p(B;R).$$

Since $f(A) \subseteq B$, the restriction of $S_p(f)$ to simplices in $A$ yields a map $S_p(f)\colon S_p(A;R) \to S_p(B;R)$ so $S_p(f)(S_p(A;R)) \subseteq S_p(B;R)$, which implies that $\varphi$ vanishes on $S_p(A;R)$. Thus $S_p(A;R) \subseteq \mathrm{Ker}\,\varphi$, which means that there is a unique homomorphism

$$f_{\sharp,p}\colon S_p(X;R)/S_p(A;R) \to S_p(Y;R)/S_p(B;R)$$

making the following diagram commute:

$$S_p(X;R) \xrightarrow{\pi_{X,A}} S_p(X;R)/S_p(A;R)$$

$$\varphi \qquad\qquad \downarrow f_{\sharp,p}$$

$$S_p(Y;R)/S_p(B;R).$$

One will verify that the maps $f_{\sharp,p}\colon S_p(X;R)/S_p(A;R) \to S_p(Y;R)/S_p(B;R)$ define a chain map $f_\sharp$ from $S_*(X,A;R) = S_*(X;R)/S_*(A;R)$ to $S_*(Y,B;R) = S_*(Y;R)/S_*(B;R)$, and this chain map induces a homomorphism $H_p(f)\colon H_p(X,A;R) \to H_p(Y,B;R)$.          □

The homomorphism $H_p(f)\colon H_p(X,A;R) \to H_p(Y,B;R)$ is also denoted by $f_{*p}\colon H_p(X,A;R) \to H_p(Y,B;R)$.

Proposition 4.5 is generalized to maps between pairs as follows.

**Proposition 4.7.** *(Homotopy Axiom) Given any two continuous maps $f,g\colon (X,A) \to (Y,B)$ if $f$ and $g$ are homotopic and $H_p(f), H_p(g)\colon H_p(X,A;R) \to H_p(Y,B;R)$ are the induced homomorphisms, then $H_p(f) = H_p(g)$ for all $p \geq 0$. As a consequence, if $(X,A)$ and $(Y,B)$ are homotopy equivalent then the homology groups $H_p(X,A;R)$ and $H_p(Y,A;R)$ are isomorphic for all $p \geq 0$,*

Each pair $(X,A)$ yields a short exact sequence of complexes

$$0 \longrightarrow S_*(A;R) \xrightarrow{i} S_*(X;R) \xrightarrow{j} S_*(X;R)/S_*(A;R) \longrightarrow 0,$$

where the second map is the inclusion map and the third map is the quotient map. Therefore, we can apply the zig-zag lemma (Theorem 2.19) to this short exact sequence. If we go back to the proof of this theorem and consider only spaces of index $p \leq 0$, then by changing each negative index $p$ to $-p$ we obtain a diagram where the direction of the arrows is reversed and where each cohomology group $H^p$ correspond to the homology group $H_{-p}$ we obtain the "zig-zag lemma" for homology. Thus we obtain the following important result.

**Theorem 4.8.** *(Long Exact Sequence of Relative Homology) For every pair $(X, A)$ of spaces, we have the following long exact sequence of homology groups*

$$\cdots \longrightarrow H_{p+2}(X, A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p+2}}$$
$$\longrightarrow H_{p+1}(A; R) \xrightarrow{i_*} H_{p+1}(X; R) \xrightarrow{j_*} H_{p+1}(X, A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p+1}}$$
$$\longrightarrow H_p(A; R) \xrightarrow{i_*} H_p(X; R) \xrightarrow{j_*} H_p(X, A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p}}$$
$$\longrightarrow H_{p-1}(A; R) \longrightarrow \cdots$$

*ending in*

$$H_0(A; R) \longrightarrow H_0(X; R) \longrightarrow H_0(X, A; R) \longrightarrow 0.$$

It is actually possible to describe the boundary maps $\partial_{*p}$ explicitly: for every relative cycle $c$, we have

$$\partial_{*p}([c]) = [\partial_p(c)].$$

To define the *reduced singular relative homology groups* $\widetilde{H}_p(X, A; R)$ when $A \neq \emptyset$, we augment the singular chain complex

$$0 \xleftarrow{\partial_0} S_0(X, A; R) \xleftarrow{\partial_1} S_1(X, A; R) \longleftarrow \cdots \xleftarrow{\partial_{p-1}} S_{p-1}(X, A; R) \xleftarrow{\partial_p} S_p(X, A; R) \xleftarrow{\partial_{p+1}} \cdots$$

of the pair $(X, A)$ by adding one more $0$ to the sequence:

$$0 \longleftarrow 0 \xleftarrow{\epsilon} S_0(X, A; R) \xleftarrow{\partial_1} S_1(X, A; R) \longleftarrow \cdots \xleftarrow{\partial_p} S_p(X, A; R) \xleftarrow{\partial_{p+1}} \cdots$$

Consequently, if $A \neq \emptyset$, we have

$$\widetilde{H}_p(X, A; R) = H_p(X, A; R) \quad \text{for all } p \geq 0.$$

In addition to the short exact sequence

$$0 \longrightarrow S_p(A; R) \longrightarrow S_p(X; R) \longrightarrow S_p(X; R)/S_p(A; R) \longrightarrow 0$$

that holds for all $p \geq 0$, we add the following exact sequence

$$0 \longrightarrow R \xrightarrow{\text{id}} R \longrightarrow 0 \longrightarrow 0$$

in dimension $-1$, and then we obtain a version of Theorem 4.9 for reduced homology.

**Theorem 4.9.** *(Long Exact Sequence of Reduced Relative Homology) For every pair $(X, A)$ of spaces, we have the following long exact sequence of reduced homology groups*

$$\cdots \longrightarrow \widetilde{H}_{p+2}(X, A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p+2}} \widetilde{H}_{p+1}(A; R) \xrightarrow{i_*} \widetilde{H}_{p+1}(X; R) \xrightarrow{j_*} \widetilde{H}_{p+1}(X, A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p+1}} \widetilde{H}_p(A; R) \xrightarrow{i_*} \widetilde{H}_p(X; R) \xrightarrow{j_*} \widetilde{H}_p(X, A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p}} \widetilde{H}_{p-1}(A; R) \longrightarrow \cdots$$

*ending in*

$$\widetilde{H}_0(A; R) \longrightarrow \widetilde{H}_0(X; R) \longrightarrow \widetilde{H}_0(X, A; R) \longrightarrow 0.$$

If we apply Theorem 4.9 to the pair $(X, \{\text{pt}\})$ where $\text{pt} \in X$, since $\widetilde{H}_p(\{\text{pt}\}; R) = (0)$ for all $p \geq 0$, we obtain the following isomorphisms:

$$H_p(X, \{\text{pt}\}; R) \cong \widetilde{H}_p(X; R) \quad \text{for all } p \geq 0.$$

The following result is proved in Hatcher [25] (Proposition 2.22).

**Proposition 4.10.** *If $(X, A)$ is a good pair, which means that $A$ is a nonempty closed subspace that is a deformation retract of some neighborhood in $X$, then*

$$H_p(X, A; R) \cong H_p(X/A, \{\text{pt}\}; R) \cong \widetilde{H}_p(X/A; R).$$

Using Proposition 4.10 we obtain the following theorem which can be used to compute the homology of a quotient space $X/A$ from the homology of $X$ and the homology of its subspace $A$ (see Hatcher [25], Theorem 2.13) .

**Theorem 4.11.** *For every pair of spaces $(X, A)$, if $(X, A)$ is a good pair, then we have the following long exact sequence of reduced homology groups*

$$\cdots \longrightarrow \widetilde{H}_{p+2}(X/A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p+2}} \widetilde{H}_{p+1}(A; R) \xrightarrow{i_*} \widetilde{H}_{p+1}(X; R) \xrightarrow{j_*} \widetilde{H}_{p+1}(X/A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p+1}} \widetilde{H}_p(A; R) \xrightarrow{i_*} \widetilde{H}_p(X; R) \xrightarrow{j_*} \widetilde{H}_p(X/A; R) \longrightarrow$$
$$\xrightarrow{\partial_{*p}} \widetilde{H}_{p-1}(A; R) \longrightarrow \cdots$$

*ending in*

$$\widetilde{H}_0(A; R) \longrightarrow \widetilde{H}_0(X; R) \longrightarrow \widetilde{H}_0(X/A; R) \longrightarrow 0.$$

## 4.3   Excision and the Mayer–Vietoris Sequence

One of the main reasons why the relative homology groups are important is that they satisfy a property known as excision.

**Theorem 4.12.** *(Excision Axiom) Given subspaces $Z \subseteq A \subseteq X$ such that the closure of $Z$ is contained in the interior of $A$, then the inclusion $(X - Z, A - Z) \longrightarrow (X, A)$ induces isomorphisms of singular homology*

$$H_p(X - Z, A - Z; R) \cong H_p(X, A; R), \quad \text{for all } p \geq 0.$$

*See Figure 4.10. Equivalently, for any subspaces $A, B \subseteq X$ whose interiors cover $X$, the inclusion map $(B, A \cap B) \longrightarrow (X, A)$ induces isomorphisms*

$$H_p(B, A \cap B; R) \cong H_p(X, A; R), \quad \text{for all } p \geq 0.$$

*See Figure 4.11.*
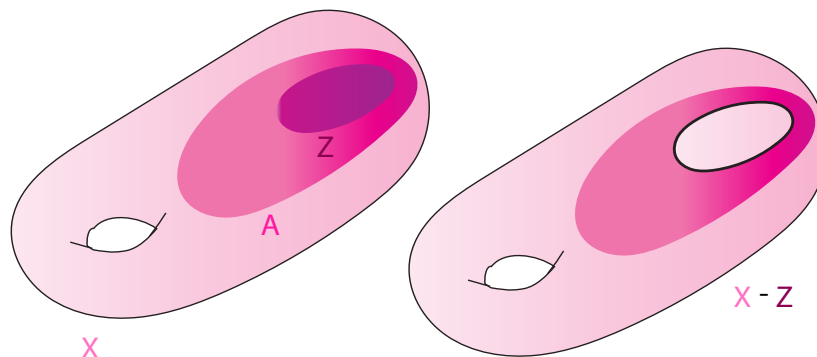


Figure 4.10: Let $X$ be the torus. This figure demonstrates the excision of the plum disk $Z$ from $X$.

The translation between the two versions is obtained by setting $B = X - Z$ and $Z = X - B$, in which case $A \cap B = A - Z$. The proof of Theorem 4.12 is rather technical and uses a technique known as barycentric subdivision. The reader is referred to Hatcher [25] (Chapter 2, Section 2.1) and Munkres [38] (Chapter 4, Section 31).

The proof of Theorem 4.12 relies on a technical lemma about the relationship between the chain complex $S_*(X; R)$ and the chain complex $S_*^{\mathcal{U}}(X; R)$ induced by a family $\mathcal{U} = (U_i)_{i \in I}$ of subsets of $X$ whose interiors form an open cover of $X$.
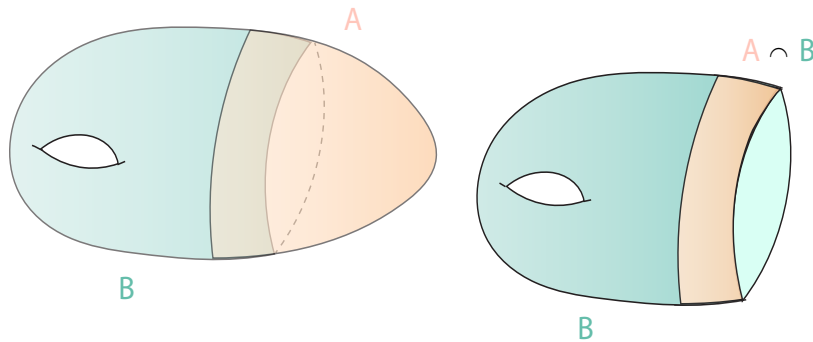
Figure 4.11: Let $X$ be the torus. This figure demonstrates the relationships between $A$, $B$ and $A \cap B$.

**Definition 4.10.** Given a topological space $X$, for any family $\mathcal{U} = (U_i)_{i \in I}$ of subsets of $X$ whose interiors form an open cover of $X$, we say that a singular $p$-simplex $\sigma \colon \Delta^p \to X$ is $\mathcal{U}$-*small* if its image is contained in one of the $U_i$. The submodule $S_p^{\mathcal{U}}(X; R)$ of $S_p(X; R)$ consists of all singular $p$-chains $\sum \lambda_k \sigma_k$ such that each $p$-simplex $\sigma_k$ is $\mathcal{U}$-small.

It is immediate that the boundary map $\partial_p \colon S_p(X; R) \to S_{p-1}(X; R)$ takes $S_p^{\mathcal{U}}(X; R)$ into $S_{p-1}^{\mathcal{U}}(X; R)$, so $S_*^{\mathcal{U}}(X; R)$ is a chain complex. The homology modules of the complex $S_*^{\mathcal{U}}(X; R)$ are denoted by $H_p^{\mathcal{U}}(X; R)$.

**Proposition 4.13.** *Given a topological space $X$, for any family $\mathcal{U} = (U_i)_{i \in I}$ of subsets of $X$ whose interiors form an open cover of $X$, the inclusions $\iota_p \colon S_p^{\mathcal{U}}(X; R) \to S_p(X; R)$ induce a chain homotopy equivalence; that is, there is a family of chain maps $\rho_p \colon S_p(X; R) \to S_*^{\mathcal{U}}(X; R)$ such that $\rho \circ \iota$ is chain homotopic to the identity map of $S_*^{\mathcal{U}}(X; R)$ and $\iota \circ \rho$ is chain homotopic to the identity map of $S_*(X; R)$. As a consequence, we have isomorphisms $H_p^{\mathcal{U}}(X; R) \cong H_p(X; R)$ for all $p \geq 0$.*

The proof of Proposition 4.13 is quite involved. It uses barycentric subdivision; see Hatcher [25] (Chapter 2, Proposition 2.21) and Munkres [38] (Chapter 4, Section 31, Theorem 31.5).

Besides playing a crucial role in proving the excision axiom, Proposition 4.13 yields simple proof of the Mayer–Vietoris sequence in singular homology. For arbitrary topological spaces, partitions of unity are not available but the set-up of Proposition 4.13 yields an alternative method of proof.

**Theorem 4.14.** *(Mayer–Vietoris in singular homology) Given any topological space $X$, for any two subsets $A, B$ of $X$ such that $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, there is a long exact sequence of*

*homology*

$$\longrightarrow H_p(A \cap B; R) \xrightarrow{\varphi_*} H_p(A; R) \oplus H_p(B; R) \xrightarrow{\psi_*} H_p(X; R) \xrightarrow{\partial_*} H_{p-1}(A \cap B; R) \longrightarrow$$

*where the maps $\varphi$ and $\psi$ are defined by*

$$\varphi_*(c) = (i_*(c), -j_*(c))$$
$$\psi_*(a, b) = k_*(a) + l_*(b),$$

*and where $i, j, k, l$ are the inclusion maps shown in the diagram below:*

$$
\begin{array}{ccc}
A \cap B & \xrightarrow{\ i\ } & A \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\
B & \xrightarrow{\ l\ } & X.
\end{array}
$$

*If $A \cap B \neq \emptyset$, a similar sequence exists in reduced homology.*

*Proof.* For simplicity of notation we suppress the ring $R$ in writing $S_p(-, R)$ or $H_p(-, R)$. We define a sequence

$$0 \longrightarrow S_p(A \cap B) \xrightarrow{\varphi} S_p(A) \oplus S_p(B) \xrightarrow{\psi} S_p(A) + S_p(B) \longrightarrow 0$$

for every $p \geq 0$, where $\varphi$ and $\psi$ are given by

$$\varphi(c) = (i_\sharp(c), -j_\sharp(c))$$
$$\psi(a, b) = k_\sharp(a) + l_\sharp(b).$$

Observe that $\psi \circ \varphi = 0$. The map $\varphi$ is injective, while $\psi$ is surjective. We have $\operatorname{Im} \varphi \subseteq \operatorname{Ker} \psi$ since $\psi \circ \varphi = 0$. The kernel of $\psi$ consists of all chains of the form $(c, -c)$ where $c \in S_p(A)$ and $-c \in S_p(B)$ so $c \in S_p(A \cap B)$ and $\varphi(c) = (c, -c)$, which shows that $\operatorname{Ker} \psi \subseteq \operatorname{Im} \varphi$. Therefore the sequence is exact, and we have a short exact sequence of chain complexes

$$0 \longrightarrow S_*(A \cap B) \xrightarrow{\varphi} S_*(A) \oplus S_*(B) \xrightarrow{\psi} S_*(A) + S_*(B) \longrightarrow 0.$$

By the long exact sequence of homology we have the long exact sequence

$$\longrightarrow H_p(A \cap B) \xrightarrow{\varphi_*} H_p(A) \oplus H_p(B) \xrightarrow{\psi_*} H_p(S_*(A) + S_*(B)) \xrightarrow{\partial_*} H_{p-1}(A \cap B) \longrightarrow .$$

However, since $X = \operatorname{Int}(A) \cup \operatorname{Int}(B)$, Proposition 4.13 implies that

$$H_p(S_*(A) + S_*(B)) \cong H_p(X),$$

and we obtain the long exact sequence

$$\cdots \longrightarrow H_p(A \cap B) \xrightarrow{\varphi_*} H_p(A) \oplus H_p(B) \xrightarrow{\psi_*} H_p(X) \xrightarrow{\partial_*} H_{p-1}(A \cap B) \longrightarrow \cdots ,$$

as desired. A similar argument applies to reduced homology by augmenting the complexes $S_*(A \cap B)$, $S_*(A) \oplus S_*(B)$, and $S_*(A) + S_*(B)$ using the maps $\epsilon \colon S_0(A \cap B) \to R$, $\epsilon \oplus \epsilon \colon S_0(A) \oplus S_0(B) \to R \oplus R$, and $\epsilon \colon S_0(A) + S_0(B) \to R$.   $\square$

The Mayer–Vietoris sequence can be used to compute the homology of spaces in terms of some of their pieces. For example, this is a way to compute the homology of the $n$-torus.

There are two more important properties of singular homology that should be mentioned:

(1) The axiom of compact support.

(2) The additivity axiom.

The additivity axiom implies that that the homology groups $H_p(X, A; R)$ are determined by the groups $H_p(C, D; R)$ where $(C, D)$ is a compact pair in $(X, A)$, which means that $D \subseteq C \subseteq X$, $D \subseteq A \subseteq X$, $C$ is compact, and $D$ is compact in $C$.

Let $\mathcal{K}(X, A)$ be the sets of all compact pairs of $(X, A)$ ordered by inclusion. It is a directed preorder.

**Proposition 4.15.** *For any pair $(X, A)$ of topological spaces with $A \subseteq X$, the following properties hold:*

(1) *Given any homology class $\alpha \in H_p(X, A)$, there is a compact pair $(C, D)$ in $(X, A)$ and a homology class $\beta \in H_p(C, D; R)$ such that $i_*(\beta) = \alpha$, where $i \colon (C, D) \to (X, A)$ is the inclusion map.*

(2) *Let $(C, D)$ be any compact pair in $(X, A)$, and let $\beta \in H_p(C, D; R)$ be any homology class such that $i_*(\beta) = 0$. Then there exists a compact pair $(C', D')$ such that $(C, D) \subseteq (C', D') \subseteq (X, A)$ and $j_*(\beta) = 0$, where $j \colon (C, D) \to (C', D')$ is the inclusion map.*

*In short, $H_p(X, A; R)$ is the direct limit*

$$H_p(X, A; R) = \varinjlim_{(C,D) \in \mathcal{K}(X,A)} H_p(C, D; R).$$

Proposition 4.15 is proved in Massey [32] (Chapter VIII, Section 6, Proposition 6.1) and Rotman [41] (Chapter 4, Theorem 4.16).

*Sketch of proof.* The proof of (1) is not difficult and relies on the fact that for any singular $p$-chain $a \in S_p(X; R)$ there is a compact subset $C$ of $X$ such that $a \in S_p(C; R)$. For simplicity of exposition assume that $A = \emptyset$. If $a = \sum_{i=1}^{k} \lambda_i \sigma_i \in S_p(X, R)$ is a cycle representing the homology class $\alpha$, with $\lambda_i \in R$ and each $\sigma_i$ a $p$-simplex $\sigma_i \colon \Delta^p \to X$, since $\Delta^p$ is compact and each $\sigma_i$ is continuous, $C = \sigma_1(\Delta^p) \cup \cdots \cup \sigma_k(\Delta^p)$ is a compact subset of $X$ and $a \in S_p(C; R)$. Let $b = \sum_{i=1}^{k} \lambda_i \sigma_i' \in S_p(C, R)$ be the $p$-chain in which $\sigma_i' \colon \Delta^p \to C$ is the corestriction of $\sigma_i$ to $C$. We need to check that $b$ is a $p$-cycle. By definition of the inclusion $i$ we have $a = i_\sharp(b)$, and since $a$ is a $p$-cycle we have

$$i_\sharp \circ \partial(b) = \partial \circ i_\sharp(b) = \partial a = 0.$$

Since $i$ is an injection, $i_\sharp$ is also an injection, thus $\partial b = 0$, which means that $b \in S_p(C; R)$ is indeed a $p$-cycle, and if $\beta$ denotes the homology class of $b$, we have $i_*(\beta) = \alpha$. The above argumemt is easily adapted to the case where $A \neq \emptyset$. The proof of (2) is similar an left as an exercise.                                                                                      □

The above fact suggests the following axiom of homology called the *Axiom of compact support*:

Given any pair $(X, A)$ with $A \subseteq X$ and given any homology class $\alpha \in H_p(X, A)$, there is a compact pair $(C, D)$ in $(X, A)$ and a homology class $\beta \in H_p(C, D; R)$ such that $i_*(\beta) = \alpha$, where $i \colon (C, D) \to (X, A)$ is the inclusion map.

This axiom is another of the axioms of a homology theory; see Munkres [38] (Chapter 3, Section 26, Axiom 8), or Spanier [47] (Chapter 4, Section 8, No. 11).

To state the additivity axiom we need to define the topological sum of a family of spaces.

**Definition 4.11.** If $(X_i)_{i \in I}$ is a family of topological spaces we define the *topological sum* $\bigsqcup_{i \in I} X_i$ of the family $(X_i)_{i \in I}$ as the disjoint union of the spaces $X_i$, and we give it the topology for which a subset $Z \subseteq \bigsqcup_{i \in I} X_i$ is open iff $Z \cap X_i$ is open for all $i \in I$.

Then the *Additivity axiom* states that for any family of topological spaces $(X_i)_{i \in I}$ there is an isomorphism

$$H_p\Big(\bigsqcup_{i \in I} X_i; R\Big) \cong \bigoplus_{i \in I} H_p(X_i; R) \quad \text{for all } p \geq 0.$$

The above axiom introduced by Milnor is stated in Bredon [4] (Chapter IV, Section 6), May [34] (Chapter 13, Section 1), and Hatcher [25] (Chapter 2, Section 2.3), where it is stated for relative homology and for a wedge sum of spaces.

The additivity axiom is a general property of chain complexes. Indeed, homology commutes with sums, products, and direct limits; see Spanier [47] (Chapter 4, Section 1, Theorem 6 and Theorem 7). This axiom is only needed for infinite sums.

## 4.4 Some Applications of Singular Homology

It is remarkable that Proposition 4.7, Theorem 4.9 or its version Theorem 4.11, and Theorem 4.12, can be used to compute the singular homology of some of the familiar simple spaces. We show below how to compute the homology groups of the spheres.

Recall that the $n$-dimensional ball $D^n$ and the the $n$-dimensional sphere $S^n$ are defined respectively as the subspaces of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ given by

$$D^n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$$
$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}.$$

Furthermore, $S^n = \partial D^{n+1}$, the boundary of $D^{n+1}$, and $D^n/\partial D^n$ is homeomorphic to $S^n$ $(n \geq 1)$. We also know that $D^n$ is contractible for all $n \geq 1$, so its homology groups are given by

$$H_0(D^n; R) = R$$
$$H_p(D^n; R) = (0), \quad p > 0,$$

or equivalently

$$\widetilde{H}_p(D^n; R) = (0), \quad p \geq 0.$$

**Proposition 4.16.** *The reduced homology of $S^n$ is given by*

$$\widetilde{H}_p(S^n; R) = \begin{cases} R & \text{if } p = n \\ (0) & \text{if } p \neq n, \end{cases}$$

*or equivalently the homology of $S^n$ is given by*

$$H_0(S^0; R) = R \oplus R$$
$$H_p(S^0; R) = (0), \quad p > 0,$$

*and for $n \geq 1$,*

$$H_p(S^n; R) = \begin{cases} R & \text{if } p = 0, n \\ (0) & \text{if } p \neq 0, n. \end{cases}$$

*Proof.* For simplicity of notation, we drop the ring $R$ in writing homology groups. Since $S^0 = \{-1, 1\}$, by the excision axiom (Theorem 4.7) we get

$$H_p(S^0, \{-1\}) \cong H_p(\{1\}, \emptyset) = H_p(\{1\})$$

for all $p \geq 0$. The long exact sequence of Theorem 4.9 for the pair $(S^0, \{-1\})$ gives the exact sequence

$$\longrightarrow H_p(\{-1\}) \longrightarrow H_p(S^0) \longrightarrow H_p(S^0, \{-1\}) \longrightarrow H_{p-1}(\{-1\}) \longrightarrow$$

If $p \geq 1$, since $H_p(\{-1\}) = H_p(\{1\}) = (0)$ and $H_p(S^0, \{-1\}) \cong H_p(\{1\})$, we get $H_p(S^0) = (0)$. If $p = 0$, since $H_0(\{-1\}) = H_0(\{1\}) = R$, we get $H_0(S^0) = R \oplus R$.

If $n \geq 1$, then since $D^n/\partial D^n$ is homeomorphic to $S^n$ and $\partial D^n = S^{n-1}$ is a deformation restract of $D^n$, the long exact sequence of Theorem 4.11 for the pair $(D^n, \partial D^n) = (D^n, S^{n-1})$ yields the exact sequence

$$\longrightarrow \widetilde{H}_p(D^n) \longrightarrow \widetilde{H}_p(D^n/S^{n-1}) = \widetilde{H}_p(S^n) \longrightarrow \widetilde{H}_{p-1}(S^{n-1}) \longrightarrow \widetilde{H}_{p-1}(D^{n-1}) \longrightarrow$$

and if $p \geq 1$, since $\widetilde{H}_p(D^n) = \widetilde{H}_{p-1}(D^{n-1}) = (0)$, we get

$$\widetilde{H}_p(S^n) \cong \widetilde{H}_{p-1}(S^{n-1}) \quad p \geq 1.$$

We conclude by induction on $n \geq 1$.  $\square$

The most convenient setting to compute homology groups is the homology of cell complexes or simplicial homology; see Chapter 6. For example, cellular homology can used to compute the homology of the real and complex projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$; see Section 6.2, and also Hatcher [25], Munkres [38], and Bredon [4].

**Example 4.1.** The real projective space $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by the equivalence relation $\sim$ defined such that for all $(u_1, \ldots, u_{n+1}) \in \mathbb{R}^{n+1} - \{0\}$ and all $(v_1, \ldots, v_{n+1}) \in \mathbb{R}^{n+1} - \{0\}$,

$$(u_1, \ldots, u_{n+1}) \sim (v_1, \ldots, v_{n+1}) \quad \text{iff} \quad (\exists \alpha \in \mathbb{R} - \{0\}) \quad (v_1, \ldots, v_{n+1}) = \alpha(u_1, \ldots, u_{n+1}).$$

Equivalently, $\mathbb{RP}^n$ is the quotient of the subset $S^n$ of $\mathbb{R}^{n+1}$ defined by

$$S^n = \{(u_1, \ldots, u_{n+1}) \in \mathbb{R}^{n+1} \mid u_1^2 + \cdots + u_{n+1}^2 = 1\},$$

in other words, the $n$-sphere, by the equivalence relation $\sim$ on $S^n$ defined so that for all $(u_1, \ldots, u_{n+1}) \in S^n$ and all $(v_1, \ldots, v_{n+1}) \in S^n$,

$$(u_1, \ldots, u_{n+1}) \sim (v_1, \ldots, v_{n+1}) \quad \text{iff} \quad (v_1, \ldots, v_{n+1}) = \pm(u_1, \ldots, u_{n+1}).$$

This says that two points on the sphere $S^n$ are equivalent iff they are antipodal. We have a quotient map $\pi \colon S^n \to \mathbb{RP}^n$.

The complex projective space $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation $\sim$ defined such that for all $(u_1, \ldots, u_{n+1}) \in \mathbb{C}^{n+1} - \{0\}$ and all $(v_1, \ldots, v_{n+1}) \in \mathbb{C}^{n+1} - \{0\}$,

$$(u_1, \ldots, u_{n+1}) \sim (v_1, \ldots, v_{n+1}) \quad \text{iff} \quad (\exists \alpha \in \mathbb{C} - \{0\}) \quad (v_1, \ldots, v_{n+1}) = \alpha(u_1, \ldots, u_{n+1}).$$

Equivalently, $\mathbb{CP}^n$ is the quotient of the subset $\Sigma^n$ of $\mathbb{C}^{n+1}$ defined by

$$\Sigma^n = \{(u_1, \ldots, u_{n+1}) \in \mathbb{C}^{n+1} \mid |u_1|^2 + \cdots + |u_{n+1}|^2 = 1\},$$

by the equivalence relation $\sim$ on $\Sigma^n$ defined so that for all $(u_1, \ldots, u_{n+1}) \in \Sigma^n$ and all $(v_1, \ldots, v_{n+1}) \in \Sigma^n$,

$$(u_1, \ldots, u_{n+1}) \sim (v_1, \ldots, v_{n+1}) \quad \text{iff} \quad (\exists \alpha \in \mathbb{C},\, |\alpha| = 1) \quad (v_1, \ldots, v_{n+1}) = \alpha(u_1, \ldots, u_{n+1}).$$

If we write $u_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$, we have $(u_1, \ldots, u_{n+1}) \in \Sigma^n$ iff

$$x_1^2 + y_1^2 + \cdots + x_{n+1}^2 + y_{n+1}^2 = 1,$$

iff $(x_1, y_1, \ldots, x_{n+1}, y_{n+1}) \in S^{2n+1}$. Therefore we can identify $\Sigma^n$ with $S^{2n+1}$, and we can view $\mathbb{CP}^n$ as the quotient of $S^{2n+1}$ by the above equivalence relation. We have a quotient map $\pi \colon S^{2n+1} \to \mathbb{CP}^n$.

For $R = \mathbb{Z}$, we have

$$H_p(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0, 2, 4, \ldots, 2n \\ (0) & \text{otherwise,} \end{cases}$$

and

$$H_p(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0 \text{ and for } p = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } p \text{ odd}, 0 < p < n \\ (0) & \text{otherwise.} \end{cases}$$

The homology of the $n$-torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$ exhibits a remarkable symmetry:

$$H_p(T^n; R) = \underbrace{R \oplus \cdots \oplus R}_{\binom{n}{p}}.$$

The homology of the $n$-torus $T^n$ can be computed by induction using the Mayer–Vietoris sequence (Theorem 4.14).

Surprisingly, computing the homology groups $H_p(\mathbf{SO}(n); \mathbb{Z})$ of the rotation group $\mathbf{SO}(n)$ is more difficult. It can be shown that the groups $H_p(\mathbf{SO}(n); \mathbb{Z})$ are directs sums of copies of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, but their exact structure is harder to obtain. For more on this topic, we refer the reader to Hatcher [25] (Chapter 3, Sections 3.D and 3.E).

Proposition 4.7, Theorem 4.8, and Theorem 4.12, state three of the properties that were singled out as characterizing homology theories by Eilenberg and Steenrod [12]. These properties hold for most of the known homology theories, and thus can be taken as axioms for homology theory; see Sato [43], MacLane [29], Munkres [38], or Hatcher [25].

One of the most spectacular applications of homology is a proof of a generalized version of the Jordan curve theorem. First, we need a bit of terminology.

Given two topological spaces $X$ and $Y$, an *embedding* is a homeomorphism $f \colon X \to Y$ of $X$ onto its image $f(X)$. A *$m$-cell* or *cell of dimension $m$* is any space $B$ homeomorphic to the closed ball $D^m$. A subspace $A$ of a space $X$ *separates* $X$ if $X - A$ is not connected.

**Proposition 4.17.** *Let $B$ be a $k$-cell in $S^n$. Then $S^n - B$ is acyclic, which means that $H_p(S^n - B) = (0)$ for all $p \neq 0$. In particular $B$ does not separate $S^n$.*

Proposition 4.17 is proved in Munkres [38] (Chapter 4, Section 36, Theorem 36.1). See also Bredon [4] (Chapter IV, Corollary 19.3).

**Proposition 4.18.** *Let $n > k \geq 0$. For any embedding $h \colon S^k \to S^n$ we have*

$$\widetilde{H}_p(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & \text{if } p = n - k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

*This implies that $\widetilde{H}_p(S^n - h(S^k)) \cong \widetilde{H}_p(S^{n-k-1})$.*

Proposition 4.18 is proved in Munkres [38] (Chapter 4, Section 36, Theorem 36.2) and Bredon [4] (Chapter IV, Theorem 19.4). The proof uses an induction on $k$ and a Mayer-Vietoris sequence. Proposition 4.18 implies the following generalization of the Jordan curve theorem for $n \geq 1$.

**Theorem 4.19.** *(Generalized Jordan curve theorem in $S^n$) Let $n > 0$ and let $C$ be any subset of $S^n$ homeomorphic to $S^{n-1}$. Then $S^n - C$ has precisely two components, both acyclic, and $C$ is their common topological boundary.*

Theorem 4.19 is proved in Munkres [38] (Chapter 4, Section 36, Theorem 36.3) and Bredon [4] (Chapter IV, Theorem 19.5), in which it is called the *Jordan–Brouwer separation theorem*.

The first part of the theorem is obtained by applying Proposition 4.18 in the case where $k = n - 1$. In this case we see that $\widetilde{H}_0(S^n - C) = \mathbb{Z}$, so $H_0(S^n - C) = \mathbb{Z} \oplus \mathbb{Z}$ and this implies that $S^n - C$ has precisely two path components. The proof of the second part uses Proposition 4.17.

One might think that because $C$ is homeomorphic to $S^{n-1}$ the two components $W_1$ and $W_2$ of $S^n - C$ should be $n$-cells, but this is false in general. The problem is that an embedding of $S^{n-1}$ into $S^n$ can be very complicated. There is a famous embedding of $S^2$ into $S^3$ called the *Alexander horned sphere* for which the sets $W_1$ and $W_2$ are not even simply connected; see Bredon [4] (Chapter IV, page 232) and Hatcher [25] (Chapter 2, Example 2B.2). In the case $n = 2$, things are simpler; see Hatcher [25] (Chapter 2, Section 2.B) and Bredon [4] (Chapter IV, pages 235-236).

The classical version of the Jordan curve theorem is stated for embeddings of $S^{n-1}$ into $\mathbb{R}^n$.

**Theorem 4.20.** *(Generalized Jordan curve theorem in $\mathbb{R}^n$) Let $n > 1$ and let $C$ be any subset of $\mathbb{R}^n$ homeomorphic to $S^{n-1}$. Then $\mathbb{R}^n - C$ has precisely two components, one of which is bounded and the other one is not. The bounded component is acyclic and the other has the homology of $S^{n-1}$.*

*Proof.* Using the inverse stereographic projection from the north pole $N$ we can embed $C$ into $S^n$. By Theorem 4.19 $S^n - C$ has two acyclic components. Let $V$ be the component containing $N$. Obviously the other component $U$ is bounded and acyclic. It follows that $S^n - U$ is homeomorphic to $D^n$ so we can view $V$ as being a subset of $D^n$. Next we follow Bredon [4] (Chapter IV, Corollary 19.6). Consider the piece of the long exact sequence of the pair $(V, V - \{N\})$:

$$\widetilde{H}_{p+1}(V) \longrightarrow H_{p+1}(V, V - \{N\}) \longrightarrow \widetilde{H}_p(V - \{N\}) \longrightarrow \widetilde{H}_p(V)$$

By Theorem 4.19 the homology of $V$ is acyclic, so we have the following isomorphisms

$$\begin{aligned}
\widetilde{H}_p(V - \{N\}) &\cong H_{p+1}(V, V - \{N\}) \\
&\cong H_{p+1}(D^n, D^n - \{0\}) \\
&\cong \widetilde{H}_p(D^n - \{0\}) \\
&\cong \widetilde{H}_p(S^{n-1}),
\end{aligned}$$

where the second isomorphism holds by excision since $V \subseteq D^n$, the third holds from the long exact sequence of $(D^n, D^n - \{0\})$, and the fourth by homotopy. $\square$

Later on, to define orientable manifolds we will need to compute the groups $H_p(M, M - \{x\}; R)$ where $M$ is a topological manifold and $x$ is any point in $M$.

Recall that a *topological manifold $M$ of dimension $n$*, for short an *$n$-manifold*, is a topological space such that for every $x \in M$, there is some open subset $U$ of $M$ containing $x$ and some homeomorphism $\varphi_U \colon U \to \Omega$ (called a *chart at $x$*) onto some open subset $\Omega \subseteq \mathbb{R}^n$. We have the following result.

**Proposition 4.21.** *If $M$ is a topological manifold of dimension $n$ and if $R$ is any commutative ring with a multiplicative identity element, then*

$$H_p(M, M - \{x\}; R) \cong H_p(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \cong \widetilde{H}_{p-1}(\mathbb{R}^n - \{x\}; R) \cong \widetilde{H}_{p-1}(S^{n-1})$$

*for all $p \geq 0$. Consequently*

$$H_p(M, M - \{x\}; R) \cong \begin{cases} R & \text{if } p = n \\ (0) & \text{if } p \neq n. \end{cases}$$

*Proof.* By shrinking $U$ is necessary we may assume that $U$ is homeomorphic to $\mathbb{R}^n$, so by excision with $X = M, A = U$, and $Z = M - U$ (see Theorem 4.12), we obtain

$$H_p(M, M - \{x\}; R) \cong H_p(U, U - \{x\}; R) \cong H_p(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R).$$

By Theorem 4.9 the long exact sequence of homology yields an exact sequence

$$\widetilde{H}_{p+1}(\mathbb{R}^n; R) \longrightarrow \widetilde{H}_{p+1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \longrightarrow \widetilde{H}_p(\mathbb{R}^n - \{x\}; R) \longrightarrow \widetilde{H}_p(\mathbb{R}^n; R).$$

Since $\mathbb{R}^n$ is contractible, $\widetilde{H}_{p+1}(\mathbb{R}^n; R) = (0)$ and $\widetilde{H}_p(\mathbb{R}^n; R) = (0)$ so we have isomorphisms

$$\widetilde{H}_{p+1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \cong \widetilde{H}_p(\mathbb{R}^n - \{x\}; R)$$

for all $p \geq 0$. Since $\widetilde{H}_{p+1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) = H_{p+1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R)$ for $p \geq 1$, we get

$$H_p(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \cong \widetilde{H}_{p-1}(\mathbb{R}^n - \{x\}; R)$$

for all $p \geq 1$. For $p = 0$, the end of the long exact sequence given by Theorem 4.8 yields

$$H_0(\mathbb{R}^n - \{x\}; R) \longrightarrow H_0(\mathbb{R}^n; R) \longrightarrow H_0(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \longrightarrow 0,$$

and since $H_0(\mathbb{R}^n; R) = R$ and $H_0(\mathbb{R}^n - \{x\}; R) = R$ or $R \oplus R$ when $n = 1$, we obtain $H_0(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) = (0)$. Since homology (and reduced homology) of negative index are $(0)$, we obtain the isomorphisms

$$H_p(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \cong \widetilde{H}_{p-1}(\mathbb{R}^n - \{x\}; R)$$

for all $p \geq 0$. To finish the proof, observe that $S^{n-1}$ is a deformation retract fo $\mathbb{R}^n - \{x\}$, so by the homotopy axiom (Proposition 4.7) we get

$$H_p(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \cong \widetilde{H}_{p-1}(S^{n-1}; R)$$

for all $p \geq 0$. We conclude by using Proposition 4.16.                                        $\square$

If $M$ is an $n$-manifold, since the groups $H_n(M, M - \{x\}; R)$ are all isomorphic to $R$, a way to define a notion of orientation is to pick some generator $\mu_x$ from $H_n(M, M - \{x\}; R)$, for every $x \in M$. Since $H_n(M, M - \{x\}; R)$ is a ring with a unit, generators are just invertible elements. To say that $M$ is orientable means that we can pick these $\mu_x \in H_n(M, M - \{x\}; R)$ in such a way that they "vary continuously" with $x$. We will how to do this in Section 7.1.

In the next section, we show how singular homology can be generalized to deal with more general coefficients.

## 4.5 Singular Homology with $G$-Coefficients

In the previous sections, given a commutative ring $R$ with an identity element, we defined the singular chain group $S_p(X; R)$ as the free $R$-module generated by the set $S_{\Delta^p}(X)$ of singular $p$-simplices $\sigma \colon \Delta^p \to X$. Thus, a singular chain $c$ can be expressed as a formal linear combination

$$c = \sum_{k=1}^{m} \lambda_i \sigma_i,$$

for some $\lambda_i \in R$ and some $\sigma_i \in S_{\Delta^p}(X)$.

If $A$ is a subset of $X$, we defined the relative chain group $S_p(X, A; R)$ as the quotient $S_p(X; R)/S_p(A; R)$. We observed that $S_p(X, A; R)$ is also a free $R$-module, and a basis of $S_p(X, A; R)$ consists of the cosets $\sigma + S_p(A; R)$ where the image of the singular simplex $\sigma \colon \Delta^p \to X$ does not lie in $A$.

Experience shows that it is fruitful to generalize homology to allow coefficients in any $R$-module $G$. Intuitively, a chain with coefficients in $G$ is a formal linear combination

$$c = \sum_{k=1}^{m} g_i \sigma_i,$$

where the $g_i$ are elements of the module $G$. We may think of such chains as "vector-valued" as opposed to the orginal chains which are "scalar-valued." As we will see shortly, the usual convention is to swap $g_i$ and $\sigma_i$ so that these formal sums are of the form $\sum \sigma_i g_i$.

A rigorous way to proceed is to define the module $S_p(X; G)$ of singular $p$-chains with coefficients in $G$ as the tensor product

$$S_p(X; G) = S_p(X; R) \otimes_R G.$$

It is a $R$-module.

Since the $R$-module $S_p(X; R)$ is freely generated by $S_{\Delta^p}(X)$, it is a standard result of linear algebra that we have an isomorphism

$$S_p(X; R) \otimes_R G \cong \bigoplus_{\sigma \in S_{\Delta^p}(X)} G,$$

the direct sum of copies of $G$, one for each $\sigma \in S_{\Delta^p}(X)$.

Recall that this direct sum is the $R$-module of all functions $c\colon S_{\Delta^p}(X) \to G$ that are zero except for finitely many $\sigma$. For any $g \neq 0$ and any $\sigma \in S_{\Delta^p}(X)$, if we denote by $\sigma g$ the function from $S_{\Delta^p}(X)$ to $G$ which has the value 0 for all arguments except $\sigma$ where its value is $g$, then every $c \in S_p(X; R) \otimes_R G = S_p(X; G)$ which is not identically 0 can be written in a unique way as a finite sum

$$c = \sum_{k=1}^{m} \sigma_i g_i$$

for some $\sigma_i \in S_{\Delta^p}(X)$ and some nonzero $g_i \in G$. Observe that in the above expression the "vector coefficient" $g_i$ comes after $\sigma_i$, to conform with the fact that we tensor with $G$ on the right.

Since we will always tensor over the ring $R$, for simplicity of notation we will drop the subscript $R$ in $\otimes_R$. Now given the singular chain complex $(S_*(X; R), \partial_*)$ displayed below

$$0 \xleftarrow{\partial_0} S_0(X; R) \xleftarrow{\partial_1} S_1(X; R) \longleftarrow \cdots \xleftarrow{\partial_{p-1}} S_{p-1}(X; R) \xleftarrow{\partial_p} S_p(X; R) \longleftarrow \cdots ,$$

(recall that $\partial_i \circ \partial_{i+1} = 0$ for all $i \geq 0$) we can form the homology complex

$$0 \xleftarrow{\partial_0 \otimes \mathrm{id}} S_0(X; R) \otimes G \xleftarrow{\partial_1 \otimes \mathrm{id}} S_1(X; R) \otimes G \longleftarrow \cdots \xleftarrow{\partial_p \otimes \mathrm{id}} S_p(X; R) \otimes G \cdots$$

denoted $(S_*(X; R) \otimes G, \partial_* \otimes \mathrm{id})$ obtained by tensoring with $G$, and since by definition $S_p(X; G) = S_p(X; R) \otimes G$, we have the homology complex

$$0 \xleftarrow{\partial_0 \otimes \mathrm{id}} S_0(X; G) \xleftarrow{\partial_1 \otimes \mathrm{id}} S_1(X; G) \longleftarrow \cdots \xleftarrow{\partial_p \otimes \mathrm{id}} S_p(X; G) \longleftarrow \cdots$$

denoted $(S_*(X; G), \partial_* \otimes \mathrm{id})$ (of course, $G_*(X; G) = S_*(X; R) \otimes G$).

**Definition 4.12.** Let $R$ be a commutative ring with identity and let $G$ be a $R$-module. The *singular homology modules $H_p(X; G)$ with coefficients in $G$* are the homology groups of the above complex; that is,

$$H_p(X; G) = H_p(S_*(X; G)) \quad p \geq 0.$$

If $\epsilon\colon S_0(X; R) \to R$ is the map of Definition 4.7, then we obtain an augmentation map $\epsilon \otimes \mathrm{id}\colon S_0(X; R) \otimes G \to R \otimes G \cong G$, that is, a map $\epsilon \otimes \mathrm{id}\colon S_0(X; G) \to G$, and we obtain an augmented complex with $G$ is dimension $-1$.

The corresponding homology groups are denoted $\widetilde{H}_p(X; G)$ and are called the *reduced singular homology groups with coefficients in $G$*. As in Section 4.2 we can pick an injective map $s\colon R \to S_0(X; R)$ such that $\epsilon \circ s = \mathrm{id}$, and since $R \otimes G \cong G$ and the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\epsilon \longrightarrow S_0(X; R) \underset{s}{\overset{\epsilon}{\rightleftarrows}} R \longrightarrow 0$$

splits, by tensoring with $G$ we get the short split exact sequence

$$0 \longrightarrow (\operatorname{Ker} \epsilon) \otimes G \longrightarrow S_0(X;R) \otimes G \underset{s \otimes \mathrm{id}}{\overset{\epsilon \otimes \mathrm{id}}{\rightleftarrows}} R \otimes G \cong G \longrightarrow 0;$$

see Munkres [38] (Chapter 6, Section 51, Exercise 1). Thus

$$S_0(X;G) = S_0(X;R) \otimes G \cong ((\operatorname{Ker} \epsilon) \otimes G) \oplus G,$$

and since $H_0(X;G) = S_0(X;G)/\mathrm{Im}(\partial_1 \otimes \mathrm{id})$, $\widetilde{H}_0(X;G) = (\operatorname{Ker}(\epsilon \otimes \mathrm{id}))/\mathrm{Im}(\partial_1 \otimes \mathrm{id}) \cong ((\operatorname{Ker} \epsilon) \otimes G)/\mathrm{Im}(\partial_1 \otimes \mathrm{id})$, and since $\mathrm{Im}\, \partial_1 \subseteq \operatorname{Ker} \epsilon$, we get

$$S_0(X;G)/\mathrm{Im}(\partial_1 \otimes \mathrm{id}) \cong (((\operatorname{Ker} \epsilon) \otimes G)/\mathrm{Im}(\partial_1 \otimes \mathrm{id})) \oplus G,$$

which shows that

$$H_0(X;G) \cong \widetilde{H}_0(X;G) \oplus G$$
$$H_p(X;G) \cong \widetilde{H}_p(X;G), \qquad p \geq 1.$$

More generally, if $A$ is a subset of $X$, we have the chain complex $(S_*(X, A; R), \partial_*)$ displayed below

$$0 \xleftarrow{\partial_0} S_0(X, A; R) \xleftarrow{\partial_1} S_1(X, A; R) \longleftarrow \cdots \xleftarrow{\partial_{p-1}} S_{p-1}(X, A; R) \xleftarrow{\partial_p} S_p(X, A; R) \longleftarrow \cdots$$

where $S_p(X, A; R) = S_p(X; R)/S_p(A; R)$, and by tensoring with $G$ and writing

$$S_p(X, A; G) = S_p(X, A; R) \otimes G,$$

we obtain the chain complex $(S_*(X, A; R) \otimes G, \partial_* \otimes G)$

$$0 \xleftarrow{\partial_0 \otimes \mathrm{id}} S_0(X, A; G) \xleftarrow{\partial_1 \otimes \mathrm{id}} S_1(X, A; G) \longleftarrow \cdots \xleftarrow{\partial_p \otimes \mathrm{id}} S_p(X, A; G) \longleftarrow \cdots$$

denoted $(S_*(X, A; G), \partial_* \otimes G)$.

**Definition 4.13.** Let $R$ be a commutative ring with identity and let $G$ be a $R$-module. For any subset $A$ of the space $X$, the *relative singular homology modules $H_p(X, A; G)$ with coefficients in $G$* are the homology groups of the above complex; that is,

$$H_p(X, A; G) = H_p(S_*(X, A; G)) \quad p \geq 0.$$

Similarly, the *reduced relative singular homology modules $\widetilde{H}_p(X, A; G)$ with coefficients in $G$* are the homology groups of the complex obtained by tensoring the reduced homology complex of $(X, A)$ with $G$. As in Section 4.2, if $A \neq \emptyset$ we have

$$H_p(X, A; G) \cong \widetilde{H}_p(X, A; G), \qquad p \geq 0.$$

A continuous map $h\colon (X, A) \to (Y, B)$ gives rise to a chain map

$$h_\sharp \otimes \mathrm{id}\colon S_*(X, A; R) \otimes G \to S_*(Y, B; R) \otimes G$$

which induces a homology homomorphism

$$h_*\colon H_*(X, A; G) \to H_*(Y, B; G).$$

As we know, we have a short exact sequence

$$0 \longrightarrow S_p(A; R) \longrightarrow S_p(X; R) \longrightarrow S_p(X, A; R) \longrightarrow 0,$$

and since $S_p(X, A; R)$ is free, it is a split exact sequence. Therefore, by tensoring with $G$ we obtain another short exact sequence

$$0 \longrightarrow S_p(A; R) \otimes G \longrightarrow S_p(X; R) \otimes G \longrightarrow S_p(X, A; R) \otimes G \longrightarrow 0;$$

that is, a short exact sequence

$$0 \longrightarrow S_p(A; G) \longrightarrow S_p(X; G) \longrightarrow S_p(X, A; G) \longrightarrow 0,$$

By Theorem 2.19, we obtain a long exact sequence of homology.

**Theorem 4.22.** *(Long Exact Sequence of Relative Homology) For every pair $(X, A)$ of spaces, for any $R$-module $G$, we have the following long exact sequence of homology groups*



*ending in*

$$H_0(A; G) \longrightarrow H_0(X; G) \longrightarrow H_0(X, A; G) \longrightarrow 0.$$

The version of Theorem 4.22 for reduced homology also holds.

It is easily checked that if $x \in X$ is a point then

$$H_p(\{x\}; G) = \begin{cases} G & \text{if } p = 0 \\ (0) & \text{if } p \neq 0. \end{cases}$$

It is quite easy to see that the homotopy axiom also holds for homology with coefficients in $G$ (see Munkres [38], Chapter 6, Section 51).

**Proposition 4.23.** *(Homotopy Axiom) Given any two continuous maps $f, g \colon (X, A) \to (Y, B)$ if $f$ and $g$ are homotopic and $H_p(f), H_p(g) \colon H_p(X, A; G) \to H_p(Y, B; G)$ are the induced homomorphisms, then $H_p(f) = H_p(g)$ for all $p \geq 0$. As a consequence, if $(X, A)$ and $(Y, B)$ are homotopy equivalent then for any $R$-module $G$ the homology groups $H_p(X, A; G)$ and $H_p(Y, A; G)$ are isomorphic for all $p \geq 0$,*

The excision axiom also holds but the proof requires a little more work (see Munkres [38], Chapter 6, Section 51).

**Theorem 4.24.** *(Excision Axiom) Given subspaces $Z \subseteq A \subseteq X$ such that the closure of $Z$ is contained in the interior of $A$, then for any $R$-module $G$ the inclusion $(X - Z, A - Z) \longrightarrow (X, A)$ induces isomorphisms of singular homology*

$$H_p(X - Z, A - Z; G) \cong H_p(X, A; G), \quad \text{for all } p \geq 0.$$

*Equivalently, for any subspaces $A, B \subseteq X$ whose interiors cover $X$, the inclusion map $(B, A \cap B) \longrightarrow (X, A)$ induces isomorphisms*

$$H_p(B, A \cap B; G) \cong H_p(X, A; G), \quad \text{for all } p \geq 0.$$

As a consequence, since the homotopy axiom, the excision axiom and the long exact sequence of homology exists, the proof of Proposition 4.16 goes through with $G$-coefficients. The homology of $D^n$ is given by

$$H_0(D^n; G) = G$$
$$H_p(D^n; G) = (0), \quad p > 0,$$

or equivalently

$$\widetilde{H}_p(D^n; G) = (0), \quad p \geq 0,$$

and we have the following result.

**Proposition 4.25.** *For any $R$-module $G$ the reduced homology of $S^n$ is given by*

$$\widetilde{H}_p(S^n; G) = \begin{cases} G & \text{if } p = n \\ (0) & \text{if } p \neq n, \end{cases}$$

*or equivalently the homology of $S^n$ is given by*

$$H_0(S^0; G) = G \oplus G$$
$$H_p(S^0; G) = (0), \quad p > 0,$$

*and for $n \geq 1$,*

$$H_p(S^n; G) = \begin{cases} G & \text{if } p = 0, n \\ (0) & \text{if } p \neq 0, n. \end{cases}$$

Relative singular homology with coefficients in $G$ satisfies the axioms of homology theory singled out by Eilenberg and Steenrod [12]. The Universal Coefficient Theorem for homology (Theorem 12.42) shows that if $R$ is a PID, then the module $H_p(X, A; G)$ can be expressed in terms of the modules $H_p(X, A; R)$ and $H_{p-1}(X, A; R)$ for any $R$-module $G$.

For example, we find that the homology groups of the real projective space with values in an $R$-module $G$ are given by

$$H_p(\mathbb{RP}^n; G) = \begin{cases} G & \text{for } p = 0, n \\ G/2G & \text{for } p \text{ odd, } 0 < p < n \\ \text{Ker}\,(G \xrightarrow{\ 2\ } G) & \text{for } p \text{ even } 0 < p < n \\ (0) & \text{otherwise} \end{cases}$$

if $n$ is odd and

$$H_p(\mathbb{RP}^n; G) = \begin{cases} G & \text{for } p = 0 \\ G/2G & \text{for } p \text{ odd, } 0 < p < n \\ \text{Ker}\,(G \xrightarrow{\ 2\ } G) & \text{for } p \text{ even } 0 < p \leq n \\ (0) & \text{otherwise.} \end{cases}$$

if $n$ is even, where the map $G \xrightarrow{\ 2\ } G$ is the map $g \mapsto 2g$.

Although homology theory is a very interesting subject, we proceed with cohomology, which is our primary focus.

## 4.6   Singular Cohomology

Roughly, to obtain cohomology from homology we dualize everything.

**Definition 4.14.** Given a topological space $X$ and a commutative ring $R$, for any $p \geq 0$ we define the *singular cochain group* $S^p(X; R)$ as the dual $\text{Hom}_R(S_p(X; R), R)$ of the $R$-module $S_p(X; R)$, namely the space of all $R$-linear maps from $S_p(X; R)$ to $R$. The elements of $S^p(X; R)$ are called *singular p-cochains*. We set $S^p(X; R) = (0)$ for $p < 0$.

Since $S_p(X; R)$ is the free $R$-module generated by the set $S_{\Delta^p}(X)$ of singular $p$-simplices, every linear map from $S_p(X; R)$ to $R$ is completely determined by its restriction to $S_{\Delta^p}(X)$, so we may view an element of $S^p(X; R)$ as an *arbitrary function* $f \colon S_{\Delta^p}(X) \to R$ assigning some element of $R$ to every singular $p$-simplex $\sigma$. Recall that the set of functions from $S_{\Delta^p}(X)$ to $R$ forms a $R$-module under the operations of multiplication by a scalar and addition given by

$$(\lambda f)(\sigma) = \lambda(f(\sigma))$$
$$(f + g)(\sigma) = f(\sigma) + g(\sigma)$$

for any singular $p$-simplex $\sigma \in S_{\Delta^p}(X)$ and any scalar $\lambda \in R$. Any singular $p$-cochain $f \colon S_{\Delta^p}(X) \to R$ can be evaluated on any singular $p$-chain $\alpha = \sum_{i=1}^{m} \lambda_i \sigma_i$, where the $\sigma_i$ are singular $p$-simplices in $S_{\Delta^p}(X)$, by

$$f(\sigma) = \sum_{i=1}^{m} \lambda_i f(\sigma_i).$$

All we need to get a chain complex is to define the coboundary map $\delta^p \colon S^p(X; R) \to S^{p+1}(X; R)$.

It is quite natural to say that for any singular $p$-cochain $f \colon S_{\Delta^p}(X) \to R$, the value $\delta^p f$ should be the function whose value $(\delta^p f)(\alpha)$ on a singular $(p+1)$-chain $\alpha$ is given by

$$(\delta^p f)(\alpha) = \pm f(\partial_{p+1} \alpha).$$

If we write $\langle g, \beta \rangle = g(\beta)$ for the result of evaluating the singular $p$-cochain $g \in S^p(X; R)$ on the singular $p$-chain $\beta \in S_p(X; R)$, then the above is written as

$$\langle \delta^p f, \alpha \rangle = \pm \langle f, \partial_{p+1} \alpha \rangle,$$

which is reminiscent of an adjoint. It remains to pick the sign of the right-hand side. Bott and Tu [2] and Warner [50] pick the $+$ sign, whereas Milnor and Stasheff [35] pick the sign $(-1)^{p+1}$, so that

$$\langle \delta^p f, \alpha \rangle + (-1)^p \langle f, \partial_{p+1} \alpha \rangle = 0.$$

Milnor and Stasheff explain that their choice of sign agrees with the convention that whenever two symbols of dimension $m$ and $n$ are permuted, the sign $(-1)^{mn}$ should be introduced. Here $\delta$ is considered to have sign $+1$ and $\partial$ is considered to have sign $-1$. MacLane explains that the choice of the sign $(-1)^{p+1}$ is desirable if a generalization of cohomology is considered; see MacLane [29] (Chapter II, Section 3).

Regarless of the choice of sign, $\delta^{p+1} \circ \delta^p = 0$. Since the $+$ sign is simpler, this is the one that we adopt. Thus, $\delta^p f$ is defined by

$$\delta^p f = f \circ \partial_{p+1} \quad \text{for all } f \in S^p(X; R).$$

If we let $A = S_{p+1}(X; R)$, $B = S_p(X; R)$ and $\varphi = \partial_{p+1}$, we see that the definition of $\delta^p$ is equivalent to

$$\delta^p = \partial_{p+1}^{\top}.$$

The cohomology complex is indeed obtained from the homology complex by dualizing spaces and maps.

We define the direct sum $S^*(X; R)$ as

$$S^*(X; R) = \bigoplus_{p \geq 0} S^p(X; R).$$

**Definition 4.15.** Given a topological space $X$ and a commutative ring $R$, for any $p \geq 0$, the *coboundary homomorphism*

$$\delta^p \colon S^p(X; R) \to S^{p+1}(X; R)$$

is defined by

$$\langle \delta^p f, \alpha \rangle = \langle f, \partial_{p+1} \alpha \rangle,$$

for every singular $p$-cochain $f \colon S_{\Delta^p}(X) \to R$ and every singular $(p+1)$-chain $\alpha \in S_{p+1}(X; R)$; equivalently,

$$\delta^p f = f \circ \partial_{p+1} \quad \text{for all } f \in S^p(X; R).$$

We obtain a coboundary map

$$\delta \colon S^*(X; R) \to S^*(X; R).$$

The following proposition is immediately obtained.

**Proposition 4.26.** *Given a topological space $X$ and a commutative ring $R$, the coboundary map $\delta \colon S^*(X; R) \to S^*(X; R)$ satisfies the equation*

$$\delta \circ \delta = 0.$$

We now have all the ingredients to define cohomology groups. Since the $S^p(X; R)$ together with the coboundary maps $\delta^p$ form the chain complex

$$0 \xrightarrow{\delta^{-1}} S^0(X; R) \xrightarrow{\delta^0} S^1(X; R) \longrightarrow \cdots \xrightarrow{\delta^{p-1}} S^p(X; R) \xrightarrow{\delta^p} S^{p+1}(X; R) \xrightarrow{\delta^{p+1}} \cdots$$

as in Section 2.3, we obtain the familar spaces $Z^p(X; R) = \operatorname{Ker} \delta^p$ of *singular $p$-cocycles*, and $B^p(X; R) = \operatorname{Im} \delta^{p-1}$ of *singular $p$-coboundaries*. By Proposition 4.26, $B^p(X; R)$ is a submodule of $Z^p(X; R)$ so we obtain cohomology spaces:

**Definition 4.16.** Given a topological space $X$ and a commutative ring $R$, for any $p \geq 0$ the *singular cohomology module* $H^p(X; R)$ is defined by

$$H^p(X; R) = \ker \delta^p / \operatorname{Im} \delta^{p-1} = Z^p(X; R) / B^p(X; R).$$

We set $H^p(X; R) = (0)$ if $p < 0$ and define $H^*(X; R)$ as the direct sum

$$H^*(X; R) = \bigcup_{p \geq 0} H^p(X; R)$$

and call it the *singular cohomology of $X$ with coefficients in $R$*.

It is common practice to refer to the spaces $H^p(X; R)$ as groups even though they are $R$-modules.

Until now we have been very compulsive in adding the term *singular* in front of every notion (chain, cochain, cycle, cocycle, boundary, coboundary, *etc.*). From now on we will drop this term unless confusion may arise. We may also drop $X$ or $R$ in $H^p(X; R)$ *etc.* whenever possible (that is, not causing confusion).

At this stage, one may wonder if there is any connection between the homology groups $H_p(X; R)$ and the cohomology groups $H^p(X; R)$. The answer is yes and it is given by the *Universal Coefficient Theorem*. However, even to state the universal coefficient theorem requires a fair amount of homological algebra, so we postpone this topic until Section 12.5. Let us just mention the following useful isomorphisms in dimension 0 and 1:

$$H^0(X; R) = \operatorname{Hom}_R(H_0(X; R), R)$$
$$H^1(X; R) = \operatorname{Hom}_R(H_1(X; R), R).$$

It is not hard to see that $H^0(X; R)$ consists of those functions from $X$ to $R$ that are constant on path-components. Readers who want to learn about Universal Coefficient Theorems should consult Section 12.5. If $R$ is a PID, then the following result proved in Milnor and Stasheff [35] (Appendix A, Theorem A.1) gives a very clean answer.

**Theorem 4.27.** *Let $X$ be a topological space $X$ and let $R$ be a PID. If the homology group $H_{p-1}(X; R)$ is a free $R$-module or $(0)$, then the cohomology group $H^p(X; R)$ is canonically isomorphic to the dual $\operatorname{Hom}_R(H_p(X; R), R)$ of $H_p(X; R)$.*

In particular, Theorem 4.27 holds if $R$ is a field.

There is a generalization of singular cohomology which is useful for certain applications. The idea is to use more general coefficients. We can use a $R$-module $G$ as the set of coefficients.

**Definition 4.17.** Given a topological space $X$, a commutative ring $R$, and a $R$-module $G$, for any $p \geq 0$ the *singular cochain group* $S^p(X; G)$ is the $R$-module $\operatorname{Hom}_R(S_p(X; R), G)$ of $R$-linear maps from $S_p(X; R)$ to $G$. We set $S^p(X; G) = (0)$ for $p < 0$.

Following Warner [50], since $S_p(X; R)$ is the free $R$-module generated by the set $S_{\Delta^p}(X)$ of singular $p$-simplices, we can view $S^p(X; G)$ as the set of all functions $f \colon S_{\Delta^p}(X) \to G$. This is also a $R$-module. As a special case, if $R = \mathbb{Z}$, then $G$ can be any abelian group. As before, we obtain $R$-modules $Z^p(X; G)$ and $B^p(X; G)$ and coboundary maps $\delta^p \colon S^p(X; G) \to S^{p+1}(X; G)$ defined by

$$\delta^p f = f \circ \partial_{p+1} \quad \text{for all } f \in S^p(X; G).$$

We get the chain complex

$$0 \xrightarrow{\delta^{-1}} S^0(X; G) \xrightarrow{\delta^0} S^1(X; G) \longrightarrow \cdots \xrightarrow{\delta^{p-1}} S^p(X; G) \xrightarrow{\delta^p} S^{p+1}(X; G) \xrightarrow{\delta^{p+1}} \cdots$$

and we obtain cohomology groups.

**Definition 4.18.** Given a topological space $X$, a commutative ring $R$, and a $R$-module $G$, for any $p \geq 0$ the *singular cohomology module* $H^p(X;G)$ is defined by

$$H^p(X;G) = \ker \delta^p / \operatorname{Im} \delta^{p-1} = Z^p(X;G)/B^p(X;G).$$

We set $H^p(X;G) = (0)$ if $p < 0$ and define $H^*(X;G)$ as the direct sum

$$H^*(X;G) = \bigcup_{p \geq 0} H^p(X;G)$$

and call it the *singular cohomology of $X$ with coefficients in $G$*.

Warner uses the notation $H^p_\Delta(X;G)$ instead of $H^p(X;G)$. When more than one cohomology theory is used, this is a useful device to distinguish among the various cohomology groups.

Cohomology is also functorial, If $f\colon X \to Y$ is a continuous map, then we know from Proposition 4.3 that there is a chain map $f_{\sharp,p}\colon S_p(X;R) \to S_p(Y;R)$, so by applying $\operatorname{Hom}_R(-,G)$ we obtain a cochain map $f^{\sharp,p}\colon S^p(Y;G) \to S^p(X;G)$ which commutes with coboundaries, and thus a homomorphism $H^p(f)\colon H^p(Y;G) \to H^p(X;G)$. This fact is recorded as the following proposition.

**Proposition 4.28.** *If $X$ and $Y$ are two topological spaces and if $f\colon X \to Y$ is a continuous function between them, then there are homomorphisms $H^p(f)\colon H^p(Y;G) \to H^p(X;G)$ for all $p \geq 0$.*

The map $H^p(f)\colon H^p(Y;G) \to H^p(X;G)$ is also denoted by $f^{*p}\colon H^p(Y;G) \to H^p(X;G)$.

We also have the following version of Proposition 4.5 for cohomology.

**Proposition 4.29.** *Given any two continuous maps $f,g\colon X \to Y$ (where $X$ and $Y$ are topological spaces), if $f$ and $g$ are homotopic and $H^p(f), H^p(g)\colon H^p(Y;G) \to H^p(X;G)$ are the induced homomorphisms, then $H^p(f) = H^p(g)$ for all $p \geq 0$. As a consequence, if $X$ and $Y$ are homotopy equivalent then the cohomology groups $H^p(X;G)$ and $H^p(Y;G)$ are isomorphic for all $p \geq 0$,*

For any PID $R$, there is a Universal Coefficient Theorem for cohomology that yields an expression for $H^p(X;G)$ in terms of $H_{p-1}(X;R)$ and $H_p(X;R)$; see Theorem 12.48. There is also a version of the Mayer–Vietoris exact sequence for singular cohomology.

Given any topological space $X$, for any two subsets $A, B$ of $X$ such that $X = \operatorname{Int}(A) \cup \operatorname{Int}(B)$, recall from Theorem 4.14 that we have a short exact sequence

$$0 \longrightarrow S_p(A \cap B) \xrightarrow{\varphi} S_p(A) \oplus S_p(B) \xrightarrow{\psi} S_p(A) + S_p(B) \longrightarrow 0$$

for every $p \geq 0$, where $\varphi$ and $\psi$ are given by

$$\varphi(c) = (i_\sharp(c), -j_\sharp(c))$$
$$\psi(a, b) = k_\sharp(a) + l_\sharp(b).$$

Because $S_p(A) \oplus S_p(B)$ is free and because $S_p(A \cap B)$ is a submodule of both $S_p(A)$ and $S_p(B)$, we can choose bases in $S_p(A)$ and $S_p(B)$ by completing a basis of $S_p(A \cap B)$, and as a consequence we can define a map $p \colon S_p(A) \oplus S_p(B) \to S_p(A \cap B)$ such that $p \circ \varphi = \mathrm{id}$. Therefore the above sequence splits, and if we apply $\mathrm{Hom}_R(-, R)$ to it we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(S_p(A) + S_p(B), R) \overset{\psi^\perp}{\longrightarrow} S^p(A) \oplus S^p(B) \overset{\varphi^\perp}{\longrightarrow} S^p(A \cap B) \longrightarrow 0 \qquad (*)$$

where $\varphi^\perp = \mathrm{Hom}(\varphi, R)$ and $\psi^\perp = \mathrm{Hom}(\psi, R)$. Since the inclusions $\iota_p \colon S_p(A) + S_p(B) \to S_p(X)$ form a chain homotopy equivalence, which means that there are maps $\rho_p \colon S_p(X) \to S_p(A) + S_p(B)$ such that $\rho \circ \iota$ and $\iota \circ \rho$ are chain homotopic to id, by applying $\mathrm{Hom}_R(-, R)$ we see that there is also a chain homotopy equivalence between $\mathrm{Hom}(S_p(A) + S_p(B), R)$ ans $\mathrm{Hom}(S_p(X), R) = S^p(X)$, so the long exact sequence associated with the short exact sequence $(*)$ yields the following result.

**Theorem 4.30.** *(Mayer–Vietoris in singular cohomology) Given any topological space $X$, for any two subsets $A, B$ of $X$ such that $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, there is a long exact sequence of cohomology*

$$\longrightarrow H^p(X; R) \longrightarrow H^p(A; R) \oplus H^p(B; R) \longrightarrow H^p(A \cap B; R) \longrightarrow H^{p+1}(X; R) \longrightarrow \cdots$$

*If $A \cap B \neq \emptyset$, a similar sequence exists in reduced cohomology.*

There is a notion of singular cohomology with compact support and generalizations of Poincaré duality. Some of the steps still use the Mayer–Vietoris sequences and the five lemma, but the proof is harder and requires two kinds of induction. Basically, Poincaré duality asserts that for any orientable manifold $M$ of dimension $n$ and any commutative ring $R$ with an identity element, there are isomorphisms

$$H_c^p(M; R) \cong H_{n-p}(M; R).$$

On left-hand side $H_c^p(M; R)$ denotes the $p$th singular cohomology group with compact support, and on the right-hand side $H_{n-p}(M; R)$ denotes the $(n-p)$th singular homology group. By manifold, we mean a topological manifold (thus, Hausdorff and paracompact), not necessarily a smooth manifold, so this is a very general theorem. For details, the interested reader is referred to Chapter 7 (Theorem 7.13), and for comprehensive presentations including proof, to Milnor and Stasheff [35] (Appendix A), Hatcher [25] (Chapter 3), and Munkres [38] (Chapter 8).

If $M$ is a smooth manifold and if $R = \mathbb{R}$, a famous result of de Rham states that *de Rham cohomology and singular cohomology are isomorphic*, that is

$$H_{\mathrm{dR}}(M) \cong H^*(M; \mathbb{R}).$$

This is a hard theorem to prove. A complete proof can be found Warner [50] (Chapter 5). Another proof can be found in Morita [36] (Chapter 3). These proofs use Čech cohomology, which we discuss next. It should be pointed that Chapter 5 of Warner [50] covers far more than the de Rham theorem. It provides a very thorough presentation of sheaf cohomology from an axiomatic point of view and shows the equivalence of four "classical" cohomology theories for smooth manifolds: Alexander-Spanier, de Rham, Singular, and Čech cohomology. Warner's presentation is based on an approach due to Henri Cartan written in the early 1950's and based on fine sheaves. In Chapter 13 we develop sheaf cohomology using a more general and more powerful approach due to Grothendieck based on derived functors and $\delta$-functors. This material is very technical; don't give up, it will probably require many passes to be digested.

## 4.7   Relative Singular Cohomology Groups

In this section $R$ is any commutative with unit 1 and $G$ is any $R$-module.

Reduced singular cohomology groups $\widetilde{H}^p(X; G)$ are defined by dualizing the augmented chain complex

$$0 \xleftarrow{\quad} R \xleftarrow{\ \epsilon\ } S_0(X; R) \xleftarrow{\ \partial_1\ } S_1(X; R) \xleftarrow{\quad} \cdots \xleftarrow{\ \partial_{p-1}\ } S_{p-1}(X; R) \xleftarrow{\ \partial_p\ } S_p(X; R) \xleftarrow{\ \partial_{p+1}\ } \cdots$$

by applying $\mathrm{Hom}_R(-, G)$. We have

$$\widetilde{H}^0(X; G) = \mathrm{Hom}_R(\widetilde{H}_0(X; R), G)$$
$$\widetilde{H}^p(X; G) = H^p(X; G) \quad p \geq 1.$$

In fact, it can be shown that

$$H^0(X; G) \cong \widetilde{H}^0(X; G) \oplus G;$$

see Munkres [38] (Chapter 5, Section 44).

To obtain the relative cohomology groups we dualize the chain complex of relative homology

$$0 \xleftarrow{\ \partial_0\ } S_0(X, A; R) \xleftarrow{\ \partial_1\ } S_1(X, A; R) \xleftarrow{\quad} \cdots \xleftarrow{\ \partial_{p-1}\ } S_{p-1}(X, A; R) \xleftarrow{\ \partial_p\ } S_p(X, A; R) \xleftarrow{\ \partial_{p+1}\ } \cdots$$

by applying $\mathrm{Hom}_R(-, G)$, where $S_p(X, A; R) = S_p(X, R)/S_p(A, R)$. We obtain the chain complex

$$0 \xrightarrow{\ \delta^{-1}\ } S^0(X, A; G) \xrightarrow{\ \delta^0\ } S^1(X, A; G) \xrightarrow{\quad} \cdots \xrightarrow{\ \delta^{p-1}\ } S^p(X, A; G) \xrightarrow{\ \delta^p\ } S^{p+1}(X, A; G) \xrightarrow{\ \delta^{p+1}\ } \cdots$$

with $S^p(X, A; G) = \mathrm{Hom}_R(S_p(X, A; R), G)$ and $\delta^p = \mathrm{Hom}_R(\partial_p, G)$ for all $p \geq 0$ (and $\delta^{-1}$ is the zero map). More explicitly

$$\delta^p(f) = f \circ \partial_{p+1} \quad \text{for all } f \in S^p(X, A; G);$$

that is

$$\delta^p(f)(\sigma) = f(\partial_{p+1}(\sigma)) \quad \text{for all } f \in S^p(X, A; G) = \mathrm{Hom}_R(S_p(X, A; R), G)$$
$$\text{and all } \sigma \in S_{p+1}(X; A; R);$$

Note that $S^p(X, A; G) = \mathrm{Hom}_R(S_p(X; R)/S_p(A; R), G)$ is isomorphic to the submodule of $S^p(X; G) = \mathrm{Hom}_R(S_p(X; R), G)$ consisting of all linear maps with values in $G$ defined on singular simplices in $S_p(X; R)$ that vanish on singular simplices in $S_p(A; R)$. Consequently, the coboundary map

$$\delta^p \colon S^p(X, A; G) \to S^{p+1}(X, A; G)$$

is the restriction of $\delta_X^p \colon S^p(X; G) \to S^{p+1}(X; G)$ to $S^p(X, A; G)$ where $\delta_X^p$ is the coboundary map of absolute cohomology.

**Definition 4.19.** Given a pair of spaces $(X, A)$, the *singular relative cohomology groups* $H^p(X, A; G)$ of $(X, A)$ arise from the chain complex

$$0 \xrightarrow{\delta^{-1}} S^0(X, A;G) \xrightarrow{\delta^0} S^1(X, A; G) \longrightarrow \cdots \xrightarrow{\delta^{p-1}} S^p(X, A; G) \xrightarrow{\delta^p} S^{p+1}(X, A; G) \xrightarrow{\delta^{p+1}} \cdots$$

and are given by

$$H^p(X, A; G) = \mathrm{Ker}\, \delta^p / \mathrm{Im}\, \delta^{p-1}, \quad p \geq 0.$$

As in the case of absolute singular cohomology, a continuous map $f \colon (X, A) \to (Y, B)$ induces a homomorphism of relative cohomology $f^* \colon H^*(Y, B) \to H^*(X, A)$. This is because by Proposition 4.6 the map $f$ induces a chain map $f_\sharp \colon S_*(X, A; R) \to S_*(Y, B; R)$, and by applying $\mathrm{Hom}_R(-, G)$ we obtain a cochain map $f^\sharp \colon S^*(Y, B; G) \to S^*(X, A; G)$ which commutes with coboundaries, and thus induces homomorphisms $H^p(f) \colon H^p(Y, B; G) \to H^p(X, A; G)$.

**Proposition 4.31.** *If $(X, A)$ and $(Y, B)$ are pairs of topological spaces and if $f \colon (X, A) \to (Y, B)$ is a continuous function between them, then there are homomorphisms $H^p(f) \colon H^p(Y, B; G) \to H^p(X, A; G)$ for all $p \geq 0$.*

The map $H^p(f) \colon H^p(Y, B; G) \to H^p(X, A; G)$ is also denoted by $f^{*p} \colon H^p(Y, B; G) \to H^p(X, A; G)$.

We also have the following version of Proposition 4.5 for relative cohomology which is the cohomological version of Proposition 4.7.

**Proposition 4.32.** *(Homotopy Axiom) Given any two continuous maps $f, g\colon (X, A) \to (Y, B)$, if $f$ and $g$ are homotopic and $H^p(f), H^p(g)\colon H^p(Y, B; G) \to H^p(X, A; G)$ are the induced homomorphisms, then $H^p(f) = H^p(g)$ for all $p \geq 0$. As a consequence, if $(X, A)$ and $(Y, B)$ are homotopy equivalent then the cohomology groups $H^p(X, A; G)$ and $H^p(Y, B; G)$ are isomorphic for all $p \geq 0$,*

To obtain the long exact sequence of relative cohomology we dualize the short exact sequence

$$0 \longrightarrow S_*(A; R) \overset{i}{\longrightarrow} S_*(X; R) \overset{j}{\longrightarrow} S_*(X, A; R) \longrightarrow 0$$

where $S_*(X, A; R) = S_*(X, R)/S_*(A, R)$ by applying $\mathrm{Hom}(-, G)$ and we obtain the sequence

$$0 \longrightarrow S^*(X, A; G) \overset{j^\top}{\longrightarrow} S^*(X; G) \overset{i^\top}{\longrightarrow} S^*(A; G) \longrightarrow 0,$$

where by definition $S^*(X, A; G) = \mathrm{Hom}_R(S_*(X; R)/S_*(A; R), G)$, and as before $S^*(A; G) = \mathrm{Hom}_R(S_*(A; R), G)$ and $S^*(X; G) = \mathrm{Hom}_G(S_*(X; R), G)$.

Since $S_p(X, A; R) = S_p(X, R)/S_p(A, R)$ is a free module for every $p$, by Proposition 2.6 the sequence of chain complexes

$$0 \longrightarrow S^*(X, A; G) \overset{j^\top}{\longrightarrow} S^*(X; G) \overset{i^\top}{\longrightarrow} S^*(A; G) \longrightarrow 0$$

is exact (this can also be verified directly; see Hatcher [25], Section 3.1). Therefore, we can apply the zig-zag lemma for cohomology (Theorem 2.19) to this short exact sequence and we obtain the following cohomological version of Theorem 4.8.

**Theorem 4.33.** *(Long Exact Sequence of Relative Cohomology) For every pair $(X, A)$ of spaces, we have the following long exact sequence of cohomology groups*



There is also a version of Theorem 4.33 for reduced relative cohomology with $A \neq \emptyset$. As in the case of reduced homology with $A \neq \emptyset$, we have

$$\widetilde{H}^p(X, A, G) = H^p(X, A, G) \quad \text{for all } p \geq 0.$$

By setting $A = \{\text{pt}\}$, the version of Theorem 4.33 for relative cohomology yields the isomorphisms

$$H^p(X, \{\text{pt}\}; G) \cong \widetilde{H}^p(X; G) \quad \text{for all } p \geq 0.$$

Finally, the excision property also holds for relative cohomology.

**Theorem 4.34.** *(Excision Axiom) Given subspaces $Z \subseteq A \subseteq X$ such that the closure of $Z$ is contained in the interior of $A$, then the inclusion $(X - Z, A - Z) \longrightarrow (X, A)$ induces isomorphisms of singular cohomology*

$$H^p(X - Z, A - Z; G) \cong H^p(X, A; G), \quad \text{for all } p \geq 0.$$

*Equivalently, for any subspaces $A, B \subseteq X$ whose interiors cover $X$, the inclusion map $(B, A \cap B) \longrightarrow (X, A)$ induces isomorphisms*

$$H^p(B, A \cap B; G) \cong H^p(X, A; G), \quad \text{for all } p \geq 0.$$

The proof of Theorem 4.34 does not follow immediately by dualization of Theorem 4.12. For details the reader is referred to Munkres [38] (Chapter 5, §44) or Hatcher [25] (Section 3.1).

Proposition 4.32, Theorem 4.33, and Theorem 4.34 state three of the properties that were singled out as characterizing cohomology theories by Eilenberg and Steenrod [12]. As in the case of homology, these properties hold for most of the known cohomology theories, and thus can be taken as axioms for cohomology theory; see Sato [43], MacLane [29], Munkres [38], or Hatcher [25].

For any PID $R$, there is a Universal Coefficient Theorem for cohomology that yields an expression for $H^p(X, A; G)$ in terms of $H_{p-1}(X, A; R)$ and $H_p(X, A; R)$; see Theorem 12.48.

# 4.8 The Cup Product and The Cohomology Ring

We will see later in Section 12.5 (the Universal Coefficient Theorem for Cohomology, Theorem 12.48) that the homology groups of a space with values in a PID $R$ determine its cohomology groups with values in any $R$-module $G$. One might then think that cohomology groups are not useful, but this is far from the truth for several reasons.

First, cohomology groups arise naturally as various "obstructions," such as the Ext-groups discussed in Section 12.5, or in the problem of classifying, up to homotopy, maps from one space into another. We will also see that in some cases only cohomology can be defined, as in the case of sheaves. But another reason why cohomology is important is that there is a natural way to define a multiplication operation on cohomology classes that makes the direct sum of the cohomology modules into a (graded) algebra. This additional structure allows the distinction between spaces that would not otherwise be distinguished by their homology (and cohomology).

We would like to define an operation $\smile$ that takes two cochains $c \in S^p(X; R)$ and $d \in S^q(X; R)$ and produces a cochain $c \smile d \in S^{p+q}(X; R)$. For this, we define two affine maps $\lambda_p \colon \Delta^p \to \Delta^{p+q}$ and $\rho_q \colon \Delta^q \to \Delta^{p+q}$ by

$$\lambda_p(e_i) = e_i \quad 1 \leq i \leq p+1$$
$$\rho_q(e_i) = e_{p+i}, \quad 1 \leq i \leq q+1.$$

For any singular $(p+q)$-simplex $\sigma \colon \Delta^{p+q} \to X$, observe that $\sigma \circ \lambda_p \colon \Delta^p \to X$ is a singular $p$-simplex and $\sigma \circ \rho_q \colon \Delta^q \to X$ is a singular $q$-simplex. See Figure 4.12.
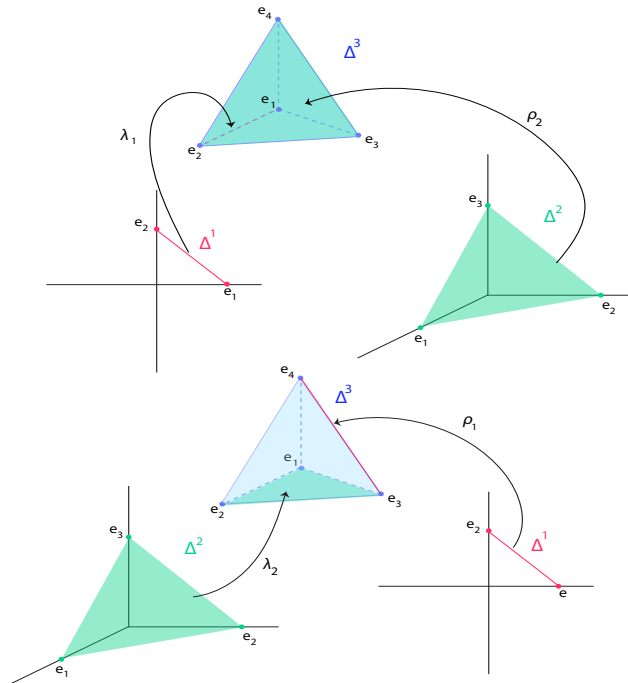


Figure 4.12: Two ways of embedding a 1-simplex and a 2-simplex into a 3-simplex. For the top figure, $p = 1$ and $q = 2$, while for the bottom figure, $p = 2$ and $q = 1$.

Recall from Definition 4.14 that a singular $p$-cochain is a $R$-linear map from $S_p(X; R)$ to $R$, where $S_p(X; R)$ is the $R$-module of singular $p$-chains. Since $S_p(X; R)$ is the free $R$-module generated by the set $S_{\Delta^p}(X)$ of singular $p$-simplices, every singular $p$-cochain $c$ is completely determined by its restriction to $S_{\Delta^p}(X)$, and thus can be viewed as a function from $S_{\Delta^p}(X)$ to $R$.

**Definition 4.20.** If $\sigma \colon \Delta^{p+q} \to X$ is a singular simplex, we call $\sigma \circ \lambda_p$ the *front $p$-face* of $\sigma$, and $\sigma \circ \rho_q$ the *back $q$-face* of $\sigma$. See Figure 4.13. Given any two cochains $c \in S^p(X; R)$ and

$d \in S^q(X; R)$, their *cup product* $c \smile d \in S^{p+q}(X; R)$ is the cochain defined by

$$(c \smile d)(\sigma) = c(\sigma \circ \lambda_p) d(\sigma \circ \rho_q)$$

for all singular simplices $\sigma \in S_{\Delta^{p+q}}(X)$. The above defines a function
$\smile \colon S^p(X; R) \times S^q(X; R) \to S^{p+q}(X; R)$.

Since $c(\sigma \circ \lambda_p) \in R$ and $d(\sigma \circ \rho_q) \in R$, we have $(c \smile d)(\sigma) \in R$, as desired.
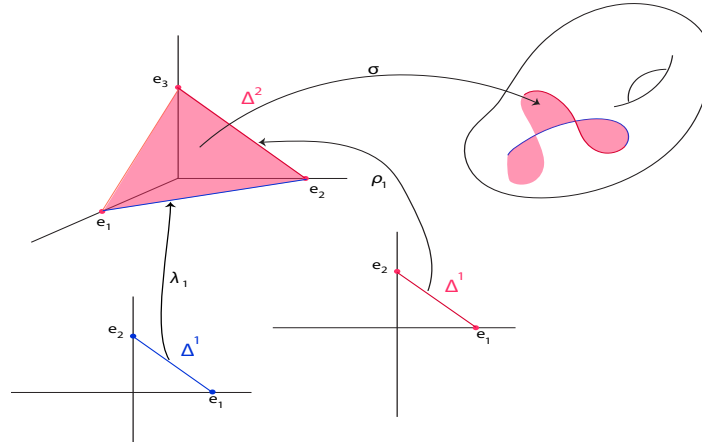


Figure 4.13: A 2-simplex embedded in a torus, where $p = 1 = q$. The front 1-face is the blue edge while the back 1-face is the maroon edge.

**Remark:** Other authors, including Milnor and Stasheff [35], add the sign $(-1)^{pq}$ to the formula in the definition of the cup product.

The reader familar with exterior algebra and differential forms will observe that the cup product can be viewed as a generalization of the wedge product.

Recall that $S^*(X; R)$ is the $R$-module $\bigoplus_{p \geq 0} S^p(X; R)$, and that $\epsilon \colon S_0(X; R) \to R$ is the unique homomorphism such that $\epsilon(x) = 1$ for every point $x \in S_0(X; R)$. Thus $\epsilon \in S^0(X; R)$ and since $\partial^0 \epsilon = \epsilon \circ \partial_1 = 0$, the cochain $\epsilon$ is actually a cocycle and its cohomology class $[\epsilon] \in H^0(X; R)$ is denoted by 1.

The following proposition is immediate from the definition of the cup-product.

**Proposition 4.35.** *The cup product operation $\smile$ in $S^*(X; R)$ is bilinear, associative, and has the cocycle $\epsilon$ as identity element. Thus $S^*(X; R)$ is an associative graded ring with unit element.*

The following technical property implies that the cup product is well defined on cocycles.

**Proposition 4.36.** *For any two cochains $c \in S^p(X; R)$ and $d \in S^q(X; R)$ we have*

$$\delta(c \smile d) = (\delta c) \smile d + (-1)^p c \smile (\delta d).$$

Again, note the analogy with the exterior derivative on differential forms. A proof of Proposition 4.35 can be found in Hatcher [25] (Chapter 3, Section 3.2, Lemma 3.6) and Munkres [38] (Chapter 6, Theorem 48.1).

The formula of Proposition 4.36 implies that the cup product of cocycles is a cocycle, and that the cup product of a cocycle with a coboundary in either order is a coboundary, so we obtain an induced cup product on cohomology classes

$$\smile \colon H^p(X; R) \times H^q(X; R) \to H^{p+q}(X; R).$$

The cup product is bilinear, associative, and has 1 has identity element.

A continuous map $f \colon X \to Y$ induces a homomorphisms of cohomology $f^{p*} \colon H^p(X; R) \to H^p(y; R)$ for al $p \geq 0$, and the cup product behaves well with respect to these maps.

**Proposition 4.37.** *Given any continuous map $f \colon X \to Y$, for all $\omega \in H^p(X; R)$ and all $\eta \in H^q(X; R)$, we have*

$$f^{(p+q)*}(\omega \smile_X \eta) = f^{p*}(\omega) \smile_Y f^{q*}(\eta).$$

*Thus, $f^* = (f^{p*})_{p \geq 0}$ is a homomorphism between the graded rings $H^*(X; R)$ (with the cup product $\smile_X$) and $H^*(Y; R)$ (with the cup product $\smile_Y$).*

Proposition 4.37 is proved in Hatcher [25] (Chapter 3, Section 3.2, Proposition 3.10) and and Munkres [38] (Chapter 6, Theorem 48.3).

**Definition 4.21.** Given a topological space $X$, its *cohomology ring* $H^*(X; R)$ is the graded ring $\bigoplus_{p \geq 0} H^p(X; R)$ equipped with the multiplication operation $\smile$ induced by the operations $\smile \colon H^p(X; R) \times H^q(X; R) \to H^{p+q}(X; R)$ for all $p, q \geq 0$.[1] An element $\omega \in H^p(X; R)$ is said to be of *degree* (or *dimension*) $p$, and we write $p = \deg(\omega)$.

Although the cup product is not commutative in general, it is skew-commutative in the following sense.

**Proposition 4.38.** *For all $\omega \in H^p(X; R)$ and all $\eta \in H^q(X; R)$, we have*

$$\omega \smile \eta = (-1)^{pq}(\eta \smile \omega).$$

---

[1] To be very precise, we have a family of multiplications $\smile_{p,q} \colon H^p(X; R) \times H^q(X; R) \to H^{p+q}(X; R)$, but this notation is too heavy and never used.

The proof of Proposition 4.38 is more complicated than the proofs of the previous propositions. It can be found in Hatcher [25] (Chapter 3, Section 2, Theorem 3.14). Another way to prove Proposition 4.38 is to first define the notion of cross-product and to define the cup product in terms of the cross-product. This is the approach followed by Bredon [4] (Chapter VI, Sections 3 and 4), and Spanier [47] (Chapter 5, Section 6).

The cohomology ring of most common spaces can be determined explicitly, but in some cases requires more machinery (such as Poincaré duality). Let us mention four examples.

**Example 4.2.** In the case of the sphere $S^n$, the cohomology ring $H^*(S^n; R)$ is the graded ring generated by one element $\alpha$ of degree $n$ subject to the single relation $\alpha^2 = 0$.

The cohomology ring $H^*(T^n; R)$ of the $n$-torus $T^n$ (with $T^n = S^1 \times \cdots \times S^1$ $n$ times) is isomorphic to the exterior algebra $\bigwedge R^n$, with $n$-generators $\alpha_1, \ldots, \alpha_n$ of degree 1 satisfying the relations $\alpha_i \alpha_j = -\alpha_j \alpha_i$ for all $i \neq j$ and $\alpha_i^2 = 0$.

The cohomology ring $H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$ of real projective space $\mathbb{RP}^n$ with respect to $R = \mathbb{Z}/2\mathbb{Z}$ is isomorphic to the truncated polynomial ring $\mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1})$, with $\alpha$ an element of degree 1. It is also possible to determine the cohomology ring $H^*(\mathbb{RP}^n, \mathbb{Z})$, but it is more complicated; see Hatcher [25] (Chapter 3, Theorem 3.12, and before Example 3.13).

The cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$ of complex projective space $\mathbb{CP}^n$ with respect to $R = \mathbb{Z}$ is isomorphic to the truncated polynomial ring $\mathbb{Z}[\alpha]/(\alpha^{n+1})$, with $\alpha$ an element of degree 2; see Hatcher [25] (Chapter 3, Theorem 3.12).

The cup product can be generalized in various ways. A first generalization is the cup product

$$\smile : S^p(X; R) \times S^q(X; G) \to S^{p+q}(X; G)$$

where $G$ is any $R$-module, using the exact same formula

$$(c \smile d)(\sigma) = c(\sigma \circ \lambda_p) d(\sigma \circ \rho_q)$$

with $c \in S^p(X; R)$ and $d \in S^q(X; G)$, for all singular simplices $\sigma \in S_{\Delta^{p+q}}(X)$. Since $c(\sigma \circ \lambda_p) \in R$ and $d(\sigma \circ \rho_q) \in G$, their product is in $G$ so the above definition makes sense.

The formula

$$\delta(c \smile d) = (\delta c) \smile d + (-1)^p c \smile (\delta d)$$

of Proposition 4.36 still holds, but associativity only holds in a restricted fashion. Still, we obtain a cup product

$$\smile : H^p(X; R) \times H^q(X; G) \to H^{p+q}(X; G)$$

Another generalization involves relative cohomology. For example, if $A$ and $B$ are open subset of a manifold $X$, there is a well-defined cup product

$$\smile : H^p(X, A; R) \times H^q(X, B; R) \to H^{p+q}(X, A \cup B; R);$$

see Hatcher [25] (Chapter 3, Section 3.2) and Milnor and Stasheff [35] (Appendix A, pages 264-265).

There are a number of interesting applications of the cup product but we will not go into this here, and instead refer the reader to Hatcher [25] (Chapter 3, Section 3.2), Bredon [4] (Chapter VI), and Spanier [47] (Chapter 5).

# Chapter 5

# Simplicial Homology and Cohomology

In Chapter 4 we introduced the singular homology groups and the singular cohomology groups and presented some of their properties. Historically, singular homology and cohomology was developed in the 1940's, starting with a seminal paper of Eilenberg published in 1944 (building up on work by Alexander and Lefschetz among others), but it was not the first homology theory. Simplicial homology emerged in the early 1920's, more than thirty years after the publication of Poincaré's first seminal paper on "analysis situ" in 1892. Until the early 1930's, homology groups had not been defined and people worked with numerical invariants such as Betti numbers and torsion numbers. Emmy Noether played a significant role in introducing homology groups as the main objects of study.

One of the main differences between singular homology and simplicial homology is that singular homology groups can be assigned to *any* topological space $X$, but simplicial homology groups are defined for certain combinatorial objects called *simplicial complexes*. A simplicial complex is a combinatorial object that describes how to contstruct a space from simple building blocks generalizing points, line segments, triangles, and tetrahedra, called *simplices*. These building blocks are required to be glued in a "nice" way. Thus, simplicial homology is not as general as singular homology, but it is less abstract, and more computational. The crucial connection betweeen simplicial homology and singular homology is that the simplicial homology groups of a simplicial complex $K$ are isomorphic to the singular homology groups of the space $K_g$ built up from $K$, called its *geometric realization*.

Proving this result takes a fair amount of work and the introduction of various techniques (Mayer–Vietoris sequences, categories with models and acyclic models; see Spanier [47] Chapter 4). As a consequence, if two simplicial complexes $K$ and $K'$ have homeomorphic geometric realizations $K_k$ and $K'_g$, then the simplicial homology groups of $K$ and $K'$ are isomorphic. Thus, simplicial homology is subsumed by singular homology, but the more computational flavor of simplicial homology should not be overlooked as it provides techniques not offered by singular homology. In Chapter 6 we will present another homology theory based on spaces called CW complexes built up from spherical cells. This homology theory is also equivalent to singular homology but it is more computational.

## 5.1   Simplices and Simplicial Complexes

In this section we define simplicial complexes. A simplicial complex is a combinatorial object which describes how to build a space by putting together some basic building blocks called simplices. The building blocks are required to be "glued" nicely, which means roughly that they can only be glued along faces (a notion to be define rigoroulsy). The building blocks (simplices) are generalizations of points, line segments, triangles, tetrahedra. Simplices are very triangular in nature; in fact, they can be defined rigorously as convex hulls of affinely independent points.

To be on firm grounds we need to review some basics of affine geometry. For more comprehensive expositions the reader should consult Munkres [38] (Chapter 1, Section 1), Rotman [41] (Chapter 2), or Gallier [16] (Chapter 2). The basic idea is that an affine space is a vector space without a prescribed origin. So properties of affine spaces are invariant not only under linear maps but also under translations. When we view $\mathbb{R}^n$ as an affine space we often refer to the vectors in $\mathbb{R}^n$ as *points*.

Given $n + 1$ points, $a_0, a_1, \ldots, a_n \in \mathbb{R}^m$, these points are *affinely independent* iff the $n$ vectors, $(a_1 - a_0 \ldots, a_n - a_0)$, are linearly independent.

Note that Munkres uses the terminology *geometrically independent* instead of affinely independent.

Given any sequence of $n$ points $a_1, \ldots, a_n$ in $\mathbb{R}^m$, an *affine combination* of these points is a linear combination

$$\lambda_1 a_1 + \cdots + \lambda_n a_n,$$

with $\lambda_i \in \mathbb{R}$, and with the restriction that

$$\lambda_1 + \cdots + \lambda_n = 1. \tag{$*$}$$

Condition $(*)$ ensures that an affine combination does not depend on the choice of an origin. An affine combination is a *convex combination* if the scalars $\lambda_i$ satisfy the extra conditions $\lambda_i \geq 0$, in addition to $\lambda_1 + \cdots + \lambda_n = 1$.

A function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is *affine* if $f$ preserves affine combinations, that is,

$$f(\lambda_1 a_1 + \cdots + \lambda_p a_p) = \lambda_1 f(a_1) + \cdots + \lambda_p f(a_p),$$

for all $a_1, \ldots, a_p \in \mathbb{R}^n$ and all $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ with $\lambda_1 + \cdots + \lambda_p = 1$.

A simplex is just the convex hull of a finite number of affinely independent points, but we also need to define faces, the boundary, and the interior, of a simplex.

**Definition 5.1.** Given any $n + 1$ affinely independent points, $a_0, \ldots, a_n$ in $\mathbb{R}^m$, the *n-simplex (or simplex) $\sigma$ defined by* $a_0, \ldots, a_n$ is the convex hull of the points $a_0, \ldots, a_n$, that is, the set of all convex combinations $\lambda_0 a_0 + \cdots + \lambda_n a_n$, where $\lambda_0 + \cdots + \lambda_n = 1$, and $\lambda_i \geq 0$ for

all $i$, $0 \leq i \leq n$. The scalars $\lambda_0, \ldots, \lambda_n$ are called *barycentric coordinates*. We call $n$ the *dimension* of the $n$-simplex $\sigma$, and the points $a_0, \ldots, a_n$ are the *vertices* of $\sigma$.

Given any subset $\{a_{i_0}, \ldots, a_{i_k}\}$ of $\{a_0, \ldots, a_n\}$ (where $0 \leq k \leq n$), the $k$-simplex generated by $a_{i_0}, \ldots, a_{i_k}$ is called a *face* of $\sigma$. A face $s$ of $\sigma$ is a *proper face* if $s \neq \sigma$ (we agree that the empty set is a face of any simplex). For any vertex $a_i$, the face generated by $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ (i.e., omitting $a_i$) is called the *face opposite* $a_i$. Every face which is a $(n-1)$-simplex is called a *boundary face*.

The union of the boundary faces is the *boundary of* $\sigma$, denoted as $\partial\sigma$, and the complement of $\partial\sigma$ in $\sigma$ is the *interior* $\overset{\circ}{\sigma} = \sigma - \partial\sigma$ of $\sigma$. The interior $\overset{\circ}{\sigma}$ of $\sigma$ is sometimes called an *open simplex*.

It should be noted that for a 0-simplex consisting of a single point $\{a_0\}$, $\partial\{a_0\} = \emptyset$, and $\overset{\circ}{\{a_0\}} = \{a_0\}$. Of course, a 0-simplex is a single point, a 1-simplex is the line segment $(a_0, a_1)$, a 2-simplex is a triangle $(a_0, a_1, a_2)$ (with its interior), and a 3-simplex is a tetrahedron $(a_0, a_1, a_2, a_3)$ (with its interior), as illustrated in Figure 5.1.
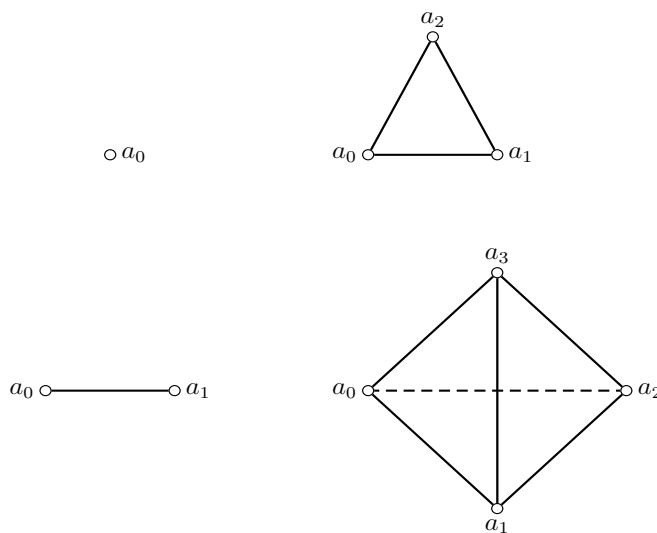


Figure 5.1: Examples of simplices.

We now state a number of properties of simplices whose proofs are left as an exercise. Clearly, a point $x$ belongs to the boundary $\partial\sigma$ of $\sigma$ iff at least one of its barycentric coordinates $(\lambda_0, \ldots, \lambda_n)$ is zero, and a point $x$ belongs to the interior $\overset{\circ}{\sigma}$ of $\sigma$ iff all of its barycentric coordinates $(\lambda_0, \ldots, \lambda_n)$ are positive, i.e., $\lambda_i > 0$ for all $i, 0 \leq i \leq n$. Then, for every $x \in \sigma$, there is a unique face $s$ such that $x \in \overset{\circ}{s}$, the face generated by those points $a_i$ for which $\lambda_i > 0$, where $(\lambda_0, \ldots, \lambda_n)$ are the barycentric coordinates of $x$.

A simplex $\sigma$ is convex, arcwise connected, compact, and closed. The interior $\overset{\circ}{\sigma}$ of a simplex is convex, arwise connected, open, and $\sigma$ is the closure of $\overset{\circ}{\sigma}$.

We now need to put simplices together to form more complex shapes. We define abstract simplicial complexes and their geometric realizations. This seems easier than defining simplicial complexes directly, as for example, in Munkres [38].

**Definition 5.2.** An *abstract simplicial complex* (for short *simplicial complex*) is a pair, $K = (V, \mathcal{S})$, consisting of a (finite or infinite) nonempty set $V$ of *vertices*, together with a family $\mathcal{S}$ of finite subsets of $V$ called *abstract simplices* (for short *simplices*), and satisfying the following conditions:

(A1) Every $x \in V$ belongs to at least one and at most a finite number of simplices in $\mathcal{S}$.

(A2) Every subset of a simplex $\sigma \in \mathcal{S}$ is also a simplex in $\mathcal{S}$.

If $\sigma \in \mathcal{S}$ is a nonempty simplex of $n + 1$ vertices, then its dimension is $n$, and it is called an *n-simplex*. A 0-simplex $\{x\}$ is identified with the vertex $x \in V$. The *dimension of an abstract complex* is the maximum dimension of its simplices if finite, and $\infty$ otherwise.

We will often use the abbreviation complex for abstract simplicial complex, and simplex for abstract simplex. Also, given a simplex $s \in \mathcal{S}$, we will often use the notation $s \in K$.

The purpose of Condition (A1) is to insure that the geometric realization of a complex is locally compact. Recall that given any set $I$, the real vector space $\mathbb{R}^{(I)}$ freely generated by $I$ is defined as the subset of the cartesian product $\mathbb{R}^I$ consisting of families $(\lambda_i)_{i \in I}$ of elements of $\mathbb{R}$ with finite support, which means that $\lambda_i = 0$ for all but finitely many indices $i \in I$ (where $\mathbb{R}^I$ denotes the set of all functions from $I$ to $\mathbb{R}$). Then every abstract complex $(V, \mathcal{S})$ has a geometric realization as a topological subspace of the normed vector space $\mathbb{R}^{(V)}$ with the norm

$$\|(\lambda_v)_{v \in V}\| = \left( \sum_{v \in V} \lambda_v^2 \right)^{1/2}.$$

Since $\lambda_v = 0$ for all but finitely many indices $v \in V$ this sum is well defined.

**Definition 5.3.** Given a simplicial complex, $K = (V, \mathcal{S})$, its *geometric realization* (also called the *polytope of* $K = (V, \mathcal{S})$) is the subspace $K_g$ of $\mathbb{R}^{(V)}$ defined as follows: $K_g$ is the set of all families $\lambda = (\lambda_a)_{a \in V}$ with finite support, such that:

(B1) $\lambda_a \geq 0$, for all $a \in V$;

(B2) The set $\{a \in V \mid \lambda_a > 0\}$ is a simplex in $\mathcal{S}$;

(B3) $\sum_{a \in V} \lambda_a = 1$.

The term *polyhedron* is sometimes used instead of polytope, and the notation $|K|$ is also used instead of $K_g$.

For every simplex $s \in \mathcal{S}$, we obtain a subset $s_g$ of $K_g$ by considering those families $\lambda = (\lambda_a)_{a \in V}$ in $K_g$ such that $\lambda_a = 0$ for all $a \notin s$. In particular, every vertex $v \in V$ is realized as the point $v_g \in K_g$ whose coordinates $(\lambda_a)_{a \in V}$ are given by $\lambda_v = 1$ and $\lambda_a = 0$ for all $a \neq v$. We sometimes abuse notation and denote $v_g$ by $v$. By (B2), we note that

$$K_g = \bigcup_{s \in \mathcal{S}} s_g.$$

It is also clear that for every $n$-simplex $s$, its geometric realization $s_g$ can be identified with an $n$-simplex in $\mathbb{R}^n$.

Figure 5.2 illustrates the definition of a complex, where $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{S} = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$. For clarity, the two triangles (2-simplices) are drawn as disjoint objects even though they share the common edge, $(v_2, v_3)$ (a 1-simplex) and similarly for the edges that meet at some common vertex.
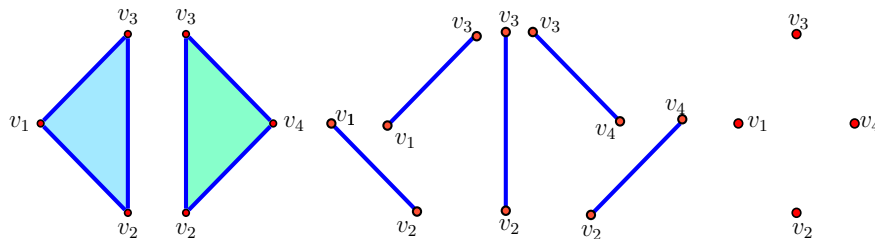


Figure 5.2: A set of simplices forming a complex.

The geometric realization of the complex from Figure 5.2 is shown in Figure 5.3.
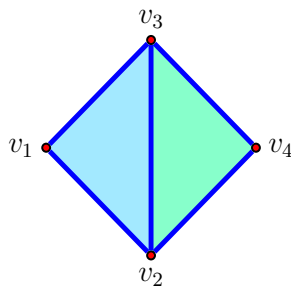


Figure 5.3: The geometric realization of the complex of Figure 5.2.
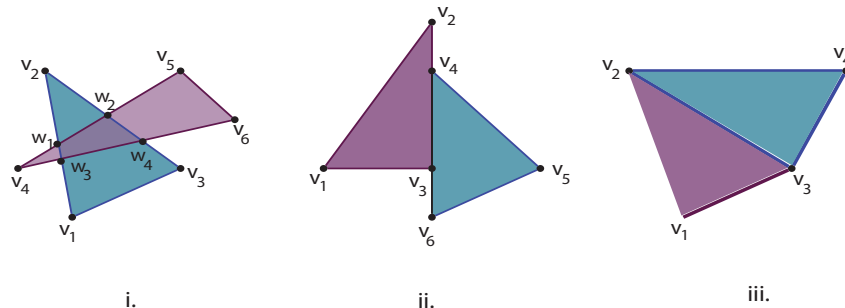
Figure 5.4: Collections of simplices not forming a complex.

Some collections of simplices violating Condition (A2) of Definition 5.2 are shown in Figure 5.4. In Figure (i), $V = \{v_1, v_2, v_3, v_4, v_5, v_6, w_1, w_2, w_3, w_4\}$ and $\mathcal{S}$ contains the two 2-simplices $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$, neither of which intersect at along an edge or at a vertex of either triangle. In other words, $\mathcal{S}$ does not contain the 2-simplex $\{w_1, w_2, w_3\}$, a violation of Condition (A2). In Figure (ii), $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\mathcal{S} = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\},$ $\{v_5\}, \{v_6\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_4, v_6\}, \{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$. Note that the two 2-simplices meet along an edge $\{v_3, v_4\}$ which is not contained in $\mathcal{S}$, another violation of Condition (A2). In Figure (iii), $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{S}$ contains the two 2-simplices $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$ but does not contain the edge $\{v_1, v_2\}$ and the vertex $v_1$.

Some geometric realizations of "legal" complexes are shown in Figure 5.5.
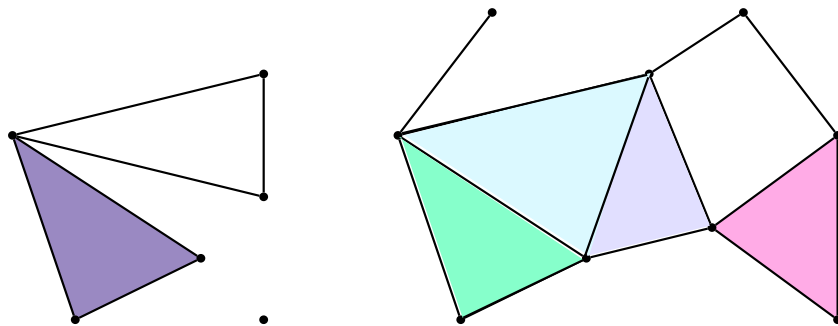


Figure 5.5: Examples of geometric realizations of complexes.

Note that distinct complexes may have the same geometric realization. In fact, all the complexes obtained by subdividing the simplices of a given complex yield the same geometric realization.

Given a vertex $a \in V$, we define the *star of $a$*, denoted as St $a$, as the finite union of the interiors $\overset{\circ}{s}_g$ of the geometric simplices $s_g$ such that $a \in s$. Clearly, $a \in$ St $a$. The *closed star of $a$*, denoted as $\overline{\text{St}}\, a$, is the finite union of the geometric simplices $s_g$ such that $a \in s$.

We define a topology on $K_g$ by defining a subset $F$ of $K_g$ to be closed if $F \cap s_g$ is closed in $s_g$ for all $s \in \mathcal{S}$. It is immediately verified that the axioms of a topological space hold.

**Definition 5.4.** A topological space $X$ is *triangulable* if it is homeomorphic to the geometric realization $K_g$ (with the above topology) of some simplicial complex $K$.

Actually, we can find a nice basis for this topology, as shown in the next proposition.

**Proposition 5.1.** *The family of subsets $U$ of $K_g$ such that $U \cap s_g = \emptyset$ for all but finitely many $s \in \mathcal{S}$, and such that $U \cap s_g$ is open in $s_g$ when $U \cap s_g \neq \emptyset$, forms a basis of open sets for the topology of $K_g$. For any $a \in V$, the star $\mathrm{St}\, a$ of $a$ is open, the closed star $\overline{\mathrm{St}}\, a$ is the closure of $\mathrm{St}\, a$ and is compact, and both $\mathrm{St}\, a$ and $\overline{\mathrm{St}}\, a$ are arcwise connected. The space $K_g$ is locally compact, locally arcwise connected, and Hausdorff.*

We also observe that for any two simplices $s_1, s_2$ of $\mathcal{S}$, we have

$$(s_1 \cap s_2)_g = (s_1)_g \cap (s_2)_g.$$

We say that a complex $K = (V, \mathcal{S})$ is connected if it is not the union of two complexes $(V_1, \mathcal{S}_1)$ and $(V_2, \mathcal{S}_2)$, where $V = V_1 \cup V_2$ with $V_1$ and $V_2$ disjoint, and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with $\mathcal{S}_1$ and $\mathcal{S}_2$ disjoint. The next proposition shows that a connected complex contains countably many simplices.

**Proposition 5.2.** *If $K = (V, \mathcal{S})$ is a connected complex, then $\mathcal{S}$ and $V$ are countable.*

Next we give several examples of simplicial complexes whose geometric realizations are classical surfaces. These complexes have additional properties that make them triangulations but we will not discuss triangulations here. Figure 5.6 shows a triangulation of the *sphere*.
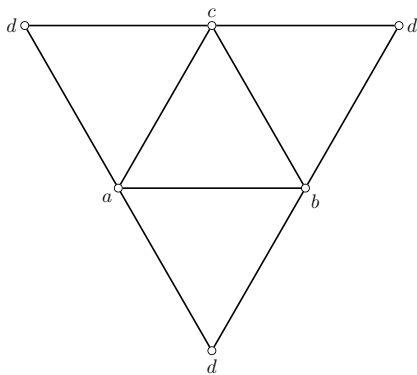


Figure 5.6: A triangulation of the sphere.

The geometric realization of the above triangulation is obtained by pasting together the pairs of edges labeled $(a, d)$, $(b, d)$, $(c, d)$. The geometric realization is a tetrahedron.
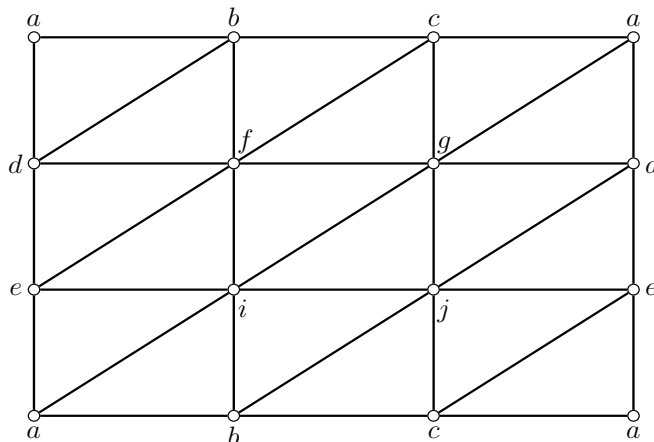
Figure 5.7: A triangulation of the torus.

Figure 5.7 shows a triangulation of a surface called a *torus*.

The geometric realization of the above triangulation is obtained by pasting together the pairs of edges labeled $(a, d)$, $(d, e)$, $(e, a)$, and the pairs of edges labeled $(a, b)$, $(b, c)$, $(c, a)$.

Figure 5.8 shows a triangulation of a surface called the *projective plane* and denoted by $\mathbb{RP}^2$.
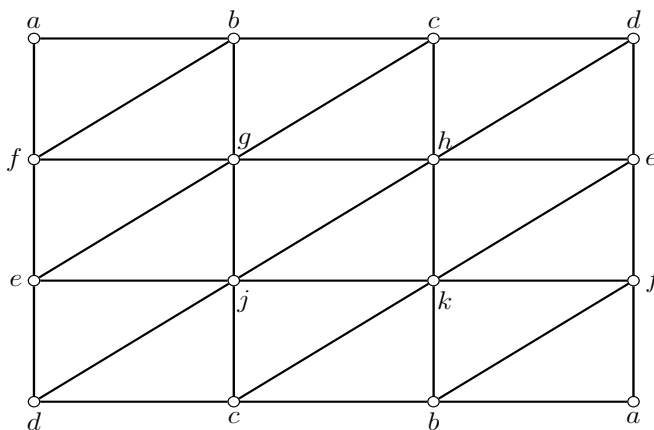


Figure 5.8: A triangulation of the projective plane.

The geometric realization of the above triangulation is obtained by pasting together the pairs of edges labeled $(a, f)$, $(f, e)$, $(e, d)$, and the pairs of edges labeled $(a, b)$, $(b, c)$, $(c, d)$. This time, the gluing requires a "twist", since the the paired edges have opposite orientation. Visualizing this surface in $\mathbb{R}^3$ is actually nontrivial.

Figure 5.9 shows a triangulation of a surface called the *Klein bottle*.
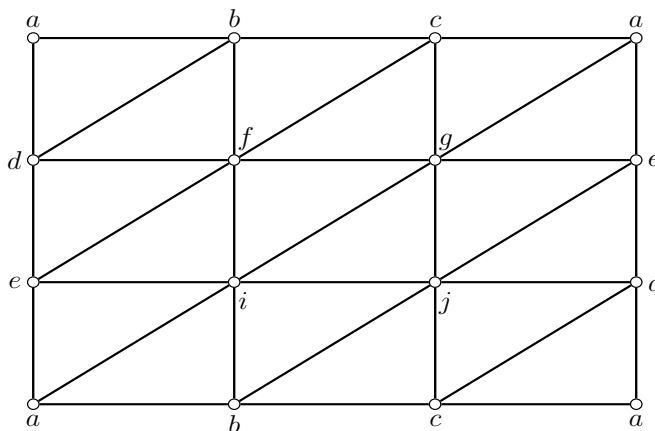
Figure 5.9: A triangulation of the Klein bottle.

The geometric realization of the above triangulation is obtained by pasting together the pairs of edges labeled $(a, d)$, $(d, e)$, $(e, a)$, and the pairs of edges labeled $(a, b)$, $(b, c)$, $(c, a)$. Again, some of the gluing requires a "twist", since some paired edges have opposite orientation. Visualizing this surface in $\mathbb{R}^3$ not too difficult, but self-intersection cannnot be avoided.

The notion of subcomplex is defined as follows.

**Definition 5.5.** Given a simplicial complex $K = (V, \mathcal{S})$, a *subcomplex* $L$ of $K$ is a simplicial complex $L = (V_L, \mathcal{S}_L)$ such that $V_L \subseteq V$ and $\mathcal{S}_L \subseteq \mathcal{S}$

Finally, the notion of map between simplicial complexes is defined as follows.

**Definition 5.6.** Given two simplicial complexes and $K_1 = (V_1, \mathcal{S}_1)$ and $K_2 = (V_2, \mathcal{S}_2)$, a *simplicial map* $f \colon K_1 \to K_2$ is a function $f \colon V_1 \to V_2$ such that whenever $\{v_1, \ldots, v_k\}$ is a simplex in $\mathcal{S}_1$, then $\{f(v_1,) \ldots, f(v_k)\}$ is simplex in $\mathcal{S}_2$. Note that the $f(v_i)$ are not necessarily distinct. If $L_1$ is a subcomplex of $K_1$ and $L_2$ is a subcomplex of $K_2$, a *simplicial map* $f \colon (K_1, L_1) \to (K_2, L_2)$ is a simplicial map $f \colon K_1 \to K_2$ which carries every simplex of $L_1$ to a simplex of $L_2$.

A simplicial map $f \colon K_1 \to K_2$ induces a continuous map $\widehat{f} \colon (K_1)_g \to (K_2)_g$, namely the function $\widehat{f}$ whose restriction to every simplex $s_g \in (K_1)_g$ is the unique affine map mapping $v_i$ to $f(v_i)$ in $(K_2)_g$, where $s = \{v_1, \ldots, v_k\} \in \mathcal{S}_1$.

## 5.2 Simplicial Homology Groups

In order to define the simplicial homology groups we need to describe how a chain complex $C_*(K)$, called a simplicial chain complex, is associated to a simplicial complex $K$. First, we assume that the ring of homology coefficients is $R = \mathbb{Z}$.

Let $K = (V, \mathcal{S})$ be a simplicial complex, for short a complex. The chain complex $C_*(K)$ associated with $K$ consists of free abelian groups $C_p(K)$ made out of oriented $p$-simplices. Every oriented $p$-simplex $\sigma$ is assigned a *boundary* $\partial_p \sigma$. Technically, this is achieved by defining homomorphisms,

$$\partial_p \colon C_p(K) \to C_{p-1}(K),$$

with the property that $\partial_{p-1} \circ \partial_p = 0$. As in the case of singular homology, if we let $Z_p(K)$ be the kernel of $\partial_p$ and

$$B_p(K) = \partial_{p+1}(C_{p+1}(K))$$

be the image of $\partial_{p+1}$ in $C_p(K)$, since $\partial_p \circ \partial_{p+1} = 0$, the group $B_p(K)$ is a subgroup of the group $Z_p(K)$, and we define the simplicial homology group $H_p(K)$ as the quotient group

$$H_p(K) = Z_p(K)/B_p(K).$$

What makes the homology groups of a complex interesting is that they only depend on the geometric realization $K_g$ of the complex $K$ and not on the various complexes representing $K_g$. We will return to this point later.

The first step is to define oriented simplices. Given a complex $K = (V, \mathcal{S})$, recall that an $n$-simplex is a subset $\sigma = \{\alpha_0, \dots, \alpha_n\}$ of $V$ that belongs to the family $\mathcal{S}$. Thus, the set $\sigma$ corresponds to $(n+1)!$ linearly ordered sequences $s \colon \{1, 2, \dots, n+1\} \to \sigma$, where each $s$ is a bijection. We define an equivalence relation on these sequences by saying that two sequences $s_1 \colon \{1, 2, \dots, n+1\} \to \sigma$ and $s_2 \colon \{1, 2, \dots, n+1\} \to \sigma$ are *equivalent* iff $\pi = s_2^{-1} \circ s_1$ is a permutation of even signature ($\pi$ is the product of an even number of transpositions).

**Definition 5.7.** The two equivalence classes associated with a simplex $\sigma$ are called *oriented simplices*, and if $\sigma = \{\alpha_0, \dots, \alpha_n\}$, we denote the equivalence class of $s$ as $[s(1), \dots, s(n+1)]$, where $s$ is one of the sequences $s \colon \{1, 2, \dots, n+1\} \to \sigma$. We also say that the two classes associated with $\sigma$ are the *orientations of $\sigma$*.

Two oriented simplices $\sigma_1$ and $\sigma_2$ are said to have *opposite orientation* if they are the two classes associated with some simplex $\sigma$. Given an oriented simplex, $\sigma$, we denote the oriented simplex having the opposite orientation by $-\sigma$, with the convention that $-(-\sigma) = \sigma$.

For example, if $\sigma = \{a_0, a_1, a_2\}$ is a 2-simplex (a triangle), there are six ordered sequences, the sequences $\langle a_2, a_1, a_0 \rangle$, $\langle a_1, a_0, a_2 \rangle$, and $\langle a_0, a_2, a_1 \rangle$, are equivalent, and the sequences $\langle a_0, a_1, a_2 \rangle$, $\langle a_1, a_2, a_0 \rangle$, and $\langle a_2, a_0, a_1 \rangle$, are also equivalent. Thus, we have the two oriented simplices, $[a_0, a_1, a_2]$ and $[a_2, a_1, a_0]$. We now define $p$-chains.

**Definition 5.8.** Given a complex, $K = (V, \mathcal{S})$, a *simplicial $p$-chain* on $K$ is a function $c$ from the set of oriented $p$-simplices to $\mathbb{Z}$, such that

(1)  $c(-\sigma) = -c(\sigma)$, iff $\sigma$ and $-\sigma$ have opposite orientation;

(2)  $c(\sigma) = 0$, for all but finitely many simplices $\sigma$.

We define addition of $p$-chains pointwise, i.e., $c_1 + c_2$ is the $p$-chain such that $(c_1 + c_2)(\sigma) = c_1(\sigma) + c_2(\sigma)$, for every oriented $p$-simplex $\sigma$. The group of simplicial $p$-chains is denoted by $C_p(K)$. If $p < 0$ or $p > \dim(K)$, we set $C_p(K) = \{0\}$.

To every oriented $p$-simplex $\sigma$ is associated an *elementary $p$-chain $c$*, defined such that

$c(\sigma) = 1$,

$c(-\sigma) = -1$, where $-\sigma$ is the opposite orientation of $\sigma$, and

$c(\sigma') = 0$, for all other oriented simplices $\sigma'$.

We will often denote the elementary $p$-chain associated with the oriented $p$-simplex $\sigma$ also by $\sigma$.

The following proposition is obvious, and simply confirms the fact that $C_p(K)$ is indeed a free abelian group.

**Proposition 5.3.** *For every complex, $K = (V, \mathcal{S})$, for every $p$, the group $C_p(K)$ is a free abelian group. For every choice of an orientation for every $p$-simplex, the corresponding elementary chains form a basis for $C_p(K)$.*

The only point worth elaborating is that except for $C_0(K)$, where no choice is involved, there is no canonical basis for $C_p(K)$ for $p \geq 1$, since different choices for the orientations of the simplices yield different bases.

If there are $m_p$ $p$-simplices in $K$, the above proposition shows that $C_p(K) = \mathbb{Z}^{m_p}$.

As an immediate consequence of Proposition 5.3, for any abelian group $G$ and any function $f$ mapping the oriented $p$-simplices of a complex $K$ to $G$ and such that $f(-\sigma) = -f(\sigma)$ for every oriented $p$-simplex $\sigma$, there is a unique homomorphism, $\widehat{f} \colon C_p(K) \to G$, extending $f$.

We now define the boundary maps $\partial_p \colon C_p(K) \to C_{p-1}(K)$.

**Definition 5.9.** Given a complex, $K = (V, \mathcal{S})$, for every oriented $p$-simplex,

$$\sigma = [\alpha_0, \ldots, \alpha_p],$$

we define the *boundary, $\partial_p \sigma$, of $\sigma$* by

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i [\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p],$$

where $[\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p]$ denotes the oriented $(p-1)$-simplex obtained by deleting vertex $\alpha_i$. The *boundary map, $\partial_p \colon C_p(K) \to C_{p-1}(K)$*, is the unique homomorphism extending $\partial_p$ on oriented $p$-simplices. For $p \leq 0$, $\partial_p$ is the null homomorphism.

One must verify that $\partial_p(-\sigma) = -\partial_p\sigma$, but this is immediate.

If $\sigma = [\alpha_0, \alpha_1]$, then

$$\partial_1\sigma = \alpha_1 - \alpha_0.$$

If $\sigma = [\alpha_0, \alpha_1, \alpha_2]$, then

$$\partial_2\sigma = [\alpha_1, \alpha_2] - [\alpha_0, \alpha_2] + [\alpha_0, \alpha_1] = [\alpha_1, \alpha_2] + [\alpha_2, \alpha_0] + [\alpha_0, \alpha_1].$$

If $\sigma = [\alpha_0, \alpha_1, \alpha_2, \alpha_3]$, then

$$\partial_3\sigma = [\alpha_1, \alpha_2, \alpha_3] - [\alpha_0, \alpha_2, \alpha_3] + [\alpha_0, \alpha_1, \alpha_3] - [\alpha_0, \alpha_1, \alpha_2].$$

If $\sigma$ is the chain

$$\sigma = [\alpha_0, \alpha_1] + [\alpha_1, \alpha_2] + [\alpha_2, \alpha_3],$$

shown in Figure 5.10 (a), then

$$\begin{aligned} \partial_1\sigma &= \partial_1[\alpha_0, \alpha_1] + \partial_1[\alpha_1, \alpha_2] + \partial_1[\alpha_2, \alpha_3] \\ &= \alpha_1 - \alpha_0 + \alpha_2 - \alpha_1 + \alpha_3 - \alpha_2 \\ &= \alpha_3 - \alpha_0. \end{aligned}$$

On the other hand, if $\sigma$ is the closed cycle,

$$\sigma = [\alpha_0, \alpha_1] + [\alpha_1, \alpha_2] + [\alpha_2, \alpha_0],$$

shown in Figure 5.10 (b), then

$$\begin{aligned} \partial_1\sigma &= \partial_1[\alpha_0, \alpha_1] + \partial_1[\alpha_1, \alpha_2] + \partial_1[\alpha_2, \alpha_0] \\ &= \alpha_1 - \alpha_0 + \alpha_2 - \alpha_1 + \alpha_0 - \alpha_2 \\ &= 0. \end{aligned}$$
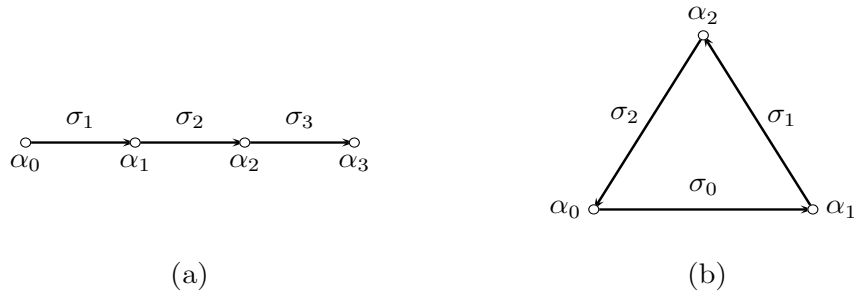


Figure 5.10: (a) A chain with boundary $\alpha_3 - \alpha_0$. (b) A chain with 0 boundary.

We have the following fundamental property:

**Proposition 5.4.** *For every complex,* $K = (V, \mathcal{S})$*, for every* $p$*, we have* $\partial_{p-1} \circ \partial_p = 0$*.*

*Proof.* For any oriented $p$-simplex, $\sigma = [\alpha_0, \ldots, \alpha_p]$, we have

$$
\begin{aligned}
\partial_{p-1} \circ \partial_p \sigma &= \sum_{i=0}^{p} (-1)^i \partial_{p-1} [\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p], \\
&= \sum_{i=0}^{p} \sum_{j=0}^{i-1} (-1)^i (-1)^j [\alpha_0, \ldots, \widehat{\alpha_j}, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p] \\
&\quad + \sum_{i=0}^{p} \sum_{j=i+1}^{p} (-1)^i (-1)^{j-1} [\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_p] \\
&= 0.
\end{aligned}
$$

The rest of the proof follows from the fact that $\partial_p \colon C_p(K) \to C_{p-1}(K)$ is the unique homomorphism extending $\partial_p$ on oriented $p$-simplices. $\qquad\square$

Proposition 5.4 shows that the family $(C_p(K))_{p \geq 0}$ together with the boundary maps $\partial_p \colon C_p(K) \to C_{p-1}(K)$ form a chain complex

$$
0 \xleftarrow{\partial_0} C_0(K) \xleftarrow{\partial_1} C_1(K) \xleftarrow{\quad} \cdots \xleftarrow{\partial_{p-1}} C_{p-1}(K) \xleftarrow{\partial_p} C_p(K) \xleftarrow{\partial_{p+1}} \cdots
$$

denoted $C_*(K)$ called the *(oriented) simplicial chain complex* associated with the complex $K$.

**Definition 5.10.** Given a complex, $K = (V, \mathcal{S})$, the kernel $\mathrm{Ker}\, \partial_p$ of the homomorphism $\partial_p \colon C_p(K) \to C_{p-1}(K)$ is denoted by $Z_p(K)$, and the elements of $Z_p(K)$ are called *p-cycles*. The image $\partial_{p+1}(C_{p+1})$ of the homomorphism $\partial_{p+1} \colon C_{p+1}(K) \to C_p(K)$ is denoted by $B_p(K)$, and the elements of $B_p(K)$ are called *p-boundaries*. The *p-th (oriented) simplicial homology group* $H_p(K)$ is the quotient group

$$
H_p(K) = Z_p(K)/B_p(K).
$$

Two $p$-chains $c, c'$ are said to be *homologous* if there is some $(p+1)$-chain $d$ such that $c = c' + \partial_{p+1} d$.

We will often omit the subscript $p$ in $\partial_p$.

As an example, consider the simplicial complex $K_1$ displayed in Figure 5.11. This complex consists of 6 vertices $\{v_1, \ldots, v_6\}$ and 8 oriented edges (1-simplices)

$$
\begin{array}{llll}
a_1 = [v_2, v_1] & a_2 = [v_1, v_4] & b_1 = [v_2, v_3] & b_2 = [v_3, v_4] \\
c_1 = [v_2, v_5] & c_2 = [v_5, v_4] & d_1 = [v_2, v_6] & d_2 = [v_6, v_4].
\end{array}
$$
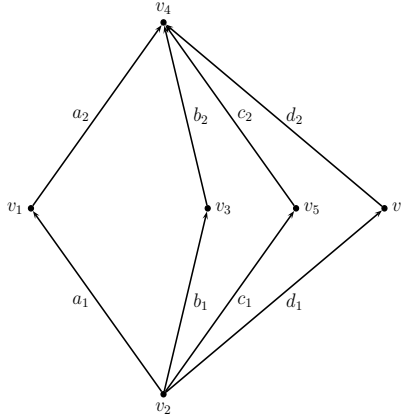
Figure 5.11: A 1-dimensional simplicial complex.

Since this complex is connected, we claim that

$$H_0(K_1) = \mathbb{Z}.$$

Indeed, given any two vertices, $u, u'$ in $K_1$, there is a path

$$\pi = [u_0, u_1], [u_1, u_2], \ldots, [u_{n-1}, u_n],$$

where each $u_i$ is a vertex in $K_1$, with $u_0 = u$ and $u_n = u'$, and we have

$$\partial_1(\pi) = u_n - u_0 = u' - u,$$

which shows that $u$ and $u'$ are equivalent. Consequently, any 0-chain $\sum n_i v_i$ is equivalent to $\left(\sum n_i\right) v_0$, which proves that

$$H_0(K_1) = \mathbb{Z}.$$

If we look at the 1-cycles in $C_1(K_1)$, we observe that they are not all independent, but it is not hard to see that the three cycles

$$a_1 + a_2 - b_1 - b_2 \qquad\qquad b_1 + b_2 - c_1 - c_2 \qquad\qquad c_1 + c_2 - d_1 - d_2$$

form a basis of $C_1(K_1)$. It follows that

$$H_1(K_1) = \operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2 = \operatorname{Ker} \partial_1 \approx \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

This reflects the fact that $K_1$ has three one-dimensional holes.

Next, consider the 2-dimensional simplicial complex $K_2$ displayed in Figure 5.12. This complex consists of 6 vertices $\{v_1, \ldots, v_6\}$, 9 oriented edges (1-simplices)

$$\begin{aligned}
&a_1 = [v_2, v_1] & &a_2 = [v_1, v_4] & &b_1 = [v_2, v_3] & &b_2 = [v_3, v_4] \\
&c_1 = [v_2, v_5] & &c_2 = [v_5, v_4] & &d_1 = [v_2, v_6] & &d_2 = [v_6, v_4] \\
&e_1 = [v_1, v_3],
\end{aligned}$$

and two oriented triangles (2-simplices)

$$A_1 = [v_2, v_1, v_3] \qquad\qquad A_2 = [v_1, v_4, v_3].$$

We have

$$\partial_2 A_1 = a_1 + e_1 - b_1 \qquad\qquad \partial_2 A_2 = a_2 - b_2 - e_1.$$

It follows that

$$\partial_2(A_1 + A_2) = a_1 + a_2 - b_1 - b_2,$$

and $A_1 + A_2$ is a diamond with boundary $a_1 + a_2 - b_1 - b_2$. Since there are no 2-cycles,
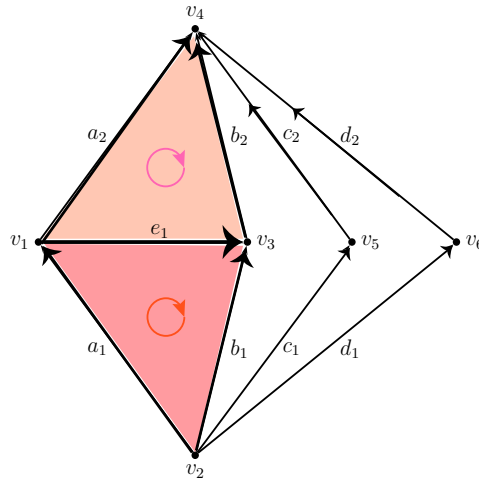


Figure 5.12: A 2-dimensional simplicial complex with a diamond.

$$H_2(K_2) = 0.$$

In order to compute

$$H_1(K_2) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2,$$

we observe that the cycles in $\operatorname{Im}\partial_2$ belong to the diamond $A_1 + A_2$, and so the only cycles in $C_1(K_2)$ whose equivalence class is nonzero must contain either $c_1 + c_2$ or $d_1 + d_2$. Then, any two cycles containing $c_1 + c_2$ (resp. $d_1 + d_2$) and passing through $A_1 + A_2$ are equivalent. For example, the cycles $a_1 + a_2 - c_1 - c_2$ and $b_1 + b_2 - c_1 - c_2$ are equivalent since their difference

$$a_1 + a_2 - c_1 - c_2 - (b_1 + b_2 - c_1 - c_2) = a_1 + a_2 - b_1 - b_2$$

is the boundary $\partial_2(A_1 + A_2)$. Similarly, the cycles $a_1 + e_1 + b_2 - c_1 - c_2$ and $a_1 + a_2 - c_1 - c_2$ are equivalent since their difference is

$$a_1 + e_1 + b_2 - c_1 - c_2 - (a_1 + a_2 - c_1 - c_2) = e_1 + b_2 - a_2 = \partial_2(-A_2).$$

Generalizing this argument, we can show that every cycle is equivalent to either a multiple of $a_1 + a_2 - c_1 - c_2$ or a multiple of $a_1 + a_2 - d_1 - d_2$, and thus

$$H_1(K_2) \approx \mathbb{Z} \oplus \mathbb{Z},$$

which reflects the fact that $K_2$ has two one-dimenensional holes. Observe that one of the three holes of the complex $K_1$ has been filled in by the diamond $A_1 + A_2$. Since $K_2$ is connected, $H_0(K_2) = \mathbb{Z}$.

Now, consider the 2-dimensional simplicial complex $K_3$ displayed in Figure 5.13. This complex consists of 8 vertices $\{v_1, \ldots, v_8\}$, 16 oriented edges (1-simplices)

$$
\begin{array}{llll}
a_1 = [v_5, v_1] & a_2 = [v_1, v_6] & b_1 = [v_5, v_3] & b_2 = [v_3, v_6] \\
c_1 = [v_5, v_7] & c_2 = [v_7, v_6] & d_1 = [v_5, v_8] & d_2 = [v_8, v_6] \\
e_1 = [v_1, v_2] & e_2 = [v_2, v_3] & f_1 = [v_1, v_4] & f_2 = [v_4, v_3] \\
g_1 = [v_5, v_2] & g_2 = [v_2, v_6] & h_1 = [v_5, v_4] & h_2 = [v_4, v_6],
\end{array}
$$

and 8 oriented triangles (2-simplices)

$$
\begin{array}{llll}
A_1 = [v_5, v_1, v_2] & A_2 = [v_5, v_2, v_3] & A_3 = [v_1, v_6, v_2] & A_4 = [v_2, v_6, v_3] \\
B_1 = [v_5, v_1, v_4] & B_2 = [v_5, v_4, v_3] & B_3 = [v_1, v_6, v_4] & B_4 = [v_4, v_6, v_3].
\end{array}
$$

It is easy to check that

$$
\begin{array}{ll}
\partial_2 A_1 = a_1 + e_1 - g_1 & \partial_2 A_2 = g_1 + e_2 - b_1 \\
\partial_2 A_3 = a_2 - g_2 - e_1 & \partial_2 A_4 = g_2 - b_2 - e_2 \\
\partial_2 B_1 = a_1 + f_1 - h_1 & \partial_2 B_2 = h_1 + f_2 - b_1 \\
\partial_2 B_3 = a_2 - h_2 - f_1 & \partial_2 B_4 = h_2 - b_2 - f_2.
\end{array}
$$

If we let

$$A = A_1 + A_2 + A_3 + A_4 \quad \text{and} \quad B = B_1 + B_2 + B_3 + B_4,$$

then we get

$$\partial_2 A = \partial_2 B = a_1 + a_2 - b_1 - b_2,$$

and thus,

$$\partial_2 (B - A) = 0.$$

Thus, $D = B - A$ is a 2-chain, and as we can see, it represents an octahedron. Observe that the chain group $C_2(K_3)$ is the eight-dimensional abelian group consisting of all linear combinations of $A_i$s and $B_j$s, and the fact that $\partial_2(B - A) = 0$ means that the kernel of the boundary map

$$\partial_2 \colon C_2(K_3) \to C_1(K_3)$$

is nontrivial. It follows that $B - A$ generates the homology group

$$H_2(K_3) = \operatorname{Ker} \partial_2 \approx \mathbb{Z}.$$
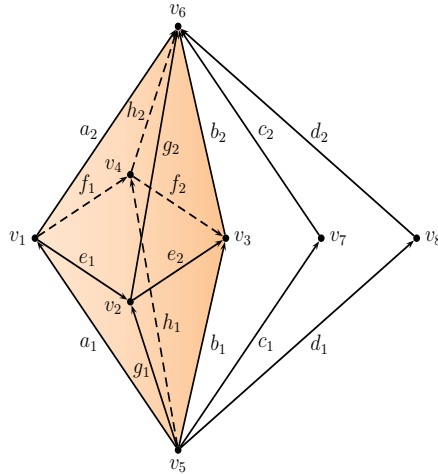
Figure 5.13: A 2-dimensional simplicial complex with an octahedron.

This reflects the fact that $K_3$ has a single two-dimenensional hole. The reader should check that as before,

$$H_1(K_3) = \operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2 \approx \mathbb{Z} \oplus \mathbb{Z}.$$

Intuitively, this is because every cycle outside of the ocahedron $D$ must contain either $c_1 + c_2$ or $d_1 + d_2$, and the "rest" of the cycle belongs to $D$. It follows that any two distinct cycles involving $c_1 + c_2$ (resp. $d_1 + d_2$) can be deformed into each other by "sliding" over $D$. The complex $K_3$ also has two one-dimensional holes. Since $K_3$ is connected, $H_0(K_3) = \mathbb{Z}$.

Finally, consider the 3-dimensional simplicial complex $K_4$ displayed in Figure 5.14 obtained from $K_3$ by adding the oriented edge

$$k = [v_2, v_4]$$

and the four oriented tetrahedra (3-simplices)

$$\begin{aligned} T_1 &= [v_1, v_2, v_4, v_6] & T_2 &= [v_3, v_4, v_2, v_6] \\ T_3 &= [v_1, v_4, v_2, v_5] & T_4 &= [v_3, v_2, v_4, v_5]. \end{aligned}$$

We get

$$\begin{aligned} \partial_3 T_1 &= [v_2, v_4, v_6] - [v_1, v_4, v_6] + [v_1, v_2, v_6] - [v_1, v_2, v_4] \\ \partial_3 T_2 &= [v_4, v_2, v_6] - [v_3, v_2, v_6] + [v_3, v_4, v_6] - [v_3, v_4, v_2] \\ \partial_3 T_3 &= [v_4, v_2, v_5] - [v_1, v_2, v_5] + [v_1, v_4, v_5] - [v_1, v_4, v_2] \\ \partial_3 4_3 &= [v_2, v_4, v_5] - [v_3, v_4, v_5] + [v_3, v_2, v_5] - [v_3, v_2, v_4]. \end{aligned}$$
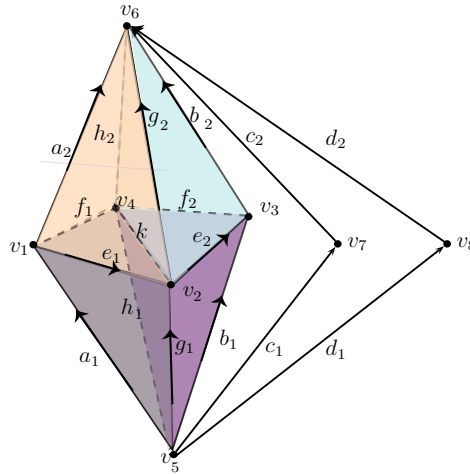
Figure 5.14: A 3-dimensional simplicial complex with a solid octahedron.

Observe that

$$
\begin{aligned}
\partial(T_1 + T_2 + T_3 + T_4) &= -[v_1, v_4, v_6] + [v_1, v_2, v_6] - [v_3, v_2, v_6] + [v_3, v_4, v_6] \\
&\quad - [v_1, v_2, v_5] + [v_1, v_4, v_5] - [v_3, v_4, v_5] + [v_3, v_2, v_5] \\
&= B_3 - A_3 - A_4 + B_4 - A_1 + B_1 + B_2 - A_2 \\
&= B_1 + B_2 + B_3 + B_4 - (A_1 + A_2 + A_3 + A_4) \\
&= B - A.
\end{aligned}
$$

It follows that

$$
\partial_3 \colon C_3(K_4) \to C_2(K_4)
$$

maps the solid octahedron $T = T_1 + T_2 + T_3 + T_4$ to $B - A$, and since $\mathrm{Ker}\,\partial_2$ is generated by $B - A$, we get

$$
H_2(K_4) = \mathrm{Ker}\,\partial_2/\mathrm{Im}\,\partial_3 = 0.
$$

We also have

$$
H_3(K_4) = \mathrm{Ker}\,\partial_3/\mathrm{Im}\,\partial_3 = \mathrm{Ker}\,\partial_3 = 0,
$$

and as before,

$$
H_0(K_4) = \mathbb{Z} \quad \text{and} \quad H_1(K_4) = \mathbb{Z} \oplus \mathbb{Z}
$$

The complex $K_4$ still has two one-dimensional holes but the two-dimensional hole of $K_3$ has been filled up by the solid octahedron.

For another example of a 2-dimensional simplicial complex with a hole, consider the complex $K_5$ shown in Figure 5.15. This complex consists of 16 vertices, 32 edges (1-simplicies) oriented as shown in the Figure, and 16 triangles (2-simplicies) oriented according to the direction of their boundary edges. The boundary of $K_5$ is

$$
\partial_2(K_5) = a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + c_1 + c_2 + c_3 + d_1 + d_2 + d_3 + e + f + g + h.
$$
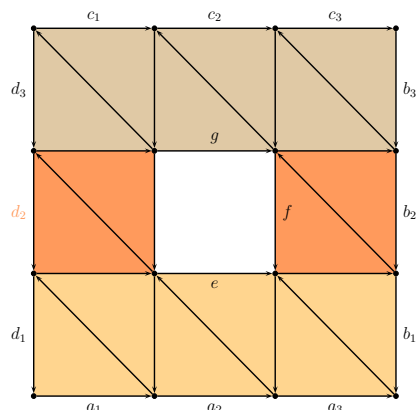
Figure 5.15: A 2-dimensional simplicial complex with a hole.

As a consequence, the outer boundary $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + c_1 + c_2 + c_3 + d_1 + d_2 + d_3$ is equivalent to the inner boundary $-(e + f + g + h)$. It follows that all cycles in $C_2(K_5)$ not equivalent to zero are equivalent to a multiple of $e + f + g + h$, and thus

$$H_1(K_5) = \mathbb{Z},$$

indicating that $K_5$ has a single one-dimensional hole. Since $K_5$ is connected, $H_0(K_5) = \mathbb{Z}$, and $H_2(K_5) = 0$ since $\operatorname{Ker} \partial_2 = 0$.

If $K = (V, \mathcal{S})$ is a finite dimensional complex, as each group $C_p(K)$ is free and finitely generated, the homology groups $H_p(K)$ are all finitely generated.

As we said in the introduction, the simplicial homology groups have a computational flavor, and this is one of the main reasons why they are attractive and useful. In fact, if $K$ is any finite simplicial complex, there is an algorithm for computing the simplicial homology groups of $K$. This algorithm relies on a matrix reduction method (The Smith Normal Form) involving some simple row operations reminiscent of row-echelon reduction. This algorithm is described in detail in Munkres [38] (Chapter 1, Section 11) and Rotman [41] (Chapter 7).

The generalization of simplicial homology to coefficients in any $R$-module $G$ is immediate, where $R$ is any commutative ring with an identity element. Simply define the chain group $C_p(K; G)$ as the $R$-module of functions $c$ from the set of oriented $p$-simplices to $G$, such that

(1) $c(-\sigma) = -c(\sigma)$, iff $\sigma$ and $-\sigma$ have opposite orientation;

(2) $c(\sigma) = 0$, for all but finitely many simplices $\sigma$.

A $p$-chain in $C_p(K; G)$ is a "vector-valued" formal finite linear combination

$$\sum_i \sigma_i g_i,$$

with $g_i \in G$ and $\sigma_i$ an oriented $p$-simplex. Equivalently we can define the complex $C_*(K;G)$ as the complex $C_*(K) \otimes_R G$. When $G = R$, each module $C_p(K;R)$ is a free module.

Then we have the simplicial chain complex $C_*(K;G)$ and the corresponding simplicial homology groups $H_p(K;G)$.

Given two simplicial complexes $K_1$ and $K_2$, a simplicial map $f\colon K_1 \to K_2$ induces a homomorphism $f_{\sharp,p}\colon C_p(K_1;G) \to C_p(K_2;G)$ between the modules of oriented $p$-chains defined as follows: For any $p$-simplex $\{v_0, \ldots, v_p\}$ in $K_1$, we set

$$f_\sharp([v_0, \ldots, v_p]) = \begin{cases} [f(v_0), \ldots, f(v_p)] & \text{if the } f(v_i) \text{ are pairwise distinct} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the $f_{\sharp,p}$ commute with the boundary maps, so $f_\sharp = (f_{\sharp,p})_{p \geq 0}$ is a chain map between the chain complexes $C_*(K_1;G)$ and $C_*(K_2;G)$ which induces homomorphisms

$$f_{*,p}\colon H_p(K_1;G) \to H_p(K_2;G) \quad \text{for all } p \geq 0.$$

This assignment is functorial; see Munkres [38] (Chapter I, Section 12).

The relative simplicial homology groups are also easily defined (by analogy with relative singular homology). Given a complex $K$ and a subcomplex $L$ of $K$, we define the *relative simplicial chain complex* $C_*(K,L;R)$ by

$$C_p(K,L;R) = C_p(K;G)/C_p(L;R).$$

As in the case of singular homology, $C_p(K,L;R)$ is a free $R$-module, because it has a basis consisting of the cosets of the form

$$\sigma + C_p(L;R),$$

where $\sigma$ is an oriented $p$-simplex of $K$ that is *not* in $L$. We obtain the *relative simplicial homology groups* $H_p(K,L;R)$. We define the chain complex $C_p(K,L;G)$ as $C_p(K,L;R) \otimes_R G$, and we obtain *relative simplicial homology groups* $H_p(K,L;G)$ with coefficients in $G$.

Given two pairs of simplicial complexes $(K_1, L_1)$ and $(K_2, L_2)$, where $L_1$ is a subcomplex of $K_1$ and $L_2$ is a subcomplex of $K_2$, as in the absolute case a simplicial map $f\colon (K_1, L_1) \to (K_2, L_2)$ induces a homomorphism $f_{\sharp,p}\colon C_p(K_1, L_1;G) \to C_p(K_2, L_2;G)$ between the modules of oriented $p$-chains, and thus homomorphisms

$$f_{*,p}\colon H_p(K_1, L_1;G) \to H_p(K_2, L_2;G) \quad \text{for all } p \geq 0.$$

Again, this assignment is functorial.

One can prove a version of the excision axiom for relative simplicial homology see Munkres [38] (Chapter I, Section 9). One can also prove a version of the homotopy axiom (see Munkres [38] (Chapter II), and that we have a long exact sequence of homology of a pair $(K, L)$; see Munkres [38] (Chapter III, Section 23).

Simplicial homology assigns homology groups to a simplicial complex $K$, not to a topological space. We can view the groups $H_p(K)$ as groups assigned to the geometric realization $K_g$ of $K$, which is a space. Let us temporarily denote these groups by $H_p^\Delta(K_g)$. Now the following question arises:

If $K$ and $K'$ are two simplicial complexes whose geometric realizations $K_g$ and $K'_g$ are homeomorphic, are the groups $H_p^\Delta(K_g)$ and $H_p^\Delta(K'_g)$ isomorphic, that is, are the groups $H_p(K)$ and $H_p(K')$ isomorphic?

If the answer to this question was no, then the simplicial homology groups would not be useful objects for classifying spaces up to homeomorphism, but fortunately the answer is yes. However, the proof of this fact is quite involved. This can be proved directly as in Munkres [38] (Chapter II), or by proving that the simplicial homology group $H_p(K)$ is isomorphic to the singular homology group $H_p(K_g)$ of the geometric realization of $K$. The proof of this isomorphism also requires a lot of work.

In order to prove the equivalence of simplicial homology with singular homology we introduce a variant of the simplicial homology groups called *ordered simplicial homology groups*.

**Definition 5.11.** Let $K = (V, \mathcal{S})$ be a simplicial complex. An *ordered p-simplex* of $K$ is a $(p+1)$-tuple $(v_0, \ldots, v_p)$ of vertices in $V$, where the $v_i$ are vertices of some simplex $\sigma$ of $K$ *but need not be distinct*.

For example, if $\{v, w\}$ is a 1-simplex, then $(v, w, w, v)$ is an ordered 3-simplex.

Let $C'_p(K; R)$ be the free $R$-module generated by the ordered $p$-simplices, called the group of *ordered p-chains*, and define the boundary map $\partial'_p \colon C'_p(K; R) \to C'_{p-1}(K; R)$ by

$$\partial'_p(v_0, \ldots, v_p) = \sum_{i=0}^{p} (-1)^i (v_0, \ldots, \widehat{v_i}, \ldots, v_p),$$

where $(v_0, \ldots, \widehat{v_i}, \ldots, v_p)$ denotes the ordered $(p-1)$-simplex obtained by deleting vertex $v_i$.

It is easily checked that $\partial'_p \circ \partial'_{p+1} = 0$, so we obtain a chain complex $C'_*(K; R)$ called the *ordered simplicial chain complex* of $K$. This is a huge and redundant complex, but it is useful to prove the equivalence of simplicial homology and singular homology.

Given a simplicial complex $K$ and a subcomplex $L$, the *relative ordered simplicial chain complex* $C'_*(K, L; R)$ of $(K, L)$ is defined by

$$C'_*(K, L; R) = C'_*(K; R)/C'_*(L; R).$$

We obtain the *ordered relative simplicial homology groups* $H'_p(K, L; R)$.

Theorem 5.5 below is proved in Munkres [38] (Chapter I, Section 13, Theorem 13.6) and in Spanier [47] (Chapter 4, Section 3, Theorem 8, and Section 5, Corollary 12). The proof

uses a techniques known as "categories with models" and "acyclic models." These results are proved for $R = \mathbb{Z}$, but because the oriented chain modules $C_p(K, L; R)$ and the ordered chain modules $C'_p(K, L; R)$ are free $R$-modules, it can be checked that the constructions and the proofs go through for any commutative ring with an identity element 1.

Assuming for simplicity that $L = \emptyset$, the idea is to define two chain maps $\varphi\colon C_p(K; R) \to C'_p(K; R)$ and $\psi\colon C'_p(K; R) \to C_p(K; R)$ that are chain homotopy inverses. To achieve this, pick a partial order $\leq$ of the vertices of $K = (V, \mathcal{S})$ that induces a total order on the vertices of every simplex in $\mathcal{S}$, and define $\varphi$ by

$$\varphi([v_0, \ldots, v_p]) = (v_0, \ldots, v_p) \quad \text{if } v_0 < v_1 < \cdots < v_p,$$

and $\psi$ by

$$\psi((w_0, \ldots, w_p)) = \begin{cases} [w_0, \ldots, w_p] & \text{if the } w_i \text{ are pairwise distinct} \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be shown that $\varphi$ and $\psi$ are natural transformations (with respect to simplicial maps) and that they are chain homotopy inverses. The maps $\varphi$ and $\psi$ can also be defined for pairs of complexes $(K, L)$, as chain maps $\varphi\colon C_p(K, L; R) \to C'_p(K, L; R)$ and $\psi\colon C'_p(K, L; R) \to C_p(K, L; R)$ which are chain homotopic.

**Theorem 5.5.** *For any simplicial complex $K$ and any subcomplex $L$ of $K$, there are (natural) isomorphims*

$$H_p(K, L; R) \cong H'_p(K, L; R) \quad \text{for all } p \geq 0$$

*between the relative simplicial homology groups and the ordered relative simplicial homology groups.*

Theorem 5.5 follows from the special case of the theorem in which $L = \emptyset$ by the five lemma (Proposition 2.23). This is a common trick in the subject which is used over and over again (see the proof of Theorem 5.6).

By naturality of the long exact sequence of homology of the pair $(K, L)$, the chain map $\varphi\colon C_*(K, L; R) \to C'_*(K, L; R)$ yields the following commutative diagram:

$$\cdots \longrightarrow H_p(L; R) \longrightarrow H_p(K; R) \longrightarrow H_p(K, L; R) \longrightarrow H_{p-1}(L; R) \longrightarrow H_{p-1}(K; R) \longrightarrow \cdots$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\cdots \longrightarrow H'_p(L; R) \longrightarrow H'_p(K; R) \longrightarrow H'_p(K, L; R) \longrightarrow H'_{p-1}(L; R) \longrightarrow H'_{p-1}(K; R) \longrightarrow \cdots$$

in which the horizontal rows are exact. If we assume that the isomorphisms of the theorem hold in the absolute case, then all vertical arrows except the middle one are isomorphisms, and by the five lemma (Proposition 2.23), the middle arrow is also an isomorphism.

The proof that the simplicial homology group $H_p(K; \mathbb{Z})$ is isomorphic to the singular homology group $H_p(K_g; \mathbb{Z})$ is nontrivial. Proofs can be found in Munkres [38] (Chapter 4,

Section 34), Spanier [47] (Chapter 4, Sections 4 and 6), Hatcher [25] (Chapter II, Section 2.1), and Rotman [41] (Chapter 7). These proofs use variants of acyclic models, Mayer–Vietoris sequences, and the five lemma.

Given a simplicial complex $K$, the idea is to define a chain map $\theta\colon C'_*(K;\mathbb{Z}) \to S_*(K_g;\mathbb{Z})$ that induces isomorphisms $\theta_{*,p}\colon H'_p(K;\mathbb{Z}) \to H_p(K_g;\mathbb{Z})$ for all $p \geq 0$. This can be done as follows: let $\ell(e_1,\ldots,e_{p+1})$ be the unique affine map from $\Delta^p$ to $K_g$ such that $\ell(e_{i+1}) = (v_i)_g$ for $i = 0,\ldots,p$. Then let

$$\theta((v_0,\ldots,v_p)) = \ell(e_1,\ldots,e_{p+1}).$$

It is also easy to define $\theta\colon C'_*(K,L;\mathbb{Z}) \to S_*(K_g,L_g;\mathbb{Z})$ for pairs of complexes $(K,L)$ with $L$ a subcomplex of $K$. Then we define the chain map $\eta\colon C_*(K,L;\mathbb{Z}) \to S_*(K_g,L_g;\mathbb{Z})$ as the composition $\eta = \theta \circ \varphi$, where $\varphi\colon C_*(K,L;\mathbb{Z}) \to C'_*(K,L;\mathbb{Z})$ is the chain map between oriented and ordered homology discussed earlier. The following important theorem shows that $\eta$ induces an isomorphism between simplicial homology and singular homology.

**Theorem 5.6.** *Given any pair of simplicial complexes* $(K,L)$*, where* $L$ *is a subcomplex of* $K$*, the chain map* $\eta\colon C_*(K,L;\mathbb{Z}) \to S_*(K_g,L_g;\mathbb{Z})$ *induces isomorphisms*

$$H_p(K,L;\mathbb{Z}) \cong H_p(K_g,L_g;\mathbb{Z}) \quad \text{for all } p \geq 0.$$

*Proof sketch.* By Theorem 5.5 it suffices to prove that the homology groups $H'_p(K,L;\mathbb{Z})$ and the singular homology groups $H_p(K_k,L_g;\mathbb{Z})$ are isomorphic. Again, we use the trick which consists in showing that Theorem 5.6 follows from the special case of the theorem in which $L = \emptyset$ by the five lemma (Proposition 2.23). Indeed, by naturality of the long exact sequence of homology of the pair $(K,L)$, the chain map $\theta\colon C'_*(K,L;\mathbb{Z}) \to S_*(K_g,L_g;\mathbb{Z})$ yields the following commutative diagram

$$\cdots \longrightarrow H'_p(L;\mathbb{Z}) \longrightarrow H'_p(K;\mathbb{Z}) \longrightarrow H'_p(K,L;\mathbb{Z}) \longrightarrow H'_{p-1}(L;\mathbb{Z}) \longrightarrow H'_{p-1}(K;\mathbb{Z}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_p(L_g;\mathbb{Z}) \longrightarrow H_p(K_g;\mathbb{Z}) \longrightarrow H_p(K_g,L_g;\mathbb{Z}) \longrightarrow H_{p-1}(L_g;\mathbb{Z}) \longrightarrow H_{p-1}(K_g;\mathbb{Z}) \longrightarrow \cdots$$

in which the horizontal rows are exact. If we assume that the isomorphisms of the theorem hold in the absolute case, then all vertical arrows except the middle one are isomorphisms, and by the five lemma (Proposition 2.23), the middle arrow is also an isomorphism.

The proof of the isomorphims $H'_p(K;\mathbb{Z}) \cong H_p(K_g;\mathbb{Z})$ proceed in two steps. We follow Spanier's proof Spanier [47] (Theorem 8, Chapter 4, Section 6). Rotman's proof is nearly the same; see Rotman [41] (Chapter 7), but beware that there appears to be some typos at the bottom of page 151.

*Step 1*. We prove our result for a finite simplicial complex $K$ by induction on the number $n$ of simplices on $K$.

*Base case, $n = 1$*. For any abstract simplex $s$, let $\overline{s}$ be the simplicial complex consisting of all the faces of $s$ (including $s$ itself). The following result will be needed.

**Proposition 5.7.** *Given any abstact simplex s, there are isomorphisms*

$$H'_p(\overline{s}; \mathbb{Z}) \cong H_p(\overline{s}_g; \mathbb{Z}) \quad \text{for all } p \geq 0.$$

Proposition 5.7 is Corollary 4.4.2 in Spanier [47] (Chapter 4, Section 4). Intuitively, Proposition 5.7 is kind of obvious, since $\overline{s}$ corresponds to the combinatorial decomposition of a simplex, and $\overline{s}_g$ is a convex body homeomorphic to some ball $D^m$. Their corresponding homology should be (0) for $p > 0$ and $\mathbb{Z}$ for $p = 0$.

A rigorous proof of Proposition 5.7 uses the following results:

(1) We have the following isomorphisms between unreduced and reduced homology:

$$H'_0(K; \mathbb{Z}) \cong \widetilde{H'_0}(K; \mathbb{Z}) \oplus \mathbb{Z}$$
$$H'_p(K; \mathbb{Z}) \cong \widetilde{H'_p}(K; \mathbb{Z}) \quad p \geq 1$$

in ordered homology, and

$$H_0(K_g; \mathbb{Z}) \cong \widetilde{H}_0(K_g; \mathbb{Z}) \oplus \mathbb{Z}$$
$$H_p(K_g; \mathbb{Z}) \cong \widetilde{H}_p(K_g; \mathbb{Z}) \quad p \geq 1$$

in singular homology. This is Lemma 4.3.1 in Spanier [47] (Chapter 4, Section 3).

(2) For any abstract simplex $s$, the reduced chain complex of ordered homology of $\overline{s}$ is acyclic; that is,

$$\widetilde{H'_p}(\overline{s}; \mathbb{Z}) = (0) \quad \text{for all } p \geq 0.$$

This is Corollary 4.3.7 in Spanier [47] (Chapter 4, Section 3). A more direct proof of the second fact (oriented simplicial homology) is given in Rotman [41] (Chapter 7, Corollary 7.18). It is easily adapted to ordered homology.

(3) A chain complex $C$ is said to be *contractible* if there is a chain homotopy between the identity chain map $\text{id}_C$ of $C$ and the zero chain map $0_C$ of $C$. Then a contractible chain complex is acyclic; that is, $H_p(C) = (0)$ for all $p \geq 0$. This is Corollary 4.2.3 in Spanier [47] (Chapter 4, Section 2).

(4) Let $X$ be any star-shaped subset of $\mathbb{R}^n$. Then the reduced singular complex of $X$ is chain contractible. This is Lemma 4.4.1 in Spanier [47] (Chapter 4, Section 4).

*Induction step, $n > 1$.* We will need the following facts:

(1) The Mayer–Vietoris sequence holds in ordered homology. This is not hard to prove; see Spanier [47] (Chapter 4, Section 6).

(2) The Mayer–Vietoris sequence holds in reduced singular homology; this is Theorem 4.14.

(3) If $K_1$ and $K_2$ are are subcomplexes of a simplicial complex $K$, then the Mayer–Vietoris sequence of singular homology holds for $(K_1)_g$ and $(K_2)_g$. This is Lemma 4.6.7 in Spanier [47] (Chapter 4, Section 6). Actually, the above result is only needed in the following situation: if $s$ is any simplex of $K$ of highest dimension, then $K_1 = K - \{s\}$ and $K_2 = \overline{s}$; this is Lemma 7.20 in Rotman [41] (Chapter 7). Since a Mayer–Vietoris sequence arises from a long exact sequence of homology, the chain map $\theta \colon C'_*(K, L; \mathbb{Z}) \to S_*(K_g, L_g; \mathbb{Z})$ induces a commutative diagram in which the top and bottom arrows are Mayer–Vietoris sequences and the vertical maps are induced by $\theta$; see below.

Assume inductively that our result holds for any simplicial complex with less than $n > 1$ simplices. Pick any simplex $s$ of maximal dimension, and let $K_1 = K - \{s\}$ and $K_2 = \overline{s}$, so that $K = K_1 \cup K_2$. Since $n > 1$ and $s$ has maximal dimension, both $K_1$ and $K_1 \cap K_2$ are complexes (Condition (A2) is satisfied) and have less than $n$ simplices, so by the induction hypothesis

$$H'_p(K_1; \mathbb{Z}) \cong H_p((K_1)_g; \mathbb{Z}) \quad \text{for all } p \geq 0$$

and

$$H'_p(K_1 \cap K_2; \mathbb{Z}) \cong H_p((K_1 \cap K_2)_g; \mathbb{Z}) \quad \text{for all } p \geq 0.$$

By Proposition 5.7 we also have

$$H'_p(K_2; R) = H'_p(\overline{s}; \mathbb{Z}) \cong H_p(\overline{s}_g; \mathbb{Z}) = H_p((K_2)_g; R) \quad \text{for all } p \geq 0.$$

Now Fact (3) (of the induction step) implies that we have the following diagram in which the horizontal rows are exact Mayer–Vietoris sequences (for a more direct argument, see Rotman [41] (Chapter 7, Proposition 7.21)), and where we have suppressed the ring $\mathbb{Z}$ to simplify notation.

$$\begin{array}{ccccccccc}
H'_p(K_1 \cap \overline{s}) & \longrightarrow & H'_p(K_1) \oplus H'_p(\overline{s}) & \longrightarrow & H'_p(K) & \longrightarrow & H'_{p-1}(K_1 \cap \overline{s}) & \longrightarrow & H'_{p-1}(K_1) \oplus H'_{p-1}(\overline{s}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_p((K_1 \cap \overline{s})_g) & \rightarrow & H_p((K_1)_g) \oplus H_p(\overline{s}_g) & \rightarrow & H_p(K_g) & \rightarrow & H_{p-1}((K_1 \cap \overline{s})_g) & \rightarrow & H_{p-1}((K_1)_g) \oplus H_{p-1}(\overline{s}_g)
\end{array}$$

Since all horizontal arrows except the middle one are isomorphisms, by the five lemma (Proposition 2.23) the middle vertical arrow is also an isomorphism, which establishes the induction hypothesis. Therefore, we proved Theorem 5.6 for finite simplicial complexes.

*Step 2*. We prove our result for an infinite simplicial complex $K$. We resort to a direct limit argument. Let $(K_\alpha)$ be the family of finite subcomplexes of $K$ under the inclusion ordering. It is a directed family. A version of this argument is given in Munkres [38] (Chapter 4, Section 34, Lemma 44.2). Spanier proves that

$$H'_p(K; \mathbb{Z}) \cong \varinjlim H'_p(K_\alpha; \mathbb{Z})$$

and that

$$H_p(K_g; \mathbb{Z}) \cong \varinjlim H_p((K_\alpha)_g; \mathbb{Z}).$$

The first result is Theorem 4.3.11 in Spanier [47] (Chapter 4, Section 3). This is an immediate consequence of the fact that homology commutes with direct limits; see Spanier [47] (Theorem 4.1.7, Chapter 4, Section 1). The second result is the axiom of compact support for singular homology (Theorem 4.15). This completes the proof.    $\square$

Theorem 5.6 proves the claim we made earlier that any two complexes $K$ and $K'$ that have homeomorphic geometric realizations have isomorphic simplicial homology groups, a result first proved by Alexander and Veblen.

The proofs of Theorem 5.6 found in the references cited earlier all assume that the ring of coefficients is $R = \mathbb{Z}$. However, close examination of Spanier's proof shows that the only result that makes use of the fact that $R = \mathbb{Z}$ is Proposition 5.7. If Proposition 5.7 holds for any commutative ring $R$ with an identity element, then so does the theorem.

Fact (1) of Step 1 holds for any ring, in fact for any $R$-module $G$.

Fact (2) of Step 1 is a corollary of Theorem 4.3.6, which itself depends on Lemma 4.3.2; see Spanier [47] (Chapter 4, Section 3). One needs to find right inverses to the augmentation maps $\epsilon \colon C_0'(K; R) \to R$ and $\epsilon \colon C_0'(K * w; R) \to R$, where $K * w$ is the cone with base $K$ and vertex $w$; see Spanier [47] (Chapter 3, Section 2). This is essentially the argument we gave in Section 4.2 just after Definition 4.7.

Actually, this argument can be generalized to any $R$-module $G$, as explained in Section 4.5 just after Definition 4.12, so we have the following generalization of Proposition 5.7: For any abstract simplex $s$ and any $R$-module $G$, we have

$$H_p'(\overline{s}; G) \cong H_p(\overline{s}_g; G) \quad \text{for all } p \geq 0.$$

By tensoring with $G$, the chain map $\theta$ yields a chain map (also denoted $\theta$) $\theta \colon C_*'(K, L; G) \to S_*(K_g, L_g; G)$. The chain map $\varphi \colon C_*(K, L; R) \to C_*'(K, L; R)$ can also be generalized to a chain map (also denoted $\varphi$) $\varphi \colon C_*(K, L; G) \to C_*'(K, L; G)$ by tensoring with $G$. We define $\varphi \colon C_*(K, L; G) \to S_*(K_g, L_g; G)$ as $\eta = \theta \circ \varphi$. Then we obtain a more general version of the isomorphism between simplicial homology and singular homology.

**Theorem 5.8.** *For any commutative ring $R$ with an identity element $1$ and for any $R$-module $G$, given any pair of simplicial complexes $(K, L)$, where $L$ is a subcomplex of $K$, the chain map $\eta \colon C_*(K, L; G) \to S_*(K_g, L_g; G)$ induces isomorphisms*

$$H_p(K, L; G) \cong H_p(K_g, L_g; G) \quad \text{for all } p \geq 0.$$

In summary, singular homology subsumes simplicial homology. Still, simplicial homology is much more computational.

## 5.3 The Euler–Poincaré Characteristic of a Simplicial Complex

In this section we assume that we are considering simplicial homology groups with coefficients in $\mathbb{Z}$. A fundamental invariant of finite complexes is the Euler–Poincaré characteristic. We saw earlier that the simplicial homology groups of a finite simplicial complex $K$ are finitely generated abelian groups. We can assign a number $\chi(K)$ to $K$ by making use of the fact that the structure of finitely generated abelian groups can be completely described. It turns out that every finitely generated abelian group can be expressed as the sum of the special abelian groups $\mathbb{Z}^r$ and $\mathbb{Z}/m\mathbb{Z}$. The crucial result is the following.

**Proposition 5.9.** *Let $G$ be a free abelian group finitely generated by $(a_1, \ldots, a_n)$ and let $H$ be any subgroup of $G$. Then $H$ is a free abelian group and there is a basis, $(e_1, ..., e_n)$, of $G$, some $q \leq n$, and some positive natural numbers, $n_1, \ldots, n_q$, such that $(n_1 e_1, \ldots, n_q e_q)$ is a basis of $H$ and $n_i$ divides $n_{i+1}$ for all $i$, with $1 \leq i \leq q - 1$.*

A neat proof of Proposition 5.9 can be found in Samuel [42]; see also Dummit and Foote [11] (Chapter 12, Theorem 4).

**Remark:** Actually, Proposition 5.9 is a special case of the structure theorem for finitely generated modules over a principal ring. Recall that $\mathbb{Z}$ is a principal ring, which means that every ideal $\mathcal{I}$ in $\mathbb{Z}$ is of the form $d\mathbb{Z}$, for some $d \in \mathbb{N}$.

We abbreviate the direct sum $\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m}$ of $m$ copies of $\mathbb{Z}$ as $\mathbb{Z}^m$. Using Proposition 5.9, we can also show the following useful result:

**Theorem 5.10.** *(Structure theorem for finitely generated abelian groups) Let $G$ be a finitely generated abelian group. There is some natural number, $m \geq 0$, and some natural numbers $n_1, \ldots, n_q \geq 2$, such that $H$ is isomorphic to the direct sum*

$$\mathbb{Z}^m \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_q\mathbb{Z},$$

*and where $n_i$ divides $n_{i+1}$ for all $i$, with $1 \leq i \leq q - 1$.*

*Proof.* Assume that $G$ is generated by $A = (a_1, \ldots, a_n)$ and let $F(A)$ be the free abelian group generated by $A$. The inclusion map $i \colon A \to G$ can be extended to a unique homomorphism $f \colon F(A) \to G$ which is surjective since $A$ generates $G$, and thus $G$ is isomorphic to $F(A)/f^{-1}(0)$. By Proposition 5.9, $H = f^{-1}(0)$ is a free abelian group and there is a basis $(e_1, ..., e_n)$ of $G$, some $p \leq n$, and some positive natural numbers $k_1, \ldots, k_p$, such that $(k_1 e_1, \ldots, k_p e_p)$ is a basis of $H$, and $k_i$ divides $k_{i+1}$ for all $i$, with $1 \leq i \leq p - 1$. Let $r$, $0 \leq r \leq p$, be the largest natural number such that $k_1 = \ldots = k_r = 1$, rename $k_{r+i}$ as $n_i$, where $1 \leq i \leq p - r$, and let $q = p - r$. Then, we can write

$$H = \mathbb{Z}^{p-q} \oplus n_1\mathbb{Z} \oplus \cdots \oplus n_q\mathbb{Z},$$

and since $F(A)$ is isomorphic to $\mathbb{Z}^n$, it is easy to verify that $F(A)/H$ is isomorphic to

$$Z^{n-p} \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_q\mathbb{Z},$$

which proves the proposition.                                                    $\square$

Observe that $G$ is a free abelian group iff $q = 0$, and otherwise $\mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_q\mathbb{Z}$ is the torsion subgroup of $G$. Thus, as a corollary of Proposition 5.10, we obtain the fact that every finitely generated abelian group $G$ is a direct sum, $G = Z^m \oplus T$, where $T$ is the torsion subroup of $G$ and $Z^m$ is the free abelian group of dimension $m$.

One verifies that $m$ is the rank (the maximal dimension of linearly independent sets in $G$) of $G$, denoted rank($G$). The number $m = \text{rank}(G)$ is called the *Betti number* of $G$ and the numbers $n_1, \ldots, n_q$ are the *torsion numbers* of $G$. It can also be shown that $q$ and the $n_i$ only depend on $G$.

In the early days of algebraic topology (between the late 1890' and the early 1930's), an aera of mathematics started by Henri Poincaré in the late 1890's, homology groups had not been defined and people worked with Betti numbers and torsion coefficients. Emmy Noether played a crucial role in introducing homology groups into the field.



Figure 5.16: Leonhard Euler, 1707–1783 (left), and Henri Poincaré, 1854–1912 (right).

**Definition 5.12.** Given a finite complex $K = (V, \mathcal{S})$ of dimension $m$, if we let $m_p$ be the number of $p$-simplices in $K$, we define the *Euler–Poincaré characteristic* $\chi(K)$ *of* $K$ by

$$\chi(K) = \sum_{p=0}^{m} (-1)^p \, m_p.$$

In order to prove Theorem 5.12 we make use of Proposition 5.11 stated below.

Proposition 5.11 follows from the fact that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. By tensoring with $\mathbb{Q}$ with obtain an exact sequence in which the spaces $E \otimes_{\mathbb{Z}} \mathbb{Q}$, $F \otimes_{\mathbb{Z}} \mathbb{Q}$, and $G \otimes_{\mathbb{Z}} \mathbb{Q}$, are vector spaces over $\mathbb{Q}$ whose dimensions are equal to the ranks of the abelian groups being tensored with; see Proposition 12.9. A proof of Proposition 5.11 is also given in Greenberg and Harper [19] (Chapter 20, Lemma 20.7 and Lemma 20.8).

**Proposition 5.11.** *If*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

*is a short exact sequence of homomorphisms of abelian groups and if $F$ has finite rank, then*

$$\text{rank}(F) = \text{rank}(E) + \text{rank}(G).$$

*In particular, if $G$ is an abelian group of finite rank and if $H$ is a subgroup of $G$, then $\text{rank}(G) = \text{rank}(H) + \text{rank}(G/H)$.*

The following remarkable theorem holds:

**Theorem 5.12.** *Given a finite complex $K = (V, \mathcal{S})$ of dimension $m$, we have*

$$\chi(K) = \sum_{p=0}^{m} (-1)^p \, r(H_p(K)),$$

*the alternating sum of the Betti numbers (the ranks) of the homology groups of $K$.*

*Proof.* We know that $C_p(K)$ is a free group of rank $m_p$. Since $H_p(K) = Z_p(K)/B_p(K)$, by Proposition 5.11, we have

$$r(H_p(K)) = r(Z_p(K)) - r(B_p(K)).$$

Since we have a short exact sequence

$$0 \longrightarrow Z_p(K) \longrightarrow C_p(K) \xrightarrow{\partial_p} B_{p-1}(K) \longrightarrow 0,$$

again, by Proposition 5.11, we have

$$r(C_p(K)) = m_p = r(Z_p(K)) + r(B_{p-1}(K)).$$

Also, note that $B_m(K) = 0$, and $B_{-1}(K) = 0$. Then, we have

$$\begin{aligned}
\chi(K) &= \sum_{p=0}^{m} (-1)^p \, m_p \\
&= \sum_{p=0}^{m} (-1)^p \left( r(Z_p(K)) + r(B_{p-1}(K)) \right) \\
&= \sum_{p=0}^{m} (-1)^p \, r(Z_p(K)) + \sum_{p=0}^{m} (-1)^p \, r(B_{p-1}(K)).
\end{aligned}$$

Using the fact that $B_m(K) = 0$, and $B_{-1}(K) = 0$, we get

$$
\begin{aligned}
\chi(K) &= \sum_{p=0}^{m}(-1)^p \, r(Z_p(K)) + \sum_{p=0}^{m}(-1)^{p+1} \, r(B_p(K)) \\
&= \sum_{p=0}^{m}(-1)^p \, (r(Z_p(K)) - r(B_p(K))) \\
&= \sum_{p=0}^{m}(-1)^p \, r(H_p(K)).
\end{aligned}
$$

$\square$                                                      $\square$

A striking corollary of Theorem 5.12 (together with Theorem 5.6) is that the Euler–Poincaré characteristic, $\chi(K)$, of a complex of finite dimension $m$ only depends on the geometric realization $K_g$ of $K$, since it only depends on the homology groups $H_p(K) = H_p(K_g)$ of the polytope $K_g$. Thus, the Euler–Poincaré characteristic is an invariant of all the finite complexes corresponding to the same polytope, $X = K_g$. We can say that it is *the* Euler–Poincaré characteristic of the polytope $X = K_g$, and denote it by $\chi(X)$. In particular, this is true of surfaces that admit a triangulation. The Euler–Poincaré characteristic in one of the major ingredients in the classification of the compact surfaces. In this case, $\chi(K) = m_0 - m_1 + m_2$, where $m_0$ is the number of vertices, $m_1$ the number of edges, and $m_2$ the number of triangles in $K$.

Going back to the triangulations of the sphere, the torus, the projective space, and the Klein bottle, we find that they have Euler–Poincaré characteristics 2 (sphere), 0 (torus), 1 (projective space), and 0 (Klein bottle).

## 5.4   Simplicial Cohomology

In this section $G$ is any $R$-module over a commutative ring $R$ with an identity element 1. The relative (and absolute) simplicial cohomology groups of a pair of simplicial complexes $(K, L)$ (where $L$ is a subcomplex of $K$) are defined the same way that the singular relative cohomology groups are defined from the singular homology groups by applying $\mathrm{Hom}_R(-; G)$, as in Section 4.7.

Given the chain complex of relative simplicial homology

$$
0 \xleftarrow{\partial_0} C_0(K, L; R) \xleftarrow{\partial_1} C_1(K, L; R) \xleftarrow{\quad} \cdots \xleftarrow{\partial_{p-1}} C_{p-1}(K, L; R) \xleftarrow{\partial_p} C_p(K, L; R) \xleftarrow{\partial_{p+1}} \cdots
$$

by applying $\mathrm{Hom}_R(-, G)$, where $C_p(K, L; R) = C_p(K, R)/C_p(L, R)$, we obtain the chain complex

$$
0 \xrightarrow{\delta^{-1}} C^0(K, L; G) \xrightarrow{\delta^0} C^1(K, L; G) \xrightarrow{\quad} \cdots \xrightarrow{\delta^{p-1}} C^p(K, L; G) \xrightarrow{\delta^p} C^{p+1}(K, L; G) \xrightarrow{\delta^{p+1}} \cdots
$$

with $C^p(K, L; G) = \operatorname{Hom}_R(C_p(K, L; R), G)$ and $\delta^p = \operatorname{Hom}_R(\partial_p, G)$ for all $p \geq 0$ (and $\delta^{-1}$ is the zero map). More explicitly

$$\delta^p(f) = f \circ \partial_{p+1} \quad \text{for all } f \in C^p(K, L; G);$$

that is

$$\delta^p(f)(\sigma) = f(\partial_{p+1}(\sigma)) \quad \text{for all } f \in C^p(K, L; G) = \operatorname{Hom}_R(C_p(K, L; R), G)$$
$$\text{and all } \sigma \in C_{p+1}(K; L; R);$$

**Definition 5.13.** Given a pair of complexes $(K, L)$ with $L$ a subcomplex of $K$, the *simplicial relative cohomology groups* $H^p(K, L; G)$ of $(K, L)$ arise from the chain complex

$$0 \xrightarrow{\delta^{-1}} C^0(K, L; G) \xrightarrow{\delta^0} C^1(K, L; G) \longrightarrow \cdots \xrightarrow{\delta^{p-1}} C^p(K, L; G) \xrightarrow{\delta^p} C^{p+1}(K, L; G) \xrightarrow{\delta^{p+1}} \cdots$$

and are given by

$$H^p(K, L; G) = \operatorname{Ker} \delta^p / \operatorname{Im} \delta^{p-1}, \quad p \geq 0.$$

To obtain the long exact sequence of relative simplicial cohomology we dualize the short exact sequence

$$0 \longrightarrow C_*(L; R) \xrightarrow{i} C_*(K; R) \xrightarrow{j} C_*(K, L; R) \longrightarrow 0$$

where $C_*(K, L; R) = C_*(K, R)/C_*(L, R)$ by applying $\operatorname{Hom}(-, G)$ and we obtain the sequence

$$0 \longrightarrow C^*(K, L; G) \xrightarrow{j^\top} C^*(K; G) \xrightarrow{i^\top} C^*(L; G) \longrightarrow 0,$$

where by definition $C^*(K, L; G) = \operatorname{Hom}_R(C_*(K; R)/C_*(L; R), G)$, and as before $C^*(L; G) = \operatorname{Hom}_R(C_*(L; R), G)$ and $C^*(K; G) = \operatorname{Hom}_G(C_*(K; R), G)$.

Since $C_p(K, L; R) = C_p(K, R)/C_p(L, R)$ is a free module for every $p$, by Proposition 2.6 the sequence of chain complexes

$$0 \longrightarrow C^*(K, L; G) \xrightarrow{j^\top} C^*(K; G) \xrightarrow{i^\top} C^*(L; G) \longrightarrow 0$$

is exact. A version of Theorem 4.33 for relative simplicial cohomology in then obtained.

Given two pairs of simplicial complexes $(K_1, L_1)$ and $(K_2, L_2)$, where $L_1$ is a subcomplex of $K_1$ and $L_2$ is a subcomplex of $K_2$, a simplicial map $f: (K_1, L_1) \to (K_2, L_2)$ induces a homomorphism $f_{\sharp,p}: C_p(K_1, L_1; R) \to C_p(K_2, L_2; R)$ between the modules of oriented $p$-chains, and thus by applying $\operatorname{Hom}_R(-, G)$ we get a homomorphism $f^{\sharp,p}: C^p(K_2, L_2; G) \to C^p(K_1, L_1; G)$ commuting with coboundaries which induces homomorphisms

$$f^{*,p}: H^p(K_2, L_2; G) \to H^p(K_1, L_1; G) \quad \text{for all } p \geq 0.$$

Again, this assignment is functorial.

If $R$ is a PID, then the simplicial cohomology group $H^p(K, L; G)$ is isomorphic to the singular cohomology group $H^p(K_g, L_g; G)$ for every $p \geq 0$. This result is easily obtained from the Universal Coefficient Theorem for cohomology, or by an argument about free chain complexes; see Munkres [38] (Chapter 5, Section 45, Theorem 45.5).

**Theorem 5.13.** *Let $(K, L)$ be any pair of simplicial complexes with $L$ a subcomplex of $K$. If $R$ is a PID, then for any $R$-module $G$ we have isomorphisms*

$$H^p(K, L; G) \cong H^p(K_g, L_g; G) \quad \text{for all } p \geq 0$$

*between the relative simplicial homology of the pair of complexes $(K, L)$ and the relative singular homology of the pair of geometric realizations $(K_g, L_g)$.*

*Proof.* The proof shows the stronger result that if $H_{p-1}(K, L; R) \cong H_{p-1}(K_g, L_g; R)$ and $H_p(K, L; R) \cong H_p(K_g, L_g; R)$, then $H^p(K, L; G) \cong H^p(K_g, L_g; G)$.

Let $\eta \colon C_*(K, L; R) \to S_*(K_g, L_g; R)$ be the chain map of Theorem 5.6. By the naturality part of Universal Coefficient Theorem for cohomology (Theorem 12.43), we have the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1_R(H_{p-1}(K_g, L_g; R), G) & \longrightarrow & H^p(K_g, L_g; G) & \longrightarrow & \mathrm{Hom}_R(H_p(K_g, L_g; R), G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Ext}^R_1(\eta_*)} & & \downarrow{\scriptstyle (\mathrm{Hom}_R(\eta, G))^*} & & \downarrow{\scriptstyle \mathrm{Hom}_R(\eta_*, \mathrm{id})} & & \\
0 & \longrightarrow & \mathrm{Ext}^1_R(H_{p-1}(K, L; R), G) & \longrightarrow & H^p(K, L; G) & \longrightarrow & \mathrm{Hom}_R(H_p(K, L; R), G) & \longrightarrow & 0.
\end{array}
$$

By Theorem 5.6 the chain map $\eta$ induces isomorphisms $H_{p-1}(K, L; R) \cong H_{p-1}(K_g, L_g; R)$ and $H_p(K, L; R) \cong H_p(K_g, L_g; R)$, so the first and the third map in the above diagram are isomorphisms. By the short five lemma (Proposition 2.22) we conclude that the middle map is an isomorphism. $\square$

In summary, simplicial cohomology is subsumed by singular cohomology (at least whene $R$ is a PID). Nevertheless, simplicial cohomology is much more amenable to computation than singular cohomology. In particular, simplicial cohomology can be used to compute the cohomology ring of various spaces; see Munkres [38] (Chapter 5, Section 49).

Indeed, it is possible to define a cup product on the simplicial cohomology of a complex. If $K = (V, \mathcal{S})$ is a simplicial complex, let $\leq$ be a partial order of the vertices of $K$ that induces a total order on the vertices of every simplex in $\mathcal{S}$.

**Definition 5.14.** Given a simplicial complex $K = (V, \mathcal{S})$ and a partial order of its vertices as above, define a map

$$\smile^\Delta \colon C^p(K; R) \times C^q(K; R) \to C^{p+q}(K; R)$$

by
$$(c \smile^{\Delta} d)([v_0, \ldots, v_{p+q}]) = c([v_0, \ldots, v_p]) \, d([v_p, \ldots, v_{p+q}])$$
ifr $v_0 < v_1 < \cdots < v_{p+q}$, for all simplicial $p$-cochains $c \in C^p(K; R)$ and all simplicial $q$-cochains $d \in C^q(K; R)$.

It can be shown that the map $\smile^{\Delta} \colon C^p(K; R) \times C^q(K; R) \to C^{p+q}(K; R)$ induces a *cup product*
$$\smile^{\Delta} \colon H^p(K; R) \times H^q(K; R) \to H^{p+q}(K; R)$$
which is bilinear and associative and independent of the partial order $\leq$ chosen on $V$; see Munkres [38] (Chapter 5, Section 49, Theorem 49.1 and Theorem 49.2).

It can also be shown that if $\eta \colon C_*(K; R) \to S_*(K_g; R)$ is the chain map of Theorem 5.6, then $\eta^* = \mathrm{Hom}_R(\eta, R)$ carries the cup product $\smile$ of singular cohomology to the cup product $\smile^{\Delta}$ of simplicial cohomology of Definition 4.20. If $h \colon K_1 \to K_2$ is a simplicial map between two simplicial complexes, then $h^*$ preserves cup products; see Munkres [38] (Chapter 5, Section 49, Theorem 49.1 and Theorem 49.2).

# Chapter 6

# Homology and Cohomology of CW Complexes

Computing the singular homology (or cohomology) groups of a space $X$ is generally very difficult. J.H.C. Whitehead invented a class of spaces called CW complexes for which the computation of the singular homology groups is much more tractable. Roughly speaking, a CW complex $X$ is built up inductively starting with a collection of points, in such a way that if the space $X^n$ has been obtained at stage $n$, then the space $X^{n+1}$ is obtained from $X^n$ by gluing, or as it is customary to say attaching, a collection of closed balls whose boundaries are glued to $X^n$ in a specific fashion. Every compact manifold is homotopy equivalent to a CW complex, so the class of CW complexes is quite rich. It also plays an important role in homotopy theory. In this short chapter, we describe CW complexes and explain how their homology and cohomology can be computed.

## 6.1 CW Complexes

First we define closed and open cells, and then we describe the process of attaching space (or adjunction space). Recall that the $n$-dimensional ball $D^n$, the $n$-dimensional open ball $\text{Int } D^n$, and the $n$-dimensional sphere $S^n$, are defined by

$$D^n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$$
$$\text{Int } D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 < 1\}$$
$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}.$$

Furthermore, $S^n = \partial D^{n+1} = D^{n+1} - \text{Int } D^{n+1}$, the boundary of $D^{n+1}$, and $D^n/\partial D^n$ is homeomorphic to $S^n$ ($n \geq 1$). When $n = 0$ we set $\text{Int } D^0 = D^0 = \{0\}$, and $\partial D^0 = S^{-1} = \emptyset$.

**Definition 6.1.** A *(closed) cell of dimension* $m \geq 0$ (or *closed $m$-cell*) is a space homeomorphic to $D^m$, and an *open cell of dimension* $m \geq 0$ (or *open $m$-cell*) is a space homeomorphic to $\text{Int } D^m$.

Observe that an open or closed 0-cell is a point. We will usually denote an open $m$-cell by $e^m$ (or simply $e$), and its closure by $\overline{e^m}$ (or simply $\overline{e}$). The set $\overline{e} - e$ is denoted by $\dot{e}$.

Given two topological spaces $X$ and $Y$, given a closed subset $A$ of $X$, and given a continuous map $f\colon A \to Y$, we would like to define the space $X \cup_f Y$ obtained by gluing $X$ and $Y$ "along $A$." We will define $X \cup_f Y$ as a quotient space of the disjoint union $X \sqcup Y$ of $X$ and $Y$ with the topology in which a subset $Z \in X \sqcup Y$ is open iff $Z \cap X$ is open in $X$ and $Z \cap Y$ is open in $Y$. See Figure 6.1. More generally, recall the definition of the topological sum of a family of spaces (Definition 4.11).

**Definition 6.2.** If $(X_i)_{i \in I}$ is a family of topological spaces we define the *topological sum* $\bigsqcup_{i \in I} X_i$ of the family $(X_i)_{i \in I}$ as the disjoint union of the spaces $X_i$, and we give it the topology for which a subset $Z \subseteq \bigsqcup_{i \in I} X_i$ is open iff $Z \cap X_i$ is open for all $i \in I$.

We will also need the notion of coherent union.

**Definition 6.3.** Given a topological space $X$, if $(X_i)_{i \in I}$ is a family of subspaces of $X$ such that $X = \bigcup_{i \in I} X_i$, we say that the topology of $X$ is *coherent* with the family $(X_i)_{i \in I}$ if a subset $A \subseteq X$ is open in $X$ iff $A \cap X_i$ is open in $X_i$ for all $i \in I$. We say that $X$ is the *coherent union* of the family $(X_i)_{i \in I}$ .

Given $X, Y, A$, and $f\colon A \to Y$ as above, we form the quotient space of $X \sqcup Y$ by identifying each set

$$f^{-1}(y) \cup \{y\}$$

for each $y \in Y$ to a point. This means that we form the quotient set corresponding to the partition of $X \sqcup Y$ into the subsets of the form $f^{-1}(y) \cup \{y\}$ for all $y \in Y$, and all singleton sets $\{x\}$ for all $x \in X - A$. Observe that if $y \notin f(A)$, then $f^{-1}(y) = \emptyset$, so in this case the subset $f^{-1}(y) \cup \{y\}$ reduces to $\{y\}$.

**Definition 6.4.** Given two topological spaces $X$ and $Y$, given a closed subset $A$ of $X$, and given a continuous map $f\colon A \to Y$, the *adjunction space determined by $f$* (or *attaching space determined by $f$*), denoted by $X \cup_f Y$, is the quotient space of the disjoint sum $X \sqcup Y$ corresponding to the partition of $X \sqcup Y$ into the subsets of the form $f^{-1}(y) \cup \{y\}$ for all $y \in Y$, and all singleton sets $\{x\}$ for all $x \in X - A$. The map $f$ is called the *adjunction map* (or *attaching map*). See Figure 6.1.

Observe that the adjunction map $f\colon A \to Y$ needs not be injective, that is, it could cause some collapsing of parts of $A$. For example, if $X = D^1$, $A = S^1$, $Y = \{0\}$ and $f\colon A \to Y$ is the constant function that "collapses" $S^1$ onto $\{0\}$, then the adjunction space $X \cup_f Y$ is homeomorphic to the sphere $S^2$. See Figure 6.2.

If $\pi\colon X \sqcup Y \to X \cup_f Y$ is the quotient map, then it is easy to show that $\pi$ maps $Y$ homeomorphically onto a closed subspace of $X \cup_f Y$.
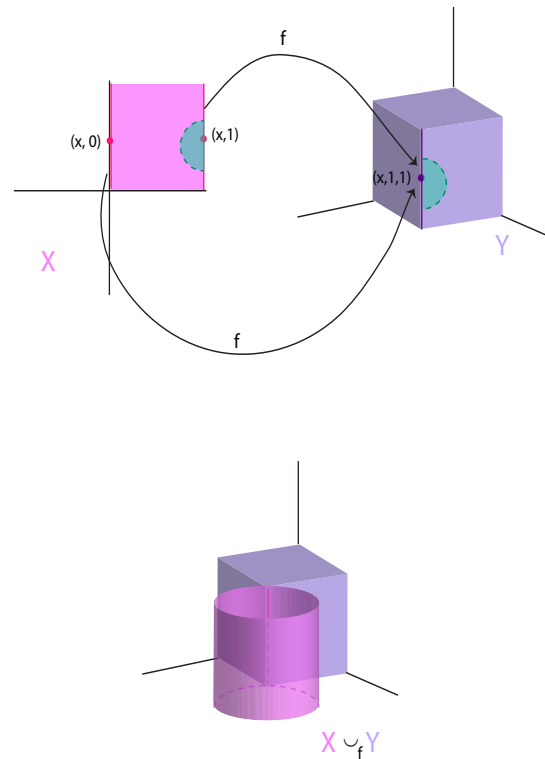
Figure 6.1: Let $X$ be the unit square in $\mathbb{R}^2$ and $Y$ be the boundary of the unit cube in $\mathbb{R}^3$. Let $A$ be the vertical lines $x = 0$ and $x = 1$. The attaching map $f\colon A \to Y$ is defined via $f(x,0) = (1,1,x) = f(x,1)$. The upper figure shows an open set in $X \cup_f Y$ as defined in Definition 4.11. The lower figure shows the three dimensional rendering of the quotient space $X \cup_f Y$.

Recall that a topological space $X$ is *normal* if the singleton subset $\{x\}$ is closed for all $x \in X$, and if for any two closed subsets $A$ and $B$ of $X$ there exist two disjoint open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. Since every singleton subset is closed, a normal space is Hausdorff.

The following result is shown in Munkres [38] (Chapter 4, Theorem 37.2).

**Proposition 6.1.** *Given $X, Y, A$, and $f\colon A \to Y$ as in Definition 6.4, if $X$ and $Y$ are normal, then $X \cup_f Y$ is also normal, and in particular Hausdorff.*

A CW complex can be defined intrinsically or by an inductive definition involving the process of attaching cells. We begin with the second definition since it is easier to grasp. To simplify matters we begin with the definition of a finite CW complex.

**Definition 6.5.** A finite *CW complex $X$ of dimension $n$* is defined inductively as follows:
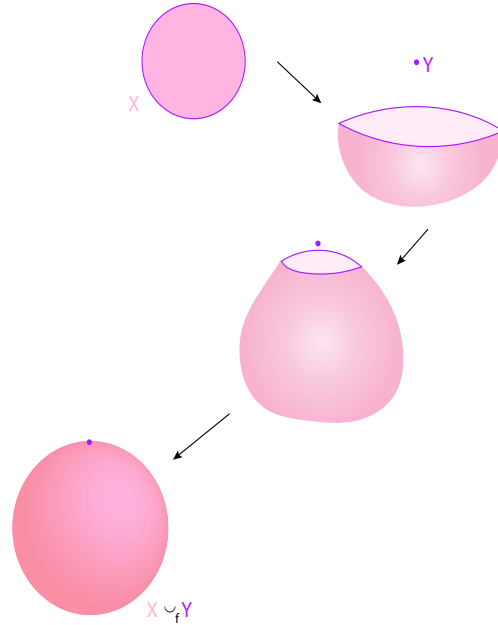
Figure 6.2: Let $X$ be the unit disk in $\mathbb{R}^2$ and $Y$ be a point. Let $A$ be the circular boundary of $X$. The attaching map $f: A \to Y$ collapses $A$ to a point and wraps the disk into a sphere as depicted by the four stage rendering of $X \cup_f Y$.

(1) Let $X^0$ be a finite set of points (0-cells) with the discrete topology.

(2) If $p < n$ and if $X^p$ has been constructed, let $I_{p+1}$ be a finite (possibly empty) index set, let $\bigsqcup_{i \in I_{p+1}} D_i^{p+1}$ be the disjoint union of closed $(p+1)$-balls, and if we write $S_i^p = \partial D_i^{p+1}$ let $g_{p+1}: \bigsqcup_{i \in I_{p+1}} S_i^p \to X^p$ be a continuous map (an *attaching map*). Then $X^{p+1}$ is the adjunction space

$$X^{p+1} = \left( \bigsqcup_{i \in I_{p+1}} D_i^{p+1} \right) \cup_{g_{p+1}} X^p.$$

Either $n = 0$ and $X = X^0$, or $n \geq 1$ in which case $X^0 \neq \emptyset$ and $I_n \neq \emptyset$, that is, there is some open $n$-cell, and we let $X = X^n$. The subspace $X^p$ is called the *p-skeleton* of $X$.

If $\pi_{p+1}^{\mathrm{CW}}$ is the quotient map

$$\pi_{p+1}^{\mathrm{CW}}: \left( \bigsqcup_{i \in I_{p+1}} D_i^{p+1} \right) \sqcup X^p \to \left( \bigsqcup_{i \in I_{p+1}} D_i^{p+1} \right) \cup_{g_{p+1}} X^p = X^{p+1},$$

then we write $e_i^{p+1} = \pi_{p+1}^{\mathrm{CW}}(\mathrm{Int}\, D_i^{p+1})$.

It is not hard to see that $e_i^{p+1}$ is an open $(p+1)$-cell (*i.e.* $\pi_{p+1}^{\mathrm{CW}}$ maps $\mathrm{Int}\, D_i^{p+1}$ homeomorphically onto $e_i^{p+1}$). Furthermore, since $\pi_{p+1}^{\mathrm{CW}}$ maps $X^p$ homeomorphically onto a subspace

of $X^{p+1}$, we can view $\pi_{p+1}^{\mathrm{CW}}$ as the inclusion on $X^p$ and as $g_{p+1}$ on $\bigsqcup_{i \in I_p} D_i^{p+1}$. It follows that the open $(p+1)$-cells $e_i^{p+1}$ are disjoint from all the open cells in $X^p$. Since $\pi_{p+1}^{\mathrm{CW}}$ is a homeomorphism on each Int $D_i^{p+1}$, we have $e_i^{p+1} \cap e_j^{p+1} = \emptyset$ for all $i \neq j$. It follows by induction that $X = X^n$ is the disjoint union of all the open cells $e_i^p$ for $p = 0, \ldots, n$ and all $i \in I_p$.

Since $X^0$ is normal, by Proposition 6.1 we conclude that $X = X^n$ is normal, thus Hausdorff. It is also clear that a finite CW complex is compact.

**Example 6.1.**

(1) A 0-dimensional CW complex is simply a discrete set of points. A 1-dimensional CW complex $X$ consists of 0-cells and 1-cells, where each 1-cell $e_i^1$ is homeomorphic to the open line segment $(-1, 1)$, whose boundaries If we view each 1-cell as a directed edge and each 0-cell as a node (or vertex), then the CW complex $X$ is a (directed) *graph* in which several edges may have the same endpoints and an edge may have identical endpoints (self-loops).

(2) The $n$-sphere $S^n$ $(n \geq 1)$ is homeomorphic to the CW complex with one 0-cell $e^0$, one $n$-cell $e^n$, and with the attaching map $g_n \colon S^{n-1} \to e^0$, the constant map, with $S^n = X^n$. See Figure 6.2. This is equivalent to viewing $S^n$ as the quotient $D^n/\partial D^n = D^n/S^{n-1}$. When $n = 0$, $S^0$ is the CW complex consisting of two disjoint 0-cells,

(3) The $n$-ball $D^n$ $(n \geq 1)$ is homeomorphic to the CW complex $X$ with one 0-cell $e^0$, one $(n-1)$-cell $e^{n-1}$, and one $n$-cell $e^n$. First, $X^{n-1} = S^{n-1}$ as explained in (2), and then $D^n = X^n$ is obtained using as attaching map the identity map $g_n \colon S^{n-1} \to S^{n-1}$.

(4) The real projective space $\mathbb{RP}^2$ is is homeomorphic to the CW complex $X$ with one 0-cell $e^0$, one 1-cell $e^1$, and one 2-cell $e^2$. First, $X^1$ is obtained by using the constant map $g_1 \colon S^0 \to e^0$ as attaching map, and then $X^2$ is obtained by using as attaching map the map $g_2 \colon S^1 \to S^1$ that sends $S^1$ around $S^1$ twice $(g_2(e^{i\theta}) = e^{2i\theta})$. Observe that $X^1 = \mathbb{RP}^1$. This suggest a recursive method for obtaining a cell structure for $\mathbb{RP}^n$.

(5) The projective space $\mathbb{RP}^n (n \geq 0)$ is homeomorphic to the CW complex $X$ with exactly one $p$-cell $e^p$ for $p = 0, \ldots, n$; that is, the set of cells $\{e^0, e^1, \ldots, e^n\}$. We have $X^0 = \{e^0\}$, and assuming that $X^{n-1} = \mathbb{RP}^{n-1}$ has been constructed, $X^n = \mathbb{RP}^n$ is obtained by using the quotient map $g_n \colon S^{n-1} \to \mathbb{RP}^{n-1}$ that identifies two antipodal points as attaching map; see Example 4.1.

(6) The complex projective space $\mathbb{CP}^n (n \geq 0)$ is homeomorphic to the CW complex $X$ with exactly one $2p$-cell $e^{2p}$ for $p = 0, \ldots, n$; that is, the set of cells $\{e^0, e^2, \ldots, e^{2n}\}$. We have $X^0 = \{e^0\}$, and assuming that $X^{2n-2} = \mathbb{CP}^{n-1}$ has been constructed, $X^{2n} = \mathbb{CP}^n$ is obtained by using the quotient map $g_{2n} \colon S^{2n-1} \to \mathbb{CP}^{n-1}$ as attaching map; see Example 4.1.

(7) The 2-torus $T^2 = S^1 \times S^1$ is homeomorphic to the CW complex $X$ with one 0-cell $e^0$, two 1-cells $e_1^1, e_2^1$, and one 2-cell $e^2$. First $X^1$ is obtained by using the constant map $g_1\colon S^0 \sqcup S^0 \to e^0$ as attaching map. The space $X^1$ consists of two circles on a torus in $\mathbb{R}^3$ (in orthogonal planes) intersecting in a common point . Then $T^2 = X^2$ is obtained by using the map $g_2\colon S^1 \to X^1$ that "wraps" $S^1$ around the two circles of $X^1$, as attaching map; think of the construction of a torus from a square in which opposite sides are glued in two steps. See Figure 6.3.
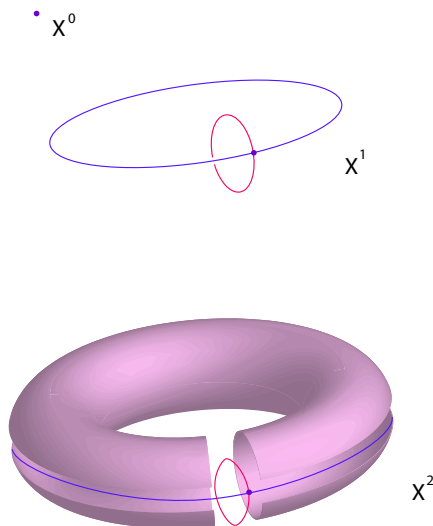


Figure 6.3: The CW complex construction of the torus $T^2$.

**Remark:** Ambitious readers should read Chapter 6 of Milnor and Stasheff [35], where a cell structure for the Grassmann manifolds is described. This is a generalization of the cell structure for $\mathbb{RP}^n$.

   The definition of a CW complex can be generalized by allowing the index sets $I_p$ to be infinite and by allowing the sequence of $p$-skeleta $X^p$ to be infinite.

**Definition 6.6.** A *CW complex* $X$ is defined inductively as follows:

(1) Let $X^0$ be a set of points (0-cells) with the discrete topology. If $X^0 = \emptyset$ then let $X = \emptyset$.

(2) If $X^p$ has been constructed ($p \geq 0$) and if $X^p \neq \emptyset$, let $I_{p+1}$ be a (possibly empty) index set, let $\bigsqcup_{i \in I_{p+1}} D_i^{p+1}$ be the disjoint union of closed $(p+1)$-balls, and if we write $S_i^p = \partial D_i^{p+1}$ let $g_{p+1}\colon \bigsqcup_{i \in I_{p+1}} S_i^p \to X^p$ be a continuous map (an *attaching map*). Then

$X^{p+1}$ is the adjunction space

$$X^{p+1} = \left( \bigsqcup_{i \in I_{p+1}} D_i^{p+1} \right) \cup_{g_{p+1}} X^p.$$

Supppose $X^0 \neq \emptyset$. If there is a smallest $n \geq 0$ such that $I_p = \emptyset$ for all $p \geq n+1$, then we let $X = X^n$ and we say that $X$ has *dimension n*. In this case, note that $X^n$ must have some open $n$-cell. Otherwise we let $X = \bigcup_{p \geq 0} X^p$, and we give $X$ the topology for which $X$ is the coherent union of the family $(X^p)_{p \geq 0}$; that is, a subset $Z$ of $X$ is open iff $Z \cap X^p$ is open in $X^p$ for all $p \geq 0$. Each subspace $X^p$ is called a *p-skeleton* of $X$.

As before if $\pi_{p+1}^{\mathrm{CW}}$ is the quotient map $\pi_{p+1}^{\mathrm{CW}} \colon \left( \bigsqcup_{i \in I_{p+1}} D_i^{p+1} \right) \sqcup X^p \to X^{p+1}$, then we write

$$e_i^{p+1} = \pi_{p+1}^{\mathrm{CW}}(\mathrm{Int}\, D_i^{p+1}),$$

and it is not hard to see that $e_i^{p+1}$ is an open $(p+1)$-cell (*i.e.* $\pi_{p+1}^{\mathrm{CW}}$ maps $\mathrm{Int}\, D_i^{p+1}$ homeomorphically onto $e_i^{p+1}$). It follows that $X$ is the disjoint union of the cells $e_i^p$ for all $p \geq 0$ and all $i \in I_p$.

For every $p$-ball $D_i^p$, the restriction to $D_i^p$ of the composition of the quotient map $\pi_p^{\mathrm{CW}}$ from $\left( \bigsqcup_{i \in I_p} D_i^p \right) \sqcup X^{p-1}$ to $X^p$ with the inclusion $X^p \longrightarrow X$ is a map from $D_i^p$ to $X$ denoted by $f_i$ (or $f_i^p$ if we want to be very precise) and called the *characteristic map* of $e_i^p = \pi_p^{\mathrm{CW}}(\mathrm{Int}\, D_i^p)$. It is not hard to show that $f_i(D_i^p) = \overline{(e_i^p)}$, $f_i(S_i^{p-1}) = (\dot{e}_i^p)$, and $f_i$ is a homeomorphism of $\mathrm{Int}\, D_i^p$ onto $e_i^p$.

One should be careful that the terminology "open cell" is slightly misleading. Although an open cell $e_i^p$ is open in $X^p$, it may *not* be open in $X$. Consider the example of the torus $T^2$ from Example 6.1(7). The open cell $e_1^1 = \pi_1(\mathrm{Int}\, D_1^1)$ of $X^1$ is not open in $T^2$.

**Example 6.2.** The infinite union $X = \mathbb{RP}^\infty = \bigcup_{n \geq 0} \mathbb{RP}^n$ is an infinite CW complex whose $n$-skeleton $X^n$ is $\mathbb{RP}^n$. The CW complex $\mathbb{RP}^\infty$ has infinitely many $n$-cells $e^n$, one for each dimension.

Similarly, the infinite union $X = \mathbb{CP}^\infty = \bigcup_{n \geq 0} \mathbb{CP}^n$ is an infinite CW complex whose $2n$-skeleta $X^{2n}$ and $X^{2n+1}$ are both $\mathbb{CP}^n$. The CW complex $\mathbb{CP}^\infty$ has infinitely many $n$-cells $e^{2n}$, one for each even dimension.

**Definition 6.7.** A *subcomplex* of a CW complex $X$ is a subspace $A$ of $X$ which is a union of open cells $e_i$ of $X$ such that the closure $\overline{e_i}$ of each open cell $e_i$ in $A$ is also in $A$.

It is easy to show by induction over skeleta that a subcomplex is a closed subspace; see Munkres [38] (Chapter 4, Section 38, page 217). The following proposition states a crucial compactness property of CW complexes.

**Proposition 6.2.** *If $X$ is a CW complex then the following properties hold and are all equivalent.*

(1) *If a subspace $A$ of $X$ has no two points in the same open cell, then $A$ is closed and discrete.*

(2) *If a subspace $C$ of $X$ is compact, then $C$ is contained in a finite union of open cells.*

(3) *Each open cell of $X$ is contained in a finite subcomplex of $X$.*

Proposition 6.2 is proved in Bredon [4] (Chapter IV, Section 8, Proposition 8.1). As a collollary we have the following result.

**Proposition 6.3.** *If $X$ is a CW complex then any compact subset $C$ of $X$ is contained in a finite subcomplex.*

*Proof.* By Proposition 6.2(2) the compact subset $C$ is contained in a union of a finite number of open cells of $X$. By Proposition 6.2(3) each of these open cells is contained in a finite subcomplex. But the union of this finite number of finite subcomplexes is a finite subcomplex which contains $C$. $\square$

It can be shown that a CW complex $X$ is normal; see Munkres [38] (Chapter 4, Section 38). In fact, more can be proved.

**Proposition 6.4.** *Let $X$ be a CW complex as defined in Definition 6.6. Then the following properties hold:*

(1) *The space $X$ is the disjoint union of a collection of open cells.*

(2) *$X$ is Hausdorff.*

(3) *For each open $p$-cell $e_i$ of the collection, there is a continuous map $f_i \colon D^p \to X$ that maps $\operatorname{Int} D^p$ homeomorphically onto $e_i$ and carries $S^{p-1} = \partial D^p$ into a finite union of open cells $e_j^k$, each of dimension $k < p$.*

(4) *A set $Z$ is closed in $X$ iff $Z \cap \overline{e_i}$ is closed in $\overline{e_i}$ for all open cells $e_i$.*

Proposition 6.4 is proved in Munkres [38] (Chapter 4, Section 38, Theorem 38.2 and Theorem 38.3). A similar development can be found in Hatcher [25] (Appendix, Topology of cell complexes).

Property (3) is what is referred to as "closure-finiteness" by J.H.C. Whitehead. Property (4) expresses the fact that $X$ has the "weak topology." This explains the CW in CW complexes!

It is easy to see that Properties (3) and (4) imply that $f_i(D^p) = \overline{e_i}$ and $f_i(S^{p-1}) = \dot{e}_i$. The map $f_i$ is called a *characteristic map* for the open cell $e_i$.

The properties of Proposition 6.4 can be taken as the definition of a CW complex. This is what J.H.C. Whitehead did originally, and this is the definition used by Munkres [38] and Milnor and Stasheff [35]. Then it can be shown that this alternate definition is equivalent to our previous definition (Definition 6.6). This is proved in Munkres [38] (Chapter 4, Section 38, Theorem 38.2 and Theorem 38.3).

Since our primary goal is to determine the homology (and cohomology) groups of CW complexes, we will not go into a more detailed study of these spaces. Let us just mention that every CW complex $X$ is normal, paracompact, compactly generated (which means that $X$ is the union of its compact subsets and that a set $A \subseteq X$ is closed in $X$ iff $A \cap C$ is closed in $C$ for every compact subset $C$ of $X$), and a finite CW complex is an ENR (Euclidean neighborhood retract).

We will also need the fact that a subcomplex $A$ of a CW complex is a deformation retract of a neighborhood of $X$. The following result is proved in Hatcher [25] (Appendix, Proposition A.5).

**Proposition 6.5.** *For any CW complex $X$ and any subcomplex $A$ of $X$, there is a neighborhood $N(A)$ of $X$ that deformation retracts onto $A$. In other words, $(X, A)$ is a good pair.*

For a more comprehensive exposition of CW complexes we refer the interested reader to Hatcher [25] (Appendix, Topology of cell complexes), Bredon [4] (Chapter IV, Sections 8-14), and Massey [32] (Chapter IX). Rotman [41] also contains a rather thorough yet elementary treatment.

## 6.2 Homology of CW Complexes

One of the nice features of CW complexes is the fact that it is possible to assign to each CW complex $X$ a chain complex $S_*^{\mathrm{CW}}(X; R)$ called its cellular chain complex, where

$$S_p^{\mathrm{CW}}(X; R) = H_p(X^p, X^{p-1}; R),$$

the relative $p$-th singular homoloy group of the pair $(X^p, X^{p-1})$, where $X^p$ is the $p$-skeleton of $X$ (by convention $X^{-1} = \emptyset$). The module $H_p(X^p, X^{p-1}; R)$ is a free $R$-module whose dimension (when finite) is equal to the number of $p$-cells in $X$. This means that we can view $H_p(X^p, X^{p-1}; R)$ as the set of formal linear combinations $\sum_i \lambda_i e_i^p$, where $\lambda_i \in R$ and the $e_i^p$ are open $p$-cells. Furthermore, the homology of the cellular complex agrees with the singular homology. That is, if we write $H_p^{\mathrm{CW}}(X; R) = H_p(S_*^{\mathrm{CW}}(X; R))$, then

$$H_p^{\mathrm{CW}}(X; R) \cong H_p(X; R) \quad \text{for all } p \geq 0,$$

where $H_p(X; R)$ is the $p$th singular homology module of $X$. In many practical cases, the number of $p$-cells is quite small so the cellular complex $S_*^{\mathrm{CW}}(X; R)$ is much more manageable than the singular complex $S_*(X; R)$.

We will need of few properties of the modules $H_k(X^p, X^{p-1}; R)$. By convention, if $X$ is a CW complex we set $X^{-1} = \emptyset$. Then $H_0(X^0, X^{-1}; R) = H_0(X^0; R)$.

**Proposition 6.6.** *If $X$ is a CW complex, then the following properties hold:*

*(a) We have $H_k(X^p, X^{p-1}; R) = (0)$ for $k \neq p$ and $H_p(X^p, X^{p-1}; R)$ is a free $R$-module with a basis in one-to-one correspondence with the p-cells of $X$.*

*(b) $H_k(X^p; R) = (0)$ for all $k > p$. In particular, if $X$ has finite dimension $n$ then $H_p(X; R) = (0)$ for all $p > n$.*

*Sketch of proof.* To prove (a) we use Proposition 6.5 which says that $(X^p, X^{p-1})$ is a good pair. By Proposition 4.10

$$H_k(X^p, X^{p-1}; R) \cong H_k(X^p/X^{p-1}, \{\mathrm{pt}\}; R) \cong \widetilde{H}_k(X^p/X^{p-1}; R).$$

Then we use Corollary 2.25 from Hatcher [25] (Chapter 2, Section 2.1), the fact that $X^p/X^{p-1}$ is the wedge sum of $p$-spheres (the disjoint sum of $p$-spheres glued at the south pole, the *basepoint*), and Proposition 4.16.

To prove (b) first observe that $H_k(X^0; R) = (0)$ for all $k > 0$. Next consider the following piece of the long exact sequence of homology of the pair $(X^p, X^{p-1})$:

$$H_{k+1}(X^p, X^{p-1}; R) \longrightarrow H_k(X^{p-1}; R) \longrightarrow H_k(X^p; R) \longrightarrow H_k(X^p, X^{p-1}; R).$$

If $k \neq p, p-1$, then the first and the fouth groups are zero by (a), so we have isomorphisms

$$H_k(X^p; R) \cong H_k(X^{p-1}; R) \quad k \neq p, p-1.$$

Thus if $k > p$, by induction we get

$$H_k(X^p) \cong H_k(X^0) = (0),$$

proving (b). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Proposition 6.6(a) implies that we can view $H_p(X^p, X^{p-1}; R)$ as the set of formal linear combinations $\sum_i \lambda_i e_i^p$, where $\lambda_i \in R$ and the $e_i^p$ are open $p$-cells.

**Proposition 6.7.** *If $X$ is a CW complex, then we have $H_k(X^p; R) \cong H_k(X; R)$ for all $k < p$.*

*Sketch of proof.* Consider the following piece of the long exact sequence of homology of the pair $(X^p, X^{p-1})$:

$$H_{k+1}(X^{p+1}, X^p; R) \longrightarrow H_k(X^p; R) \longrightarrow H_k(X^{p+1}; R) \longrightarrow H_k(X^{p+1}, X^p; R).$$

If $k < p$ then $k+1 < p+1$ so the first and fourth groups are zero and we have isomorphisms

$$H_k(X^p; R) \cong H_k(X^{p+1}; R) \quad k < p.$$

By induction, if $k < p$ then

$$H_k(X^p; R) \cong H_k(X^{p+m}; R) \quad \text{for all } m \geq 0.$$

If $X$ is finite-dimensional, we are done. Otherwise, following Milnor and Stasheff [35] (Appendix A, Corollary A.3), we use the fact that

$$H_k(X; R) \cong \varinjlim_{r \geq 0} H_k(H^r; R),$$

because every singular simplex of $X$ is contained in a compact subset, and hence in some $X^r$. A similar proof is given in Hatcher [25] (Chapter 2, Lemma 2.34). $\square$

We now show that we can form a chain complex with the modules $H_p(X^p, X^{p-1}; R)$.

Recall that $S_k(X^p, X^{p-1}; G) = S_k(X^p; G)/S_k(X^{p-1}; G)$, so we have the quotient map $\pi_k \colon S_k(X^p; G) \to S_k(X^p, X^{p-1}; G)$ which yields the map $j_k \colon H_k(X^p; G) \to H_k(X^p, X^{p-1}; G)$. Consider the following pieces of the long exact sequence of homology of the pairs $(X^{p+1}, X^p)$, $(X^p, X^{p-1})$, and $(X^{p-1}, X^{p-2})$:

$$H_{p+1}(X^{p+1}, X^p; R) \xrightarrow{\partial_{p+1}} H_p(X^p; R) \longrightarrow H_p(X^{p+1}; R) \longrightarrow H_p(X^{p+1}, X^p; R)$$

$$H_p(X^{p-1}; R) \longrightarrow H_p(X^p; R) \xrightarrow{j_p} H_p(X^p, X^{p-1}; R) \xrightarrow{\partial_p} H_{p-1}(X^{p-1}; R)$$

$$H_{p-1}(X^{p-2}; R) \longrightarrow H_{p-1}(X^{p-1}; R) \xrightarrow{j_{p-1}} H_{p-1}(X^{p-1}, X^{p-2}; R) \longrightarrow H_{p-2}(X^{p-2}; R).$$

Observe that by Proposition 6.6 the modules showed in red are $(0)$; that is, we have

$$H_p(X^{p+1}, X^p; R) = H_p(X^{p-1}; R) = H_{p-1}(X^{p-2}; R) = (0),$$

and by Proposition 6.7 we have $H_p(X^{p+1}; R) \cong H_p(X; R)$. We form the following diagram

in which for simplicity of notation we omitted the ring $R$, and where $d_{p+1} = j_p \circ \partial_{p+1}$ and $d_p = j_{p-1} \circ \partial_p$. Since $\partial_p \circ j_p = 0$ (because the sequence on that descending diagonal is exact), we have

$$d_p \circ d_{p+1} = j_{p-1} \circ \partial_p \circ j_p \circ \partial_{p+1} = 0.$$

Therefore, the modules $H_p(X^p, X^{p-1}; R)$ together with the boundary maps $d_p \colon H_p(X^p, X^{p-1}; R) \to H_{p-1}(X^{p-1}, X^{p-2}; R)$ form a chain complex. Recall that we set $X^{-1} = \emptyset$.

**Definition 6.8.** Given a CW complex $X$, the *cellular chain complex* $S_*^{\mathrm{CW}}(X; R)$ associated with $X$ is the chain complex where $S_p^{\mathrm{CW}}(X; R) = H_p(X^p, X^{p-1}; R)$ and the boundary maps $d_p \colon H_p(X^p, X^{p-1}; R) \to H_{p-1}(X^{p-1}, X^{p-2}; R)$ are given by $d_p = j_{p-1} \circ \partial_p$ as in the diagram above. We denote the *cellular homology module* $H_p(S_*^{\mathrm{CW}}(X; R))$ of the chain complex $S_*^{\mathrm{CW}}(X; R)$ by $H_p^{\mathrm{CW}}(X; R)$.

The reason for introducing the modules $H_p^{\mathrm{CW}}(X; R)$ is that they are isomorphic to the singular homology modules $H_p(X; R)$, and in practice they are usually much easier to compute.

**Theorem 6.8.** *Let $X$ be a CW complex. There are isomorphisms*

$$H_p^{\mathrm{CW}}(X; R) \cong H_p(X; R) \quad \text{for all } p \geq 0$$

*between the cellular homology modules and the singular homology modules of $X$.*

*Proof.* Exactness of the left ascending diagonal sequence in the diagram above shows that

$$H_p(X; R) \cong H_p(X^p; R)/\mathrm{Im}\, \partial_{p+1}.$$

Since $j_p$ is injective, it maps $\mathrm{Im}\, \partial_{p+1}$ isomorphically onto $\mathrm{Im}\, j_p \circ \partial_{p+1} = \mathrm{Im}\, d_{p+1}$ and it maps $H_p(X^p; R)$ isomorphically onto $\mathrm{Im}\, j_p = \mathrm{Ker}\, \partial_p$, so

$$H_p(X; R) \cong \mathrm{Ker}\, \partial_p/\mathrm{Im}\, d_{p+1}.$$

Since $j_{p-1}$ is injective, $\mathrm{Ker}\, \partial_p = \mathrm{Ker}\, d_p$, thus we obtain an isomorphism

$$H_p(X; R) \cong \mathrm{Ker}\, d_p/\mathrm{Im}\, d_{p+1} = H_p^{\mathrm{CW}}(X; R),$$

as claimed. $\qquad\square$

Theorem 6.8 has the following immediate corollaries:

(1) If the CW complex $X$ has no $p$-cells then $H_p(X; R) = (0)$.

(2) If the CW complex $X$ has $k$ $p$-cells, then $H_p(X; R)$ is generated by at most $k$ elements.

(3) If the CW complex $X$ has no two of its cells in adjacent dimensions, then $H_p(X; R)$ is a free $R$-module with a basis in one-to-one correspondence with the $p$-cells in $X$. This is because whenever there is some $p$-cell, then there are no $(p-1)$-cells and no $(p+1)$-cells so $X^{p-2} = X^{p-1}$ and $X^p = X^{p+1}$, which implies that $H_{p-1}(X^{p-1}, X^{p-2}) = H_{p+1}(X^{p+1}, X^p) = (0)$ and then we have the piece of the cellular chain complex

$$H_{p+1}(X^{p+1}, X^p) = (0) \xrightarrow{\ 0\ } H_p(X^p, X^{p-1}) \xrightarrow{\ d_p\ } (0) = H_{p-1}(X^{p-1}, X^{p-2}),$$

and $H_p(X; R) = \operatorname{Ker} d_p = H_p(X^p, X^{p-1})$.

Property (3) immediately yields the homology of $\mathbb{CP}^n$. Indeed, recall from Example 6.1 that as a CW complex $\mathbb{CP}^n$ has $n+1$ cells

$$e^0, e^2, e^4, \cdots, e^{2n}.$$

Therefore, we get

$$H_p(\mathbb{CP}^n; R) = \begin{cases} R & \text{for } p = 0, 2, 4, \ldots, 2n \\ (0) & \text{otherwise.} \end{cases}$$

We also get the homology of $\mathbb{CP}^\infty$:

$$H_p(\mathbb{CP}^\infty; R) = \begin{cases} R & \text{for } p \text{ even} \\ (0) & \text{otherwise.} \end{cases}$$

Computing the homology of $\mathbb{RP}^n$ is more difficult. The problem is to figure out what are the boundary maps $d_p$.

Generally, in order to be able to compute the cellular homology groups, we need a method to "compute" the boundary maps $d_p$. This can indeed be done in principle, and often in practice although this can be tricky, using the notion of degree of a map of the sphere to itself. To simplify matters assume that $R = \mathbb{Z}$, although any abelian group $G$ will do.

Let $f \colon S^n \to S^n$ be a continuous map. We have the homomorphism $f_* \colon \widetilde{H}_n(S^n; \mathbb{Z}) \to \widetilde{H}_n(S^n; \mathbb{Z})$, and since $\widetilde{H}_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$, the homomorphism $f_*$ must be of the form $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$.

The integer $d$ is called the *degree* of $f$ and is denoted by $\deg f$. The degree is an important invariant of a map $f \colon S^n \to S^n$. Intuitively, the degree $d = \deg f$ measures how many times $f$ wraps around $S^n$ (and preserves or reverses direction). For example, it can be shown that the degree of the antipodal map $-\mathbb{1} \colon S^n \to S^n$ given by $-\mathbb{1}(x) = -x$ is $(-1)^{n+1}$.

Our intention is not to discuss degree theory, but simply to point out that this notion can be used to determine the boundary maps $d_n$. Detailed expositions about degrees of maps can be found in Hatcher [25] (Chapter 2, Section 2.2), Bredon [4] (Chapter IV, Sections 6 and 7), and Rotman [41] (Chapter 6).

To compute $d_p$, for every open $p$-cell $e_i^p \in X$ considered as a chain in $H_p(X^p, X^{p-1}; R)$ and for any open $(p-1)$-cell $e_j^{p-1} \in X$ considered as a chain in $H_{p-1}(X^{p-1}, X^{p-2}; R)$, we define a map $f_{ij} \colon S^{p-1} \to S^{p-1}$ as follows:

1. Let $q_{p-1} \colon X^{p-1} \to X^{p-1}/X^{p-2}$ be the quotient map.

2. Recall that $X^{p-1}/X^{p-2}$ is homeomorphic to the wedge sum of $(p-1)$-spheres $S^{p-1}$, one for each $j \in I_{p-1}$ (this is the disjoint sum of $(p-1)$-spheres with their south pole identified). Let $q_j \colon X^{p-1}/X^{p-2} \to S^{p-1}$ be the projection onto the $j$th sphere. It is the map that collapses all the other spheres in the wedge sum except the $j$th one onto a point (the south pole). Then we let

$$f_{ij} = q_j \circ q_{p-1} \circ f_i | S^{p-1},$$

where $f_i \colon D_i^p \to X$ is the characteristic map of the cell $e_i^p$ and $f_i | S^{p-1}$ is the restriction of $f_i$ to $S^{p-1}$.

The following proposition is proved in Hatcher [25] (Chapter 2, Section 2.2, after Theorem 2.35) and in Bredon [4] (Chapter IV, Section 10, Theorem 10.3).

**Proposition 6.9.** *Let $X$ be a CW complex. Then the boundary map $d_p \colon H_p(X^p, X^{p-1}; \mathbb{Z}) \to H_{p-1}(X^{p-1}, X^{p-2}; \mathbb{Z})$ of the cellular complex $S_*^{\mathrm{CW}}(X; \mathbb{Z})$ associated with $X$ is given by*

$$d_p(e_i^p) = \sum_j d_{ij} e_j^{p-1}$$

*where $d_{ij} = \deg f_{ij}$ is the degree of the map $f_{ij} \colon S^{p-1} \to S^{p-1}$ defined above as the composition $f_{ij} = q_j \circ q_{p-1} \circ f_i | S^{p-1}$.*

The sum in Proposition 6.9 is finite because $f_i$ maps $S^{p-1}$ into a the union of a finite number of cells of dimension at most $p-1$ (by Proposition 6.4(3)). The degrees $d_{i,j}$ are often called *incidence numbers*.

The boundary map $d_1 \colon H_1(X^1, X^0; \mathbb{Z}) \to H_0(X^0; \mathbb{Z})$ is much easier to compute than it appears. Recall that $X^1$ is a graph in which every 1-cell $e_i$ (an edge) is attached to some 0-cells (nodes) $x$ and $y$, with $x$ attached to $-1$ and $y$ attached to $+1$ ($x$ and $y$ may be identical). Then it is not hard to show that

$$d_1(e) = y - x.$$

Details of this computation are given in Bredon [4] (Chapter IV, Section 10).

As an ilustration of Proposition 6.9 we can compute the homology groups of $\mathbb{RP}^n$.

**Example 6.3.** Recall that as a CW complex $\mathbb{RP}^n$ has a cell structure with $n+1$ cells

$$e^0, e^1, e^2, \ldots, e^n.$$

It follows that the cellular cell complex is of the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0.$$

One finds that $d_{k,k_1} = 1 + (-1)^k$; see Hatcher [25] (Chapter 2, Example 2.43). It follows that $d_k$ is either 0 of multiplication by 2 according to the parity of $k$. Thus if $n$ is even we have the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

and if $n$ is odd we have the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

From this we get

$$H_p(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0 \text{ and for } p = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } p \text{ odd, } 0 < p < n \\ (0) & \text{otherwise,} \end{cases}$$

as stated in Section 4.4.

Similarly we find that the homology of $\mathbb{RP}^\infty$ is given by

$$H_p(\mathbb{RP}^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{for } p \text{ odd} \\ (0) & \text{otherwise,} \end{cases}$$

Other examples are given in Hatcher [25] (Chapter 2, Section 2.2). A slightly different approach to incidence numbers is presented in Massey [32] (Chapter IX, Sections 5-7). Massey shows that for special types of CW complexes called *regular complexes* there is a procedure for computing the incidence numbers (see Massey [32] (Chapter IX, Section 7).

The generalization of cellular homology to coefficients in an $R$-module $G$ is immediate. We define the $R$-modules $S_p^{\mathrm{CW}}(X; G)$ by

$$S_p^{\mathrm{CW}}(X; G) = H_p(X^p, X^{p-1}; G),$$

where as before we set $X^{-1} = \emptyset$. The only change in Proposition 6.6 is that

$$H_p(X^p, X^{p-1}; G) \cong \bigoplus_{e_i^p \mid i \in I_p} G$$

is the direct sum of copies of $G$, one for each open $p$-cell of $X$. This means that we can view $H_p(X^p, X^{p-1}; R)$ as the set of formal "vector-valued" linear combinations $\sum_i e_i^p g_i$, where $g_i \in G$ and the $e_i^p$ are open $p$-cells. Then Proposition 6.7 goes through, the boundary maps are defined as before and we get the following theorem.

**Theorem 6.10.** *Let $X$ be a CW complex. For any $R$-module $G$ there are isomorphisms*

$$H_p^{\mathrm{CW}}(X; G) \cong H_p(X; G) \quad \text{for all } p \geq 0$$

*between the cellular homology modules and the singular homology modules of $X$.*

In the next section we take a quick look at cellular cohomology.

## 6.3   Cohomology of CW Complexes

In this section since we will make use of the Universal Coefficient Theorem for cohomology we assume that $R$ is a PID, and we let $G$ be any $R$-module.

By the version of the Universal Coefficient Theorem for cohomology given by Theorem 12.45, since the modules $H_p(X^p, X^{p-1}; R)$ are free (with $X^{-1} = \emptyset$ as before) we have

$$H^p(X^p, X^{p-1}; G) \cong \mathrm{Hom}_R(H_p(X^p, X^{p-1}; R), G)$$
$$H^k(X^p, X^{p-1}; G) = (0) \qquad k \neq p.$$

**Proposition 6.11.** *If $X$ is a CW complex, then the following properties hold:*

(a)  *We have $H^k(X^p, X^{p-1}; G) = (0)$ for all $k \neq p$, and*
$H^p(X^p, X^{p-1}; G) \cong \mathrm{Hom}_R(H_p(X^p, X^{p-1}; R), G).$

(b)  *We have $H^k(X^p; G) \cong (0)$ for all $k > p$.*

(c)  *We have $H^k(X^p; G) \cong H^k(X; G)$ for all $k < p$.*

*Proof.* (a) has already been proved.

(b) We have the following piece of the long exact sequence of cohomology for the pair $(X^p, X^{p-1})$:

$$H^k(X^p, X^{p-1}; G) \longrightarrow H^k(X^p; G) \longrightarrow H^k(X^{p-1}; G) \longrightarrow H^{k+1}(X^p, X^{p-1}; G),$$

and if $k \neq p - 1, p$ we know that $H^k(X^p, X^{p-1}; G) = H^{k+1}(X^p, X^{p-1}; G) \cong (0)$, so we have isomorphisms
$$H^k(X^p; G) \cong H^k(X^{p-1}; G) \quad \text{for all } k \neq p - 1, p.$$
If we assume that $k > p$, then by induction on $p$ we get

$$H^k(X^p; G) \cong H^k(X^0; G) \cong (0).$$

(c) To prove (c) we will use the fact that $H_k(X, X^p; G) = (0)$ for all $k \leq p$. This is proved in Hatcher [25] (Chapter 2, Lemma 2.34) using a construction known as the "mapping telescope." In Milnor and Stasheff [35] it is is claimed that $H_k(X, X^p; G) \cong H_k(X^{p+1}, X^p; G)$, and since $H_k(X^{p+1}, X^p; G) = (0)$ for all $k \neq p + 1$ we conclude that $H_k(X, X^p; G) = (0)$ for all $k \leq p$.

By the Universal Coefficient Theorem for cohomology (Theorem 12.45) we deduce that

$$H^k(X, X^p; G) = (0) \quad \text{for all } k \leq p.$$

Consider the following piece of the long exact sequence of cohomology of the pair $(X, X^p)$:

$$H^k(X, X^p; G) \longrightarrow H^k(X; G) \longrightarrow H^k(X^p; G) \longrightarrow H^{k+1}(X, X^p; G).$$

If $k < p$ then $k + 1 \leq p$ and we know that $H^k(X, X^p; G) = H^{k+1}(X, X^p; G) = (0)$, so we get isomorphisms

$$H^k(X; G) \cong H^k(X^p; G) \quad \text{for all } k < p,$$

as claimed $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In particular, Proposition 6.11 implies that $H^p(X; G) \cong H^p(X^{p+1}; G)$.

Recall that $S_k(X^p, X^{p-1}; G) = S_k(X^p; G)/S_k(X^{p-1}; G)$, so we have the quotient map $\pi_k \colon S_k(X^p; G) \to S_k(X^p, X^{p-1}; G)$ which yields the map $j^k \colon H^k(X^p, X^{p-1}; G) \to H^k(X^p; G)$. Consider the following pieces of the long exact sequences of cohomology for the pairs $(X^{p-1}, X^{p-2})$, $(X^p, X^{p-1})$, and $(X^{p+1}, X^p)$:

$$H^{p-2}(X^{p-2}; G) \longrightarrow H^{p-1}(X^{p-1}, X^{p-2}; G) \overset{j^{p-1}}{\longrightarrow} H^{p-1}(X^{p-1}; G) \longrightarrow H^{p-1}(X^{p-2}; G)$$

$$H^{p-1}(X^{p-1}; G) \overset{\delta^{p-1}}{\longrightarrow} H^p(X^p, X^{p-1}; G) \overset{j^p}{\longrightarrow} H^p(X^p; G) \longrightarrow H^p(X^{p-1}; G)$$

$$H^p(X^{p+1}, X^p; G) \longrightarrow H^p(X^{p+1}; G) \longrightarrow H^p(X^p; G) \overset{\delta^p}{\longrightarrow} H^{p+1}(X^{p+1}, X^p; G).$$

Since by Proposition 6.11 we also have

$$H^{p-1}(X^{p-2}; G) = H^p(X^{p-1}; G) = H^p(X^{p+1}, X^p; G) = (0),$$

and $H^p(X; G) \cong H^p(X^{p+1}; G)$, we have the following diagram:



in which for simplicity of notation we omitted the module $G$, and where $d^{p-1} = \delta^{p-1} \circ j^{p-1}$ and $d^p = \delta^p \circ j^p$. Since $j^p \circ \delta^{p-1} = 0$ (because the sequence on that diagonal is exact), we have

$$d^p \circ d^{p-1} = \delta^p \circ j^p \circ \delta^{p-1} \circ j^{p-1} = 0.$$

**Definition 6.9.** Given a CW complex $X$, the modules $H^p(X^p, X^{p-1}; G)$ together with the coboundary maps $d^p \colon H^p(X^p, X^{p-1}; G) \to H^{p+1}(X^{p+1}, X^p; G)$ form a cochain complex

$S^*_{\mathrm{CW}}(X; G)$ called the *cellular cochain complex* associated with $X$. The cohomlogy modules associated with the cochain complex $S^*_{\mathrm{CW}}(X; G)$ are denoted by

$$H^p_{\mathrm{CW}}(X; G) = H^p(S^*_{\mathrm{CW}}(X; G))$$

and called the *cellular cohomology modules* of the cochain complex $S^*_{\mathrm{CW}}(X; G)$.

**Theorem 6.12.** *Let $X$ be a CW complex. For any PID $R$ and any $R$-module $G$ there are isomorphisms*

$$H^p_{\mathrm{CW}}(X; G) \cong H^p(X; G) \quad \text{for all } p \geq 0$$

*between the cellular cohomology modules and the singular cohomology modules of $X$. Furthermore, the cellular cochain complex $S^*_{\mathrm{CW}}(X; G)$ is isomorphic to the cochain complex $\mathrm{Hom}_R(S^{\mathrm{CW}}_*(X; R), G)$ (the dual of the cellular chain complex $S^{\mathrm{CW}}_*(X; R)$ with respect to $G$).*

*Proof.* The above diagram shows that

$$H^p(X; G) \cong \mathrm{Ker}\, \delta^p.$$

We will need the following simple proposition

**Proposition 6.13.** *If the following diagram is commutative and if $j \colon A \to B$ is surjective*



*then*

$$\mathrm{Ker}\, \delta = \mathrm{Ker}\, d / \mathrm{Ker}\, j.$$

*Proof.* Define a map $\varphi \colon \mathrm{Ker}\, \delta \to \mathrm{Ker}\, d / \mathrm{Ker}\, j$ as follows: for any $b \in \mathrm{Ker}\, \delta$, let

$$\varphi(b) = a + \mathrm{Ker}\, j$$

for any $a \in \mathrm{Ker}\, d$ such that $j(a) = b$. Since $j$ is surjective, there is some $a \in A$ such that $j(a) = b$. Furthermore, for any $a \in A$ such that $j(a) = b \in \mathrm{Ker}\, \delta$, since $d = \delta \circ j$ we have $d(a) = \delta(j(a)) = \delta(b) = 0$, so $a \in \mathrm{Ker}\, d$. This map is well defined because if another $a' \in \mathrm{Ker}\, d$ is chosen such that $j(a') = b$, then $j(a') = j(a)$ so $j(a' - a) = 0$, that is, $a' - a \in \mathrm{Ker}\, j$, so $a + \mathrm{Ker}\, j = a' + \mathrm{Ker}\, j$.

The map $\varphi$ is injective because if $\varphi(b) = \mathrm{Ker}\, j$, since $\varphi(b) = a + \mathrm{Ker}\, j$ for any $a \in \mathrm{Ker}\, d$ such that $j(a) = b$, we have $a + \mathrm{Ker}\, j = \mathrm{Ker}\, j$, which implies that $a \in \mathrm{Ker}\, j$ so $b = j(a) = 0$. The map $\varphi$ is surjective because for any $a + \mathrm{Ker}\, j$ with $a \in \mathrm{Ker}\, d$, by definition of $\varphi$ we have $\varphi(j(a)) = a + \mathrm{Ker}\, j$. Therefore $\varphi \colon \mathrm{Ker}\, \delta \to \mathrm{Ker}\, d / \mathrm{Ker}\, j$ is an isomorphism. $\qquad\square$

Since $j^p$ is surjective Proposition 6.13 show that

$$\operatorname{Ker} \delta^p = \operatorname{Ker} d^p / \operatorname{Ker} j^p,$$

which yields $H^p(X;G) \cong \operatorname{Ker} d^p / \operatorname{Ker} j^p$. But $\operatorname{Ker} j^p = \operatorname{Im} \delta^{p-1}$ so

$$H^p(X;G) \cong \operatorname{Ker} d^p / \operatorname{Im} \delta^{p-1}.$$

Since $j^{p-1}$ is surjective, $\operatorname{Im} \delta^{p-1} = \operatorname{Im} d^{p-1}$, and finally we obtain

$$H^p(X;G) \cong \operatorname{Ker} d^p / \operatorname{Im} d^{p-1} = H^p_{\mathrm{CW}}(X;G),$$

as claimed.

By the naturality part of the Universal Coefficient Theorem for cohomology (Theorem 12.43) applies to the chain map $\pi \colon S_*(X^p;R) \to S_*(X^p, X^{p-1};R)$ and the naturality part of the long exact sequence of relative cohomology of the pair $(X^p, X^{p-1})$, since by definition

$$H_p(X^p;R) = H_p(S_*(X^p;R)), \quad H_p(X^p, X^{p-1};R) = H_p(S_*(X^p, X^{p-1};R)),$$
$$S^*(X^p;G) = \operatorname{Hom}_R(S_*(X^p;R), G), \quad S^*(X^p, X^{p-1};G) = \operatorname{Hom}_R(S_*(X^p, X^{p-1};R), G),$$
$$H^p(X^p;G) = H^p(S^*(X^p;G)), \quad H^p(X^p, X^{p-1};G) = H^p(S^*(X^p, X^{p-1};G)),$$

we obtain the following diagram:

$$
\begin{array}{ccccc}
H^p(X^p, X^{p-1};G) & \xrightarrow{\ j^p\ } & H^p(X^p;G) & \xrightarrow{\ \delta^p\ } & H^{p+1}(X^{p+1}, X^p;G) \\
\downarrow{\scriptstyle h^p} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h^{p+1}} \\
\operatorname{Hom}_R(H_p(X^p, X^{p-1};R), G) & \xrightarrow[\ j_p^*\ ]{} & \operatorname{Hom}_R(H_p(X^p;R), G) & \xrightarrow[\ \partial_{p+1}^*\ ]{} & \operatorname{Hom}_R(H_{p+1}(X^{p+1}, X^p;R), G).
\end{array}
$$

The left square commutes due to naturality (Theorem 12.43), and the right square also commutes due to naturality (of the long exact sequence of relative cohomology), so the big rectangle commutes. Furthermore, by Theorem 12.45 the maps $h^p$ and $h^{p+1}$ are isomorphisms. But the composition of the two maps on the top row is $d^p$, the cellular coboundary map, and the composition of the two maps on the bottom row is $d_{p+1}^* = \operatorname{Hom}_R(d_{p+1};G)$ since $d_{p+1} = j_p \circ \partial_{p+1}$ which implies that $d_{p+1}^* = \partial_{p+1}^* \circ j_p^*$, so we have the commutative diagram

$$
\begin{array}{ccc}
H^p(X^p, X^{p-1};G) & \xrightarrow{\ d^p\ } & H^{p+1}(X^{p+1}, X^p;G) \\
\downarrow{\scriptstyle h^p} & & \downarrow{\scriptstyle h^{p+1}} \\
\operatorname{Hom}_R(H_p(X^p, X^{p-1};R), G) & \xrightarrow[\ d_{p+1}^*\ ]{} & \operatorname{Hom}_R(H_{p+1}(X^{p+1}, X^p;R), G),
\end{array}
$$

which shows that the cellular cochain complex $S^*_{\mathrm{CW}}(X;G)$ is isomorphic to the cochain complex $\operatorname{Hom}_R(S^{\mathrm{CW}}_*(X;R), G)$. $\qquad \square$

As a consequence, although this is not obvious a priori, the cellular cochain complex $S^*_{\mathrm{CW}}(X;G)$ is isomorphic to the cochain complex obtained by applying $\operatorname{Hom}_R(-, G)$ to the cellular chain complex $S^{\mathrm{CW}}_*(X;R)$. Also, the cellular cohomology modules "compute" the singular cohomology modules.

## 6.4   The Euler–Poincaré Characteristic of a CW Complex

In this section we generalize the Euler–Poincaré formula obtained for simplicial complexes in Section 5.3 to CW complexes. Let us assume that our ring $R$ is $R = \mathbb{Z}$ and that $G = \mathbb{Z}$. In this case we abbreviate $H_p(X; \mathbb{Z})$ as $H_p(X)$. We know that if $X$ is a finite CW complex then its homology groups $H_p(X; \mathbb{Z})$ are finitely generated abelian groups. More generally we have the following definition.

**Definition 6.10.** Let $X$ be a topological space. We say that $X$ is of *finite type* if $H_p(X)$ if a finitely generated abelian group for all $p \geq 0$, and $X$ is of *bounded finite type* if it is of finite type and $H_p(X) = 0$ for all but a finite number of indices $p$.

We can now define a famous invariant of a space.

**Definition 6.11.** If $X$ is a space of bounded finite type, then its *Euler–Poincaré characteristic* $\chi(X)$ is defined as

$$\chi(X) = \sum_p (-1)^p \operatorname{rank} H_p(X).$$

Since $X$ is of finite bounded type the above sum contains only finitely many nonzero terms. The natural number $\operatorname{rank} H_p(X) = \operatorname{rank} H_p(X; \mathbb{Z})$ is called the *p-th Betti number of $X$* and is denoted by $b_p$.

If $X$ is a finite CW complex of dimension $n$, then each $p$-skeleton has a finite number of $p$-cells, say $a_p$. Remarkably $\chi(X) = \sum_{p=0}^n (-1)^p a_p$, a formula generalizing Euler's formula in the case of a convex polyhedron. We can now prove the following beautiful result.

**Theorem 6.14.** *(Euler–Poincaré) Let $X$ be a finite CW complex of dimension $n$ and let $a_p$ be the number of $p$-cells in $X$. We have*

$$\chi(X) = \sum_p (-1)^p \operatorname{rank} H_p(X) = \sum_{p=0}^n (-1)^p a_p.$$

*Proof.* As usual let $B_p = \operatorname{Im} d_{p+1} \subseteq S_p^{\mathrm{CW}}(X)$ be the group of $p$-boundaries and let $Z_p = \operatorname{Ker} d_p \subseteq S_p^{\mathrm{CW}}(X)$ be the group of $p$-cycles. By definition $H_p^{\mathrm{CW}}(X) = Z_p / B_p$, by Theorem 6.8 we have $H_p^{\mathrm{CW}}(X) \cong H_p(X)$, and $S_p^{\mathrm{CW}}(X)$ is a free abelian group of rank $a_p$ (the number of $p$-cells). Observe that $B_n = B_{-1} = (0)$. We have the exact sequence

$$0 \longrightarrow Z_p \longrightarrow S_p^{\mathrm{CW}}(X) \longrightarrow B_{p-1} \longrightarrow 0$$

which (by Proposition 5.11) shows that

$$a_p = \operatorname{rank}(S_p^{\mathrm{CW}}(X)) = \operatorname{rank}(Z_p) + \operatorname{rank}(B_{p-1}), \tag{$*$}$$

and the exact sequence

$$0 \longrightarrow B_p \longrightarrow Z_p \longrightarrow H_p(X) \longrightarrow 0$$

which (by Proposition 5.11) shows that

$$\text{rank}(Z_p) = \text{rank}(B_p) + \text{rank}(H_p(X)). \tag{$**$}$$

From equation $(**)$ we obtain

$$\sum_p (-1)^p (\text{rank}(B_p) + \text{rank}(H_p(X))) = \sum_p (-1)^p \text{rank}(Z_p),$$

and from equation $(*)$ we obtain

$$\sum_p (-1)^p \text{rank}(Z_p) = \sum_p (-1)^p (a_p - \text{rank}(B_{p-1})),$$

so we obtain

$$\sum_p (-1)^p (\text{rank}(B_p) + \text{rank}(H_p(X))) = \sum_p (-1)^p (a_p - \text{rank}(B_{p-1}))$$

$$\sum_p (-1)^p \text{rank}(B_p) + \sum_p (-1)^p \text{rank}(H_p(X)) = \sum_p (-1)^p a_p + \sum_p (-1)^{p-1} \text{rank}(B_{p-1}).$$

The sums involving the $B_*$ cancel out because $B_n = B_{-1} = (0)$, and we obtain

$$\sum_p (-1)^p a_p = \sum_p (-1)^p \text{rank}(H_p(X))) = \chi(X),$$

as claimed. $\square$

Theorem 6.14 proves that the number $\sum_{p=0}^n (-1)^p a_p$ is the same for *all* cell structures (of CW complexes) defining a given space $X$. It is a topological invariant.

For example, if $X = S^2$, we know that as a CW complex $S^2$ has two cells $e^0$ and $e^2$, so we get

$$\chi(S^2) = 1 + (-1)^2 \times 1 = 2.$$

As a consequence, if $X$ is any CW complex homeomorphic to $S^2$ with $V$ 0-cells, $E$ 1-cells and $F$ 2-cells, we must have

$$F - E + V = 2,$$

a famous equation due to Euler (for convex polyhedra in $\mathbb{R}^3$). More generally, since the $n$-sphere $S^n$ has a structure with one 0-cell and one $n$-cell, we see that

$$\chi(S^n) = 1 + (-1)^n.$$

This is the Euler–Poincaré characteristic of any convex polytope in $\mathbb{R}^{n+1}$, a formula proved by Poincaré.

For the the real projective plane $\mathbb{RP}^2$ we have a CW cell structure with three cells $e^0, e^1, e^2$, so we get

$$\chi(\mathbb{RP}^2) = 1.$$

In general

$$\chi(\mathbb{RP}^{2n}) = 1 \quad \text{and} \quad \chi(\mathbb{RP}^{2n+1}) = 0.$$

For the torus $T^2$, we have a CW cell structure with four cells $e^0, e_1^1, e_2^1, e^2$, so we get

$$\chi(T^2) = 0.$$

More generally, since the homology groups of the $n$-torus $T^n$ are given by

$$H_p(T^n) = \mathbb{Z}^{\binom{n}{p}},$$

using the fact that $0 = (1-1)^n = \sum_{p=0}^n (-1)^p \binom{n}{p}$, we have

$$\chi(T^n) = \sum_{p=0}^n (-1)^p \binom{n}{p} = 0.$$

If $R$ is any ring and if $X$ is a space of bounded finite type, then its *Euler–Poincaré characteristic* $\chi_R(X)$ is defined as

$$\chi_R(X) = \sum_p (-1)^p \operatorname{rank} H_p(X; R),$$

where $\operatorname{rank} H_p(X; R)$ is the rank of $R$-module $H_p(X; R)$. Since Proposition 5.11 actually holds for finitely generated modules over an integral domain $R$ (see Proposition 12.9), and since the rest of the proof of Theorem 6.14 does not depend on the ring $R$, we have the following slight generalization of Theorem 6.14.

**Theorem 6.15.** *(Euler–Poincaré) Let $X$ be a finite CW complex of dimension $n$ and let $a_p$ be the number of $p$-cells in $X$. For any integral domain $R$, we have*

$$\chi_R(X) = \sum_p (-1)^p \operatorname{rank} H_p(X; R) = \sum_{p=0}^n (-1)^p a_p.$$

Thus, for finite CW complexes, the Euler–Poincaré characteristic

$$\chi_R(X) = \sum_p (-1)^p \operatorname{rank} H_p(X; R)$$

is independent of the ring $R$, as long as it is an integral domain. This fact is also noted in Greenberg and Harper in the special case where $R$ is a PID; see [19] (Chapter 20, Remark 20.19).

We also have the following proposition showing that for any space $X$ of bounded finite type, the Euler–Poincaré characteristic $\chi_R(X) = \sum_p (-1)^p \operatorname{rank} H_p(X; R)$ is independent the ring $R$, provided that it is a PID.

**Proposition 6.16.** *Let $X$ be any space of bounded finite type and let $R$ be any PID. Then we have*

$$\chi_R(X) = \sum_p (-1)^p \operatorname{rank} H_p(X; R) = \chi(X) = \sum_p (-1)^p \operatorname{rank} H_p(X; \mathbb{Z}).$$

*Proof.* We use the Universal Coefficient Theorem for homology (Theorem 12.38) and the following two facts:

$$\operatorname{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \cong \operatorname{Ker}(A \xrightarrow{m} A),$$

and

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/mA,$$

where $A$ is any abelian group. Since

$$H_p(X; R) \cong (H_p(X; \mathbb{Z}) \otimes_{\mathbb{Z}} R) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{p-1}(X; \mathbb{Z}), R),$$

every term $\mathbb{Z}^k$ in $H_p(X; \mathbb{Z})$ after being tensored with $R$ yields the term $R^k$ in $H_p(X; R)$, and every term $\mathbb{Z}/m\mathbb{Z}$ in $H_p(X; \mathbb{Z})$ after being tensored with $R$ yields the term $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} R \cong R/mR$ in $H_p(X; R)$, and the term $\operatorname{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, R) \cong \operatorname{Ker}(R \xrightarrow{m} R)$ in $H_{p+1}(X; R)$. Since $R$ is a PID, we have $\operatorname{Ker}(R \xrightarrow{m} R) = sR$ for some natural number $s$, so we have the exact sequence

$$0 \longrightarrow sR \longrightarrow R \longrightarrow mR \longrightarrow 0,$$

and since $R$ is a PID it is an integral domain so the module $mR$ is free over $R$ and the above sequence splits, which implies that

$$R \cong sR \oplus mR,$$

and thus

$$R/mR \cong sR.$$

Either $sR \not\cong R$, in which case $R/mR \cong sR$ is a torsion term that does not contribute to the sum $\sum_p (-1)^p \operatorname{rank} H_p(X; R)$, or $R/mR \cong sR \cong R$, in which case the contributions of the terms $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} R \cong R$ and $\operatorname{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, R) \cong R$ to the sum $\sum_p (-1)^p \operatorname{rank} H_p(X; R)$ cancel out since they have the signs $(-1)^p$ and $(-1)^{p+1}$, which proves that

$$\sum_p (-1)^p \operatorname{rank} H_p(X; R) = \sum_p (-1)^p \operatorname{rank} H_p(X; \mathbb{Z}),$$

as claimed. $\square$

Proposition 6.16 justifies using the ring $\mathbb{Z}$ in the definition of the Euler–Poincaré characteristic. This remark is also made in Greenberg and Harper; see [19] (Chapter 20, Remark 20.19).

# Chapter 7

# Poincaré Duality

Our goal is to state a version of the Poincaré duality for singular homology and cohomology. The basic version is that if $M$ is a "nice" $n$-manifold, then there are isomorphisms

$$H^p(M; \mathbb{Z}) \cong H_{n-p}(M; \mathbb{Z}) \qquad (*)$$

for all $p \geq 0$. Here, nice means compact and orientable, a notion that will be defined in Section 7.1.

The isomorphisms $(*)$ are actually induced by an operation

$$\frown \colon S^p(M; \mathbb{Z}) \times S_n(M; \mathbb{Z}) \to S_{n-p}(M; \mathbb{Z})$$

combining a chain and a cochain to make a chain, called *cap product*, which induces an operation

$$\frown \colon H^p(M; \mathbb{Z}) \times H_n(M; \mathbb{Z}) \to H_{n-p}(M; \mathbb{Z})$$

combining a homology class and a cohomology class to make a homology class. Furthermore, if $M$ is orientable, then there is a unique special homology class $\mu_M \in H_n(X; \mathbb{Z})$ called the *fundamental class* of $M$, and Poincaré duality means that the map

$$c \mapsto c \frown \mu_M$$

is an isomorphism between $H^p(M; \mathbb{Z})$ and $H_{n-p}(M; \mathbb{Z})$.

All this can be generalized to coefficients in any commutative ring $R$ with an identity element and to compact manifolds that are $R$-orientable, a notion defined in Section 7.1.

It is even possible to generalize Poincaré duality to noncompact $R$-orientable manifolds, by replacing singular cohomology by the more general notion of singular cohomology with compact support. We will sketch all this in the following sections. We begin with the notion of orientation.

# 7.1  Orientations of a Manifold

Since 0-dimensional manifolds constitute a degenerate case of little interest (discrete sets of points), we assume that $n > 0$.

If $M$ is a topological manifold of dimension $n$ and if $R$ is any commutative ring with multiplicative unit, we saw in Proposition 4.21 that

$$H_p(M, M - \{x\}; R) \cong \begin{cases} R & \text{if } p = n \\ (0) & \text{if } p \neq n. \end{cases}$$

Since the groups $H_n(M, M - \{x\}; R)$ are all isomorphic to $R$, a way to define a notion of orientation is to pick some generator $\mu_x$ from $H_n(M, M - \{x\}; R)$, for every $x \in M$. Since $H_n(M, M - \{x\}; R)$ is a ring with a unit, generators are just invertible elements. To say that $M$ is orientable means that we can pick these $\mu_x \in H_n(M, M - \{x\}; R)$ in such a way that they "vary continuously" with $x$.

A way to achieve this is introduce the notion of fundamental class of $M$ at a subspace $A$.

**Definition 7.1.** Given an $n$-manifold $M$ and any subset $A$ of $M$, a *R-fundamental (homology) class of $M$ at the subspace $A$* is a homology class $\mu_A \in H_n(M, M - A; R)$ such that

$$\rho_x^A(\mu_A) = \mu_x \in H_n(M, M - \{x\}; R)$$

is a generator of $H_n(M, M - \{x\}; R)$ for all $x \in A$, where $\rho_x^A \colon H_n(M, M - A; R) \to H_n(M, M - \{x\}; R)$ is the homomorphism induced by the inclusion $M - A \subseteq M - \{x\}$. If $A = M$, we call $\mu_M$ a *R-fundamental (homology) class of $M$.*

An *R-orientation* of $M$ is an open cover $\mathcal{U} = (U_i)_{i \in I}$ together with a family $(\mu_{U_i})_{i \in I}$ of fundamental classes of $M$ at $U_i$ such that whenever $U_i \cap U_j \neq \emptyset$, then

$$\rho_{U_i \cap U_j}^{U_i}(\mu_{U_i}) = \rho_{U_i \cap U_j}^{U_j}(\mu_{U_j}), \tag{$\dagger$}$$

where $\rho_{U_i \cap U_j}^{U_i} \colon H_n(M, M - U_i; R) \to H_n(M, M - U_i \cap U_j; R)$ and $\rho_{U_i \cap U_j}^{U_j} \colon H_n(M, M - U_j; R) \to H_n(M, M - U_i \cap U_j; R)$ are the homomorphisms induced by the inclusions $U_i \cap U_j \subseteq U_i$ and $U_i \cap U_j \subseteq U_j$.

When $R = \mathbb{Z}$, we use the terminology fundamental classes and orientations (we drop the prefix $R$). For simplicity of notation, we write $\mu_i$ instead of $\mu_{U_i}$.

Observe that if $(\mathcal{U} = (U_i)_{i \in I}, (\mu_i)_{i \in I})$ is a $R$-orientation of $M$, since $\rho_x^{U_i} = \rho_x^{U_i \cap U_j} \circ \rho_{U_i \cap U_j}^{U_i}$ and $\rho_x^{U_j} = \rho_x^{U_i \cap U_j} \circ \rho_{U_i \cap U_j}^{U_j}$, the condition $\rho_{U_i \cap U_j}^{U_i}(\mu_i) = \rho_{U_i \cap U_j}^{U_j}(\mu_j)$ implies that

$$\rho_x^{U_i}(\mu_i) = \rho_x^{U_j}(\mu_j) \quad \text{for all } x \in U_i \cap U_j,$$

that is, the $R$-orientation is indeed consistent.

**Remark:** Readers familiar with differential geometry will observe the analogy between a fundamental class and a (global) volume form in the case where the $n$-manifold is smooth. In the smooth case, there is a tangent space at every point $x \in M$, and an orientation is given by a nonzero global section $\omega$ of the bundle $\bigwedge^n T^*M$. In the absence of the tangent bundle, the substitute is the orientation bundle whose fibres are the homology rings $H_n(M, M - \{x\}; R)$.

For any $x \in M$, for any chart $\varphi_U \colon U \to \Omega$ where $U$ is an open subset of $M$ containing $x$, if $D$ is a closed ball of center $\varphi_U(x)$ contained in $\Omega \subseteq \mathbb{R}^n$, then $B = \varphi_U^{-1}(D)$ is a compact subset of $M$ and we call it a *compact and convex subset of $M$ at $x$*. Then, a minor modification of Proposition 4.21 can be used to show the following fact (which is proved in Bredon [4], Chapter VI, Proposition 7.1).

**Proposition 7.1.** *Given a topological $n$-manifold $M$, for any point $x \in M$ and any compact and convex subset $B$ of $M$ at $x$, the homomorphism $\rho_x^B \colon H_n(M, M - B; R) \to H_n(M, M - x; R)$ induced by the inclusion $M - B \subseteq M - x$ is an isomorphism.*

Proposition 7.1 shows that for any small enough compact subset $B$, the manifold $M$ has an $R$-fundamental class at $B$. It is also easy to sbow that Proposition 7.1 implies that condition (†) in Definition 7.1 can be replaced by the condition

$$\rho_x^{U_i}(\mu_i) = \rho_x^{U_j}(\mu_j) \quad \text{for all } x \in U_i \cap U_j.$$

Some textbooks use this condition instead of (†).

If a manifold $M$ has an $R$-fundamental class, then it has an $R$-orientation, since for any open cover $\mathcal{U} = (U_i)_{i \in I}$ of $M$ we have $\rho_x^M = \rho_x^{U_i} \circ \rho_{U_i}^M$, so we can take $\mu_i = \rho_{U_i}^M(\mu_M) \in H_n(M, M - U_i; R)$. The converse holds if either $M$ is compact or if $R = \mathbb{Z}/2\mathbb{Z}$. In the latter case, since $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, the only generator is 1 so this case is trivial.

**Remark:** There are other ways of defining $R$-orientability. One can define the *orientation bundle $M_R$* of $M$ by taking the disjoint union of the groups $H^n(M, M - \{x\}; R)$ where $x$ ranges over $M$, and giving it a suitable topology that amounts to a local consistency condition for $R$-orientatbility. Then, an $R$-orientation is a continuous section $s \colon M \to M_R$ that picks a generator of $H^n(M, M - \{x\}; R)$ for every $x \in M$. We refer the reader to Hatcher [25] (Chapter 3, Section 3.3), Bredon [4] (Chapter VI, Section 7), and Spanier [47] (Chapter 6, Sections 2 and 3). The notion of $R$-orientation in Definition 7.1 corresponds to the notion of a $\mathcal{U}$-compatible family in Spanier [47] (Chapter 6, Sections 3). Milnor and Stasheff [35] use a condition using the notion of a small cell, as defined in Spanier [47] (Chapter 6, Sections 3). The equivalence of the condition of Definition 7.1 with the orientation bundle condition amounts to the proof of Theorem 4 in Spanier [47] (Chapter 6, Sections 3); see also Proposition 7.3 in Bredon [4] (Chapter VI, Section 7).

It can also be shown that a connected nonorientable $n$-manifold has a two-sheeted connected covering space which is orientable. This implies that every simply connected manifold is orientable; see Hatcher [25] (Chapter 3, Section 3.3, Proposition 3.25).

We see that we are naturally led to the study of the groups $H_n(M, M - K; R)$, where $K$ is a compact subset of $M$. We have the following theorems.

**Theorem 7.2.** *(Vanishing) Let $M$ be an $n$-manifold. We have $H_p(M; R) = (0)$ if $p > n$.*

Theorem 7.2 is proved in Hatcher [25] (Chapter 3, Theorem 3.26 and Proposition 3.39), May [34] (Chapter 20, Section 4), and Bredon [4] (Chapter VI, Theorem 7.8). The proof is quite technical.

**Theorem 7.3.** *Let $M$ be an $n$-manifold. For any compact subset $K$, we have $H_p(M, M - K; R) = (0)$ if $p > n$. For any homology class $\alpha \in H_p(M, M - K; R)$, we have $\alpha = 0$ iff $\rho_x^K(\alpha) = 0$ for all $x \in K$, where $\rho_x^K \colon H_p(M, M - K; R) \to H_p(M, M - x; R)$ is the homomorphism induced by the inclusion $M - K \subseteq M - x$.*

Theorem 7.3 is proved in Hatcher [25] (Chapter 3, Lemma 3.27), in May [34] (Chapter 20, Section 3), in Milnor and Stasheff [35] (Appendix A, Lemma A.7), and Bredon [4] (Chapter VI, Theorem 7.8).

The proof technique used to prove Theorems 7.2 and 7.3 as well as a number of other results, is a type of induction on compact subsets involving some limit argument. It is nicely presented in Bredon [4] (Chapter VI, Section VI), where it is called the *Bootstrap Lemma*. Omitting proofs, here is a presentation of this method.

### The Bootstrap Method

Given an $n$-manifold $M$, we would like to prove some property $P_M(A)$ about closed subsets $A$ of $M$. Consider the following five properties:

(i) If $A$ is a compact and convex subset of $M$, then $P_M(A)$ holds.

(ii) If $P_M(A), P_M(B)$ and $P_M(A \cap B)$ hold for some closed subsets $A$ and $B$, then $P_M(A \cup B)$ holds.

(iii) if $A_1 \supseteq A_2 \supseteq \cdots A_i \supseteq A_{i+1} \supseteq \cdots$ is a sequence of compact subsets and if $P_M(A_i)$ holds for all $i$, then $P\left(\bigcap_i A_i\right)$ holds.

(iv) If $(A_i)_{i \in I}$ is a family of disjoint compact subsets with disjoint neigborhoods and if $P_M(A_i)$ holds for all $i$, then $P\left(\bigcup_i A_i\right)$ holds.

(v) For any closed subset $A$, if $P_M(A \cap W)$ holds for all open subsets $W$ of $M$ such that the closure of $W$ is compact, then $P_M(A)$ holds.

We have the following proposition shown in Bredon [4] (Chapter VI, Section 7, Lemma 7.9) and called the Bootstrap Lemma.

**Proposition 7.4.** *(Bootstrap Lemma) Let $M$ be any $n$-manifold.*

(1) *Let $P_M(A)$ be a property about compact subsets $A$ of $M$. If (i), (ii), and (iii) hold, then $P_M(A)$ holds for all compact subsets $A$ of $M$.*

(2) *If $M$ is a separable metric space, $P_M(A)$ be a property about closed subsets $A$ of $M$, and all four statements (i)–(iv) hold, then $P_M(A)$ holds for all closed subsets $A$ of $M$.*

(3) *Let $P_M(A)$ be a property about closed subsets $A$ of $M$. If all five statements (i)–(v) hold, then $P_M(A)$ holds for all closed subsets $A$ of $M$.*

When applying Proposition 7.4 to prove Theorems 7.2 and 7.3, Property (ii) is proved using a Mayer–Vietoris sequence and the five lemma (see Bredon [4], Chapter VI, Section 7).

The next theorem tells us what the group $H_n(M; R)$ looks like.

**Theorem 7.5.** *Let $M$ be an n-manifold. If $M$ is connected then*

$$H_n(M; R) = \begin{cases} R & \text{if } M \text{ is compact and orientable} \\ \text{Ker}\,(R \xrightarrow{2} R) & \text{if } M \text{ is compact and not orientable} \\ (0) & \text{if } M \text{ is not compact.} \end{cases}$$

*Here, the map $R \xrightarrow{2} R$ is the map $r \mapsto 2r$.*

Theorem 7.5 is proved in [4] (Chapter VI, Corollary 7.12) and Hatcher [25] (Chapter 3, Therorem 3.26, Lemma 3.27 and Proposition 3.29), In particular, Theorem 7.5 shows that if $R = \mathbb{Z}$ and if $M$ is compact and not orientable then $H_n(M; R) = (0)$, and that if $M$ is compact then $H_n(M; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Theorem 7.5 yields a crisp characterization of the orientability of a compact $n$-manifold (when $R = \mathbb{Z}$) in terms of the vanishing of $H_n(M; \mathbb{Z})$.

**Proposition 7.6.** *If $M$ is a connected and compact n-manifold, then either $H_n(M; \mathbb{Z}) = (0)$ and $M$ is not orientable, or $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$, $M$ is orientable, and the homomorphisms $H_n(M; \mathbb{Z}) = H_n(M, \emptyset; \mathbb{Z}) \longrightarrow H_n(M, M - \{x\}; \mathbb{Z})$ are isomorphisms for all $x \in M$.*

Proposition 7.6 is a special case of Corollary 8 in Spanier [47] (Chapter 6, Section 3). It is also proved in May [34] (Chapter 20, Section 3). This second proof only uses Theorem 7.5 together with the Universal Coefficient Theorem for homology (Theorem 12.42), but it is a nice proof worth presenting.

*Proof.* Since $M$ is a compact manifold, for any $x \in M$, the manifold $M - \{x\}$ is not compact. By Theorem 7.5, we have $H_n(M - \{x\}; R) = (0)$. The long exact sequence of relative homology of the pair $(M, M - \{x\})$ yields the exact sequence

$$H_n(M - \{x\}; R) \longrightarrow H_n(M; R) \longrightarrow H_n(M, M - \{x\}; R),$$

and since $H_n(M - \{x\}; R) = (0)$ we deduce that

$$H_n(M; R) \longrightarrow H_n(M, M - \{x\}; R) \cong R$$

is an injective homomorphism for every ring $R$. We would like to conclude that if $R = \mathbb{Z}$ and if $H_n(M; \mathbb{Z}) \neq (0)$, then $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ and the above map is an isomorphism.

Since $\text{Tor}^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = (0)$ (see the discussion after Theorem 12.42), by Theorem 12.41 we have

$$H_n(M; \mathbb{Z}/p\mathbb{Z}) \cong H_n(M; \mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z}$$

and similarly

$$H_n(M, M - \{x\}; \mathbb{Z}/p\mathbb{Z}) \cong H_n(M, M - \{x\}; \mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z}$$

for all $p > 0$. Since $H_n(M; R) \longrightarrow H_n(M, M - \{x\}; R) \cong R$ is an injective homomorphism for every ring $R$, the homomorphism

$$H_n(M; \mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z} \longrightarrow H_n(M, M - \{x\}; \mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \qquad (*)$$

is injective for all $p > 0$. If $H_n(M; \mathbb{Z}) \neq (0)$, then we must have $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$, since every abelian subgroup of $\mathbb{Z}$ is of the form $m\mathbb{Z}$ for some $m > 0$, but $m\mathbb{Z} \cong \mathbb{Z}$ (both are freely generated, send $m$ to 1). Finally, the map $H_n(M; \mathbb{Z}) \longrightarrow H(M, M - \{x\}; \mathbb{Z})$ must be an isomorphism since otherwise 1 would be mapped to some $m > 1$, but then the map $(*)$ would not be injective for $p = m$ (since $m \otimes z = 0$ for all $z \in \mathbb{Z}/m\mathbb{Z}$). $\qquad\square$

Finally, we have our major result.

**Theorem 7.7.** *Let $M$ be an $n$-manifold. For any compact subset $K$ of $M$, if $M$ is $R$-orientable, then there is a unique $R$-fundamental class $\mu_K$ of $M$ at $K$. In particular, if $M$ is compact or if $R = \mathbb{Z}/2\mathbb{Z}$, then $M$ has a unique $R$-fundamental class $\mu_M$.*

Theorem 7.7 is proved in Hatcher [25] (Chapter 3, Lemma 3,27), in May [34] (Chapter 20, Section 3), and in Milnor and Stasheff [35] (Appendix A, Theorem A.8).

The fundamental class of a compact orientable manifold $M$ is often denoted by $[M]$.

An important (and deep fact) about a compact manifold $M$ is that its homology groups are finitely generated. This is not easy to prove; see Bredon [4] (Appendix E, Corollary E.5), and Hatcher [25] (Appendix, Topology of Cell Complexes, Corollaries A.8 and A.9). As a consequence, using the Universal Coefficient Theorem for cohomology (Theorem 12.48) we have the following result about the cohomology group $H^n(M; R)$ (see Bredon [4], Chapter VI, Section 7, Corollary 7.14).

**Proposition 7.8.** *For any $n$-manifold $M$, if $M$ is compact, then*

$$H^n(M; R) = \begin{cases} R & \text{if } M \text{ is orientable} \\ R/2R & \text{if } M \text{ is not orientable.} \end{cases}$$

It should also be noted that if $M$ is a smooth manifold, then the notion of orientability in terms of Jacobians of transition functions or the existence of a volume form, as defined for instance in Warner [50] or Tu [49], is equivalent to the notion of orientability given in Definition 7.1. This is proved (with a bit of handwaving) in Bredon [4] (Chapter VI, Section 7, Theorem 7.15).

The second step to state the Poincaré duality Theorem is to define the cap-product.

## 7.2   The Cap Product

Recall the definition of the maps $\lambda_p \colon \Delta^p \to \Delta^{p+q}$ and $\rho_q \colon \Delta^q \to \Delta^{p+q}$ defined in Section 4.8.

**Definition 7.2.** Given a cochain $c \in S^p(X; R)$ and a chain $\sigma \in S_n(X; R)$ (with $n \geq p \geq 0$), define the *cap product* $c \frown \sigma$ as the chain in $S_{n-p}(X; R)$ given by

$$c \frown \sigma = c(\sigma \circ \rho_p)(\sigma \circ \lambda_{n-p})$$

where $\sigma \circ \lambda_{n-p}$ is the front $(n-p)$-face of $\Delta^n$ and $\sigma \circ \rho_p$ is the back $p$-face of $\Delta^n$.

Since $\sigma \circ \rho_p \in S_p(X; R)$ and $\sigma \circ \lambda_{n-p} \in S_{n-p}(X; R)$ we have $c(\sigma \circ \rho_p) \in R$, and indeed $c(\sigma \circ \rho_p)(\sigma \circ \lambda_{n-p}) \in S_{n-p}(X; R)$.

Definition 7.2 is designed so that

$$a(b \frown \sigma) = (a \smile b)(\sigma)$$

for all $a \in S^{n-p}(X; R)$, all $b \in S^p(X; R)$, and all $\sigma \in S_n(X; R)$, or equivalently using the bracket notation for evaluation as

$$\langle a, b \frown \sigma \rangle = \langle a \smile b, \sigma \rangle,$$

which shows that $\frown$ is the adjoint of $\smile$ with respect to the evaluation pairing $\langle -, - \rangle$.

The reader familiar with exterior algebra and differential forms will observe that the cap product is type a of contraction (or hook).

**Remark:** There are several variants of Definition 7.2. Our version is the one adopted by Munkres [38] (Chapter 8, Section 66). Milnor and Stasheff [35] use the same formula except for the presence of the sign $(-1)^{p(n-p)}$ (also recall their sign convention for the coboundary operator). Hatcher [25] (Chapter 3, Section 3.3) uses the formula

$$c \frown \sigma = c(\sigma \circ \lambda_p)(\sigma \circ \rho_{n-p}).$$

Bredon [4] (Chapter VI, Section 5) uses the above formula, with the sign $(-1)^{p(n-p)}$ as in Milnor and Stasheff. In the end, this makes no difference but one has to be very careful about signs when stating the formula for $\partial(c \frown \sigma)$.

**Proposition 7.9.** *For any* $c \in S^p(X; R)$ *and any* $\sigma \in S_n(X; R)$, *we have*

$$\partial(c \frown \sigma) = (-1)^{n-p}(\delta c \frown \sigma) + c \frown \partial\sigma.$$

*Furthermore, we have*

$$\epsilon \frown \sigma = \sigma$$

*for all* $\sigma \in S_n(X; R)$, *and*

$$c \frown (d \frown \sigma) = (c \smile d) \frown \sigma,$$

*for all* $c \in S^p(X; R)$, *all* $d \in S^q(X; R)$, *and all* $\sigma \in S_{p+q+r}(X; R)$.

Proposition 7.9 is from Munkres [38] (Chapter 8, Section 66, Theorem 66.1). As a consequence of the first formula, we see that the cap product induces an operation on cohomology and homology classes

$$\frown: H^p(X; R) \times H_n(X; R) \to H_{n-p}(X; R)$$

(if $0 \le p \le n$), also called *cap product*. The following properties are immediate consequences of Proposition 7.9.

**Proposition 7.10.** *For any* $a \in H_n(X; R)$ *we have*

$$1 \frown a = a,$$

*and*

$$\omega \frown (\eta \frown a) = (\omega \smile \eta) \frown a,$$

*for all* $\omega \in H^p(X; R)$, *all* $\eta \in H^q(X; R)$, *and all* $a \in H_{p+q+r}(X; R)$.

Recall that $\epsilon: S_0(X; R) \to R$ is the unique homomorphism such that $\epsilon(x) = 1$ for every point $x \in S_0(X; R)$. Given any cochain $c \in S^p(X; R)$ and any chain $\sigma \in S_p(X; R)$, the operation (evaluation) $(c, \sigma) \mapsto c(\sigma)$ is bilinear, and it is easy to check that it induces a bilinear map $\langle -, - \rangle: H^p(X; R) \times H_p(X; R) \longrightarrow R$ called the *Kronecker index*.

**Proposition 7.11.** *Let* $M$ *be an* $n$-*manifold. For all* $\omega \in H^p(X; R)$ *and all* $a \in H_p(X; R)$, *we have*

$$\epsilon^*(\omega \frown a) = \langle \omega, a \rangle,$$

*with* $0 \le p \le n$.

Proposition 7.11 is proved in Munkres [38] (Chapter 8, Section 66, Theorem 66.3). There is also a version of the cap product for relative homology and cohomology,

$$\frown: H^p(X, A; R) \times H_n(X, A \cup B; R) \to H_{n-p}(X, B; R),$$

where $A$ and $B$ are open in $X$. We will need the version where $B = \emptyset$ in the proof of the Poincaré duality theorem, namely

$$\frown: H^p(X, A; R) \times H_n(X, A; R) \to H_{n-p}(X; R).$$

Proposition 7.10 also holds for this version of the cap product.

# 7.3 Cohomology with Compact Support

We define a subcomplex $S_c^*(X; R)$ of $S^*(X; R)$ where each module $S_c^p(X; R)$ consists of cochains with compact support as follows.

**Definition 7.3.** A cochain $c \in S^p(X; R)$ is said to have *compact support* if there is some compact subset $K \subseteq X$ such that such that $c \in S^p(X, X - K; R)$, or equivalently if $c$ has value zero on every singular simplex in $X - K$. For such a cochain $c$ we see that $\delta c$ also vanishes on all singular simplices in $X - K$, so the modules $S_c^p(X; R)$ of cochains with compact support form a subcomplex $S_c^*(X; R)$ of $S^*(X; R)$ whose cohomology modules are denoted $H_c^p(X; R)$ and called *cohomology groups with compact support*.

It turns out that the group $H_c^p(X; R)$ can be conveniently expressed as the direct limit of the groups of the form $H^p(X, X - K; R)$ where $K$ is compact. Observe that if $K$ and $L$ are any two compact subsets of $X$ and if $K \subseteq L$, then $S^p(X; X - K; R) \subseteq S^p(X, X - L; R)$ , so we have a module homomorphism $\rho_L^K \colon H^p(X, X - K; R) \to H^p(X, X - L; R)$. The family $\mathcal{K}$ of all compact subsets of $X$ ordered by inclusion is a directed set since the union of two compact sets is compact, so the direct limit

$$\varinjlim_{K \in \mathcal{K}} H^p(X, X - K; R)$$

of the mapping family $(H^p(X, X - K; R)_{K \in \mathcal{K}}, (\rho_L^K)_{K \subseteq L})$ is well-defined; see Section 9.3.

**Proposition 7.12.** *We have isomorphisms*

$$H_c^p(X; R) \cong \varinjlim_{K \in \mathcal{K}} H^p(X, X - K; R)$$

*for all $p \geq 0$. Furthermore, if $X$ is compact, then $H_c^p(X; R) \cong H^p(X; R)$.*

Proposition 7.12 is actually not hard to prove; see Hatcher [25] (Chapter, Section 3.3, just after Proposition 3.33). Intuitively, $X$ is approximated by larger and larger compact subsets $K$. If $K$ is very large, $X - K$ is very small, so the group $H^p(X, X - K; R)$ is a "good" approximation of $H_c^p(X; R)$.

**Remark:** Unlike the case for ordinary singular cohomology, if $f \colon X \to Y$ is a continuous map, there is not necessarily an induced map $f^* \colon H_c^p(X; R) \to H_c^p(Y; R)$. The problem is that if $K$ is a compact subset of $Y$, then $f^{-1}(K)$ is not necessarily compact. However, *proper* maps have this property and induce a corresponding map between cohomology groups with compact support. Fortunately, the maps invoved in Poincaré duality are inclusions and they are proper.

We know from Theorem 7.7 that if $K$ is compact and if the $n$-manifold $M$ is $R$-orientable, then there is a unique $R$-fundamental class $\mu_K \in H_n(M, M - K; R)$ of $M$ at $K$. In particular,

if $M$ itself is compact and $R$-orientable, then there is a $R$-fundamental class $\mu_M$. In this case (if $0 \leq p \leq n$) we have a map

$$D_M \colon H^p(M; R) \to H_{n-p}(M; R)$$

given by

$$D_M(\omega) = \omega \frown \mu_M.$$

Poincaré duality asserts that this map is an isomorphism.  To extend this isomorphism to cohomology with compact support when $M$ is $R$-orientable we need to define $D_M$ for noncompact spaces. We do this as follows.

Recall that there is a cap product

$$\frown \colon H^p(M, M - K; R) \times H_n(M, M - K; R) \to H_{n-p}(M; R).$$

Since there is an isomorphism

$$H^p_c(M; R) \cong \varinjlim_{K \in \mathcal{K}} H^p(M, M - K; R),$$

for any $\omega \in H^p_c(M; R)$ we pick some representative $\omega'$ in the equivalence class defining $\omega$ in $\varinjlim H^p(M, M - K; R)$, namely some $\omega' \in H^p(M, M - K; R)$ for some compact subset $K$, and since $\mu_K \in H_n(M, M - K; R)$ we set

$$D_M(\omega) = \omega' \frown \mu_K \in H_{n-p}(M; R).$$

We need to prove that the above definition does not depend on the choice of the representative $\omega' \in H^p(M, M - K; R)$. If $\omega'' \in H^p(M, M - L; R)$ is another representative for some compact subset such that $K \subseteq L$, then it is easy to show that the diagram

$$
\begin{array}{ccc}
H^p(M, M - K; R) & \longrightarrow & H^p(M, M - L; R) \\
 & \searrow{\scriptstyle -\frown\mu_K} \quad \swarrow{\scriptstyle -\frown\mu_L} & \\
 & H_{n-p}(M; R) &
\end{array}
$$

is commutative, and thus

$$D_M \colon H^p_c(M; R) \to H_{n-p}(M; R)$$

as specified above is indeed well-defined.

## 7.4   The Poincaré Duality Theorem

The following theorem is a very general version of Poincaré duality applying to compact as well as noncompact manifolds.

**Theorem 7.13.** *(Poincaré Duality Theorem) Let $M$ be an $n$-manifold and let $R$ be a PID. If $M$ is $R$-orientable, then the map*

$$D_M \colon H_c^p(M; R) \to H_{n-p}(M; R)$$

*defined in Section 7.3 is an isomorphism for all $p \geq 0$. In particular, if $R = \mathbb{Z}/2\mathbb{Z}$, the above map is an isomorphism whether $M$ is orientable or not.*

*If $M$ is compact and $R$-orientable, then the map*

$$D_M \colon \omega \mapsto \omega \frown \mu_M$$

*is an isomorphism between $H^p(M; R)$ and $H_{n-p}(M; R)$.*

A proof of Theorem 7.4 can be found in Milnor and Stasheff [35] (Appendix A, pages 277-279), Hatcher [25] (Chapter 3, Theorem 3.35), and Greenberg and Harper [19] (Part III, Section 26, Theorem 26.6). Although the proof in these texts is not presented as an application of the Bootstrap Lemma, it is.

The proof of Case (1) of the Bootstrap Lemma is instructive (see Milnor and Stasheff [35], Appendix A, page 278). Consider case where $M = \mathbb{R}^n$ and $B$ is any closed ball. Then we know by Proposition 7.1 that $H_n(\mathbb{R}^n, \mathbb{R}^n - B; R) \cong R$ with generator $\mu_B$, and $H_p(\mathbb{R}^n, \mathbb{R}^n - B; R) = (0)$ for all $p \neq n$. By Theorem 12.48 or Theorem 4.27 we have

$$H^n(\mathbb{R}^n, \mathbb{R}^n - B; R) \cong \operatorname{Hom}_R(H_n(\mathbb{R}^n, \mathbb{R}^n - B; R), R) \cong \operatorname{Hom}_R(R, R) \cong R$$

with a generator $a$ such that $\langle a, \mu_B \rangle = 1$. Now Proposition 7.10 applied to the cap product

$$\frown \colon H^n(\mathbb{R}^n, \mathbb{R}^n - B; R) \times H_n(\mathbb{R}^n, \mathbb{R}^n - B; R) \to H_0(\mathbb{R}^n; R)$$

implies that

$$1 = \langle a, \mu_B \rangle = \langle 1 \smile a, \mu_B \rangle = \langle 1, a \frown \mu_B \rangle,$$

and by definition of 1 (as the cohomology class of $\epsilon$), $a \frown \mu_B$ is a generator of $H_0(\mathbb{R}^n; R) \cong R$. Thus $- \frown \mu_B$ maps $H^n(\mathbb{R}^n, \mathbb{R}^n - B; R)$ isomorphically to $H_0(\mathbb{R}^n; R)$, and since all the other modules are zero for $p \neq n$, by passing to the direct limit over the balls $B$ as they become larger it follows that $D_M$ maps $H_c^*(\mathbb{R}^n; R)$ onto $H_*(\mathbb{R}^n; R)$.

Theorem 7.13 actually holds for any commutative ring $R$ with an identity element, not necessarily a PID. The only change in the proof occurs in Step (1); see Hatcher [25] (Chapter 3, Case (1) in the proof of Theorem 3.35), May [34] (Chapter 20, Section 5, page 159), and Greenberg and Harper [19] (Part III, Section 26, page 221).

Since the sphere $S^n$ is compact and orientable, we can obtain its cohomology from its homology. Recall from Proposition 4.16 that for $n \geq 1$ we have

$$H_p(S^n; R) = \begin{cases} R & \text{if } p = 0, n \\ (0) & \text{if } p \neq 0, n. \end{cases}$$

Thus we obtain

$$H^p(S^n; R) = \begin{cases} R & \text{if } p = 0, n \\ (0) & \text{if } p \neq 0, n. \end{cases}$$

Similarly, since the $n$-torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$ is compact and orientable, its cohomology is given by

$$H^p(T^n; R) = \underbrace{R \oplus \cdots \oplus R}_{\binom{n}{p}}.$$

As in the case of the sphere, it is identical to its homology, which reconfirms that these spaces are very symmetric.

Applications of Poincaré duality often involve the Universal Coefficient Theorems (see Section 12.5). The reader is referred to Hatcher [25] (Chapter 3) for some of these applications. In particular, one will find a proof of the fact that the cohomology ring $H^*(\mathbb{CP}^n; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^{n+1})$, with $\alpha$ of degree 2. As an application of Poincaré duality, we prove an important fact about compact manifolds of odd dimension.

Recall from Section 6.4 that the *Euler–Poincaré characteristic* $\chi(M)$ of a compact $n$-dimensional manifold is defined by

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \operatorname{rank} H_p(M; \mathbb{Z}).$$

The natural numbers rank $H_p(M; \mathbb{Z})$ are the *Betti numbers* of $M$ and are denoted by $b_p$.

**Proposition 7.14.** *If $M$ is a compact topological manifold (orientable or not) of odd dimension, then its Euler–Poincaré characteristic is zero, that is, $\chi(M) = 0$.*

*Proof.* Let $\dim M = 2m + 1$. If $M$ is orientable, by Poincaré duality $H^{2m+1-p}(M; \mathbb{Z}) \cong H_p(M; \mathbb{Z})$ for $p = 0, \ldots, 2m + 1$, so $\operatorname{rank}(H_p(M; \mathbb{Z})) = \operatorname{rank}(H^{2m+1-p}(M; \mathbb{Z}))$, but by Proposition 12.49, we have

$$H^n(M; \mathbb{Z}) \cong F_n \oplus T_{n-1}$$

where $H_n(M; \mathbb{Z}) = F_n \oplus T_n$ with $F_n$ free and $T_n$ a torsion abelian group, so $\operatorname{rank}(H^{2m+1-p}(M; \mathbb{Z})) = \operatorname{rank}(H_{2m+1-p}(M; \mathbb{Z}))$. Therefore,

$$\operatorname{rank}(H_p(M; \mathbb{Z})) = \operatorname{rank}(H_{2m+1-p}(M; \mathbb{Z})),$$

and since $2m + 1$ is odd we get

$$
\begin{aligned}
\chi(M) &= \sum_{p=0}^{2m+1} (-1)^p \operatorname{rank} H_p(M; \mathbb{Z}) \\
&= \sum_{p=0}^{2m+1} (-1)^p \operatorname{rank} H_{2m+1-p}(M; \mathbb{Z}) \\
&= -\sum_{p=0}^{2m+1} (-1)^{2m+1-p} \operatorname{rank} H_{2m+1-p}(M; \mathbb{Z}) \\
&= -\chi(M),
\end{aligned}
$$

so $\chi(M) = 0$.

If $M$ is not orientable we apply Poincaré duality with $R = \mathbb{Z}/2\mathbb{Z}$. In this case each $H_p(M; \mathbb{Z}/2\mathbb{Z})$ and each $H^{2m+1-p}(M; \mathbb{Z}/2\mathbb{Z})$ is a vector space and their rank is just their dimension. Because $\mathbb{Z}/2\mathbb{Z}$ is a field, $H^{2m+1-p}(M; \mathbb{Z}/2\mathbb{Z})$ and $H_{2m+1-p}(M; \mathbb{Z}/2\mathbb{Z})$ are dual spaces of the same dimension, and as above we conclude that

$$
\sum_{p=0}^{2m+1} (-1)^p \dim H_p(M; \mathbb{Z}/2\mathbb{Z}) = 0.
$$

If we can show that

$$
\chi(M) = \sum_{p=0}^{2m+1} (-1)^p \dim H_p(M; \mathbb{Z}/2\mathbb{Z}),
$$

we are done. Since $\mathbb{Z}/2\mathbb{Z}$ is a field it is a PID, and the above equation follows from Proposition 6.16. For the sake of those readers who have not read Chapter 6 we provide the proof in the special case $R = \mathbb{Z}/2\mathbb{Z}$.

By the Universal Coefficient Theorem for Homology (Theorem 12.38) and the fact that

$$
\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \operatorname{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z},
$$

every term $\mathbb{Z}^k$ in $H_p(M; \mathbb{Z})$ when tensored with $\mathbb{Z}/2\mathbb{Z}$ gives a term $(\mathbb{Z}/2\mathbb{Z})^k$ in $H_p(M; \mathbb{Z}/2\mathbb{Z})$, every term $\mathbb{Z}/q\mathbb{Z}$ in $H_p(M; \mathbb{Z})$ with $q > 2$ when tensored with $\mathbb{Z}/2\mathbb{Z}$ yields $(0)$, and every term $(\mathbb{Z}/2\mathbb{Z})^h$ in $H_p(M; \mathbb{Z})$ when tensored with $\mathbb{Z}/2\mathbb{Z}$ yields a term $(\mathbb{Z}/2\mathbb{Z})^h$ in $H_p(M; \mathbb{Z}/2\mathbb{Z})$, and the same term $(\mathbb{Z}/2\mathbb{Z})^h$ in $H_{p+1}(M; \mathbb{Z}/2\mathbb{Z})$ as the contribution of $\operatorname{Tor}_1^{\mathbb{Z}}((\mathbb{Z}/2\mathbb{Z})^h, \mathbb{Z}/2\mathbb{Z})$. The contribution of the two terms $(\mathbb{Z}/2\mathbb{Z})^h$ to the sum $\sum_{p=0}^{2m+1} (-1)^p \dim H_p(M; \mathbb{Z}/2\mathbb{Z})$ cancel out since their respective signs are $(-1)^p$ and $(-1)^{p+1}$, so

$$
\chi(M) = \sum_{p=0}^{2m+1} (-1)^p \dim H_p(M; \mathbb{Z}/2\mathbb{Z}),
$$

which concludes the proof.                                                                    $\square$

In the next section we present an even more general version of Poincaré Duality for cohomology and homology with coefficients in any $R$-module $G$ and any commutative ring with identity element 1.

## 7.5   The Poincaré Duality Theorem with Coefficients in $G$

The first step is to define a version of the cap product that accomodates coefficients in $G$. This is easy to do, simply define the cap product

$$\frown : S^p(X;G) \times S_n(X;R) \to S_{n-p}(X;G)$$

using a variant of the formula of Definition 7.2, namely

$$c \frown \sigma = (\sigma \circ \lambda_{n-p})c(\sigma \circ \rho_p),$$

where we switched the order of the two expressions on the right-hand side to conform with the convention that a chain in $S_{n-p}(X;G)$ is a formal combination of the form $\sum \sigma_i g_i$ with $g_i \in G$ and $\sigma_i$ a $(n-p)$-simplex. Since $\sigma \circ \rho_p \in S_p(X;R)$, $\sigma \circ \lambda_{n-p} \in S_{n-p}(X;R)$, and $c \in S^p(X;G)$, we have $c(\sigma \circ \rho_p) \in G$, and indeed $(\sigma \circ \lambda_{n-p})c(\sigma \circ \rho_p) \in S_{n-p}(X;G)$.

If $a \in S^{n-p}(X;R)$, $b \in S^p(X;G)$ and $\sigma \in S_n(X;R)$, by the definition at the end of Section 4.8 we have

$$\langle a \smile b, \sigma \rangle = a(\sigma \circ \lambda_{n-p})b(\sigma \circ \rho_p)$$

and

$$b \frown \sigma = (\sigma \circ \lambda_{n-p})b(\sigma \circ \rho_p),$$

so if $\langle f, s \rangle$ with $f \in S^p(X;R)$ and $s \in S_p(X;G)$ is defined the right way, the identity

$$\langle a, b \frown \sigma \rangle = \langle a \smile b, \sigma \rangle$$

will hold. But the definition of a pairing $\langle -, - \rangle \colon S^p(X;R) \times S_p(X;G) \to G$ is standard, namely

$$\langle f, \sum \sigma_i g_i \rangle = \sum f(\sigma_i) g_i,$$

where $f \in S^p(X;R)$ and $\sum \sigma_i g_i$ is a singular $p$-simplex in $S_p(X;G)$ (where the $\sigma_i$ are $p$-simplices).

It is even possible to define a paring $\langle -, - \rangle \colon S^p(X;G) \times S_p(X;G') \to G \otimes G'$, where $G$ and $G'$ are two different $R$-modules; see Spanier [47] (Chapter 5, Section 5, page 243). In summary, the equation

$$\langle a, b \frown \sigma \rangle = \langle a \smile b, \sigma \rangle$$

holds for this more general version of cup products and cap products.

The formula
$$\partial(c \frown \sigma) = (-1)^{n-p}(\delta c \frown \sigma) + c \frown \partial\sigma.$$
of Proposition 7.9 still holds for any $c \in S^p(X; G)$ and any $\sigma \in S_n(X; R)$, so we obtain a cap product
$$\frown \colon H^p(X; G) \times H_n(X; R) \to H_{n-p}(X; G);$$
see Munkres [38] (Chapter 8, Section 66).

There is also a relative version of the cup product
$$\frown \colon H^p(X, A; G) \times H_n(X, A \cup B; R) \to H_{n-p}(X, B; G)$$
with $A$ and $B$ two open subsets of $X$ which will be used in the version of Poincaré duality with coefficients in $G$; see May [34] (Chapter 20, Section 2).

Next we promote singular cohomology with coefficients in $G$ to cohomology with compact support. All one has to do is replace $R$ by $G$ everywhere. We obtain the *cohomology groups with compact support* $H_c^p(X; G)$. It is easy to verify that that Proposition 7.12 also holds.

**Proposition 7.15.** *We have isomorphisms*
$$H_c^p(X; G) \cong \varinjlim_{K \in \mathcal{K}} H^p(X, X - K; G)$$
*for all $p \geq 0$. Furthermore, if $X$ is compact, then $H_c^p(X; G) \cong H^p(X; G)$.*

Given a $R$-orientable manifold $M$ we also have to generalize the mapping $D_M \colon H_c^p(M; R) \to H_{n-p}(M; R)$ to a mapping
$$D_M \colon H_c^p(M; G) \to H_{n-p}(M; G),$$
and for this we use the cup product
$$\frown \colon H^p(M, M - K; G) \times H_n(M, M - K; R) \to H_{n-p}(X; G).$$
Since there is an isomorphism
$$H_c^p(M; G) \cong \varinjlim_{K \in \mathcal{K}} H^p(M, M - K; G),$$
for any $\omega \in H_c^p(M; G)$ we pick some representative $\omega'$ in the equivalence class defining $\omega$ in $\varinjlim H^p(M, M - K; G)$, namely some $\omega' \in H^p(M, M - K; G)$ for some compact subset $K$, and since $\mu_K \in H_n(M, M - K; R)$ we set
$$D_M(\omega) = \omega' \frown \mu_K \in H_{n-p}(M; G).$$
Then we prove that the above definition does not depend on the choice of the representative $\omega' \in H^p(M, M - K; G$ just as in the case where $G = R$. In conclusion, we obtain our map
$$D_M \colon H_c^p(M; G) \to H_{n-p}(M; G).$$
Using this map, the following version Poincaré duality can be proved.

**Theorem 7.16.** *(Poincaré Duality Theorem for Coefficients in a Module) Let $M$ be an $n$-manifold, let $R$ be any commutative ring with unit $1$, and let $G$ be any $R$-module. If $M$ is $R$-orientable, then the map*

$$D_M \colon H_c^p(M;G) \to H_{n-p}(M;G)$$

*defined above is an isomorphism for all $p \geq 0$. In particular, if $R = \mathbb{Z}/2\mathbb{Z}$, the above map is an isomorphism whether $M$ is orientable or not.*

*If $M$ is compact and $R$-orientable, then the map*

$$D_M \colon \omega \mapsto \omega \frown \mu_M$$

*is an isomorphism between $H^p(M;G)$ and $H_{n-p}(M;G)$.*

Theorem 7.16 is proved in May [34] (Chapter 20, Section 5). The proof also implicitly uses the Bootstrap Lemma. Except for Case (1), it is basically identical to the proofs in Milnor and Stasheff [35] (Appendix A, pages 277-279) and Hatcher [25] (Chapter 3, Theorem 3.35).

We will se later on that there is an even more general version of duality known as Alexander–Lefschetz duality; see Chapter 14.

# Chapter 8

# Persistent Homology

# Chapter 9

# Presheaves and Sheaves; Basics

## 9.1 Presheaves

Roughly speaking, presheaves (and sheaves) are a way of packaging local information about a topological space $X$ in a way that is mathematically useful. We can imagine that above every open subset $U$ of $X$ there is a "balloon" $\mathcal{F}(U)$ of information about $U$, often a set of functions, and that this information is compatible with restriction; namely if $V$ is another open set contained in $U$, then the balloon of information $\mathcal{F}(V)$ is obtained from $\mathcal{F}(U)$ by some restriction function $\rho_V^U$.

The typical example is as follows: given a topological space $X$ (for simplicity, you may assume that $X = \mathbb{R}$, or $X = \mathbb{R}^n$), for every (nonempty) open subset $U$ of $X$, let $C^0(U)$ be the set of all real-valued functions $f \colon U \to \mathbb{R}$. For any open subset $V \subseteq U$, we obtain a function $\rho_V^U \colon C^0(U) \to C^0(V)$ by restricting any function $f \colon U \to \mathbb{R}$ to $V$. See Figure 9.1.

Observe that if $W \subseteq V \subseteq U$, then

$$\rho_W^U = \rho_W^V \circ \rho_V^U$$

and

$$\rho_U^U = \mathrm{id}_U.$$

See Figure 9.2.

The assignment $U \mapsto C^0(U)$ is a presheaf on $X$. In the above example each $C^0(U)$ can be viewed as a set, but also as a real vector space, or a ring, or even as an algebra, since functions can be added, rescaled, and multiplied pointwise.

More generally, we can pick a class of structures, say sets, vector spaces, $R$-modules (where $R$ is a commutative ring with a multiplicative identity), groups, commutative rings, $R$-algebra, *etc.*, and assign an object $\mathcal{F}(U)$ in this class to every open subset $U$ of $X$, and for every pair of open subsets $U, V$ such that $V \subseteq U$, if we write $i \colon V \to U$ for the inclusion map from $V$ to $U$, then we assign to $i$ a map $\mathcal{F}(i) \colon \mathcal{F}(U) \to \mathcal{F}(V)$ which is a morphism of the class of of objects under consideration. This means that if the $\mathcal{F}(U)$ are sets, then the
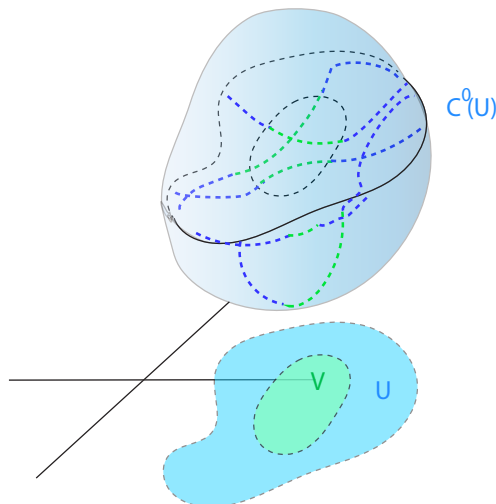
Figure 9.1: The elevated blue balloon is schematic representation of a presheaf of real valued functions over the open set $U \subseteq \mathbb{R}^2$. Each "function" is represented as blue and green dotted lines, where the green dash is the restriction of the function on $V$.

$\mathcal{F}(i)$ are just functions; if the $\mathcal{F}(U)$ are $R$-modules then the $\mathcal{F}(i)$ are $R$-linear maps; if the $\mathcal{F}(U)$ are groups then the $\mathcal{F}(i)$ are group homomorphisms; if the $\mathcal{F}(U)$ are rings then the $\mathcal{F}(i)$ are ring homomorphisms, *etc.*

A fancy way to proceed would be assume that we have a category $\mathbf{C}$ and that objects of $\mathbf{C}$ are assigned to open subsets of $X$ and morphisms of $\mathbf{C}$ are assigned to inclusion maps, so that a presheaf is a contravariant functor. For our purposes it is not necessary to assume such generality.

**Definition 9.1.** Given a topological space $X$ and a class $\mathbf{C}$ of structures (a category), say sets, vector spaces, $R$-modules, groups, commutative rings, *etc.*, a *presheaf on $X$ with values in $\mathbf{C}$* consists of an assignment of some object $\mathcal{F}(U)$ in $\mathbf{C}$ to every open subset $U$ of $X$ and of a map $\mathcal{F}(i) \colon \mathcal{F}(U) \to \mathcal{F}(V)$ of the class of structures in $\mathbf{C}$ to every inclusion $i \colon V \to U$ of open subsets $V \subseteq U \subseteq X$, such that

$$\mathcal{F}(i \circ j) = \mathcal{F}(j) \circ \mathcal{F}(i)$$
$$\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)},$$

for any two inclusions $i \colon V \to U$ and $j \colon W \to V$, with $W \subseteq V \subseteq U$.

Note that the order of composition is switched in $\mathcal{F}(i \circ j) = \mathcal{F}(j) \circ \mathcal{F}(i)$.

Intuitively, the map $\mathcal{F}(i) \colon \mathcal{F}(U) \to \mathcal{F}(V)$ is a restriction map if we think of $\mathcal{F}(U)$ and $\mathcal{F}(V)$ as a sets of functions (which is often the case). For this reason, the map $\mathcal{F}(i) \colon \mathcal{F}(U) \to$

Figure 9.2: A schematic representation of the nested presheaves of continuous functions associated with the open subsets $W \subseteq V \subseteq U \subseteq \mathbb{R}^2$. The wavy plane with the bold dashed line represents the graph of a continuous real-valued function with domain in $U$. If this function is restricted to the different colored "balloons," (which have been opened to show the graph of the continuous function), the domain is restricted appropriately, namely to either $V$ or $W$, as evidenced by the color change.

$\mathcal{F}(V)$ is also denoted by $\rho_V^U \colon \mathcal{F}(U) \to \mathcal{F}(V)$, and the first equation of Definition 9.1 is expressed by

$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

See Figure 9.1 and 9.2. Here are some examples of presheaves.

**Example 9.1.**

(1) The *constant presheaf* $G_X$ with values in $G \in C$, defined such that $G_X(U) = G$ for all open subsets $U$ of $X$, and $\rho_V^U$ is the identity function of $G$ for all open subsets $U, V$. A variant of the constant presheaf which comes up in cohomology has $G_X(\emptyset) = (0)$ instead of $G_X(\emptyset) = G$ when $G$ is an algebraic structure with an identity element $0$.

(2) If $Y$ is another topological space, then $\mathcal{C}_Y^0$ is the presheaf defined so that $\mathcal{C}_Y^0(U)$ is the set of all continuous functions $f \colon U \to Y$ from the open subset $U$ of $X$ to $Y$.

(3) If $Y = (\mathbb{R}, +, \text{usual metric topology})$, then $\mathcal{C}_Y^0$ is the presheaf of real-valued continuous functions on $X$. It is presheaf of $\mathbb{R}$-algebras.

(4) If $Y = (\mathbb{R}, +, \text{trivial topology})$, then $\mathcal{C}_Y^0$ is the presheaf of all real-valued functions on $X$. It is presheaf of $\mathbb{R}$-algebras.

(5) If $M$ is a smooth manifold, then then $\mathcal{C}^\infty$ is the presheaf defined so that $\mathcal{C}^\infty(U)$ is the set of all smooth real-valued functions $f \colon U \to \mathbb{R}$ from the open subset $U$ of $M$.

A map between two presheaves is defined as follows.

**Definition 9.2.** Given a topological space $X$ and a fixed class $\mathbf{C}$ of structures (a category), say sets, vector spaces, $R$-modules, groups, commutative rings, *etc.*, a *map (or morphism)* $\varphi \colon \mathcal{F} \to \mathcal{G}$ of presheaves $\mathcal{F}$ and $\mathcal{G}$ on $X$ consists of a family of maps $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ of the class of structures in $\mathbf{C}$, for any open subset $U$ of $X$, such that

$$\varphi_V \circ (\rho_\mathcal{F})_V^U = (\rho_\mathcal{G})_V^U \circ \varphi_U$$

for every pair of open subsets $U, V$ such that $V \subseteq U \subseteq X$. Equivalently, the following diagrams commute for every pair of open subsets $U, V$ such that $V \subseteq U \subseteq X$ (and $i \colon V \to U$ is the corresponding inclusion map):

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\;\varphi_U\;} & \mathcal{G}(U) \\
{\scriptstyle \mathcal{F}(i)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{G}(i)} \\
\mathcal{F}(V) & \xrightarrow[\;\varphi_V\;]{} & \mathcal{G}(V),
\end{array}
$$

or using the restriction notation $(\rho_\mathcal{F})_V^U$ for $\mathcal{F}(i)$ and $(\rho_\mathcal{G})_V^U$ for $\mathcal{G}(i)$,

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\;\varphi_U\;} & \mathcal{G}(U) \\
{\scriptstyle (\rho_\mathcal{F})_V^U}\Big\downarrow & & \Big\downarrow{\scriptstyle (\rho_\mathcal{G})_V^U} \\
\mathcal{F}(V) & \xrightarrow[\;\varphi_V\;]{} & \mathcal{G}(V).
\end{array}
$$

See Figure 9.3.

**Remark:** In fancy terms, a map of presheaves is a natural transformation.

Given three presheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ on $X$ and two maps of presheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ and $\varphi \colon \mathcal{G} \to \mathcal{H}$, the *composition* $\psi \circ \varphi$ of $\varphi$ and $\psi$ is defined by the family of maps

$$(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$$

for all open subsets $U$ of $X$. It is easily checked that $\psi \circ \varphi$ is indeed a map of presheaves from $\mathcal{F}$ to $\mathcal{H}$.

**Definition 9.3.** Given two presheaves $\mathcal{F}$ and $\mathcal{G}$ on $X$, a presheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$ is *injective* if every map $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective, *surjective* if every map $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective (for each open subset $U$ of $X$). Two presheaves $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* if there exists some presheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$ and $\psi \colon \mathcal{G} \to \mathcal{F}$ such that $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$.
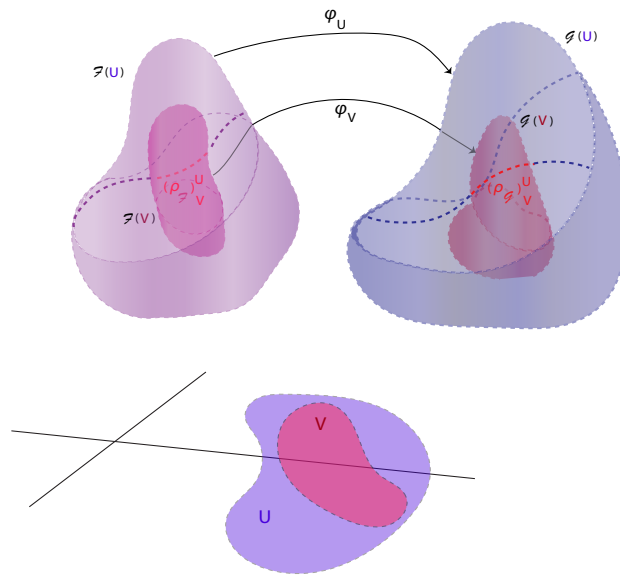
Figure 9.3: The two purple "eggplants" represent the elements of the presheaves $\mathcal{F}$ and $\mathcal{G}$. The presheaf map $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ maps the left "eggplant" to the right "eggplant" in a manner which preserves restrictions associated with the inclusion $V \subseteq U \subseteq \mathbb{R}^2$.

It is not hard to see that a presheaf map is an isomorphism iff it is injective and surjective.

If $\mathcal{F}$ and $\mathcal{G}$ are presheaves of algebraic structures (modules, groups, commutative rings, *etc.*) then there is a notion of kernel, image, and cokernel of a map of presheaves. This allows the definition of exact sequences of presheaves. We will come back to this point later on.

## 9.2   Sheaves

In Section 9.1 we defined the notion of a presheaf. Presheaves are typically used to keep track of local information assigned to a global object (the space $X$). It is usually desirable to use to consistent local information to recover some global information, but this requires a sharper notion, that of a sheaf.

Expositions on the subject of sheaves tend to be rather abstract and assume a significant amount of background. Our goal is to provide just enough background to have a good understanding of the sheaficiation process and of the subtleties involving exact sequences of presheaves and sheaves. We should mention some of the classics, including (in alphabetic order) Bredon [4], Eisenbud and Harris [14], Forster [15], Godement [18], Griffith and Harris [20], Gunning [23], Hartshorne [24], Hirzebruch [26], Kashiwara and Shapira [27], MacLane

and Moerdijk [30], Mumford [37], Narasimham [39], Serre FAC [44], Shafarevich [45], Spanier [47]. One of the most accessible (and quite thorough) presentations is found in Tennison [48].

The motivation for the extra condition that a sheaf should satisfy is this. Suppose we consider the presheaf of continuous functions on a topological space $X$. If $U$ is any open subset of $X$ and if $(U_i)_{i \in I}$ is an open cover of $U$, for any family $(f_i)_{i \in I}$ of continuous functions $f_i \colon U_i \to \mathbb{R}$, if $f_i$ and $f_j$ agree on every overlap $U_i \cap U_j$, then they $f_i$ patch to a unique continuous function $f \colon U \to \mathbb{R}$ whose restriction to $U_i$ is $f_i$.

**Definition 9.4.** Given a topological space $X$ and a class $\mathbf{C}$ of structures (a category), say sets, vector spaces, $R$-modules, groups, commutative rings, *etc.*, a *sheaf on $X$ with values in* $\mathbf{C}$ is a presheaf $\mathcal{F}$ on $X$ such that for any open subset $U$ of $X$, for every open cover $(U_i)_{i \in I}$ of $U$ (that is, $U = \bigcup_{i \in I} U_i$ for some open subsets $U_i \subseteq U$ of $X$), the following conditions hold:

(G) (Gluing condition) For every family $(f_i)_{i \in I}$ with $f_i \in \mathcal{F}(U_i)$, if the $f_i$ are consistent, which means that

$$\rho^{U_i}_{U_i \cap U_j}(f_i) = \rho^{U_j}_{U_i \cap U_j}(f_j) \quad \text{for all } i, j \in I,$$

then there is some $f \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(f) = f_i$ for all $i \in I$. See Figure 9.4.

(M) (Monopresheaf condition) For any two elements $f, g \in \mathcal{F}(U)$, if $f$ and $g$ agree on all the $U_i$, which means that

$$\rho^U_{U_i}(f) = \rho^U_{U_i}(g) \quad \text{for all } i \in I,$$

then $f = g$.

Obviously, Condition (M) implies that in Condition (G) the element $f$ obtained by patching the $f_i$ is unique.

Another notation often used for $\mathcal{F}(U)$ is $\Gamma(U, \mathcal{F})$. An element of $\Gamma(U, \mathcal{F})$ is called a *section above $U$*, and elements of $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ are called *global sections*. This terminology is justified by the fact that many sheaves arise as continuous sections of some surjective continuous map $p \colon E \to X$; that is, continuous functions $s \colon U \to E$ such that $p \circ s = \mathrm{id}_U$; see Example 9.2 (1).

For any two open subsets $U$ and $V$ with $V \subseteq U$, for any $s \in \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, it is often convenient to abbreviate $\rho^U_V(s)$ by $s|V$.

**Remarks:**

1. If $\mathcal{F}(U) = \emptyset$ for some open subset $U$ of $X$, then $\mathcal{F}$ is the trivial sheaf such that $\mathcal{F}(V) = \emptyset$ for all open subsets $V$ of $X$. This is because there is a restriction function $\rho^X_U \colon \mathcal{F} \to \emptyset$, but the only function with range $\emptyset$ is the empty function with domain $\emptyset$ so $\mathcal{F}(X) = \emptyset$. Since there is restriction function $\rho^X_V \colon \mathcal{F}(X) \to \mathcal{F}(V)$ for every open subset $V$ of $X$, we deduce that $\mathcal{F}(V) = \emptyset$ for all open subsets of $X$. This observation is due to Godement [18]. From now on, we rule out the above possibility. Note that it is ruled out automatically for sheaves of algebraic structures having an identity element.
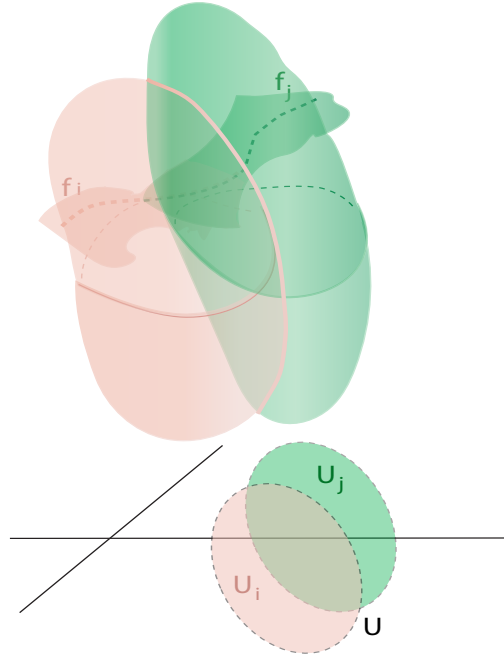
Figure 9.4: A schematic representation Condition (G) for the set $U = U_i \cup U_j \subseteq \mathbb{R}^2$. The element $f_i \in \mathcal{F}(U_i)$ is represented by the wavy peach plane with the bold peach dotted line in the peach "balloon" while the element $f_j \in \mathcal{F}(U_j)$ is represented by the wavy green plane with the bold green dotted line. Where the two "balloons" intersect, the peach plane overlaps the green plane. In other words $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$.

2. Assuming that $\mathcal{F}$ is not the trivial sheaf, then Conditions (G) and (M) apply to all open subsets $U$ of $X$ and all families of open covers $(U_i)_{i \in I}$ of $U$, including the case where $U = \emptyset$ and $I = \emptyset$. In this case, Conditions (G) and (M) implies that $\mathcal{F}(\emptyset)$ is a one-element set. In the case of groups, modules, groups, commutative rings, *etc.*, we have $\mathcal{F}(\emptyset) = \{0\}$.

3. Condition (G) applies to open subsets $U$ that are the *disjoint union* of open subsets $U_i \subseteq U$. In this case, every family $(f_i)_{i \in I}$ with $f_i \in \mathcal{F}(U_i)$ must patch to yield some global element $f \in \mathcal{F}(U)$ such that $\rho_{U_i}^{U}(f) = f_i$. Thus, the gluing condition imposes some *consistency* among the local pieces $f_i \in \mathcal{F}(U_i)$, even if the $U_i$ are pairwise disjoint. This is a major difference with presheaves, where unrelated and inconsistent objects may be assigned to disjoint open subsets.

4. If $\mathcal{F}$ is a sheaf of $R$-modules or commutative rings, then Condition (M) can be replaced by the following condition which is often more convenient:

(M) (Monopresheaf condition) For any element $f \in \mathcal{F}(U)$, if $f$ is zero on the $U_i$, which

means that
$$\rho^U_{U_i}(f) = 0 \quad \text{for all } i \in I,$$

then $f = 0$.

5. If $\mathcal{F}$ is a sheaf of $R$-modules or commutative rings, then Conditions (M) and (G) can be stated as an exactness condition. For any nonempty subset $U$ of $X$, for any open cover $(U_i)_{i \in I}$ of $U$, define the maps $f \colon \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ and $g \colon \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$ by

$$f(s) = (\rho^U_{U_i}(s))_{i \in I}$$
$$g((s_i)_{i \in I}) = (\rho^{U_i}_{U_i \cap U_j}(s_i) - \rho^{U_j}_{U_i \cap U_j}(s_j))_{(i,j) \in I \times I}.$$

Then Conditions (M) and (G) are equivalent to the hypothesis that the sequence

$$0 \longrightarrow \mathcal{F}(U) \overset{f}{\longrightarrow} \prod_{i \in I} \mathcal{F}(U_i) \overset{g}{\longrightarrow} \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j)$$

is exact.

6. Intuitively, we may think of the elements $f \in \mathcal{F}(U)$ (the sections above $U$) as abstract functions. In fact, this point of view can be justified rigorously. For every sheaf $\mathcal{F}$ on a space $X$, we can construct a "big" space $E$ with a continuous projection function $p \colon E \to X$ so that for every open subset $U$ of $X$, every $s \in \mathcal{F}(U)$ can be viewed as a function $\widetilde{s} \colon U \to E$ (a section of $p$, see Example 9.2 (1) below). In fact, $p$ is a local homeomorphism. We will investigate the construction of $E$ in Section 11.1.

Here are some examples of sheaves.

**Example 9.2.**

1. Let $p \colon E \to X$ be a surjective continuous map between two topological spaces $E$ and $X$. We define the *sheaf* $\Gamma[E, p]$ *of (continuous) sections of $p$ on $X$* as follows: for every open subset $U$ of $X$,

$$\Gamma[E, p](U) = \Gamma(U, \Gamma[E, p]) = \{s \colon U \to E \mid p \circ s = \text{id} \text{ and } s \text{ is continuous}\};$$

equivalently, the following diagram commutes:

$$
\begin{array}{ccc}
 & & E \\
 & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\
U & \hookrightarrow & X
\end{array}
$$

where the horizontal arrow is inclusion; see Figure 9.5. For the sake of notational simplicity, the sheaf $\Gamma[E, p]$ is often denoted by $\Gamma E$.
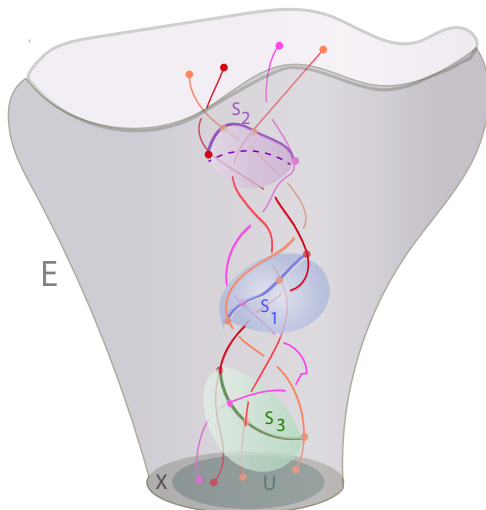
Figure 9.5: A schematic representation of the sheaf of sections $\Gamma[E, p]$ where $X$ is the circular base and $E$ be the solid gray upside-down "lamp shade". Note $p^{-1}(x)$ for $x \in X$ is a twisted orange "spaghetti strand." We illustrate three elements of $\Gamma[E, p](U)$ as bold blue, green, and purple wavy disks which connect the various preimages in a continuous manner, namely $p \circ s(u) = u$ for $u \in U$.

2.  If $Y$ is another topological space, $E = X \times Y$, and $p \colon X \times Y \to X$ is the first projection, then the sheaf $\Gamma[E, p]$ in (1) corresponds to the presheaf on $X$ of Example 9.1 (2–4), which is actually a sheaf. Indeed, since $p$ is the map $(x, y) \mapsto x$, every continuous section $s$ of $p$ above $U$ is a function of the form $x \mapsto (x, f(x))$, where $f \colon U \to Y$ is a continuous function. Therefore, there is a bijection between the set of continuous sections of $p$ above $U$ and the set of continuous functions from $U$ to $Y$. See Figure 9.6.

3.  If $Y$ is given the discrete topology, $E = X \times Y$, and $p \colon X \times Y \to X$ is the first projection, then the sheaf $\Gamma[E, p]$ in (1) corresponds to the sheaf of locally constant functions with values in $Y$, because every continuous section $s$ of $p$ above $U$ is a function of the form $x \mapsto (x, f(x))$, where $f \colon U \to Y$ is a locally constant function. Recall that a function $f \colon U \to Y$ is locally constant if for every $x \in U$ there is some open subset $V$ of $U$ containing $x$ such that $f$ is constant on $V$. For any $x \in U$, since $Y$ is discrete the set $\{f(x)\}$ is open, and since $f$ is continuous $V = f^{-1}(f(x))$ is some open subset of $U$ containing $x$ and $f$ is constant on $V$ (with value $f(x)$). A locally constant function must have a constant value on a connected open subset. See Figure 9.7.

    The sheaf of locally constant functions on $X$ with values in $Y$ is denoted $\widetilde{Y}_X$ (or $Y_X^+$ if the "tilde" notation is already used). Beware that in general this is *not* the constant presheaf $Y_X$ with values in $Y$. Indeed if $X$ is the union of two disjoint open subsets $U_1$ and $U_2$ and if $Y$ has at least two distinct elements $y_1, y_2$, then we can pick the family $(y_1, y_2)$ with $y_1 \in Y_X(U_1) = Y$ and $y_2 \in Y_X(U_2) = Y$, and since $U_1 \cap U_2 = \emptyset$, by
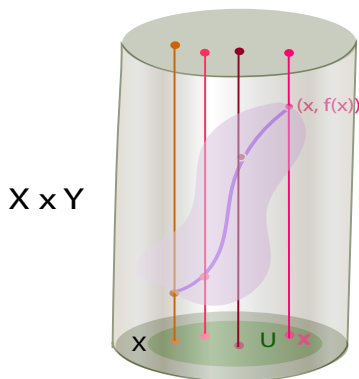
Figure 9.6: Let $X$ be the closed unit disk, $Y = [0,1]$, and $E = X \times Y$ be the solid grey cylinder. Each $p^{-1}(x)$ is straight orange "spaghetti strand." We illustrate an element of $\Gamma[E,p](U)$ associated with the function $f \colon U \to Y$ as a wavy purple disk.

Condition $(G)$ there should be some element $y \in Y_X(X) = Y$ such that $\rho^X_{U_1}(y) = y_1$ and $\rho^X_{U_2}(y) = y_2$. But since $Y_X$ is the constant presheaf, $\rho^X_{U_1} = \rho^X_{U_2} = \mathrm{id}$, so we should have $y = y_2 = y_2$, which is impossible since $y_1 \neq y_2$. The sheaf $\widetilde{Y}_X$ of locally constant functions with values in $Y$ is usually called (confusingly) the *constant sheaf with values in $Y$*.

4. Given a smooth manifold $M$, the smooth real-valued functions on $M$ form a sheaf $\mathcal{C}^\infty$. For every open subset $U$ of $M$, let $\mathcal{C}^\infty(U)$ be the $\mathbb{R}$-algebra of smooth functions on $U$.

5. Given a smooth manifold $M$, the differential forms on $M$ form a sheaf $\mathcal{A}^*_X$. For every open subset $U$ of $M$, let $\mathcal{A}^p(U)$ be the vector space of of $p$-forms on $U$, and let $\mathcal{A}^*_X(U) = \mathcal{A}^p(U)$. Then it is easy to check that we obtain a sheaf of vector spaces; the restriction maps are the pullbacks of forms.

We just observed that in general the constant presheaf with values in $Y$ in not a sheaf. Here is another example of a presheaf which is not a sheaf.

**Example 9.3.** Let $X$ be any topological space with at least two points (for example, $X = \{0,1\}$), and let $\mathcal{F}_1$ be the presheaf given by

$$\mathcal{F}_1(U) = \begin{cases} \mathbb{Z} & \text{if } U = X \\ (0) & \text{if } U \neq X \text{ is an open subset,} \end{cases}$$

with all $\rho^U_V$ equal to the zero map except if $U = V = X$ (in which case it is the identity). It is easy to check that Condition (M) fails. In particular if $X = \{0,1\}$ with the discrete

X x Y
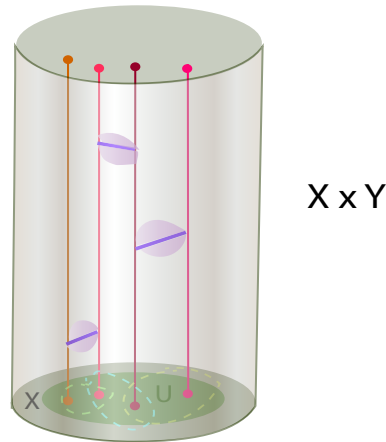
Figure 9.7: Let $X$ be the closed unit disk, $Y = [0, 1]$, and $E = X \times Y$ be the solid grey cylinder. Each $p^{-1}(x)$ is straight orange "spaghetti strand" composed of disjoint open points. An element of $\Gamma[E, p](U)$ is illustrated as the purple "jump" function.

topology, then $X = \{0\} \cup \{1\}$, where $\{0\}$ and $\{1\}$ are open sets in $X$. Let $f \in \mathcal{F}_1(X)$ be $f = 1$, while $g \in \mathcal{F}_1(X)$ is $g = -1$. Then

$$\rho^X_{\{0\}}(f) = 0 = \rho^X_{\{1\}}(g),$$

where $f \neq g$.

The notion of a map $\varphi \colon \mathcal{F} \to \mathcal{G}$ between two sheaves $\mathcal{F}$ and $\mathcal{G}$ is exactly as in Definition 9.2. Two sheaves $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* if there exist some sheaf morphisms $\varphi \colon \mathcal{F} \to \mathcal{G}$ and $\psi \colon \mathcal{G} \to \mathcal{F}$ such that $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$.

It turns out that every sheaf is isomorphic to a sheaf of sections as in Example 9.2(1), but to prove this we need the notion of direct limit; see Section 9.6.

**Definition 9.5.** Given a topological space $X$, for every (nonempty) open subset $U$ of $X$, for every presheaf (or sheaf) $\mathcal{F}$ on $X$, the *restriction* $\mathcal{F}|U$ of $\mathcal{F}$ to $U$ is defined so that for every open subset $V$ to $U$,

$$(\mathcal{F}|U)(V) = \mathcal{F}(V).$$

If $\mathcal{F}$ is a sheaf, it is immediate that $\mathcal{F}|U$ is a also a sheaf. Given two preshaves (or sheaves) $\mathcal{F}$ and $\mathcal{G}$ on $X$, the presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is defined by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|U, \mathcal{G}|U)$$

for every open subset $U$ of $X$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, it is easy to see that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is also a sheaf.

The next section is devoted to direct limits, an indispensible tool in sheaf theory and the cohomology of sheaves.

## 9.3   Direct Mapping Families and Direct Limits

We begin our study of direct limits with the following two definitions.

**Definition 9.6.** A *directed set* is a set $I$ equipped with a preorder $\leq$ (where $\leq$ is a reflexive and transitive relation) such that for all $i, j \in I$, there is some $k \in I$ such that $i \leq k$ and $j \leq k$. A subset $J$ of $I$ is said to be *cofinal* in $I$ if for every $i \in I$ there is some $j \in J$ such that $i \leq j$. For example, $2\mathbb{Z}$ is cofinal in $\mathbb{Z}$, where $2\mathbb{Z} = \{2x \mid x \in \mathbb{Z}\}$.

**Definition 9.7.** A *direct mapping family* of sets (or $R$-modules, or commutative rings, *etc.*) is a pair $((F_i)_{i \in I}, (\rho_j^i)_{i \leq j})$ where $(F_i)_{i \in I}$ is a family of sets ($R$-modules, commutative rings, *etc.*) $F_i$ whose index set $I$ is a directed set, and for all $i, j \in I$ with $i \leq j$, $\rho_j^i \colon F_i \to F_j$ is a map ($R$-linear, ring homomorphism, *etc.*) so that

$$\rho_i^i = \mathrm{id}$$
$$\rho_k^i = \rho_k^j \circ \rho_j^i$$

for all $i, j, k \in I$ with $i \leq j \leq k$, as illustrated below



Here are two examples of direct mapping families.

**Example 9.4.**

1. Let $X$ be a topological space and pick any point $x \in X$. Then the family of open subsets $U$ of $X$ such that $x \in U$ forms a directed set under the preorder $U \prec V$ iff $V \subseteq U$. If $C^0(U)$ is the set of continuous $\mathbb{R}$-valued functions defined in $U$ and if $\rho_V^U \colon C^0(U) \to C^0(V)$ is the restriction map, then the family of sets (rings) $(C^0(U))_{U \ni x}$ (for all open subsets $U$ of $X$ containing $x$) forms a direct mapping family.

2. More generally, if $\mathcal{F}$ is a presheaf on $X$, then the family of sets ($R$-modules, *etc.*) $(\mathcal{F}(U))_{U \ni x}$ forms a direct mapping family, with $\rho_V^U \colon \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subseteq U$, the presheaf restriction map.

The direct limit of a direct mapping family $((F_i)_{i \in I}, (\rho_j^i)_{i \leq j})$ is obtained as a quotient of a disjoint union of the $F_i$.

**Definition 9.8.** The *direct limit* (or *inductive limit*) $\varinjlim F_i$ of the direct mapping family $((F_i)_{i \in I}, (\rho_j^i)_{i \leq j})$ of sets ($R$-modules, commutative rings, *etc.*) is defined as follows:

First form the disjoint union $\coprod_{i \in I} F_i$. Next let $\sim$ be the equivalence relation on $\coprod_{i \in I} F_i$ defined by:

$$f_i \sim f_j \quad \text{iff} \quad \rho_k^i(f_i) = \rho_k^j(f_j) \quad \text{for some } k \in I \text{ with } k \geq i, j,$$

for any $f_i \in F_i$ and any $f_j \in F_j$; see Figure 9.8. Finally the direct limit $\varinjlim F_i$ is given by

$$\varinjlim_{i \in I} F_i = \left( \coprod_{i \in I} F_i \right) / \sim .$$

It is clear that $\sim$ is reflexive and symmetric but we need to check transitivity. This is where the fact that $I$ is a directed set is used. If $f_i \sim f_j$ and $f_j \sim f_k$, then there exist $p, q \in I$ such that $i, j \leq p$, $j, k \leq q$, $\rho_p^i(f_i) = \rho_p^j(f_j)$ and $\rho_q^j(f_j) = \rho_q^k(f_k)$. Since $I$ is a directed preorder there is some $r \in I$ such that $p, q \leq r$. We claim that

$$\rho_r^i(f_i) = \rho_r^k(f_k),$$

showing that $f_i \sim f_k$. This is because

$$\rho_r^i(f_i) = \rho_r^p \circ \rho_p^i(f_i) = \rho_r^p \circ \rho_p^j(f_j) = \rho_r^j(f_j) = \rho_r^q \circ \rho_q^j(f_j) = \rho_r^q \circ \rho_q^k(f_k) = \rho_r^k(f_k),$$

as illustrated by the following diagram:
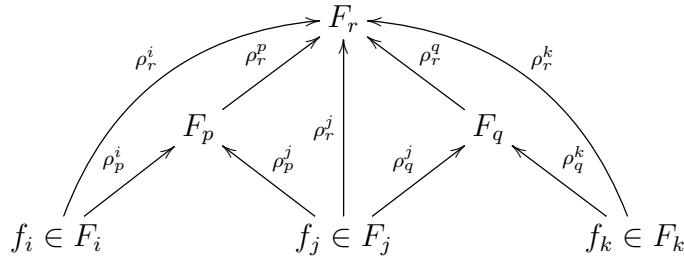


For every index $i \in I$, we have the canonical injection $\epsilon_i \colon F_i \to \coprod_{i \in I} F_i$, and thus, a canonical map $\pi_i \colon F_i \longrightarrow \varinjlim F_i$, namely

$$\pi_i \colon f \mapsto [\epsilon_i(f)]_\sim = [\epsilon_i(f)].$$

(Here, $[x]_\sim = [x]$ means equivalence class of $x$ modulo $\sim$.) It is obvious that $\pi_i = \pi_j \circ \rho_j^i$ for all $i, j \in I$ with $i \leq j$ as illustrated in the diagram below
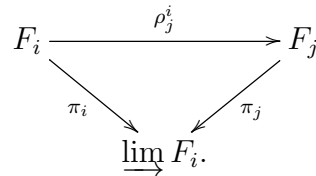


If each $F_i$ is a $R$-module, then $\varinjlim F_i$ is also a $R$-module (a ring, *etc.*). We define addition by

$$[f_i] + [f_j] = [\rho_k^i(f_i) + \rho_k^j(f_j)], \quad \text{for any } k \in I \text{ with } k \geq i, j$$
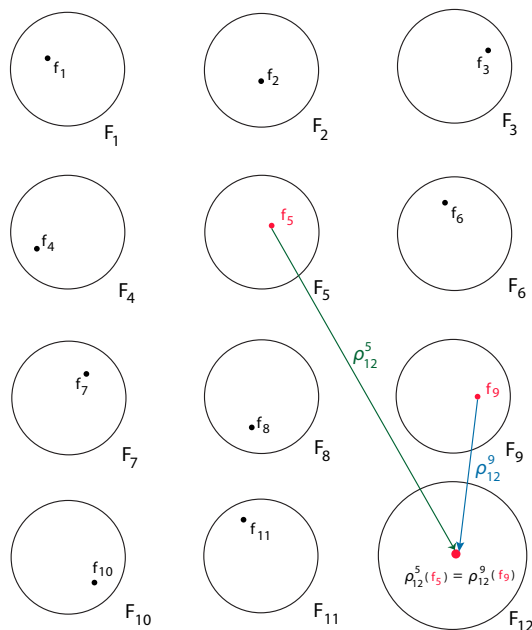
Figure 9.8: An illustration of the equivalence relation $\sim$ used in the direct limit construction. Since $\rho_{12}^5(f_5) = \rho_{12}^9(f_9)$, $f_5 \sim f_9$.

and multiplication by a scalar as

$$\lambda[f_i] = [\lambda f_i].$$

If the $F_i$ are rings, then we define multiplication by

$$[f_i] \cdot [f_j] = [\rho_k^i(f_i) \cdot \rho_k^j(f_j)], \quad \text{for any } k \in I \text{ with } k \geq i, j.$$

The direct limit $(\varinjlim F_i, (\pi_i)_{i \in I})$ is characterized by the important *universal mapping property*: For every set (R-module, commutative ring, *etc.*) $G$ and every family of maps $\theta_i \colon F_i \to G$ so that $\theta_i = \theta_j \circ \rho_j^i$, for all $i, j \in I$ with $i \leq j$ as in the diagram below

$$
\begin{array}{ccc}
F_i & \xrightarrow{\ \ \rho_j^i\ \ } & F_j \\
& \theta_i \searrow \quad \swarrow \theta_j & \\
& G, &
\end{array}
$$
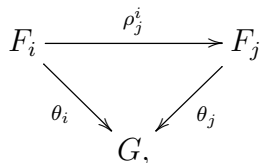
there is a unique map $\varphi \colon \varinjlim F_i \to G$, so that

$$\theta_i = \varphi \circ \pi_i, \quad \text{for all } i \in I$$

as illustrated in the diagram below

$$
\begin{array}{ccc}
F_i & \xrightarrow{\;\;\rho^i_j\;\;} & F_j \\
\end{array}
$$

$$
F_i \xrightarrow{\;\pi_i\;} \varinjlim F_i \xleftarrow{\;\pi_j\;} F_j
$$

$$
\theta_i \searrow \quad \downarrow \varphi \quad \swarrow \theta_j
$$

$$
G.
$$
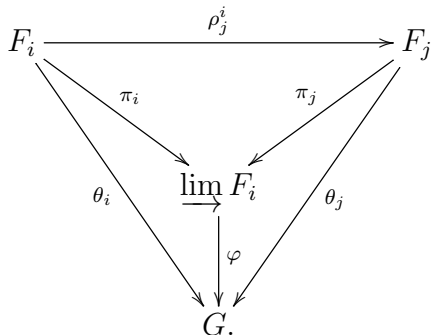
The universal mapping property of the direct limit implies that it is unique up to isomorphism.

**Remark:** The direct limit $\varinjlim F_i$ is actually a *colimit*; it is an initial object in a suitably defined category. Unfortunately, following common practice (probably due to some obscure historical tradition) it is called a *direct* limit.

The following proposition gives a useful criterion to show that an object is a direct limit.

**Proposition 9.1.** *Given a direct mapping family $((F_i)_{i \in I}, (\rho^i_j)_{i \leq j})$ of sets (R-modules, commutative rings, etc.), suppose $G$ is a set (R-module, ring, etc.) and $(\theta_i)_{i \in I}$ is a family of maps $\theta_i \colon F_i \to G$ such that $\theta_i = \theta_j \circ \rho^i_j$, for all $i, j \in I$ with $i \leq j$ as in the diagram below*

$$
\begin{array}{ccc}
F_i & \xrightarrow{\;\rho^i_j\;} & F_j \\
 & \theta_i \searrow \quad \swarrow \theta_j & \\
 & G. &
\end{array}
$$
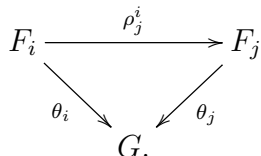
*If the following two conditions are satisfied*

(a) *For every $g \in G$, there is some $i \in I$ and some $f_i \in F_i$ such that $g = \theta_i(f_i)$*

(b) *For all $i, j \in I$, for any $f_i \in F_i$ and any $f_j \in F_j$,*

$$
\theta_i(f_i) = \theta_j(f_j) \quad iff \quad \exists k \text{ such that } i \leq k, j \leq k \text{ and } \rho^i_k(f_i) = \rho^j_k(f_j),
$$

*then $(G, (\theta_i)_{i \in I})$ is a direct limit of the direct mapping family $((F_i)_{i \in I}, (\rho^i_j)_{i \leq j})$.*

*Proof.* It suffices to prove that $(G, (\theta_i)_{i \in I})$ satisfies the universal mapping family. Let $H$ be a set (R-module, commutative ring, *etc.*) and $(\eta_i)_{i \in I}$ is a family of maps $\eta_i \colon F_i \to H$ such that $\eta_i = \eta_j \circ \rho^i_j$, for all $i, j \in I$ with $i \leq j$ as in the diagram below

$$
\begin{array}{ccc}
F_i & \xrightarrow{\;\rho^i_j\;} & F_j \\
 & \eta_i \searrow \quad \swarrow \eta_j & \\
 & H. &
\end{array}
$$
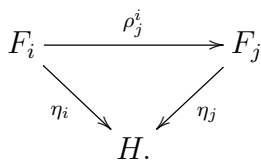
We need to prove that there is a unique map $\varphi\colon G \to H$ such that the following diagrams commute
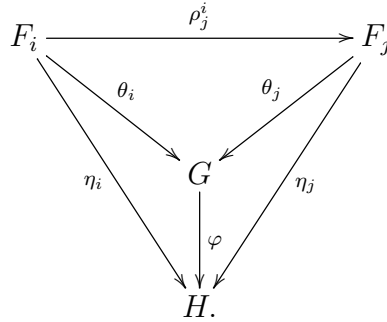


By (a), since every $g \in G$ is of the form $g = \theta_i(f_i)$ for some $f_i \in F_i$, then we must have

$$\varphi(g) = \varphi(\theta_i(f_i)) = \eta_i(f_i).$$

Thus, if $\varphi$ exists, it is unique. It remains to show that the definition of $\varphi(g)$ as $\eta_i(f_i)$ does not depend on the choice of $f_i$. If $f_j \in F_j$ is another element such that $\theta_j(f_j) = g$, then $\theta_i(f_i) = \theta_j(f_j)$, which by (b) means that there is some $k \in I$ such that, $i \leq k$, $j \leq k$ and $\rho_k^i(f_i) = \rho_k^j(f_j)$. But then since the following diagrams commute
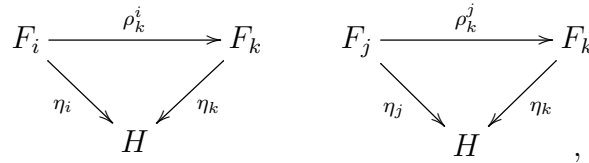


we have

$$\eta_i(f_i) = \eta_k(\rho_k^i(f_i)) = \eta_k(\rho_k^j(f_j)) = \eta_j(f_j),$$

which shows that $\varphi(g)$ is well defined.                                        □

We will also need the notion of map between two direct mapping families and of the direct limit of such a map.

**Definition 9.9.** Given any two direct mapping families $((F_i)_{i\in I}, ((\rho_F)_j^i)_{i\leq j})$ and $((G_i)_{i\in I}, ((\rho_G)_j^i)_{i\leq j})$ of sets ($R$-modules, commutative rings, *etc.*) over the same directed preorder $I$, a *map* from $((F_i)_{i\in I}, ((\rho_F)_j^i)_{i\leq j})$ to $((G_i)_{i\in I}, ((\rho_G)_j^i)_{i\leq j})$ is a family $\varphi = (\varphi_i)_{\in I}$ of maps $\varphi_i\colon F_i \to G_i$ (of sets, of $R$-modules, commutative rings, *etc.*) such that the following diagrams commute for all $i \leq j$:

Let $\varphi = (\varphi_i)_{\in I}$ be a map between two direct mapping families $((F_i)_{i \in I}, ((\rho_F)^i_j)_{i \leq j})$ and $((G_i)_{i \in I}, ((\rho_G)^i_j)_{i \leq j})$. If we write $(F = \varinjlim F_i, \theta_i \colon F_i \to F)$ for the direct limit of the first family and $(G = \varinjlim G_i, \eta_i \colon G_i \to G)$ for the direct limit of the second family, the commutativity of the following diagrams

$$
\begin{array}{ccc}
F_i & \xrightarrow{(\rho_F)^i_j} & F_j \\
\varphi_i \downarrow & & \downarrow \varphi_j \\
G_i & \xrightarrow{(\rho_G)^i_j} & G_j \\
& \eta_i \searrow \quad \swarrow \eta_j & \\
& G &
\end{array}
$$

shows that if we write $\psi_i = \eta_i \circ \varphi_i$, then following diagrams commute

$$
\begin{array}{ccc}
F_i & \xrightarrow{(\rho_F)^i_j} & F_j \\
& \psi_i \searrow \quad \swarrow \psi_j & \\
& G, &
\end{array}
$$

therefore by the universal mapping property of the direct limit $(F = \varinjlim F_i, \theta_i \colon F_i \to F)$, there is a unique map $\Phi \colon F \to G$ such that the following diagrams commute:

$$
\begin{array}{ccc}
F_i & \xrightarrow{\varphi_i} & G_i \\
\theta_i \downarrow & & \downarrow \eta_i \\
F & \xrightarrow{\Phi} & G.
\end{array}
$$
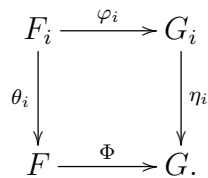
**Definition 9.10.** Let $\varphi = (\varphi_i)_{\in I}$ be a map between two direct mapping families $((F_i)_{i \in I}, ((\rho_F)^i_j)_{i \leq j})$ and $((G_i)_{i \in I}, ((\rho_G)^i_j)_{i \leq j})$. If we write $(F = \varinjlim F_i, \theta_i \colon F_i \to F)$ for the direct limit of the first family and $(G = \varinjlim G_i, \eta_i \colon G_i \to G)$ for the direct limit of the second family, the *direct limit* $\Phi = \varinjlim \varphi_i$ is the unique map $\Phi \colon \varinjlim F_i \to \varinjlim G_i$ such that all diagrams below commute:

$$
\begin{array}{ccc}
F_i & \xrightarrow{\varphi_i} & G_i \\
\theta_i \downarrow & & \downarrow \eta_i \\
F & \xrightarrow{\Phi} & G.
\end{array}
$$
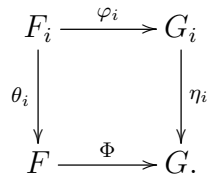
We will also need a generalization of the notion of map of direct mapping families for families indexed by different index sets. Such maps will be needed to define the notion of homomorphism induced by a continuous map in Čech cohomology.

**Definition 9.11.** Given any two direct mapping families $((F_i)_{i \in I}, ((\rho_F)^i_k)_{i \leq k})$ and $((G_j)_{j \in J}, ((\rho_G)^j_l)_{j \leq l})$ of sets ($R$-modules, commutative rings, *etc.*) over the directed preorders $I$ and $J$, a *map* from $((F_i)_{i \in I}, ((\rho_F)^i_k)_{i \leq k})$ to $((G_j)_{j \in J}, ((\rho_G)^j_l)_{j \leq l})$ is pair $(\tau, \varphi)$, where $\tau \colon I \to J$ is an order-preserving map and $\varphi$ is a family $\varphi = (\varphi_i)_{\in I}$ of maps $\varphi_i \colon F_i \to G_{\tau(i)}$ (of sets, of $R$-modules, commutative rings, *etc.*) such that the following diagrams commute for all $i \leq k$:

$$
\begin{array}{ccc}
F_i & \xrightarrow{\;(\rho_F)^i_k\;} & F_k \\
{\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi_k} \\
G_{\tau(i)} & \xrightarrow[\;(\rho_G)^{\tau(i)}_{\tau(k)}\;]{} & G_{\tau(k)}.
\end{array}
$$

Let $(\tau, \varphi = (\varphi_i)_{\in I})$ be a map between two direct mapping families $((F_i)_{i \in I}, ((\rho_F)^i_k)_{i \leq k})$ and $((G_j)_{j \in J}, ((\rho_G)^j_l)_{j \leq l})$. If we write $(F = \varinjlim F_i, \theta_i \colon F_i \to F)$ for the direct limit of the first family and $(G = \varinjlim G_j, \eta_j \colon G_j \to G)$ for the direct limit of the second family, the commutativity of the following diagrams

$$
\begin{array}{ccc}
F_i & \xrightarrow{\;(\rho_F)^i_k\;} & F_k \\
{\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi_k} \\
G_{\tau(i)} & \xrightarrow[\;(\rho_G)^{\tau(i)}_{\tau(k)}\;]{} & G_{\tau(k)} \\
\;\;\;\searrow{\scriptstyle \eta_{\tau(i)}} & & \swarrow{\scriptstyle \eta_{\tau(k)}}\;\;\; \\
& G &
\end{array}
$$

shows that if we write $\psi_i = \eta_{\tau(i)} \circ \varphi_i$, then following diagrams commute

$$
\begin{array}{ccc}
F_i & \xrightarrow{\;(\rho_F)^i_k\;} & F_k \\
\;\;\searrow{\scriptstyle \psi_i} & & \swarrow{\scriptstyle \psi_k}\;\; \\
& G, &
\end{array}
$$

therefore by the universal mapping property of the direct limit $(F = \varinjlim F_i, \theta_i \colon F_i \to F)$, there is a unique map $\Phi \colon F \to G$ such that the following diagrams commute:

$$
\begin{array}{ccc}
F_i & \xrightarrow{\;\varphi_i\;} & G_{\tau(i)} \\
{\scriptstyle \theta_i}\downarrow & & \downarrow{\scriptstyle \eta_{\tau(i)}} \\
F & \xrightarrow{\;\Phi\;} & G.
\end{array}
$$

**Definition 9.12.** Let $(\tau, \varphi = (\varphi_i)_{\in I})$ be a map between two direct mapping families $((F_i)_{i \in I},$ $((\rho_F)_k^i)_{i \leq k})$ and $((G_j)_{j \in J}, ((\rho_G)_l^j)_{j \leq l})$. If we write $(F = \varinjlim F_i, \theta_i \colon F_i \to F)$ for the direct limit of the first family and $(G = \varinjlim G_j, \eta_j \colon G_j \to G)$ for the direct limit of the second family, the *direct limit* $\Phi = \varinjlim \varphi_i$ is the unique map $\Phi \colon \varinjlim F_i \to \varinjlim G_j$ such that all diagrams below commute:

$$
\begin{array}{ccc}
F_i & \xrightarrow{\ \varphi_i\ } & G_{\tau(i)} \\
\theta_i \downarrow & & \downarrow \eta_{\tau(i)} \\
F & \xrightarrow{\ \Phi\ } & G.
\end{array}
$$

# Chapter 10

# Čech Cohomology with Values in a Presheaf

## 10.1  Čech Cohomology of a Cover

Given a topological space $X$ and a presehaf $\mathcal{F}$, there is a way of defining cohomology groups $\check{H}^p(X, \mathcal{F})$ as a limit process involving the definition of some cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ associated with open covers $\mathcal{U} = (U_j)_{j \in J}$ of the space $X$. Given two open covers $\mathcal{U}$ and $\mathcal{V}$, we can define when $\mathcal{V}$ is a refinement of $\mathcal{U}$, and then we define the cohomology group $\check{H}^p(X, \mathcal{F})$ as the direct limit of the directed system of groups $\check{H}^p(\mathcal{U}, \mathcal{F})$. When the presheaf $\mathcal{F}$ has some special properties and when nice covers exist, the limit process can be bypassed.

Throughout this chapter $R$ will denote a fixed commutative ring with unit. Let $\mathcal{F}$ be a presheaf of $R$-modules on $X$. We always assume that that $\mathcal{F}(\emptyset) = (0)$, as in the case of a sheaf. Our first goal is to define $R$-modules of cochains, $C^p(\mathcal{U}, \mathcal{F})$. Here a decision must be made, namely whether we use sequences of indices with or without repetitions allowed. This is one of the confusing aspects of the set up of Čech cohomology, as the literature uses both approaches typically without any justification. In order to deal correctly with the passage to a finer cover it is necessary to allow repetitions of indices. However, it can also be shown that using special kinds of cochains called alternating cochains, isomorphic cohomology $R$-modules are obtained. As a corollary, one may indeed assume that sequences without repetitions are used. Note that in the sequel, when we refer to "groups" we usually mean $R$-modules.

Given any finite sequence $I = (i_0, \ldots, i_p)$ of elements of some index set $J$ (where $p \geq 0$ and the $i_j$ are not necessarily distinct), we let

$$U_I = U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

Note that it may happen that $U_I = \emptyset$ (this is another confusing point: some authors only consider sequences $I = (i_0, \ldots, i_p)$ for which $U_{i_0 \cdots i_p} \neq \emptyset$). We denote by $U_{i_0 \cdots \widehat{i_j} \cdots i_p}$ the intersection

$$U_{i_0 \cdots \widehat{i_j} \cdots i_p} = U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_p}$$

of the $p$ subsets obtained by omitting $U_{i_j}$ from $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ (the intersection of the $p + 1$ subsets). See Figure 10.1.
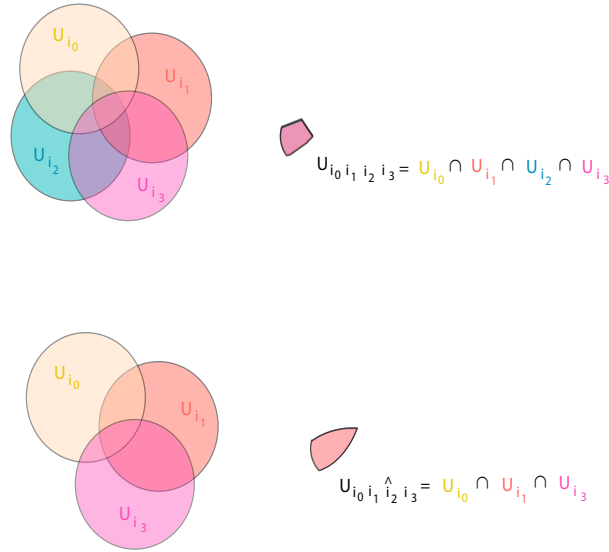


Figure 10.1: An illustration of $U_{i_0 i_1 i_2 i_3}$ and $U_{i_0 i_1 \widehat{i_2} i_3}$.

Then we have $p + 1$ inclusion maps

$$\delta_j^p \colon U_{i_0 \cdots i_p} \longrightarrow U_{i_0 \cdots \widehat{i_j} \cdots i_p}, \quad 0 \le j \le p.$$

For example, if $p = 0$ we have the map

$$\delta_0^0 \colon U_{i_0} \longrightarrow X$$

for $p = 1$, we have the two maps

$$\delta_0^1 \colon U_{i_0} \cap U_{i_1} \longrightarrow U_{i_1}, \quad \delta_1^1 \colon U_{i_0} \cap U_{i_1} \longrightarrow U_{i_0},$$

for $p = 2$, we have the three maps

$$\delta_0^2 \colon U_{i_0} \cap U_{i_1} \cap U_{i_2} \longrightarrow U_{i_1} \cap U_{i_2}, \ \delta_1^2 \colon U_{i_0} \cap U_{i_1} \cap U_{i_2} \longrightarrow U_{i_0} \cap U_{i_2}, \ \delta_2^2 \colon U_{i_0} \cap U_{i_1} \cap U_{i_2} \longrightarrow U_{i_1} \cap U_{i_2}.$$

**Definition 10.1.** Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of abelian groups $\mathcal{F}$ on $X$, the $R$-module of *Čech $p$-cochains* $C^p(\mathcal{U}, \mathcal{F})$ is the set of all functions $f$ with domain $J^{p+1}$ such that $f(i_0, \ldots, i_p) \in \mathcal{F}(U_{i_0 \cdots i_p})$; in other words,

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \ldots, i_p) \in J^{p+1}} \mathcal{F}(U_{i_0 \cdots i_p}),$$

the set of all $J^{p+1}$-indexed families $(f_{i_0, \ldots, i_p})_{(i_0, \ldots, i_p) \in J^{p+1}}$ with $f_{i_0, \ldots, i_p} \in \mathcal{F}(U_{i_0 \cdots i_p})$.

In particular, for $p = 0$ we have

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{j \in J} \mathcal{F}(U_j)$$

so a 0-cochain is a $J$-indexed family $f = (f_j)_{j \in J}$ with $f_j \in \mathcal{F}(U_j)$, and for $p = 1$ we have

$$C^1(\mathcal{U}, \mathcal{F}) = \prod_{(i,j) \in J^2} \mathcal{F}(U_i \cap U_j)$$

so a 1-cochain is a $J^2$-indexed family $f = (f_{i,j})_{(i,j) \in J^2}$ with $f_{i,j} \in \mathcal{F}(U_i \cap U_j)$.

**Remark:** Since $\mathcal{F}(\emptyset) = (0)$, for any cochain $f \in C^p(\mathcal{U}, \mathcal{F})$, if $U_{i_0 \cdots i_p} = \emptyset$ then $f_{i_0 \cdots i_p} = 0$. Therefore, we could define $C^p(\mathcal{U}, \mathcal{F})$ as the set of families $f_{i_0 \cdots i_p} \in \mathcal{F}(U_{i_0 \cdots i_p})$ corresponding to tuples $(i_0, \ldots, i_p) \in J^{p+1}$ such that $U_{i_0 \cdots i_p} \neq \emptyset$. This is the definition adopted by several authors, including Warner [50] (Chapter 5, Section 5.33).

Each inclusion map $\delta_j^p \colon U_{i_0 \cdots i_p} \longrightarrow U_{i_0 \cdots \widehat{i_j} \cdots i_p}$ induces a map

$$\mathcal{F}(\delta_j^p) \colon \mathcal{F}(U_{i_0 \cdots \widehat{i_j} \cdots i_p}) \longrightarrow \mathcal{F}(U_{i_0 \cdots i_p})$$

which is none other that the restriction map $\rho_{U_{i_0 \cdots i_p}}^{U_{i_0 \cdots \widehat{i_j} \cdots i_p}}$ which, for the sake of notational simplicity, we also denote by $\rho_{i_0 \cdots i_p}^j$.

**Definition 10.2.** Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, the *coboundary maps* $\delta_{\mathcal{F}}^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ are given by

$$\delta_{\mathcal{F}}^p = \sum_{j=1}^{p+1} (-1)^j \mathcal{F}(\delta_j^{p+1}), \quad p \geq 0.$$

More explicitly, for any $p$-cochain $f \in C^p(\mathcal{U}, \mathcal{F})$, for any sequence $(i_0, \ldots, i_{p+1}) \in J^{p+2}$, we have

$$(\delta_{\mathcal{F}}^p f)_{i_0, \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho_{i_0 \cdots i_{p+1}}^j (f_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}}).$$

Unravelling Definition 10.2, for $p = 0$ we have

$$(\delta_{\mathcal{F}}^0 f)_{i,j} = \rho_{ij}^0(f_j) - \rho_{ij}^1(f_i),$$

and for $p = 1$ we have

$$(\delta_{\mathcal{F}}^1 f)_{i,j,k} = \rho_{ijk}^0(f_{j,k}) - \rho_{ijk}^1(f_{i,k}) + \rho_{ijk}^2(f_{i,j}).$$
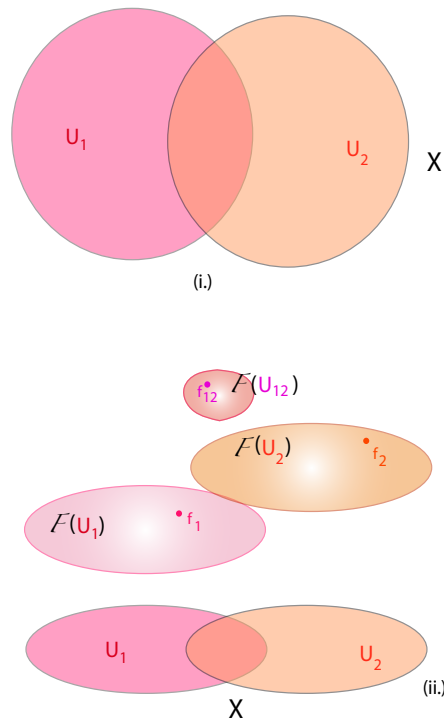
Figure 10.2: An illustration of $X$ in Example 10.1. Figure ($ii$.) illustrates the associated presheaf $\mathcal{F}$.

**Example 10.1.** As an explicit example of Definitions 10.1 and 10.2, let $X$ be the union of two open sets, namely $X = U_1 \cup U_2$. See Figure 10.2.

Then

$$C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_1) \times \mathcal{F}(U_2)$$
$$C^1(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_{11}) \times \mathcal{F}(U_{12}) \times \mathcal{F}(U_{21}) \times \mathcal{F}(U_{22})$$
$$C^2(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_{111}) \times \mathcal{F}(U_{112}) \times \mathcal{F}(U_{121}) \times \mathcal{F}(U_{122}) \times \mathcal{F}(U_{211}) \times \mathcal{F}(U_{212})$$
$$\times \mathcal{F}(U_{221}) \times \mathcal{F}(U_{222}),$$

where

$$U_{11} = U_1 \cap U_1 = U_1, \qquad U_{12} = U_1 \cap U_2 = U_{21}, \qquad U_{22} = U_2 \cap U_2 = U_2$$
$$U_{111} = U_1 \cap U_1 \cap U_1 = U_1, \qquad U_{222} = U_2 \cap U_2 \cap U_2 = U_2$$
$$U_{112} = U_{121} = U_{211} = U_1 \cap U_1 \cap U_2 = U_1 \cap U_2 \cap U_2 = U_{221} = U_{212} = U_{122}.$$

In general $C^p(\mathcal{U}, \mathcal{F})$ is a product with $2^{p+1}$ factors. A typical element of $C^0(\mathcal{U}, \mathcal{F})$ has the form $(f_1, f_2)$ where $f_1$ is an element of the group associated with $U_1$ and $f_2$ is an element of

the group associated with $U_2$. A typical element of $C^1(\mathcal{U}, \mathcal{F})$ has the form $(f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2})$ where $f_{1,1}$ is an element of the group associated with $U_{11} = U_1$, $f_{1,2}$ is an element of the group associated with $U_{12} = U_1 \cap U_2$ $f_{2,1}$ is another element of the group associated with $U_{21} = U_{12}$, and $f_{2,2}$ is an element of the group associated with $U_{22}$. In general $f_{1,2} \neq f_{2,1}$. A typical element of $C^2(\mathcal{U}, \mathcal{F})$ has the form

$$(f_{1,1,1}, f_{1,1,2}, f_{1,2,1}, f_{1,2,2}, f_{2,1,1}, f_{2,1,2}, f_{2,2,1}, f_{2,2,2}),$$

where $f_{1,1,1}$ is an element of the group associated with $U_{111} = U_1$, $f_{1,1,2}$ is an element of the group associated with $U_{112} = U_1 \cap U_2$,, $f_{1,2,1}$ is an element of the group associated with $U_{112}$, $f_{1,2,2}$ is an element of the group associated with $U_{112}$, $f_{2,1,1}$ is an element of the group associated with $U_{112}$, $f_{2,1,2}$ is an element of the group associated with $U_{112}$, $f_{2,2,1}$ is an element of the group associated with $U_{112}$, and $f_{2,2,2}$ is an element of the group associated with $U_{222} = U_2$. In general, a typical element of $C^p(\mathcal{U}, \mathcal{F})$ is a $2^{p+1}$-tuple.

The coboundary map $\delta^0_{\mathcal{F}} \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ takes $f \in C^0(\mathcal{U}, \mathcal{F})$, say $f = (f_1, f_2)$, and makes it into element of $C^1(\mathcal{U}, \mathcal{F})$ by calculating

$$(\delta^0_{\mathcal{F}} f)_{1,1} = \rho^0_{11}(f_1) - \rho^1_{11}(f_1) = 0$$
$$(\delta^0_{\mathcal{F}} f)_{1,2} = \rho^0_{12}(f_2) - \rho^1_{12}(f_1)$$
$$(\delta^0_{\mathcal{F}} f)_{2,1} = \rho^0_{21}(f_1) - \rho^1_{21}(f_2)$$
$$(\delta^0_{\mathcal{F}} f)_{2,2} = \rho^0_{22}(f_2) - \rho^1_{22}(f_2) = 0.$$

In other words

$$\delta^0_{\mathcal{F}}(f_1, f_2) = (0, \rho^0_{12}(f_2) - \rho^1_{12}(f_1), \rho^0_{21}(f_1) - \rho^1_{21}(f_2), 0) \in C^1(\mathcal{U}, \mathcal{F}).$$

The coboundary map $\delta^1_{\mathcal{F}} \colon C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ takes $f \in C^1(\mathcal{U}, \mathcal{F})$, say $f = (f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2})$, and makes it into element of $C^2(\mathcal{U}, \mathcal{F})$ by calculating

$$(\delta^1_{\mathcal{F}} f)_{1,1,1} = \rho^0_{111}(f_{1,1}) - \rho^1_{111}(f_{1,1}) + \rho^2_{111}(f_{1,1}) = \rho^2_{111}(f_{1,1})$$
$$(\delta^1_{\mathcal{F}} f)_{1,1,2} = \rho^0_{112}(f_{1,2}) - \rho^1_{112}(f_{1,2}) + \rho^2_{112}(f_{1,1}) = \rho^2_{112}(f_{1,1})$$
$$(\delta^1_{\mathcal{F}} f)_{1,2,1} = \rho^0_{121}(f_{2,1}) - \rho^1_{121}(f_{1,1}) + \rho^2_{121}(f_{1,2})$$
$$(\delta^1_{\mathcal{F}} f)_{1,2,2} = \rho^0_{122}(f_{2,2}) - \rho^1_{122}(f_{1,2}) + \rho^2_{122}(f_{1,2}) = \rho^0_{122}(f_{2,2})$$
$$(\delta^1_{\mathcal{F}} f)_{2,1,1} = \rho^0_{211}(f_{1,1}) - \rho^1_{211}(f_{2,1}) + \rho^2_{211}(f_{2,1}) = \rho^0_{211}(f_{1,1})$$
$$(\delta^1_{\mathcal{F}} f)_{2,1,2} = \rho^0_{212}(f_{1,2}) - \rho^1_{212}(f_{2,2}) + \rho^2_{212}(f_{2,1})$$
$$(\delta^1_{\mathcal{F}} f)_{2,2,1} = \rho^0_{221}(f_{2,1}) - \rho^1_{221}(f_{2,1}) + \rho^2_{221}(f_{2,2}) = \rho^2_{221}(f_{2,2})$$
$$(\delta^1_{\mathcal{F}} f)_{2,2,2} = \rho^0_{222}(f_{2,2}) - \rho^1_{222}(f_{2,2}) + \rho^2_{222}(f_{2,2}) = \rho^2_{222}(f_{2,2}).$$

In other words

$$\delta^1_{\mathcal{F}}(f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2}) = (\rho^2_{111}(f_{1,1}), \rho^2_{112}(f_{1,1}), \rho^0_{121}(f_{2,1}) - \rho^1_{121}(f_{1,1}) + \rho^2_{121}(f_{1,2}), \rho^0_{122}(f_{2,2}),$$
$$\rho^0_{211}(f_{1,1}), \rho^0_{212}(f_{1,2}) - \rho^1_{212}(f_{2,2}) + \rho^2_{212}(f_{2,1}), \rho^2_{221}(f_{2,2}), \rho^2_{222}(f_{2,2})).$$

Families of the form $(\delta^0_{\mathcal{F}} f)_{i,j}$ form the group ($R$-module) $B^1(\mathcal{U}, \mathcal{F})$ of Čech coboundaries, and the group ($R$-module) $Z^0(\mathcal{U}, \mathcal{F})$ of Čech cocycles consists of the families $(f_j)_{j \in J} \in C^0(\mathcal{U}, \mathcal{F})$ such that $(\delta^0_{\mathcal{F}} f) = 0$; that is, families $(f_j)_{j \in J} \in C^0(\mathcal{U}, \mathcal{F})$ such that

$$\rho^0_{ij}(f_j) = \rho^1_{ij}(f_i)$$

for all $i, j \in J$.

Families of the form $(\delta^1_{\mathcal{F}} f)_{i,jk}$ form the group ($R$-module) $B^2(\mathcal{U}, \mathcal{F})$ of Čech coboundaries, and the group ($R$-module) $Z^1(\mathcal{U}, \mathcal{F})$ of Čech cocycles consists of the families $(f_{ij})_{(i,j) \in J^2} \in C^1(\mathcal{U}, \mathcal{F})$ such that $(\delta^1_{\mathcal{F}} f) = 0$; that is, families $(f_{i,j})_{(i,j) \in J^2} \in C^1(\mathcal{U}, \mathcal{F})$ such that

$$\rho^1_{ijk}(f_{i,k}) = \rho^2_{ijk}(f_{i,j}) + \rho^0_{ijk}(f_{j,k})$$

for all $i, j, k \in J$.

In general the definition of $B^p(\mathcal{U}, \mathcal{F})$ and $Z^p(\mathcal{U}, \mathcal{F})$ is as follows.

**Definition 10.3.** Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, the $R$-module $B^p(\mathcal{U}, \mathcal{F})$ of *Čech $p$-boundaries* is given by $B^p(\mathcal{U}, \mathcal{F}) = \operatorname{Im} \delta^{p-1}_{\mathcal{F}}$ for $p \geq 1$ with $B^0(\mathcal{U}, \mathcal{F}) = (0)$, and the $R$-module $Z^p(\mathcal{U}, \mathcal{F})$ of *Čech $p$-cocycles* is given by $Z^p(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} \delta^p_{\mathcal{F}}$, for $p \geq 0$.

It is easy to check that $\delta^{p+1}_{\mathcal{F}} \circ \delta^p_{\mathcal{F}} = 0$ for all $p \geq 0$, so we have a chain complex of cohomology

$$0 \xrightarrow{\delta^{-1}_{\mathcal{F}}} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0_{\mathcal{F}}} C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots \xrightarrow{\delta^{p-1}_{\mathcal{F}}} C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^p_{\mathcal{F}}} C^{p+1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^{p+1}_{\mathcal{F}}} \cdots$$

and we can define the Čech cohomology groups as follows. Let $G$ be a $R$-module, and write $G_X$ for the constant presheaf on $X$ such that $G_X(U) = G$ for every nonempty open subset $U \subseteq X$ (with $G_X(\emptyset) = (0)$).

**Definition 10.4.** Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, the *Čech cohomology groups* $\check{H}^p(\mathcal{U}, \mathcal{F})$ *of the cover $\mathcal{U}$ with values in $\mathcal{F}$* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = Z^p(\mathcal{U}, \mathcal{F})/B^p(\mathcal{U}, \mathcal{F}), \quad p \geq 0.$$

The *classical Čech cohomology groups* $\check{H}^p(\mathcal{U}; G)$ *of the cover $\mathcal{U}$ with coefficients in the $R$-module $G$* are the groups $\check{H}^p(\mathcal{U}, G_X)$.

The groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ and $\check{H}^p(\mathcal{U}, G_X)$ are in fact $R$-modules.

If $\mathcal{F}$ is a sheaf, then $\check{H}^0(\mathcal{U}, \mathcal{F})$ is independent of the cover $\mathcal{U}$.

**Proposition 10.1.** *Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, if $\mathcal{F}$ is a sheaf, then*

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X) = \Gamma(X, \mathcal{F}),$$

*the module of global sections of $\mathcal{F}$.*

*Proof.* We saw earlier that a 0-cocycle is a family $(f_j)_{j \in J} \in C^0(\mathcal{U}, \mathcal{F})$ such that

$$\rho^0_{ij}(f_j) = \rho^1_{ij}(f_i)$$

for all $i, j \in J$. Since $\mathcal{F}$ is a sheaf, the $f_i$ patch to a global section $f \in \mathcal{F}(X)$ such that $\rho^X_{U_i}(f) = f_i$ for all $i \in I$. $\qquad\square$

The module of $p$-cochains $C^p(\mathcal{U}, \mathcal{F})$ consists of the set of all families $(f_{i_0,\dots,i_p})_{(i_0,\dots,i_p) \in J^{p+1}}$ with $f_{i_0,\dots,i_p} \in \mathcal{F}(U_{i_0 \cdots i_p})$. This is not a very economical definition. It turns out that the same Čech cohomology groups are obtained using the more economical notion of alternating cochain.

**Definition 10.5.** Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, a cochain $f \in C^p(\mathcal{U}, \mathcal{F})$ is *alternating* if it satisfies the following conditions:

(a) $f_{i_0,\dots,i_p} = 0$ whenever two of the indices $i_0, \dots, i_p$ are equal.

(b) $f_{\sigma(i_0),\dots,\sigma(i_p)} = \mathrm{sign}(\sigma) f_{i_0,\dots,i_p}$, for every permutation $\sigma$ of the set $\{0,\dots,p\}$ (where $\mathrm{sign}(\sigma)$ denotes the sign of the permutation $\sigma$).

The set of alternating $p$-cochains forms a submodule $C'^p(\mathcal{U}, \mathcal{F})$ of $C^p(\mathcal{U}, \mathcal{F})$.

It is easily checked that $\partial^p_{\mathcal{F}} f$ is alternating if $f$ is alternating. As a consequence the alternating cochains yield a chain complex $(C'^*(\mathcal{U}, \mathcal{F}), \delta_{\mathcal{F}})$. The corresponding cohomology groups are denoted by $\check{H}'^p(\mathcal{U}, \mathcal{F})$. The following proposition is shown in FAC [44] (Chapter 1, §3, Subsection 20).

**Proposition 10.2.** *Given a topological space $X$, an open cover $\mathcal{U} = (U_j)_{j \in J}$ of $X$, and a presheaf of $R$-modules $\mathcal{F}$ on $X$, the Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ and $\check{H}'^p(\mathcal{U}, \mathcal{F})$ are isomorphic for all $p \geq 0$.*

The proof of Proposition 10.2 consists in defining a suitable chain homotopy. It also justifies the fact that we may assume that the index set $J$ is totally ordered (say by $\leq$), and using cochains $f_{i_0,\dots,i_p}$ where the indices form a strictly increasing sequence $i_0 < i_1 < \cdots < i_p$; Bott and Tu [2] use this approach (Chapter II, §8).

Our next goal is to define Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ that are independent of the open cover $\mathcal{U}$ chosen for $X$. Such groups are obtained as direct limits of direct mapping families of modules, as defined in Section 9.6. The direct limit construction is applied to the preorder of refinement among open coverings.

## 10.2   Čech Cohomology with Values in a Presheaf

First we need to define the notion of refinement of a cover.

**Definition 10.6.** Given two covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ of a space $X$, we say that $\mathcal{V}$ *is a refinement of* $\mathcal{U}$, denoted $\mathcal{U} \prec \mathcal{V}$,[1] if there is a function $\tau \colon J \to I$ such that

$$V_j \subseteq U_{\tau(j)} \quad \text{for all } j \in J.$$

See Figure 10.3. We say that two covers $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* if $\mathcal{V} \prec \mathcal{U}$ and $\mathcal{U} \prec \mathcal{V}$.
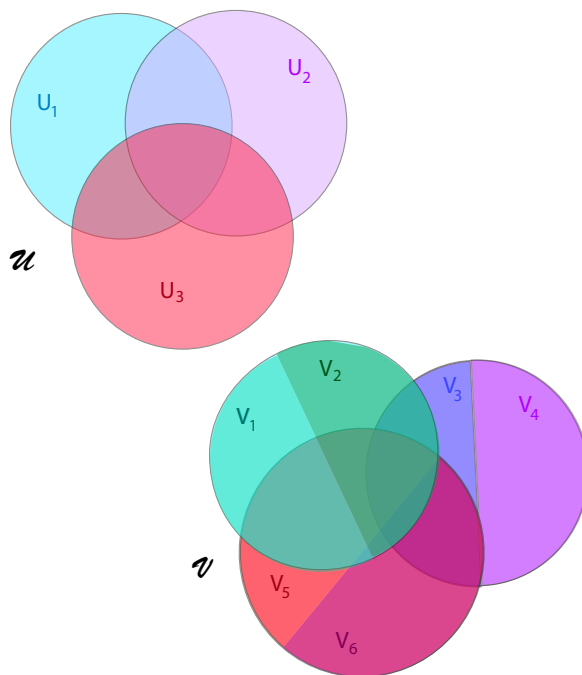


Figure 10.3: Let $\mathcal{U} = U_1 \cup U_2 \cup U_3$. Let $\mathcal{V} = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$. Then $\mathcal{U} \prec \mathcal{V}$ with $\tau \colon \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3\}$ where $\tau(1) = 1$, $\tau(2) = 1$, $\tau(3) = 2$, $\tau(4) = 2$, $\tau(5) = 3$, $\tau(6) = 3$ since $V_1 \subseteq U_1$, $V_2 \subseteq U_1$, $V_3 \subseteq U_2$, $V_4 \subseteq U_2$, $V_5 \subseteq U_3$, $V_6 \subseteq U_3$.

Let $\tau \colon J \to I$ be a function such that

$$V_j \subseteq U_{\tau(j)} \quad \text{for all } j \in J$$

as above. Then we can define a homomorphism from $C^p(\mathcal{U}, \mathcal{F})$ to $C^p(\mathcal{V}, \mathcal{F})$ denoted by $\tau^p$ as follows: for every $p$-cochain $f \in C^p(\mathcal{U}, \mathcal{F})$, let $\tau^p f \in C^p(\mathcal{V}, \mathcal{F})$ be the $p$-cochain given by

$$(\tau^p f)_{j_0 \cdots j_p} = \rho_V^U \big( f_{\tau(j_0) \cdots \tau(j_p)} \big)$$

---

[1]This is the notation used by Bott and Tu [2]. Serre uses the opposite notation $\mathcal{V} \prec \mathcal{U}$ in FAC [44] (Chapter 1, §3, Subsection 22).

for all $(j_0, \ldots, j_p) \in J^{p+1}$, where $\rho_V^U$ denotes the restriction map associated with the inclusion of $V_{j_0 \cdots j_p}$ into $U_{\tau(j_0) \cdots \tau(j_p)}$.

For example, if we take the refinement $\mathcal{U} \prec \mathcal{V}$ illustrated by Figure 10.3, set $p = 0$, and take a cochain $f = (f_1, f_2, f_3) \in C^0(\mathcal{U}, \mathcal{F})$, where $f_1$ is an element of the group associated with $U_1$, $f_2$ is an element of the group associated with $U_2$, and $f_3$ is an element of the group associated with $U_3$, we calculate $\tau^0 f \in C^0(\mathcal{V}, \mathcal{F})$ as

$$
\begin{aligned}
(\tau^0 f)_1 &= \rho_{V_1}^{U_1}(f_{\tau(1)}) = \rho_{V_1}^{U_1}(f_1) \\
(\tau^0 f)_2 &= \rho_{V_2}^{U_1}(f_{\tau(2)}) = \rho_{V_2}^{U_1}(f_1) \\
(\tau^0 f)_3 &= \rho_{V_3}^{U_2}(f_{\tau(3)}) = \rho_{V_3}^{U_2}(f_2) \\
(\tau^0 f)_4 &= \rho_{V_4}^{U_2}(f_{\tau(4)}) = \rho_{V_4}^{U_2}(f_2) \\
(\tau^0 f)_5 &= \rho_{V_5}^{U_3}(f_{\tau(5)}) = \rho_{V_5}^{U_3}(f_3) \\
(\tau^0 f)_6 &= \rho_{V_6}^{U_3}(f_{\tau(6)}) = \rho_{V_6}^{U_3}(f_3).
\end{aligned}
$$

In other words

$$
\tau^0(f_1, f_2, f_3) = (\rho_{V_1}^{U_1}(f_1), \rho_{V_2}^{U_1}(f_1), \rho_{V_3}^{U_2}(f_2), \rho_{V_4}^{U_2}(f_2), \rho_{V_5}^{U_3}(f_3), \rho_{V_6}^{U_3}(f_3)).
$$

Note that even if the $j_k$'s are distinct, $\tau$ may not be injective so the $\tau(j_k)$'s may not be distinct. This is why it is necessary to define the modules $C^p(\mathcal{U}, \mathcal{F})$ using families indexed by sequences whose elements are not necessarily distinct.

It is easy to see that the map $\tau^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ commutes with $\delta_{\mathcal{F}}$ so we obtain homomorphisms

$$
\tau^{*p} \colon H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{V}, \mathcal{F}).
$$

**Proposition 10.3.** *Given any two open covers $\mathcal{U}$ and $\mathcal{V}$ of a space $X$, if $\mathcal{U} \prec \mathcal{V}$ and if $\tau_1 \colon J \to I$ and $\tau_2 \colon J \to I$ are functions such that*

$$
V_j \subseteq U_{\tau_1(j)} \quad \text{and} \quad V_j \subseteq U_{\tau_2(j)} \quad \text{for all } j \in J,
$$

*then $\tau_1^{*p} = \tau_2^{*p}$ for all $p \geq 0$.*

*Proof Sketch.* Following Serre (see FAC [44], Chapter 1, §3, Subsection 21), given any $f \in C^p(\mathcal{U}, \mathcal{F})$, let

$$
(k^p f)_{j_0 \cdots j_{p-1}} = \sum_{h=0}^{p-1} (-1)^h \rho_h(f_{\tau_1(j_0) \cdots \tau_1(j_h) \tau_2(j_h) \cdots \tau_2(j_{p-1})})
$$

for all $(j_0, \ldots, j_{p-1}) \in J^p$, where $\rho_h$ denotes the restriction map associated with the inclusion of $V_{j_0 \cdots j_{p-1}}$ into $U_{\tau_1(j_0) \cdots \tau_1(j_h) \tau_2(j_h) \cdots \tau_2(j_{p-1})}$. Then, it can be verified that

$$
\delta_{\mathcal{F}} \circ k^p(f) + k^{p+1} \circ \delta_{\mathcal{F}}(f) = \tau_2^p(f) - \tau_1^p(f),
$$

which means that the maps $k^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{V}, \mathcal{F})$ define a chain homotopy, and by Proposition 2.17, we have $\tau_1^{*p} = \tau_2^{*p}$ for all $p \geq 0$. $\qquad \square$

Proposition 10.3 implies that if $\mathcal{U} \prec \mathcal{V}$, then there is a homomorphism

$$\rho_{\mathcal{V}}^{\mathcal{U}} \colon \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F}).$$

It is easy to check that the relation $\mathcal{U} \prec \mathcal{V}$ among covers is a directed preorder; indeed, given any two covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$, the cover $\mathcal{W} = (U_i \cap V_j)_{(i,j) \in I \times J}$ is a common refinement of both $\mathcal{U}$ and $\mathcal{V}$, so $\mathcal{U} \prec \mathcal{W}$ and $\mathcal{V} \prec \mathcal{W}$. It is also immediately verified that if $\mathcal{U} \prec \mathcal{V} \prec \mathcal{W}$, then

$$\rho_{\mathcal{W}}^{\mathcal{U}} = \rho_{\mathcal{W}}^{\mathcal{V}} \circ \rho_{\mathcal{V}}^{\mathcal{U}}$$

and that

$$\rho_{\mathcal{U}}^{\mathcal{U}} = \mathrm{id}.$$

Furthermore, if $\mathcal{U}$ and $\mathcal{V}$ are equivalent, then because

$$\rho_{\mathcal{U}}^{\mathcal{V}} \circ \rho_{\mathcal{V}}^{\mathcal{U}} = \mathrm{id} \quad \text{and} \quad \rho_{\mathcal{V}}^{\mathcal{U}} \circ \rho_{\mathcal{U}}^{\mathcal{V}} = \mathrm{id},$$

we see that

$$\rho_{\mathcal{V}}^{\mathcal{U}} \colon \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$$

is an isomorphism.

Consequently, it appears that the family $(\check{H}^p(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ is a direct mapping family of modules indexed by the directed set of open covers of $X$.

However, there is a set-theoretic difficulty, which is that the family of open covers of $X$ is **not** a set because it allows arbitrary index sets.[2]

A way to circumvent this difficulty is provided by Serre (see FAC [44], Chapter 1, §3, Subsection 22). The key observation is that any covering $(U_i)_{i \in I}$ is equivalent to a covering $(U'_\lambda)_{\lambda \in L}$ whose index set $L$ is a subset of $2^X$. Indeed, we can take for $(U'_\lambda)_{\lambda \in L}$ the *set* of all open subsets of $X$ that belong to the family $(U_i)_{i \in I}$.

As we noted earlier, if $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ are equivalent, then there is an isomorphism between $\check{H}^p(\mathcal{U}, \mathcal{F})$ and $\check{H}^p(\mathcal{V}, \mathcal{F})$, so we can define

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

with respect to coverings $\mathcal{U} = (U_i)_{i \in I}$ whose index set $I$ is a subset of $2^X$. Another way to circumvent the set theoretic difficulty is to use a device due to Godement ([18], Chapter 5, Section 5.8).

In summary, we have the following definition.

---

[2] Most textboook presentations of Čech cohomology ignore this subtle point.

**Definition 10.7.** Given a topological space $X$ and a presheaf $\mathcal{F}$ of $R$-modules on $X$, the *Čech cohomology groups* $\check{H}^p(X, \mathcal{F})$ *with values in* $\mathcal{F}$ are defined by

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \ \check{H}^p(\mathcal{U}, \mathcal{F})$$

with respect to coverings $\mathcal{U} = (U_i)_{i \in I}$ whose index set $I$ is a subset of $2^X$. The *classical Čech cohomology groups* $\check{H}^p(X; G)$ *with coefficients in the* $R$*-module* $G$ are the groups $\check{H}^p(X, G_X)$ where $G_X$ is the constant presheaf with value $G$.

**Remark:** Warner [50] and Bott and Tu [2] (second edition) define the classical Čech cohomology groups $\check{H}^p(X; G)$ as the groups $\check{H}^p(X, \widetilde{G}_X)$, where $\widetilde{G}_X$ is the sheaf of locally constant functions with values in $G$. Although this is not obvious, if $X$ is paracompact, then the groups $\check{H}^p(X, G_X)$ are $\check{H}^p(X, \widetilde{G}_X)$ are isomorphic; this is proved in Proposition 13.15. As a consequence, for manifolds (which by definition are paracompact), this makes no difference. However, Alexander–Lefschetz duality is proved for the classical definition of Čech cohomology corresponding to the case where the constant presheaf $G_X$ is used, and this is why we used it in our definition.

Next, we will investigate the relationship between de Rham cohomology and classical Čech cohomology for the constant sheaf $\widetilde{\mathbb{R}}_X$ (corresponding to coefficients in $\mathbb{R}$), and singular cohomology and classical Čech cohomology for the constant sheaf $\widetilde{\mathbb{Z}}_X$ (corresponding to coefficients in $\mathbb{Z}$). For manifolds, the de Rham cohomology and the classical Čech cohomology for the constant sheaf $\widetilde{\mathbb{R}}_X$ are isomorphic, and the singular cohomology and the classical Čech cohomology for the constant sheaf $\widetilde{\mathbb{Z}}_X$ are also isomorphic. Furthermore, we will see that if our spaces have a good cover $\mathcal{U}$, then the Čech cohomology groups $\check{H}^p(\mathcal{U}, \widetilde{\mathbb{R}}_X)$ are independent of $\mathcal{U}$ and in fact isomorphic to the de Rham cohomology groups $H^p_{\mathrm{dR}}(X)$, and similarly the Čech cohomology groups $\check{H}^p(\mathcal{U}, \widetilde{\mathbb{Z}}_X)$ are independent of $\mathcal{U}$ and in fact isomorphic to the singular cohomology groups $H^p(X; \mathbb{Z})$ (if $X$ is triangularizable).

**Theorem 10.4.** *Let $M$ be a smooth manifold. The de Rham cohomology groups are isomorphic to the Čech cohomology groups with values in the sheaf $\widetilde{\mathbb{R}}_M$, and also isomorphic to the Čech cohomology groups associated with good covers (with values in the sheaf $\widetilde{\mathbb{R}}_M$):*

$$H^p_{\mathrm{dR}}(M) \cong \check{H}^p(M, \widetilde{\mathbb{R}}_M) \cong \check{H}^p(\mathcal{U}, \widetilde{\mathbb{R}}_M),$$

*for all $p \geq 0$ and all good covers $\mathcal{U}$ of $M$.*

By a previous remark, since manifolds are paracompact, the above theorem also holds with the constant presheaf $\mathbb{R}_M$ instead of the sheaf $\widetilde{\mathbb{R}}_M$.

Theorem 10.4 is proved in Bott and Tu [2] (Theorem 8.9 and Proposition 10.6). The technique used for proving the first isomorphism is based on an idea of André Weil. The idea is to use a double complex known as the Čech–de–Rham complex. A complete exposition is given in Chapter 2, Section 8, of Bott and Tu [2], and we only give a sketch of the argument.

Let $M$ be a smooth manifold. The differential $p$-forms on $M$ form a sheaf $\mathcal{A}_X^p$ with $\Gamma(U, \mathcal{A}_X^p) = \mathcal{A}^p(U)$, the vector space of $p$-forms on the open subset $U \subseteq X$. We define the double complex $\mathcal{AC}^{*,*}$ by

$$\mathcal{AC}^{p,q} = \bigoplus_{I,|I|=p+1} \Gamma(U_I, \mathcal{A}_X^q).$$

There are two differentials

$$d\colon \mathcal{AC}^{p,q} \to \mathcal{AC}^{p,q+1} \quad \text{and} \quad \delta\colon \mathcal{AC}^{p,q} \to \mathcal{AC}^{p+1,q}$$

and we have $d \circ d = 0$ and $\delta \circ \delta = 0$. We associate to the double complex $\mathcal{AC}^{*,*}$ the single complex $\mathcal{AC}^*$ defined by

$$\mathcal{AC}^n = \bigoplus_{p+q=n} \mathcal{AC}^{p,q},$$

with the differential $D^n\colon \mathcal{AC}^n \to \mathcal{AC}^{n+1}$ given by

$$D^n = \delta + (-1)^n d.$$

It is easily verified that

$$D^{n+1} \circ D^n = 0.$$

The cohomology of the complex $(\mathcal{AC}^*, D)$ is denoted by $H_D\{\mathcal{AC}^*(\mathcal{U}, \mathcal{A}_X^*)\}$. It is shown in Bott and Tu [2] (Proposition 8.8) that there is an isomorphism

$$H_{\mathrm{dR}}^*(M) \cong H_D\{\mathcal{AC}^*(\mathcal{U}, \mathcal{A}_X^*)\}.$$

Furthermore, if $\mathcal{U}$ is a a good cover, it is shown in Bott and Tu [2] (before Theorem 8.9) that there is an isomorphism

$$\check{H}^p(\mathcal{U}, \widetilde{\mathbb{R}}_M) \cong H_D\{\mathcal{AC}^*(\mathcal{U}, \mathcal{A}_X^*)\}.$$

Consequently, we obtain an isomorphism

$$H_{\mathrm{dR}}^p(M) \cong \check{H}^p(\mathcal{U}, \widetilde{\mathbb{R}}_M)$$

for all good covers $\mathcal{U}$ and all $p \geq 0$. Since every smooth manifold has a good cover (see Theorem 3.3), and since the good covers are cofinal in the set of all covers of $M$ (with index set in $2^M$), following Bott and Tu [2] (Proposition 10.6), we obtain the isomorphism

$$\check{H}^p(M, \widetilde{\mathbb{R}}_M) \cong \check{H}^p(\mathcal{U}, \widetilde{\mathbb{R}}_M)$$

for all good covers $\mathcal{U}$ and all $p \geq 0$.

We now turn to singular cohomology.

**Theorem 10.5.** *If $X$ is a paracompact topological manifold and if $G$ is a $R$-module over a commutative ring $R$, then the singular cohomology groups $H^p(X;G)$ are isomorphic to the Čech cohomology groups $\check{H}^p(X, \widetilde{G}_X)$:*

$$H^p(X;G) \cong \check{H}^p(X, \widetilde{G}_X) \quad \text{for all } p \geq 0.$$

*If $X$ is a topological space and if $\mathcal{U}$ is a good cover of $X$, then we have isomorphisms between the singular cohomology groups $H^p(X;\mathbb{Z})$ and the Čech cohomology groups $\check{H}^p(X, \widetilde{\mathbb{Z}}_X)$ and $\check{H}^p(\mathcal{U}, \widetilde{\mathbb{Z}}_X)$:*

$$H^p(X, \mathbb{Z}) \cong \check{H}^p(\mathcal{U}, \widetilde{\mathbb{Z}}_X) \cong \check{H}^p(X, \widetilde{\mathbb{Z}}_X) \quad \text{for all } p \geq 0.$$

*In particular, the above holds if $X$ is a smooth manifold.*

By a previous remark, since our spaces are paracompact, the above theorem also holds with the constant presheaf $G_X$ (or $\mathbb{Z}_X$) instead of the sheaf $\widetilde{G}_X$ (or $\widetilde{\mathbb{Z}}_X$)).

The proof of the isomorphism $H^p(X;G) \cong \check{H}^p(X, \widetilde{G}_X)$ takes a lot of work. A version of this proof can be found in Warner [50] (Chapter 5). Another type of cohomology known as sheaf cohomology is introduced, and it is shown that both singular cohomology and Čech cohomology agree with sheaf cohomology if $X$ is paracompact and locally Euclidean. Sheaf cohomology is a special case of Grothendieck's approach to cohomology using derived functors. This is a very general and poweful approach which is discussed in Chapter 13.

The other isomorphisms involving good covers are proved in Bott and Tu [2] using double complexes; see Chapter III, §15, Theorem 15.8.

If should be noted that if the space $X$ is not well-behaved, then singular cohomology and singular homology may differ. For example, if $X$ is the *topologist's sine curve* (a space which is connected but neither locally connected nor path connected), it can be shown that

$$H^1(X;\mathbb{Z}) = (0)$$
$$\check{H}^1(X;\mathbb{Z}) = \mathbb{Z};$$

see Munkres [38] (Chapter 8, §73).

# Chapter 11

# Presheaves and Sheaves; A Deeper Look

## 11.1 Stalks and The Sheafification of a Presheaf

In the case where $\mathcal{F}$ is a presheaf on a topological space $X$ and $x$ is any given point in $X$, the direct limit $\varinjlim(\mathcal{F}(U))_{U \ni x}$ of the direct mapping family $(\mathcal{F}(U))_{U \ni x}$ plays an important role. In particular, these limits called stalks can be used to construct a sheaf $\widetilde{\mathcal{F}}$ from a presheaf $\mathcal{F}$; furthermore, the sheaf $\widetilde{\mathcal{F}}$ is the "smallest" sheaf extending $\mathcal{F}$, in a technical sense that will be explained later. If $\mathcal{F}$ is already a sheaf, then $\widetilde{\mathcal{F}}$ is isomorphic to $\mathcal{F}$.

**Definition 11.1.** If $\mathcal{F}$ is a presheaf on a topological space $X$ and $x$ is any given point in $X$, the direct limit $\varinjlim(\mathcal{F}(U))_{U \ni x}$ of the direct mapping family $(\mathcal{F}(U))_{U \ni x}$, as defined in Example 9.4 (2), is called the *stalk* of $\mathcal{F}$ at $x$, and is denoted by $\mathcal{F}_x$. For every open subset $U$ such that $x \in U$, we have a projection map $\rho_{U,x} \colon \mathcal{F}(U) \to \mathcal{F}_x$, and we write $f_x = \rho_{U,x}(f)$ for every $f \in \mathcal{F}(U)$. One calls $f_x$ the *germ* of $f$ at $x$. See Figure 11.1.

If $\mathcal{F}$ is the presheaf (actually a sheaf) of continuous functions given by $\mathcal{F}(U) = C^0(U)$, the set of continuous functions defined on an open subset $U$ containing $x$, then $\mathcal{F}_x$ is just the set of *germs* of locally defined functions near $x$. Indeed, two locally defined functions $f \in C^0(U)$ and $g \in C^0(V)$ near $x$ are equivalent iff their restrictions to $U \cap V$ agree.

For an arbitrary sheaf $\mathcal{F}$, the stalk $\mathcal{F}_x$ is the set of equivalence classes defined such that for any two open subsets $U$ and $V$ both containing $x$, the "local" sections $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ are equivalent, written $f \sim g$, iff there is some open subset $W$ containing $x$ such that $W \subseteq U \cap V$ and $\rho_W^U(f) = \rho_W^V(g)$. So, we can also think of the elements of $\mathcal{F}_x$ are "abstract germs" of local sections near $x$.

For a constant presheaf $G_X$ on $X$ with values in $G$, we have $G_{X,x} = G$ for all $x \in X$. Beware that for some pathological presheaves $\mathcal{F}$ (for example, of abelian groups), it is possible that $\mathcal{F}_x = (0)$ for all $x \in X$, even though $\mathcal{F}$ is not the constant presheaf with value 0. An
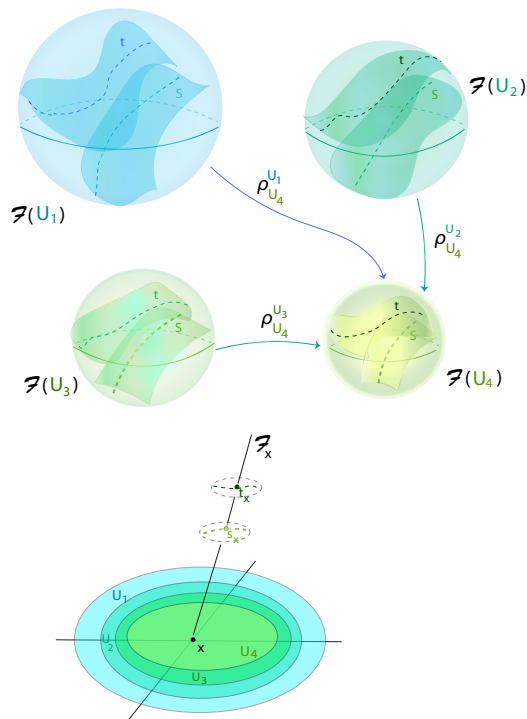
Figure 11.1: A schematic representation of $\mathcal{F}_x$ for $x \in \mathbb{R}^2$. We illustrate the direct limit construction for two germs, $s_x$ and $t_x$. Elements of the presheaf $\mathcal{F}$ are the spherical balloons. Since $U_4 \subseteq U_3 \subseteq U_2 \subseteq U_1$, the presheaf restriction maps imply that all images of $s$ are equivalent to the image of $s$ in $U_4$, and all the images of $t$ are equivalent to the image of $t$ in $U_4$. By continuing this process, we form the equivalence classes $s_x$ and $t_x$, which we illustrate as little disks centered on the radial stalk extending from $x \in \mathbb{R}^2$.

example is given by the following presheaf. Let $X$ be any topological space with at least two points (for example, $X = \{0, 1\}$), and let $\mathcal{F}_1$ be the presheaf given by

$$\mathcal{F}_1(U) = \begin{cases} \mathbb{Z} & \text{if } U = X \\ (0) & \text{if } U \neq X \text{ is an open subset,} \end{cases}$$

with all $\rho_V^U$ equal to the zero map except if $U = V = X$ (in which case it is the identity). It is easy to check that $\mathcal{F}_{1,x} = (0)$ for all $x \in X$.

The following result will be needed in Section 11.4.

**Proposition 11.1.** *Let $\mathcal{F}$ be a presheaf on a topological space $X$. If $\mathcal{F}$ satisfies Condition (M), then for any open subset $U$ of $X$, for any sections $s, t \in \mathcal{F}(U)$, we have*

$$s = t \quad \text{iff} \quad s_x = t_x \quad \text{for all } x \in U.$$

*Proof.* Obviously if $s = t$ then $s_x = t_x$ for all $x \in U$. Conversely, for every $x \in U$, there is some open subset $U_x \subseteq U$ containing $x$ such that $\rho_{U_x}^U(s) = \rho_{U_x}^U(t)$, and by Condition (M), we conclude that $s = t$. $\qquad\square$

A map $\varphi \colon \mathcal{F} \to \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space $X$ induces maps of stalks $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$. When $\mathcal{F}$ and $\mathcal{G}$ are sheaves, these maps carry a lot of information about $\varphi$.

To define $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ we proceed as follows. Any element $\gamma \in \mathcal{F}_x$ is an equivalence class $\gamma = s_x$ for some section $s \in \mathcal{F}(U)$ and some open subset $U$ of $X$ containing $x$. Let

$$\varphi_x(s_x) = (\varphi_U(s))_x,$$

where $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is the map defining $\varphi$ on $U$. We need to prove that this definition does not depend on the choice of the representative in the equivalence class $\gamma$. If $t \in \mathcal{F}(V)$ is another section such that $s \sim_{\mathcal{F}} t$, then there is some open subset $W$ such that $W \subseteq U \cap V$ and $(\rho_{\mathcal{F}})_W^U(s) = (\rho_{\mathcal{F}})_W^V(t)$. Since $\varphi$ is a map of presheaves, the following diagrams commute

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\
{\scriptstyle (\rho_{\mathcal{F}})_W^U}\downarrow & & \downarrow{\scriptstyle (\rho_{\mathcal{G}})_W^U} \\
\mathcal{F}(W) & \xrightarrow[\varphi_W]{} & \mathcal{G}(W)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \\
{\scriptstyle (\rho_{\mathcal{F}})_W^V}\downarrow & & \downarrow{\scriptstyle (\rho_{\mathcal{G}})_W^V} \\
\mathcal{F}(W) & \xrightarrow[\varphi_W]{} & \mathcal{G}(W),
\end{array}
$$

and we get

$$(\rho_{\mathcal{G}})_W^U(\varphi_U(s)) = \varphi_W((\rho_{\mathcal{F}})_W^U(s)) = \varphi_W((\rho_{\mathcal{F}})_W^V(t)) = (\rho_{\mathcal{G}})_W^V(\varphi_V(t)),$$

which shows that $\varphi_U(s) \sim_{\mathcal{G}} \varphi_V(t)$, thus $(\varphi_U(s))_x = (\varphi_V(t))_x$. Therefore, $\varphi_x$ is well defined and suggests the following definition of a map of stalks, which a special instance of Definition 9.10.

**Definition 11.2.** A map $\varphi \colon \mathcal{F} \to \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space $X$ induces *maps of stalks* $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$ defined as follows: for every $\gamma \in \mathcal{F}_x$, if $\gamma = s_x$ for some section $s \in \mathcal{F}(U)$ and some open subset $U$ of $X$ containing $x$, set

$$\varphi_x(s_x) = (\varphi_U(s))_x.$$

See Figure 11.2. By the above argument this definition does not depend on the choice of the representative chosen in the equivalence class $\gamma$.

If $\varphi \colon \mathcal{F} \to \mathcal{G}$ and $\psi \colon \mathcal{G} \to \mathcal{H}$ are two maps of presheaves, it is immediately verified that

$$(\psi \circ \varphi)_x = \psi_x \circ \varphi_x$$

and

$$(\mathrm{id}_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x},$$

for all $x \in X$ (where $\mathrm{id}_{\mathcal{F}}$ denotes the identity map of the presheaf $\mathcal{F}$).
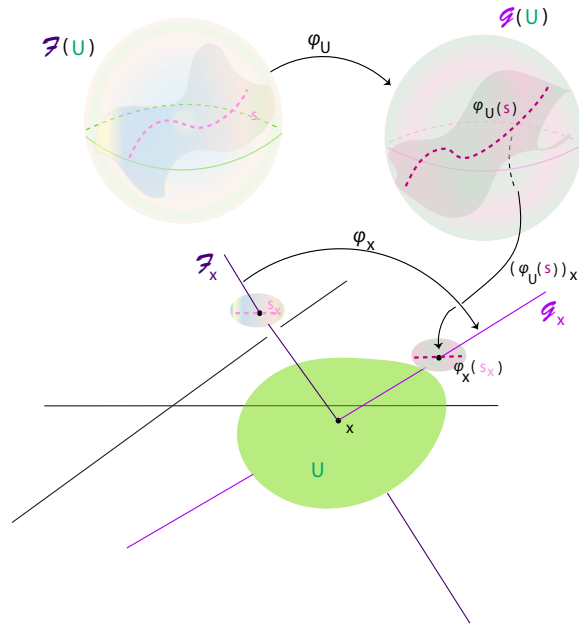
Figure 11.2: A schematic representation of $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ which maps the dark purple "stick" onto the plum "stick". The result of this stalk mapping is the same as first mapping the presheaf element $\mathcal{F}(U)$ onto $\mathcal{G}(U)$ and then using the direct limiting procedure to compute the stalk of $\varphi_U(s)$ where $s \in \mathcal{F}(U)$.

**Proposition 11.2.** *Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on a topological space $X$, and let $\varphi \colon \mathcal{F} \to \mathcal{G}$ and $\psi \colon \mathcal{F} \to \mathcal{G}$ be two maps of presheaves. If $\mathcal{G}$ satisfies Condition (M) (in particular, if $\mathcal{G}$ is a sheaf) and if $\varphi_x = \psi_x$ for all $x \in X$, then $\varphi = \psi$.*

*Proof.* We need to prove that $\varphi_U(s) = \psi_U(s)$ for any open subset $U$ of $X$ and any $s \in \mathcal{F}(U)$. Since $\varphi_x = \psi_x$ for every $x \in X$, for every $x \in U$ we have

$$\varphi_x(s_x) = \psi_x(s_x),$$

that is,

$$(\varphi_U(s))_x = (\psi_U(s))_x.$$

The above equations means that there is some open subset $U_x$ of $X$ such that $U_x \subseteq U$ and

$$(\rho_{\mathcal{G}})^U_{U_x}(\varphi_U(s)) = (\rho_{\mathcal{G}})^U_{U_x}(\psi_U(s)).$$

Since the family $(U_x)_{x \in U}$ is an open cover of $U$, Condition (M) implies that $\varphi_U(s) = \psi_U(s)$, and so $\varphi = \psi$. $\qquad\qquad\square$

Proposition 11.2 shows that if $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves, then $\varphi$ is uniquely determined by the family of stalk maps $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$.

Next, given a presheaf $\mathcal{F}$ on $X$, we construct a sheaf $\widetilde{F}$ and a presheaf map $\eta \colon \mathcal{F} \to \widetilde{\mathcal{F}}$ such that $\mathcal{F}$ satisfies Condition (M) iff $\eta$ is injective, and $\mathcal{F}$ is a sheaf iff $\eta$ is an isomorphism. We follow Godement's exposition [18] (Chapter II, Section 1.2), which we find to be one of the most lucid.

The key idea is to make the disjoint union $\coprod_{x \in X} \mathcal{F}_x$ of all the stalks into a topological space denoted $S\mathcal{F}$, with a projection function $p \colon S\mathcal{F} \to X$, and to let $\widetilde{\mathcal{F}}$ be the sheaf $\Gamma[S\mathcal{F}, p]$ of continuous sections of $p$, as in Example 9.2(1). See Figures 11.3 and 11.4.
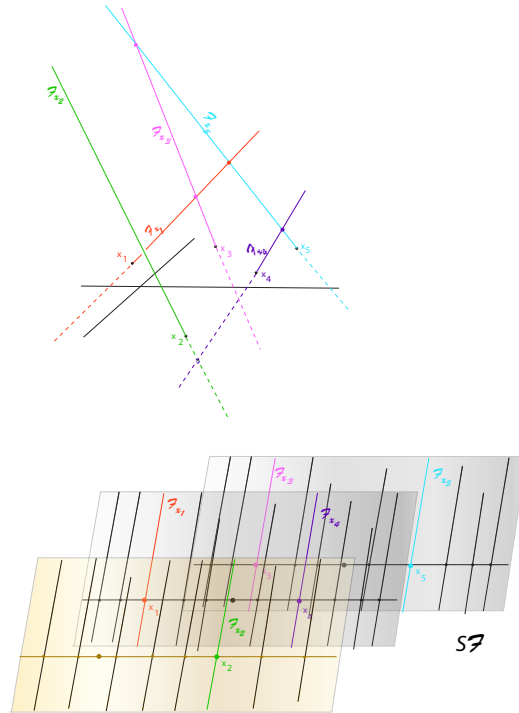


Figure 11.3: A schematic representation of $\coprod_{x \in X} \mathcal{F}_x$ for $X = \mathbb{R}^2$. The top picture illustrates five "stalks" before taking the disjoint union. Once the disjoint union is formed, the "stalks" are lined up in parallel planes.

If we let $S\mathcal{F} = \coprod_{x \in X} \mathcal{F}_x$ be the disjoint union of all the stalks, we denote by $p$ the function $p \colon S\mathcal{F} \to X$ given by $p(\gamma) = x$ for all $\gamma \in \mathcal{F}_x$. For every (nonempty) open subset $U$ of $X$, we view each "abstract" section $s \in \mathcal{F}(U)$ as the actual function $\widetilde{s} \colon U \to S\mathcal{F}$ given by

$$\widetilde{s}(x) = s_x, \quad x \in U.$$

By definition, $\widetilde{s}$ is a section of $p$. The final step is to give $S\mathcal{F}$ the finest topology which makes all the functions $\widetilde{s}$ continuous. Consequently, a subset $\Omega$ of $S\mathcal{F}$ is open iff for every

open subset $U$ of $X$ and every $s \in \mathcal{F}(U)$, the subset

$$\{x \in U \mid \widetilde{s}(x) = s_x \in \Omega\}$$

is open in $X$. The space $S\mathcal{F}$ endowed with the above topology is called the *stalk space* of the presheaf $\mathcal{F}$, and we let $\widetilde{\mathcal{F}}$ be the sheaf $\Gamma[S\mathcal{F}, p]$ of continuous sections of $p$. See Figure 11.4.
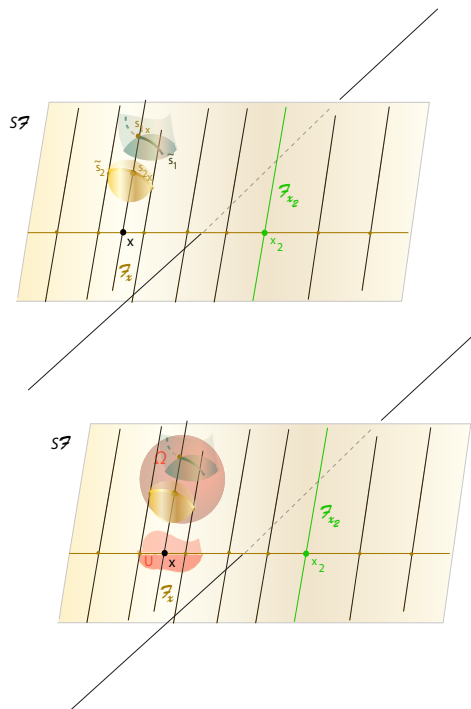


Figure 11.4: A schematic representation of the region around "first plane" in $S\mathcal{F}$ where $X = \mathbb{R}^2$. The top picture illustrates two sections $\tilde{s}_1$ and $\tilde{s}_2$. The bottom picture illustrates the relationship between $\Omega$, an open "spherical" set of $S\mathcal{F}$, and $U$, an open set of $X$ containing $x$. Both $\tilde{s}_1$ and $\tilde{s}_2$ are open sets in $S\mathcal{F}$.

We claim that $\widetilde{s}(U)$ is open in $S\mathcal{F}$. It suffices to show that for any $s_x \in \widetilde{s}(U)$ (with $x \in U$ and $s \in \mathcal{F}(U)$), for any open subset $V$ containing $x$, and for any $t \in \mathcal{F}(V)$, the subset $\{x \in U \cap V \mid \widetilde{t}(x) = t_x = s_x = \widetilde{s}(x)\}$ is open in $X$. However, $t_x = s_x$ means that there is some open subset $W \subseteq U \cap V$ containing $x$ such that $\rho_W^U(t) = \rho_W^V(s)$ on $W$, which means that $\{x \in U \cap V \mid t_x = s_x\}$ is indeed open in $X$.

Clearly, every open subset $\Omega$ of $S\mathcal{F}$ is the union of the open subsets $\widetilde{s}(U)$ such that $\widetilde{s}(U) \subseteq \Omega$, and these form a basis of this topology, because for any $s_x \in \widetilde{s}(U) \cap \widetilde{t}(V)$, some

open subset of the form $\widetilde{\rho_W^U(s)}(W) = \widetilde{\rho_W^V(t)}(W)$ is contained in $\widetilde{s}(U) \cap \widetilde{t}(V)$, for some open subset $W \subseteq U \cap V$ containing $x$.

The function $p$ is continuous because

$$p^{-1}(U) = \bigcup_{\substack{s \in \mathcal{F}(V) \\ V \subseteq U,\, V \text{ open}}} \widetilde{s}(V).$$

For any open subset $U$ of $X$ and any $s \in \mathcal{F}(U)$, since $\widetilde{s}$ is the inverse of the restriction of $p$ to $\widetilde{s}(U)$, we see that $p$ is a local homeomorphism.

In summary, we proved the following proposition.

**Proposition 11.3.** *Let $\mathcal{F}$ be a presheaf on a topological space $X$. The stalk space $S\mathcal{F}$ together with the finest topology that makes all the maps $\widetilde{s}\colon U \to S\mathcal{F}$ continuous has a basis for its topology consisting of the subsets of the form $\widetilde{s}(U)$, for all open subsets $U$ of $X$ and all $s \in \mathcal{F}(U)$. Furthermore, the projection map $p\colon S\mathcal{F} \to X$ is a local homeomorphism.*

It should be noted that the topology of $S\mathcal{F}$ is not assumed to be Hausdorff. In fact, in many interesting examples it is not. We called the space $S\mathcal{F}$ the *stalk space* of $\mathcal{F}$. In Godement [18] and most of the French literature, the space $S\mathcal{F}$ is called "espace étalé." A rough translation is "spread over space" or "laid over space."

**Definition 11.3.** Given any presheaf $\mathcal{F}$ on a topological space $X$, the map $\eta\colon \mathcal{F} \to \widetilde{\widetilde{\mathcal{F}}}$ is defined such that for every open subset $U$ of $X$, for every $s \in \mathcal{F}(U)$,

$$\eta_U(s) = \widetilde{s}.$$

It is easily checked that $\eta = (\eta_U)$ is indeed a map of presheaves. We now take a closer look a the map $\eta\colon \mathcal{F} \to \widetilde{\widetilde{\mathcal{F}}}$

**Proposition 11.4.** *Let $\mathcal{F}$ be a presheaf on a topological space $X$. The presheaf $\mathcal{F}$ satisfies Condition (M) iff the presheaf map $\eta\colon \mathcal{F} \to \widetilde{\widetilde{\mathcal{F}}}$ is injective*

*Proof.* We follow Serre's proof in FAC [44] (Chapter I, Section 3). First assume that $\mathcal{F}$ satisfies Condition (M). We have to prove that for every open subset $U$ of $X$, for any two elements $s, t \in \mathcal{F}$, if $\widetilde{s} = \widetilde{t}$, then $s = t$. Now, $\widetilde{s} = \widetilde{t}$ iff $s_x = t_x$ for all $x \in U$, which means that there is some open subset $U_x$ of $U$ containing $x$ such that

$$\rho_{U_x}^U(s) = \rho_{U_x}^U(t).$$

Since the family $(U_x)_{x \in U}$ is an open cover of $U$, by Condition (M) we must have $s = t$.

Conversely, assume that $\eta_U\colon \mathcal{F}(U) \to \widetilde{\widetilde{\mathcal{F}}}(U)$ is injective. Pick any $s, t \in \mathcal{F}(U)$, and assume there is some open cover $(U_i)_{i \in I}$ of $U$ such that $\rho_{U_i}^U(s) = \rho_{U_i}^U(t)$ for all $i \in I$. By definition of a direct limit, for any $x \in U$,

$$\widetilde{s}(x) = s_x = (\rho_{U_i}^U(s))_x \quad \text{and} \quad \widetilde{t}(x) = t_x = (\rho_{U_i}^U(t))_x,$$

so if $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$ then $\widetilde{s}(x) = \widetilde{t}(x)$ for all $x \in U$; that is, $\widetilde{s} = \widetilde{t}$. Since $\eta_U$ is injective, we conclude that $s = t$, which means that Condition (M) holds.  $\square$

The next proposition characterizes when $\eta$ is an isomorphism.

**Proposition 11.5.** *Let $\mathcal{F}$ be a presheaf on a topological space $X$ and assume that $\mathcal{F}$ satisfies Condition (M). The presheaf map $\eta\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ is surjective iff Condition (G) holds. As a consequence, $\eta$ is an isomorphism iff $\mathcal{F}$ is a sheaf.*

*Proof.* Again, we follow Serre's proof in FAC [44] (Chapter I, Section 3). By Proposition 11.4 Condition (M) holds iff $\eta$ is injective, so we may assume that $\eta$ is injective.

First assume that $\mathcal{F}$ satisfies Condition (G). For any open subset $U$ of $X$, for any continuous section $f\colon U \to S\mathcal{F}$, for any $x \in U$, since $f(x) \in \mathcal{F}_x$, there is some open subset $U_x$ of $U$ containing $x$ and some $s^x \in \mathcal{F}(U_x)$ such that $f(x) = (s^x)_x$. Since $\widetilde{s^x}$ and $f$ both invert $p$ on $U_x$, we see that the restriction of $f$ to $U_x$ agrees with $\widetilde{s^x}$. Observe that $(\rho^U_{U_x \cap U_y}(s^x))_z = (\rho^U_{U_x \cap U_y}(s^y))_z = f(z)$ for all $x, y \in U$ and all $z \in U_x \cap U_y$, that is, $\widetilde{\rho^U_{U_x \cap U_y}(s^x)} = \widetilde{\rho^U_{U_x \cap U_y}(s^y)}$. Since $\eta$ is injective, we get

$$\rho^U_{U_x \cap U_y}(s^x) = \rho^U_{U_x \cap U_y}(s^y).$$

But then, by Condition (G), the $s^x$ patch to some $s \in \mathcal{F}(U)$ such that $\rho^U_{U_x}(s) = s^x$, thus $\eta_U(s) = \widetilde{s}$ agrees with $\widetilde{s^x} = f|U_x$ on each $U_x$, which means that $\eta_U(s) = f$. See Figure 11.5.

Conversely, assume that $\eta_U$ is surjective (and injective) for every open subset $U$ of $X$. Let $(U_i)_{i \in I}$ be some open cover of $U$ and let $(s_i)_{i \in I}$ be a family of elements $s_i \in \mathcal{F}(U_i)$ such that

$$\rho^U_{U_i \cap U_j}(s_i) = \rho^U_{U_i \cap U_j}(s_j)$$

for all $i, j$. It follows that the sections $f_i = \widetilde{s_i}$ and $f_j = \widetilde{s_j}$ agree on $U_i \cap U_j$, so they patch to a continuous section $f\colon U \to S\mathcal{F}$ which agrees with $f_i$ on each $U_i$. Since $\eta_U$ is assumed to be surjective, there is some $s \in \mathcal{F}(U)$ such that $\eta_U(s) = f$. Then, if we write $s'_i = \rho^U_{U_i}(s)$, we see that $\widetilde{s'_i} = f_i$. Since $f_i = \widetilde{s_i} = \widetilde{s'_i}$ for all $i$ and since $\eta$ is injective, we conclude that $s_i = s'_i$; that is, $\rho^U_{U_i}(s) = s_i$, which shows that Condition (G) holds.  $\square$

Propositions 11.4 and 11.5 show that the Conditions (M) and (G) in the definition of a sheaf (Definition 9.4) are not as arbitrary as they might seem. They are just the conditions needed to ensure that a sheaf is isomorphic to a sheaf of sections of a certain space.

**Remark:** We proved earlier that for any open subset $U$ of $X$, for any two continuous sections $f$ and $g$ in $\Gamma(U, S\mathcal{F})$, the subset $W = \{x \in U \mid f(x) = g(x)\}$ is open. If the stalk space $S\mathcal{F}$ is Hausdorff, then $W$ is also closed. It follows that if $U$ is a connected open subset of $X$, if two continuous sections $f$ and $g$ in $\Gamma(U, S\mathcal{F})$ agree at some point, then $f = g$. In other words, the principle of analytic continuation holds. If $\mathcal{F}$ is the sheaf of continuous functions
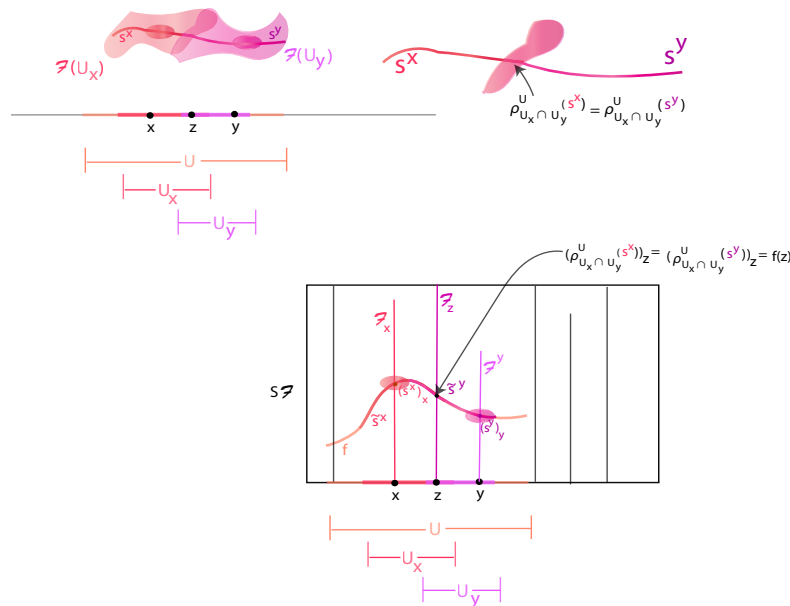
Figure 11.5: A schematic representation of the proof that a presheaf $\mathcal{F}$ which satisfies Conditions (M) and (G) implies $\eta$ is surjective. The top two diagrams are related to $\mathcal{F}$ while the bottom diagram is related to $\widetilde{\mathcal{F}}$. Note that $X$ is $\mathbb{R}$.

on $\mathbb{R}^n$, the principle of analytic continuation fails so $S\mathcal{F}$ is not Hausdorff. However, if $\mathcal{F}$ is the sheaf of holomorphic functions on a complex analytic manifold, then $S\mathcal{F}$ is Hausdorff.

If we examine more closely the construction of the sheaf $\widetilde{\mathcal{F}}$ from a presheaf $\mathcal{F}$, we see that we actually used two constructions:

(1) Given a presheaf $\mathcal{F}$, we constructed the stalk space $S\mathcal{F}$ and we gave it a topology that made the projection $p\colon S\mathcal{F} \to X$ into a local homeomorphism. This is the construction $S$ ("stalkification"), which constructs the stalk space $(S\mathcal{F}, p)$ from a presheaf $\mathcal{F}$.

(2) Given a pair $(E, p)$, where $p\colon E \to X$ is a local homeomorphism, we constructed the sheaf $\Gamma[E, p]$ (abbreviated as $\Gamma E$) of continuous sections of $p$.

Observe that the construction $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ is the composition of $S$ and $\Gamma$, that is, $\widetilde{\mathcal{F}} = \Gamma S\mathcal{F}$, and Proposition 11.5 shows that if $\mathcal{F}$ is a sheaf, then $\Gamma S\mathcal{F}$ is isomorphic to $\mathcal{F}$.

It is natural to take a closer look at the properties of a pair $(E, p)$, where $p\colon E \to X$ is a local homeomorphism, and to ask what is the effect of applying the operations $\Gamma$ and $S$ to the space $E$. We will see that the stalk space $S\Gamma E$ is isomorphic to the original space $E$.

The upshot of all this is that the constructions $S$ and $\Gamma$ are essentially inverse of each other, modulo some isomorphisms. To make this more precise we need to define what kind

of objects are in the domain of $\Gamma$, and what are the maps between such objects.[1]

## 11.2   Stalk Spaces (or Sheaf Spaces)

As we just explained, given a presheaf $\mathcal{F}$, the construction of the stalk space $S\mathcal{F}$ yields a pair $(S\mathcal{F}, p)$, where $p\colon S\mathcal{F} \to X$ is the projection, and by Proposition 11.3 the map $p$ is a local homeomorphism. This suggests the following definition.

**Definition 11.4.** A pair $(E, p)$ where $E$ is a topological space and $p\colon E \to X$ is a surjective local homeomorphism is called a *stalk space* (or *sheaf space*[2]). A *map* (or *morphism*) of stalk spaces $(E_1, p_1)$ and $(E_2, p_2)$ is a continuous map $\varphi\colon E_1 \to E_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\;\varphi\;} & E_2 \\
& \searrow{\scriptstyle p_1} \quad \swarrow{\scriptstyle p_2} & \\
& X. &
\end{array}
$$

Observe that the commutativity of the diagram implies that $\varphi$ maps fibres of $E_1$ to fibres of $E_2$.

The main construction on a stalk space $(E, p)$ is the construction $\Gamma$ described in Example 9.2 (1), which yields the sheaf $\Gamma[E, p]$ (abbreviated $\Gamma E$) of continuous sections of $p$, with

$$\Gamma[E, p](U) = \Gamma(U, \Gamma[E, p]) = \{s\colon U \to E \mid p \circ s = \mathrm{id} \text{ and } s \text{ is continuous}\}$$

for any open subset $U$ of $X$. This construction also applies to maps of stalk spaces (it is functorial).

**Definition 11.5.** Given a map $\varphi\colon E_1 \to E_2$ of stalk spaces $(E_1, p_1)$ and $(E_2, p_2)$ we obtain a map of sheaves $\Gamma\varphi\colon \Gamma E_1 \to \Gamma E_2$ defined as follows: For every open subset $U$ of $X$, the map $(\Gamma\varphi)_U\colon \Gamma(U, E_1) \to \Gamma(U, E_2)$ is given by

$$(\Gamma\varphi)_U(f) = \varphi \circ f,$$

as illustrated by the diagram below:

$$
\begin{array}{ccc}
& E_1 \xrightarrow{\;\varphi\;} E_2 & \\
{\scriptstyle f}\nearrow & \searrow{\scriptstyle p_1} \quad \swarrow{\scriptstyle p_2} & \\
U \hookrightarrow & X. &
\end{array}
$$

---

[1]Actually, $S$ and $\Gamma$ are functors between certain categories.
[2]The terminology "sheaf space" is used by Tennison [48]. Godement uses the terminology "espace étalé."

It is immediately checked that $\Gamma\varphi$ is a map of sheaves. Also, if $\varphi\colon E_1 \to E_2$ and $\psi\colon E_2 \to E_3$ are two maps of stalk spaces, then

$$\Gamma(\psi \circ \varphi) = \Gamma\psi \circ \Gamma\varphi,$$

and $\Gamma\mathrm{id}_E = \mathrm{id}_{\Gamma E}$. This means that the construction $\Gamma$ is functorial.

Here are a few useful properties of stalk spaces. In particular, we will see that the fibres of a stalk space are isomorphic to the stalks of the sheaf $\Gamma E$ of continuous sections.

**Proposition 11.6.** *Let $(E, p)$ be a stalk space. Then the following properties hold:*

(a) *The map $p$ is an open map.*

(b) *For any open subset $U$ of $X$ and any continuous section $f \in \Gamma(U, E)$, the subset $f(U)$ is open in $E$; such open subsets form a basis for the topology of $E$.*

(c) *For any commutative diagram*



*where $(E_1, p_2)$ and $(E_2, p_2)$ are stalk spaces, the map $\varphi$ is continuous iff it is an open map iff it is a local homeomorphism.*

*Proof.* (a) Let $V$ be any nonempty open subset in $E$. For any $x \in p(V)$ let $e \in E$ be any point in $E$ such that $p(e) = x$. Since $p$ is a local homeomorphism, there is some open subset $W$ of $E$ containing $e$ such that $p(W)$ is open in $X$. Then $p(W)$ is some open subset of $p(V)$ containing $x$, so $p(W)$ is open. See Figure 11.6.

(b) For any $e \in f(U)$, since $p$ is a local homeomorphism there is some open subset $W$ of $E$ containing $e$ such that $p(W)$ is open in $X$ and $p$ maps $W$ homeomorphically onto $p(W)$. It follows that $p$ maps $f(U) \cap W$ homeomorphically onto $U \cap V$, where $V = p(W)$ (since $f$ is a section of $p$). Since $U \cap V$ is open in $X$, the subset $f(U) \cap W$ is an open subset of $f(W)$ containing $e$, which shows that $f(U)$ is open. Using (a), it is easy to see that open subsets of the form $f(U)$ form a basis for the topology of $E$.

(c) A proof can be found in Tennison [48] (see Chapter 2, Lemma 3.5). $\qquad\square$

The next proposition tells us that the fibres of a stalk space are stalks of the sheaf $\Gamma E$.

**Proposition 11.7.** *Let $(E, p)$ be a stalk space. For any $x \in X$, the stalk $(\Gamma E)_x$ of the sheaf $\Gamma E$ of continuous sections of $p$ is isomorphic to the fibre $p^{-1}(x)$ at $x$. Furthermore, as a subspace of $E$, the fibre $p^{-1}(x)$ has the discrete topology.*
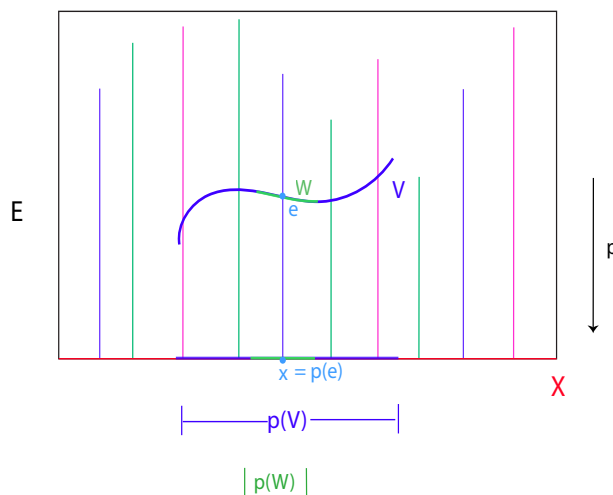
Figure 11.6: A schematic representation of the stalk space $(E, p)$ where $E$ is the rectangle and $X$ its red edge. The open set $V$ may be thought of as a section $f \in \Gamma(p(V), E)$.

*Proof.* Pick any $x \in X$. For any open subset $U$ of $X$ with $x \in U$ we have a map $\mathrm{Eval}_{U,x} \colon \Gamma(U, E) \to p^{-1}(x)$ given by

$$\mathrm{Eval}_{U,x}(f) = f(x)$$

for any continuous section $f \colon U \to E$ of $p$. For any open subset $V$ such that $V \subseteq U$ and $x \in V$ the following diagram commutes

$$
\begin{array}{ccc}
\Gamma(U, E) & \xrightarrow{\quad \theta_V^U \quad} & \Gamma(V, E) \\
& \mathrm{Eval}_{U,x} \searrow \quad \swarrow \mathrm{Eval}_{V,x} & \\
& p^{-1}(x), &
\end{array}
$$

where the map $\theta_V^U \colon \Gamma(U, E) \to \Gamma(V, E)$ is the restriction map. We use Proposition 9.1 to prove that $(p^{-1}(x), \mathrm{Eval}_{U,x})$ is a direct limit. By the universal mapping property, $p^{-1}(x)$ is isomorphic to the direct limit $(\Gamma E)_x$ of the direct mapping family $((\Gamma(U, E))_U, (\theta_V^U))$.

(a) We need to show that for every $e \in p^{-1}(x)$, there is some open subset $U$ of $X$ and some section $f \in \Gamma(U, E)$ such that $f(x) = e$. Since $p$ is a local diffeomorphism, there is some open subset $W$ of $E$ such that $e \in W$ and the restriction $p|W$ maps $W$ homeomorphically onto an open subset $U = p(W)$ of $X$. Then the inverse $f$ of $p|W$ is a continuous section in $\Gamma(U, E)$ such that $f(x) = e$. Observe that $p^{-1}(x) \cap W = \{e\}$, which shows that the fibre $p^{-1}(x)$ has the discrete topology.

(b) For any $x \in X$, suppose that $\mathrm{Eval}_{U,x}(f) = f(x) = g(x) = \mathrm{Eval}_{V,x}(g)$ where $f \in \Gamma(U, E)$ and $g \in \Gamma(V, E)$, with $x \in U \cap V$. Then, by Proposition 11.6 both $f(U)$ and $g(V)$ are open in $E$ so $W = f(U) \cap g(U)$ is open and $f$ and $g$ agree on $p(W)$ (since they are both the inverse of $p$ on $U \cap V$), which by Proposition 11.6 is open. This means that

$$\theta^U_{p(W)}(f) = \theta^U_{p(W)}(g),$$

which shows that Condition (b) of Proposition 9.1 is also satisfied.

Therefore, the stalk $(\Gamma E)_x$ of the sheaf $\Gamma E$ is isomorphic to the fibre $p^{-1}(x)$ at $x$.    $\square$

Proposition 11.7, when combined with Definition 11.3, has the following corollaries.

**Proposition 11.8.** *For any presheaf $\mathcal{F}$ on a space $X$, the map $\eta\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ induces isomorphisms of stalks $\eta_x\colon \mathcal{F}_x \to \widetilde{\mathcal{F}}_x$ for all $x \in X$.*

*Proof.* By construction the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at $x$ is equal to the fibre $p^{-1}(x)$ of the stalk space $S\mathcal{F}$, and $\widetilde{\mathcal{F}} = \Gamma S\mathcal{F}$, the sheaf of continuous sections of $p$, so $\widetilde{\mathcal{F}}_x = (\Gamma S\mathcal{F})_x$. By Proposition 11.7, we have $\mathcal{F}_x \cong \widetilde{\mathcal{F}}_x$. It remains to show that $\eta_x$ is a stalk isomorphism. The stalk map $\eta_x\colon \mathcal{F}_x \to \widetilde{\mathcal{F}}_x$ as given by Definition 9.10 is the unique map that makes the following diagrams commute

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\ \eta_U\ } & \Gamma(U, S\mathcal{F}) \\
{\scriptstyle \rho_{U,x}}\downarrow & & \downarrow{\scriptstyle \widetilde{\rho}_{U,x}} \\
\mathcal{F}_x & \xrightarrow[\ \eta_x\ ]{} & \widetilde{\mathcal{F}}_x
\end{array}
$$

for all open subsets $U$ of $X$ with $x \in U$. Since $p^{-1}(x) = \mathcal{F}_x$, by Proposition 11.7, there are isomorphisms $\theta_x\colon \widetilde{\mathcal{F}}_x \to p^{-1}(x)$ and thus $\theta_x\colon \widetilde{\mathcal{F}}_x \to \mathcal{F}_x$ such that the following diagrams commute:

$$
\begin{array}{ccc}
 & \Gamma(U, S\mathcal{F}) & \\
{\scriptstyle \widetilde{\rho}_{U,x}}\swarrow & & \searrow{\scriptstyle \mathrm{Eval}_{U,x}} \\
\widetilde{\mathcal{F}}_x \quad & \xrightarrow[\ \theta_x\ ]{} & \quad \mathcal{F}_x.
\end{array}
$$

Consequently, the diagrams

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\ \eta_U\ } & \Gamma(U, S\mathcal{F}) \\
{\scriptstyle \rho_{U,x}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eval}_{U,x}} \\
\mathcal{F}_x & \xrightarrow[\ \theta_x \circ \eta_x\ ]{} & \mathcal{F}_x
\end{array}
$$

all commute. However, for all $s \in \mathcal{F}(U)$, we have

$$\rho_{U,x}(s) = s_x = \widetilde{s}(x) = \mathrm{Eval}_{U,x}(\eta_U(s)) = (\mathrm{Eval}_{U,x} \circ \eta_U)(s),$$

so the above diagrams also commute with id instead of $\theta_x \circ \eta_x$, and by uniqueness of such a map making all these diagrams commute, we must have

$$\theta_x \circ \eta_x = \text{id}.$$

Since $\theta_x$ is an isomorphism, so must be $\eta_x$. □

**Proposition 11.9.** *For any stalk space $(E, p)$, there is a stalk space isomorphism $\epsilon \colon E \to S\Gamma E$.*

*Proof sketch.* For every $x \in X$, by Proposition 11.7 there are isomorphisms $\epsilon_x \colon p^{-1}(x) \to (\Gamma E)_x$. Since the fibre of $S\Gamma E$ at $x$ is equal to $(\Gamma E)_x$, the bijections $\epsilon_x$ define a bijection $\epsilon \colon E \to S\Gamma E$ such that $p = \Gamma p \circ \epsilon$, where $\Gamma p \colon S\Gamma E \to X$ is the projection associated with the stalk space $S\Gamma E$. It remains to check that $\epsilon$ is continuous, which is shown in Tennison [48] (Chapter II, Theorem 3.10). □

Strictly speaking the map $\epsilon \colon E \to S\Gamma E$ depends on $E$, so it should really be denoted by $\epsilon_E$. It can be shown that the family $\epsilon$ of maps $\epsilon_E$ is natural in the following sense: for every map $\varphi \colon E_1 \to E_2$ of stalk spaces $(E_1, p_1)$ and $(E_2, p_2)$, the following diagram commutes:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \epsilon_{E_1}\ } & S\Gamma E_1 \\
{\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle S\Gamma\varphi} \\
E_2 & \xrightarrow[\ \epsilon_{E_2}\ ]{} & S\Gamma E_2.
\end{array}
$$

The construction of the stalk space $S\mathcal{F}$ (and of the sheaf $\widetilde{\mathcal{F}}$) from a presheaf $\mathcal{F}$ is functorial in the following sense.

**Proposition 11.10.** *Given any map of preshaves $\varphi \colon \mathcal{F} \to \mathcal{G}$, there is a map of stalk spaces $S\varphi \colon S\mathcal{F} \to S\mathcal{G}$ induced by the stalk maps $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$, and a map of sheaves $\widetilde{\varphi} \colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$.*

*Proof.* Since $S\mathcal{F}$ is the disjoint union of the stalks $\mathcal{F}_x$ of $\mathcal{F}$ and $S\mathcal{G}$ is the disjoint union of the stalks $\mathcal{G}_x$ of $\mathcal{G}$, the stalk maps $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ define a map $S\varphi \colon S\mathcal{F} \to S\mathcal{G}$. It can be checked that the following diagram commutes

$$
\begin{array}{ccc}
S\mathcal{F} & \xrightarrow{\ \ S\varphi\ \ } & S\mathcal{G} \\
{\scriptstyle p_1}\searrow & & \swarrow{\scriptstyle p_2} \\
& X &
\end{array}
$$

and that $S\varphi$ is continuous using Proposition 11.6(c). The map $\widetilde{\varphi} \colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ is obtained from $S\varphi \colon S\mathcal{F} \to S\mathcal{G}$ by applying $\Gamma$ as in Definition 11.5 □

It is easy to check that if $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ are maps of presheaves, then $S(\psi \circ \varphi) = S\psi \circ S\varphi$ and $S\mathrm{id}_{\mathcal{F}} = \mathrm{id}_{S\mathcal{F}}$. Similarly $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$ and $\widetilde{\mathrm{id}_{\mathcal{F}}} = \mathrm{id}_{\widetilde{\mathcal{F}}}$.

Strictly speaking the map $\eta\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ depend on $\mathcal{F}$, so it should really be denoted by $\eta_{\mathcal{F}}\colon \mathcal{F} \to \widetilde{\mathcal{F}}$. It is easy to check that the family $\eta$ of maps $\eta_{\mathcal{F}}$ is natural in the following sense: given any presheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\;\eta_{\mathcal{F}}\;} & \widetilde{\mathcal{F}} \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\widetilde{\varphi}} \\
\mathcal{G} & \xrightarrow[\;\eta_{\mathcal{G}}\;]{} & \widetilde{\mathcal{G}}.
\end{array}
$$

**Remark:** If $\mathcal{F}$ is a presheaf on a space $X$, we define the presheaf $\mathcal{F}^{(+)}$ as follows: for every open subset $U$ of $X$,

$$\mathcal{F}^{(+)}(U) = \check{H}^0(U, \mathcal{F}|U),$$

where $\check{H}^0(U, \mathcal{F}|U)$ is a Čech cohomology groups defined in Section 10.1. Then it can be shown that $\mathcal{F}^{(+)}$ satisfies Condition (M), and that $\mathcal{F}^{(+)(+)}$ is isomorphic to the sheafification $\widetilde{\mathcal{F}}$ of $\mathcal{F}$.

The results of the previous sections can be put together to show that the construction $\mathcal{F} \mapsto \widetilde{\mathcal{F}} = \Gamma S \mathcal{F}$ of a sheaf from a presheaf (the *sheafification* of $\mathcal{F}$) is universal, and that the constructions $S$ and $\Gamma$ are essentially mutual inverses.

## 11.3 The Equivalence of Sheaves and Stalk Spaces

The following theorem shows the universality of the sheafification construction $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$.

**Theorem 11.11.** *Given any presheaf $\mathcal{F}$ and any sheaf $\mathcal{G}$, for any presheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$, there is a unique sheaf map $\widehat{\varphi}\colon \widetilde{\mathcal{F}} \to \mathcal{G}$ such that*

$$\varphi = \widehat{\varphi} \circ \eta_{\mathcal{F}}$$

*as illustrated by the following commutative diagram*

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\;\eta_{\mathcal{F}}\;} & \widetilde{\mathcal{F}} \\
 & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\widehat{\varphi}} \\
 & & \mathcal{G}.
\end{array}
$$

*Proof.* First we prove that if $\widehat{\varphi}\colon \widetilde{\mathcal{F}} \to \mathcal{G}$ exists, then it is unique. Since $\varphi = \widehat{\varphi} \circ \eta_{\mathcal{F}}$, for every $x \in X$, by considering the stalk maps we must have

$$\varphi_x = \widehat{\varphi}_x \circ \eta_x.$$

However, by Proposition 11.8, the map $\eta_x$ is an isomorphism, which shows that $\widehat{\varphi}_x = \varphi_x \circ \eta_x^{-1}$ is uniquely defined. Since $\mathcal{G}$ is a sheaf, by Proposition 11.2 the map $\widehat{\varphi}$ is uniquely determined.

We now show the existence of the map $\widehat{\varphi}$. By Proposition 11.10, the presheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$ yields the sheaf map $\widetilde{\varphi} \colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$. Furthermore, since $\mathcal{G}$ is a sheaf, by Proposition 11.5, the map $\eta_{\mathcal{G}} \colon \mathcal{G} \to \widetilde{\mathcal{G}}$ is an isomorphism. Therefore, we get the sheaf map $\widehat{\varphi} = \widetilde{\eta_{\mathcal{G}}}^{-1} \circ \widetilde{\varphi}$ from $\widetilde{\mathcal{F}}$ to $\mathcal{G}$. Using the naturality of $\eta$ we see that $\varphi = \widetilde{\eta_{\mathcal{G}}}^{-1} \circ \widetilde{\varphi} \circ \eta_{\mathcal{F}} = \widehat{\varphi} \circ \eta_{\mathcal{F}}$.           $\square$

We now go back to the constructions $S$ and $\Gamma$ to make the equivalence of sheaves and stalk spaces more precise. The "right" framework to do so is category theory, but we prefer to remain more informal.

The situation is that we have three kinds of objects and maps between these objects (categories):

(1) The class (category) $\mathbf{PSh}(X)$ whose objects are presheaves over a topological space $X$ and whose maps (morphisms) are maps of presheaves.

(2) The class (category) $\mathbf{Sh}(X)$ whose objects are sheaves over a topological space $X$ and whose maps (morphisms) are maps of sheaves.

(3) The class (category) $\mathbf{StalkS}(X)$ whose objects are stalk spaces over a topological space $X$ and whose maps (morphisms) are maps of stalk spaces.

Definition 11.2 implies that the operation $S$ maps an object $\mathcal{F}$ of $\mathbf{PSh}(X)$ to an object $(S\mathcal{F}, p \colon S\mathcal{F} \to X)$ of $\mathbf{StalkS}(X)$, and a map $\varphi \colon \mathcal{F} \to \mathcal{G}$ between objects of $\mathbf{PSh}(X)$ to a map $S\varphi \colon S\mathcal{F} \to S\mathcal{G}$ between objects in $\mathbf{StalkS}(X)$, in such that a way that $S(\psi \circ \varphi) = S\psi \circ S\varphi$ and $S\mathrm{id}_{\mathcal{F}} = \mathrm{id}_{S\mathcal{F}}$. In sophisticated terms,

$$S \colon \mathbf{PSh}(X) \to \mathbf{StalkS}(X)$$

is a functor from the category $\mathbf{PSh}(X)$ to the category $\mathbf{StalkS}(X)$.

Definition 11.5 implies that the operation $\Gamma$ maps an object $(E, p)$ from $\mathbf{StalkS}(X)$ to an object $\Gamma E$ in $\mathbf{Sh}(X)$, and a map $\varphi \colon E_1 \to E_2$ between two objects $(E_1, p_1)$ and $(E_2, p_2)$ in $\mathbf{StalkS}(X)$ to a map $\Gamma\varphi \colon \Gamma E_1 \to \Gamma E_2$ between objects in $\mathbf{Sh}(X)$, in such a way that $\Gamma(\psi \circ \varphi) = \Gamma\psi \circ \Gamma\varphi$ and $\Gamma\mathrm{id}_E = \mathrm{id}_{\Gamma E}$. In sophisticated terms,

$$\Gamma \colon \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$$

is a functor from the category $\mathbf{StalkS}(X)$ to the category $\mathbf{Sh}(X)$.

Note that every sheaf $\mathcal{F}$ is also a presheaf, and that every map $\varphi \colon \mathcal{F} \to \mathcal{G}$ of sheaves is also a map of presheaves. Therefore, we have an inclusion map

$$i \colon \mathbf{Sh}(X) \to \mathbf{PSh}(X),$$

which is a functor. As a consequence, $S$ restricts to an operation (functor)

$$S \colon \mathbf{Sh}(X) \to \mathbf{StalkS}(X).$$

We also defined the map $\eta$ which maps a presheaf $\mathcal{F}$ to the sheaf $\Gamma S(\mathcal{F}) = \widetilde{\mathcal{F}}$, and showed in Proposition 11.5 that this map is an isomorphism iff $\mathcal{F}$ is a sheaf. We also showed that $\eta$ is natural. This can be restated as saying that $\eta$ is a natural isomorphism between the functors id (the identity functor) and $\Gamma S$ from $\mathbf{Sh}(X)$ to itself.

We also defined the map $\epsilon$ which takes a stalk space $(E, p)$ and makes the stalk space $S\Gamma E$, and proved in Proposition 11.9 that $\epsilon \colon E \to S\Gamma E$ is an isomorphism. This can be restated as saying that $\epsilon$ is a natural isomorphism between the functors id (the identity functor) and $S\Gamma$ from $\mathbf{StalkS}(X)$ to itself. Then, we see that the two operations (functors)

$$S \colon \mathbf{Sh}(X) \to \mathbf{StalkS}(X) \quad \text{and} \quad \Gamma \colon \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$$

are almost mutual inverses, in the sense that there is a natural isomorphism $\eta$ between $\Gamma S$ and id and a natural isomorphism $\epsilon$ between $S\Gamma$ and id. In such a situation, we say that the classes (categories) $\mathbf{Sh}(X)$ and $\mathbf{StalkS}(X)$ are *equivalent*. The upshot is that it is basically a matter of taste (or convenience) whether we decide to work with sheaves or stalk spaces.[3]

We also have the operator (functor)

$$\Gamma S \colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$$

which "sheafifies" a presheaf $\mathcal{F}$ into the sheaf $\widetilde{\mathcal{F}}$. Theorem 11.11 can be restated as saying that there is an isomorphism

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(\mathcal{F}, i(\mathcal{G})) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\widetilde{\mathcal{F}}, \mathcal{G}),$$

between the set (category) of maps between the presheaves $\mathcal{F}$ and $i(\mathcal{G})$ and the set (category) of maps between the sheaves $\widetilde{\mathcal{F}}$ and $\mathcal{G}$. In fact, such an isomorphism is natural, so in categorical terms $i$ and $\widetilde{\phantom{x}} = \Gamma S$ are *adjoint functors*. This is as far as we will go with our excursion into category theory. The reader should consult Tennison [48] for a comprehensive treatment of a preshaves and sheaves in the framework of *abelian categories*.

In Sections 11.2 and 11.3 we have considered presheaves and sheaves of sets. If $\mathcal{F}$ is a sheaf of $R$-modules, then it is immediately verified that for every $x \in X$ the stalk $\mathcal{F}_x$ at $x$ is an $R$-module, and similarly if $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}_x$ is a ring.

Minor modifications need to be made to the notion of a stalk space to extend the equivalence between sheaves of $R$-modules, rings, *etc.* and stalk spaces. We simply need to assume that every fibre $p^{-1}(x)$ (with $x \in X$) is a $R$-module, ring, *etc.*, and that the $R$- module operations, ring operations, *etc.*, are continuous.

More precisely, we have the following definitions.

---

[3]Actually, if we deal with sheaves of modules or rings, it turns out that stalk spaces have a better behavior when it comes to images of morphisms, or quotients.

**Definition 11.6.** A *stalk space of R-modules* is a pair $(E, p\colon E \to X)$ where $p$ is a surjective local homeomorphism, and the following conditions hold:

(1) Every fibre $p^{-1}(x)$ (with $x \in X$) is a $R$-module.

(2) For every $\lambda \in R$, the function from $E$ to itself given by $e \mapsto \lambda \cdot e$ is continuous, where $\cdot$ is scalar multiplication in the fibre $p^{-1}(p(e))$.

(3) There is a continuous function $-_E\colon E \to E$ such that for all $e \in E$, $-_E e = -e$, where $-e$ is the additive inverse of $e$ in the fibre $p^{-1}(p(e))$.

(4) If we set $E \sqcap E = \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\}$, then there is a continuous function $+_E\colon E \sqcap E \to E$ such that $e_1 +_E e_2 = e_1 + e_2$, where $+$ is addition in the fibre $p^{-1}(p(e_1)) \,(= p^{-1}(p(e_2)))$.

A map of stalk spaces of $R$-modules $(E_1, p_1)$ and $(E_2, p_2)$ is a map $\varphi\colon (E_1, p_1) \to (E_2, p_2)$ of stalk spaces such that for every $x \in X$, the restriction of $\varphi$ to the fibre $p_1^{-1}(x)$ is a $R$-linear map between $p_1^{-1}(x)$ and $p_2^{-1}(x)$.

**Definition 11.7.** A *stalk space of commutative rings* is a pair $(E, p\colon E \to X)$ where $p$ is a surjective local homeomorphism, and the following conditions hold:

(1) Every fibre $p^{-1}(x)$ (with $x \in X$) is a commutative ring.

(2) There is a continuous function $-_E\colon E \to E$ such that for all $e \in E$, $-_E e = -e$, where $-e$ is the additive inverse of $e$ in the fibre $p^{-1}(p(e))$.

(3) If we set $E \sqcap E = \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\}$, then there is a continuous function $+_E\colon E \sqcap E \to E$ such that $e_1 +_E e_2 = e_1 + e_2$, where $+$ is addition in the fibre $p^{-1}(p(e_1)) \,(= p^{-1}(p(e_2)))$.

(4) There is a continuous function $*_E\colon E \sqcap E \to E$ such that $e_1 *_E e_2 = e_1 * e_2$, where $*$ is multiplication in the fibre $p^{-1}(p(e_1)) \,(= p^{-1}(p(e_2)))$.

A map of stalk spaces of rings $(E_1, p_1)$ and $(E_2, p_2)$ is a map $\varphi\colon (E_1, p_1) \to (E_2, p_2)$ of stalk spaces such that for every $x \in X$, the restriction of $\varphi$ to the fibre $p_1^{-1}(x)$ is a ring homomorphism between $p_1^{-1}(x)$ and $p_2^{-1}(x)$.

**Definition 11.8.** Given a stalk space of $R$-modules (or rings) $(E, p)$ over a space $X$, for every subset $Y$ of $X$, we define the *restriction* $(E, p)|Y$ of $(E, p)$ to $Y$ as the stalk space $(\pi^{-1}(Y), p|\pi^{-1}(Y))$.

The reader is referred to Tennison [48] for more details on the equivalence between sheaves with an algebraic structure and stalk spaces with the same algebraic structure on the fibres.

# 11.4 Properties of Maps of Presheaves and Sheaves

If $f \colon A \to B$ is a homomorphism between two $R$-modules $A$ and $B$, recall that the *kernel* $\mathrm{Ker}\,(f)$ of $f$ is defined by

$$\mathrm{Ker}\,(f) = \{u \in A \mid f(u) = 0\},$$

the *image* $\mathrm{Im}(f)$ of $f$ is defined by

$$\mathrm{Im}(f) = \{v \in B \mid (\exists u \in A)(v = f(u))\},$$

the *cokernel* $\mathrm{Coker}(f)$ of $f$ is defined by

$$\mathrm{Coker}(f) = B/\mathrm{Im}(f),$$

and the *coimage* $\mathrm{Coim}(f)$ of $f$ is defined by

$$\mathrm{Coim}(f) = A/\mathrm{Ker}\,(f).$$

Furthermore, $f$ is injective iff $\mathrm{Ker}\,(f) = (0)$, $f$ is surjective iff $\mathrm{Coker}(f) = (0)$, and there is an isomorphism $\mathrm{Coim}(f) \cong \mathrm{Im}(f)$. A sequence of $R$-modules

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

is *exact* at $B$ if $\mathrm{Im}(f) = \mathrm{Ker}\,(g)$.

We would like to generalize the above notions to maps of presheaves and sheaves of $R$-modules or commutative rings. In the case of presheaves, everything works perfectly, but in the case of sheaves, there are two problems:

(1) In general, the presheaf image of a sheaf is not a sheaf.

(2) In general, the presheaf quotient of two sheaves is not a sheaf.

A way to fix these problems is to apply the sheafification process to the presheaf, but in the case of the image of a sheaf morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$, this has the slightly unpleasant consequence that $\widetilde{\mathrm{Im}(\varphi)}$ is a not a subsheaf of $\mathcal{G}$. This small problem can be avoided by defining the image of a sheaf morphism as the kernel of its cokernel map (as this would be the case in an abelian category).

From now on in this section we assume that we are dealing with presheaves and sheaves of $R$-modules or commutative rings. We follow closely Tennison [48], so many proof are omitted.

We begin with kernels. If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves on a space $X$, then for every open subset $U$ of $X$, define $(\mathrm{Ker}\,\varphi)_U$ by

$$(\mathrm{Ker}\,\varphi)_U = \mathrm{Ker}\,\varphi_U = \{s \in \mathcal{F}(U) \mid \varphi_U(s) = 0\}.$$

If $V$ is some open subset of $U$, the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\
{\scriptstyle (\rho_\mathcal{F})^U_V} \downarrow & & \downarrow {\scriptstyle (\rho_\mathcal{G})^U_V} \\
\mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V)
\end{array}
$$

implies that is $s \in (\operatorname{Ker} \varphi)_U$, that is, $\varphi_U(s) = 0$, then

$$
\varphi_V((\rho_\mathcal{F})^U_V(s)) = (\rho_\mathcal{G})^U_V(\varphi_U(s)) = (\rho_\mathcal{G})^U_V(0) = 0,
$$

so $(\rho_\mathcal{F})^U_V(s) \in (\operatorname{Ker} \varphi)_V$. This shows that the $(\operatorname{Ker} \varphi)_U$ together with the restriction functions $\rho^U_V$ (as a function from $(\operatorname{Ker} \varphi)_U$ to $(\operatorname{Ker} \varphi)_V$) is a presheaf.

**Definition 11.9.** If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves on a space $X$, then for every open subset $U$ of $X$, define $(\operatorname{Ker} \varphi)_U$ by

$$
(\operatorname{Ker} \varphi)_U = \operatorname{Ker} \varphi_U = \{s \in \mathcal{F}(U) \mid \varphi_U(s) = 0\}.
$$

Then the $(\operatorname{Ker} \varphi)_U$ together with the restriction functions $\rho^U_V$ (as a function from $(\operatorname{Ker} \varphi)_U$ to $(\operatorname{Ker} \varphi)_V$) is a presheaf called the *presheaf kernel* of $\varphi$ and denoted $\operatorname{Ker} \varphi$.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\operatorname{Ker} \varphi$ is a sheaf.

**Proposition 11.12.** *If $\mathcal{F}$ is a sheaf and $\mathcal{G}$ satisfies Condition (M), then $\operatorname{Ker} \varphi$ is a sheaf. In particular, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\operatorname{Ker} \varphi$ is a sheaf.*

*Proof.* Since $\mathcal{F}$ is a sheaf, it satisfies Condition (M), and it is easy to show that $\operatorname{Ker} \varphi$ also satisfies Condition (M).

Let $U$ be any open subset of $X$, let $(U_i)_{i \in I}$ be any open cover of $U$ , and let $(s_i)_{i \in I}$ be a family of sections $s_i \in (\operatorname{Ker} \varphi)_{U_i}$ such that $(\rho_\mathcal{F})^U_{U_i}(s_i) = (\rho_\mathcal{F})^U_{U_j}(s_j)$ for all $i, j$. Since $\mathcal{F}$ is a sheaf, there is some $s \in \mathcal{F}(U)$ such that $(\rho_\mathcal{F})^U_{U_i}(s) = s_i$ for all $i \in I$. Since $\varphi_{U_i}(s_i) = 0$, the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\
{\scriptstyle (\rho_\mathcal{F})^U_{U_i}} \downarrow & & \downarrow {\scriptstyle (\rho_\mathcal{G})^U_{U_i}} \\
\mathcal{F}(U_i) & \xrightarrow[\varphi_{U_i}]{} & \mathcal{G}(U_i)
\end{array}
$$

implies that

$$
0 = \varphi_{U_i}(s_i) = \varphi_{U_i}((\rho_\mathcal{F})^U_{U_i}(s)) = (\rho_\mathcal{G})^U_{U_i}(\varphi_U(s))
$$

for all $i \in I$. Since $\mathcal{G}$ satisfies Condition (M), then $\varphi_U(s) = 0$, which means that $s \in (\operatorname{Ker} \varphi)_U$.                                                   $\square$

The next proposition generalizes the property that a module or (ring) map $f\colon A \to B$ is injective iff Ker $f = (0)$.

**Proposition 11.13.** *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a map of presheaves. The following two conditions are equivalent.*

*(i) Ker $\varphi = (0)$ (the trivial zero sheaf).*

*(ii) $\varphi_U$ is injective for all open subsets $U$ of $X$.*

*These imply*

*(iii) $\varphi_x$ is injective for all $x \in X$, which is equivalent to (i) and (ii) if $\mathcal{F}$ satisfies Condition (M).*

*Proof.* The equivalence of (i) and (ii) is immediate by definition of $(\text{Ker } \varphi)_U$.

Assume that (ii) holds, and suppose that $\varphi_x(\gamma) = 0$ for some $\gamma \in \mathcal{F}_x$ (with $x \in X$). This means that there is some open subset $U$ of $X$ containing $x$ and some $s \in \mathcal{F}(U)$ such that $s_x = \gamma$ and $(\varphi_U(s))_x = 0$, which in turn means that there is some open subset $V \subseteq U$ containing $x$ such that

$$(\rho_{\mathcal{G}})_V^U(\varphi_U(s)) = 0.$$

Since

$$(\rho_{\mathcal{G}})_V^U(\varphi_U(s)) = \varphi_V((\rho_{\mathcal{F}})_V^U(s)),$$

we get $\varphi_V((\rho_{\mathcal{F}})_V^U(s)) = 0$, and since $\varphi_V$ is injective, $(\rho_{\mathcal{F}})_V^U(s) = 0$. But, $(\rho_{\mathcal{F}})_V^U(s) = 0$ implies that $\gamma = ((\rho_{\mathcal{F}})_V^U(s))_x = 0$, so $\varphi_x$ is injective.

Conversely, assume that $\varphi_x$ is injective for all $x \in X$ and that $\mathcal{F}$ satisfies Condition (M). Suppose $\varphi_U(s) = 0$ for some $s \in \mathcal{F}(U)$ (where $U$ is any open subset of $X$). Then

$$\varphi_x(s_x) = (\varphi_U(s))_x = 0$$

for all $x \in U$, and since $\varphi_x$ is injective for all $x$, we deduce that $s_x = 0$ for all $x \in U$. Since $\mathcal{F}$ satisfies Condition (M), by Proposition 11.1 (with $t = 0$), we conclude that $s = 0$, which shows that $\varphi_U$ is injective. $\qquad\square$

**Definition 11.10.** A map of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if any of the Conditions (i) and (ii) of Proposition 11.13 holds. A map of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if any of the Conditions (i)–(iii) of Proposition 11.13 holds.

**Remark:** A presheaf or sheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$ is said to a *monomorphism* if for every presheaf $\mathcal{H}$ any two presheaf maps $\psi_1, \psi_2\colon \mathcal{H} \to \mathcal{F}$, if $\varphi \circ \psi_1 = \varphi \circ \psi_2$, then $\psi_1 = \psi_2$. It can be shown that being a monomorphism is equivalent to any of the conditions of Proposition 11.13; see Tennison [48] (Chapter III, Theorem 3.5).

The following two propositions are stated without proof; see Tennison [48] (Chapter III) for details.

**Proposition 11.14.** *If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves, then*

$$(\operatorname{Ker} \varphi)_x = \operatorname{Ker} \varphi_x$$

*for all $x \in X$.*

**Proposition 11.15.** *If $\varphi\colon (E_1, p_1) \to (E_2, p_2)$ is a map of stalk spaces, then $\Gamma\varphi\colon \Gamma E_1 \to \Gamma E_2$ is an injective map of sheaves iff $\varphi$ is injective iff $\varphi$ is a homeomorphism onto an open subspace of $E_2$.*

The notions of subpresheaves and subsheaves are defined as follows.

**Definition 11.11.** Given two presheaves $\mathcal{F}$ and $\mathcal{G}$ on a space $X$, we say that $\mathcal{F}$ *is a subpresheaf of $\mathcal{G}$* if for every open subset $U$ of $X$, the $R$-module (resp. ring) $\mathcal{F}(U)$ is a submodule (resp. subring) of $\mathcal{G}(U)$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves and the above condition holds for all open subets $U$ of $X$ we say that $\mathcal{F}$ *is a subsheaf of $\mathcal{G}$.*

**Remark:** In terms of stalk spaces, in view of Proposition 11.15, we say that $(E_1, p_1)$ is a *substalk space* of $(E_2, p_2)$ if $E_1$ is an open subset of $E_2$, $p_1$ is the restriction of $p_2$ to $E_1$, and the fibre $p_1^{-1}(x)$ is a submodule (resp. subring) of the fibre $p_2^{-1}(x)$ for all $x \in X$.

The following proposition will be needed.

**Proposition 11.16.** *Let $\mathcal{G}$ be a sheaf and assume that $\mathcal{F}$ and $\mathcal{F}'$ are two subsheaves of $\mathcal{G}$. Then $\mathcal{F} = \mathcal{F}'$ iff $\mathcal{F}_x = \mathcal{F}'_x$ for all $x \in X$ (as submodules or subrings).*

*Proof.* First, we prove that if $\mathcal{F}_x \subseteq \mathcal{F}'_x$ for all $x \in X$ (as submodules or subrings) then $\mathcal{F}$ is a subsheaf of $\mathcal{F}'$. For any open subset $U$ for $X$, for any section $s \in \mathcal{F}(U)$, since $\mathcal{F}_x \subseteq \mathcal{F}'_x$ for all $x \in U$, there is an open cover $(U_x)_{x\in U}$ of $U$ and a family of sections $t^x \in \mathcal{F}'(U_x)$ such that $(\rho_{\mathcal{F}})^U_{U_x}(s) = (\rho_{\mathcal{F}'})^U_{U_x}(t^x)$ for all $x \in U$. It follows that $(\rho_{\mathcal{F}'})^{U_x}_{U_x \cap U_y}(t^x) = (\rho_{\mathcal{F}'})^{U_y}_{U_x \cap U_y}(t^y)$ for all $x, y$ and since $\mathcal{F}'$ is a sheaf there is a unique section $t \in \mathcal{F}'(U)$ such that $(\rho_{\mathcal{F}'})^U_{U_x}(t) = t^x$ for all $x \in U$. Therefore, we obtain a map $\varphi_U\colon \mathcal{F}(U) \to \mathcal{F}'(U)$ by setting $\varphi_U(s) = t$, and it is easy to see that these maps define a sheaf map $\varphi\colon \mathcal{F} \to \mathcal{F}'$. At first glance it is not obvious that $\varphi$ is an inclusion map, but it is as the following argument shows. Note that the composition $i' \circ \varphi$ where $i'$ is the inclusion of $\mathcal{F}'$ in $\mathcal{G}$ agrees on stalks with the inclusion $i$ of $\mathcal{F}$ in $\mathcal{G}$, so by Proposition 11.2 we have $i' \circ \varphi = i$, so $\varphi$ is an inclusion.

Now, if $\mathcal{F}_x = \mathcal{F}'_x$ for all $x \in X$, by the above $\mathcal{F}$ is a subsheaf of $\mathcal{F}'$ and $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ so $\mathcal{F} = \mathcal{F}'$.

If $\mathcal{F} = \mathcal{F}'$, then obviously $\mathcal{F}_x = \mathcal{F}'_x$ for all $x \in X$.          $\square$

Let us now consider cokernels and images. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a map of presheaves. For every open subset $U$ of $X$, define $\operatorname{PCoker}_U$ by

$$\operatorname{PCoker}_U = \mathcal{G}(U)/\varphi_U(\mathcal{F}(U)) = \mathcal{G}(U)/\operatorname{Im} \varphi_U,$$

the quotient module (resp. quotient ring) of $\mathcal{G}(U)$ modulo $\varphi_U(\mathcal{F}(U))$, which is well defined since $\varphi_U(\mathcal{F}(U))$ is a submodule (resp. subring) of $\mathcal{G}(U)$ because $\varphi_U$ is a homomorphism.

For any open subset $V \subseteq U$, the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\
{\scriptstyle (\rho_{\mathcal{F}})_V^U}\big\downarrow & & \big\downarrow{\scriptstyle (\rho_{\mathcal{G}})_V^U} \\
\mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V)
\end{array}
$$

implies that for any $s \in \mathcal{F}(U)$, we have

$$(\rho_{\mathcal{G}})_V^U(\varphi_U(s)) = \varphi_V((\rho_{\mathcal{F}})_V^U(s)),$$

which shows that $(\rho_{\mathcal{G}})_V^U(\varphi_U(s)) \in \mathrm{Im}(\varphi_V)$, that is, $(\rho_{\mathcal{G}})_V^U(\mathrm{Im}(\varphi_U)) \subseteq \mathrm{Im}(\varphi_V)$. If we let $\mathrm{pcoker}_U \colon \mathcal{G}(U) \to \mathcal{G}(U)/\mathrm{Im}(\varphi_U)$ be the projection map, then $\mathrm{pcoker}_V \circ (\rho_{\mathcal{G}})_V^U \colon \mathcal{G}(U) \to \mathcal{G}(V)/\mathrm{Im}(\varphi_V)$ vanishes on $\mathrm{Im}(\varphi_U)$, which implies that there is a unique map $(\bar{\rho}_{\mathcal{G}})_V^U \colon \mathcal{G}(U)/\mathrm{Im}(\varphi_U) \to \mathcal{G}(V)/\mathrm{Im}(\varphi_V)$ making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{G}(U) & \xrightarrow{\mathrm{pcoker}_U} & \mathcal{G}(U)/\mathrm{Im}(\varphi_U) \\
{\scriptstyle (\rho_G)_V^U}\big\downarrow & & \big\downarrow{\scriptstyle (\bar{\rho}_{\mathcal{G}})_V^U} \\
\mathcal{G}(V) & \xrightarrow[\mathrm{pcoker}_V]{} & \mathcal{G}(V)/\mathrm{Im}(\varphi_V)
\end{array}
$$

Therefore, the $\mathrm{PCoker}_U$ together with the restriction functions $(\bar{\rho}_{\mathcal{G}})_V^U$ define a presheaf.

**Definition 11.12.** If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves on a space $X$, then for every open subset $U$ of $X$, define $\mathrm{PCoker}_U$ by

$$\mathrm{PCoker}_U = \mathcal{G}(U)/\varphi_U(\mathcal{F}(U)) = \mathcal{G}(U)/\mathrm{Im}\,\varphi_U.$$

Then the $\mathrm{PCoker}_U$ together with the restriction functions $(\bar{\rho}_{\mathcal{G}})_V^U$ define a presheaf called the *presheaf cokernel* of $\varphi$, and denoted $\mathrm{PCoker}(\varphi)$. The projection maps $\mathrm{pcoker}_U \colon \mathcal{G}(U) \to \mathcal{G}(U)/\mathrm{Im}(\varphi_U)$ define a presheaf map $\mathrm{pcoker}(\varphi) \colon \mathcal{G} \to \mathrm{PCoker}(\varphi)$.

Obviously, $\mathrm{pcoker}(\varphi) \circ \varphi = 0$ as illustrated in the diagram below:

$$\mathcal{F} \xrightarrow{\quad \varphi \quad} \mathcal{G} \xrightarrow{\quad \mathrm{pcoker}(\varphi) \quad} \mathrm{PCoker}(\varphi).$$

In fact, $\mathrm{pcoker}(\varphi)$ is characterized by a universal property of this kind; see Tennison [48] (Chapter III) for details.

If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves, in general the presheaf cokernel $\mathrm{PCoker}(\varphi)$ is not a sheaf. To obtain a sheaf, we sheafify it.

**Definition 11.13.** If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves on a space $X$, then the *sheaf cokernel* of $\varphi$, denoted $\mathrm{SCoker}(\varphi)$, is the sheafification $\widetilde{\mathrm{PCoker}}(\varphi)$ of the presheaf cokernel $\mathrm{PCoker}(\varphi)$ of $\varphi$. The composition

$$\mathcal{G} \xrightarrow{\mathrm{pcoker}(\varphi)} \mathrm{PCoker}(\varphi) \xrightarrow{\eta_{\mathrm{PCoker}(\varphi)}} \widetilde{\mathrm{PCoker}}(\varphi) = \mathrm{SCoker}(\varphi)$$

defines a presheaf map $\mathrm{scoker}(\varphi)\colon \mathcal{G} \to \mathrm{SCoker}(\varphi)$ ($\eta_{\mathrm{PCoker}(\varphi)}\colon \mathrm{PCoker}(\varphi) \to \widetilde{\mathrm{PCoker}}(\varphi)$ is the canonical map of Definition 11.3).

Again, $\mathrm{scoker}(\varphi) \circ \varphi = 0$ as illustrated in the diagram below:

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\mathrm{scoker}(\varphi)} \mathrm{SCoker}(\varphi).$$

In fact, $\mathrm{scoker}(\varphi)$ is characterized by a universal property of this kind; see Tennison [48] (Chapter III) for details.

The following propositions generalize the characterization of the surjectivity of a module (resp. ring) homomorphism $f\colon A \to B$ in terms of its cokernel to presheaves and sheaves.

**Proposition 11.17.** *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a map of presheaves on a space $X$. Then the following conditions are equivalent:*

*(i) $\mathrm{PCoker}(\varphi) = (0)$.*

*(ii) For every open subset $U$ of $X$, the map $\varphi_U$ is surjective.*

*Proof.* The equivalence of (i) and (ii) follows immediately from the definitions.     $\square$

**Proposition 11.18.** *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves on a space $X$. Then the following conditions are equivalent:*

*(i) $\mathrm{SCoker}(\varphi) = (0)$.*

*(ii) For every $x \in X$, $(\mathrm{PCoker}(\varphi))_x = (0)$.*

*(iii) For every $x \in X$, $\varphi_x$ is surjective.*

*(iv) For every open subset $U$ of $X$, for every $t \in \mathcal{G}(U)$, there is some open cover $(U_i)_{i\in I}$ of $U$ and a family $(s_i)_{i\in I}$ of sections $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = (\rho_{\mathcal{G}})^U_{U_i}(t)$ for all $i \in I$.*

*Any of the conditions of Proposition 11.17 implies the above conditions.*

*Proof.* The equivalence of (i) and (ii) goes as follows. Since $\mathrm{SCoker}(\varphi)$ is a sheaf, by Proposition 11.2 (with $\psi$ the zero map), $\mathrm{SCoker}(\varphi) = (0)$ iff $(\mathrm{SCoker}(\varphi))_x = (0)$ for all $x \in X$. But by Proposition 11.8 the stalks $(\mathrm{SCoker}(\varphi))_x$ and $(\mathrm{PCoker}(\varphi))_x$ are isomorphic, so $(\mathrm{SCoker}(\varphi))_x = (0)$ iff $(\mathrm{PCoker}(\varphi))_x = (0)$ for all $x \in X$.

To prove the equivalence of (ii) and (iii) we need to unwind the definitions. We have $(\mathrm{PCoker}(\varphi))_x = (0)$ iff for every open subset $U$ of containing $x$ and any $s \in \mathrm{PCoker}(\varphi)(U)$ there is some open subset $V \subseteq U$ containing $x$ such that $(\overline{\rho}_{\mathcal{G}})_V^U(s) = 0$ iff for every open subset $U$ of containing $x$ and any $t \in \mathcal{G}(U)$ there is some open subset $V \subseteq U$ containing $x$ such that $(\rho_{\mathcal{G}})_V^U(t) \in \varphi_V(\mathcal{F}(V))$ iff $\varphi_x$ is surjective.

Assume (iii) holds. For any open subset $U$ of $X$ and for any $t \in \mathcal{G}(U)$, for any $x \in U$, since $\varphi_x$ is surjective, there is some $\alpha \in \mathcal{F}_x$ such that $\varphi_x(\alpha) = t_x$. If $\alpha$ is represented by some $f^x \in \mathcal{F}(V_x)$ for some open subset $V_x$ of $U$ containing $x$, to say that $\varphi_x(\alpha) = t_x$ means that there is some open subset $U_x$ of $V_x$ containing $x$ such that $(\rho_{\mathcal{G}})_{U_x}^{V_x}(\varphi_{V_x}(f^x)) = (\rho_{\mathcal{G}})_{U_x}^U(t)$. However, the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{F}(V_x) & \xrightarrow{\varphi_{V_x}} & \mathcal{G}(V_x) \\
{\scriptstyle (\rho_{\mathcal{F}})_{U_x}^{V_x}} \downarrow & & \downarrow {\scriptstyle (\rho_{\mathcal{G}})_{U_x}^{V_x}} \\
\mathcal{F}(U_x) & \xrightarrow[\varphi_{U_x}]{} & \mathcal{G}(U_x)
\end{array}
$$

shows that $(\rho_{\mathcal{G}})_{U_x}^{V_x}(\varphi_{V_x}(f^x)) = \varphi_{U_x}((\rho_{\mathcal{F}})_{U_x}^{V_x}(f^x))$, and thus

$$\varphi_{U_x}((\rho_{\mathcal{F}})_{U_x}^{V_x}(f^x)) = (\rho_{\mathcal{G}})_{U_x}^U(t).$$

If we let $s^x = (\rho_{\mathcal{F}})_{U_x}^{V_x}(f^x)$, then we have a family $(s^x)_{x \in U}$ of sections $s^x \in \mathcal{F}(U_x)$ such that the $U_x$ form an open cover of $U$ and $\varphi_{U_x}(s^x) = (\rho_{\mathcal{G}})_{U_x}^U(t)$ for all $x \in U$, which is (iv).

The implication $(iv) \implies (iii)$ is immediate. Indeed, any $\gamma \in \mathcal{G}_x$ is represented by some section $t \in \mathcal{G}(U)$ for some open subset $U$ containing $x$, and by (iv), we have $\varphi_x((s_i)_x) = (\varphi_{U_i}(s_i))_x = t_x$ for any of the $s_i \in \mathcal{F}(U_i)$ given by (iv) since $\varphi_{U_i}(s_i) = (\rho_{\mathcal{G}})_{U_i}^U(t)$ for all $i \in I$. $\qquad\square$

It is important to note that in the case of a map of sheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$, unlike the case of presheaves, Condition (i) $(\mathrm{SCoker}(\varphi) = (0))$ *does not* imply that the maps $\varphi_U$ are surjective for all open subsets $U$. We can only assert a *local version* of the surjectivity of the $\varphi_U$, as in condition (iv).

An example of the failure of surjectivity of the $\varphi_U$ is provided by $X = \mathbb{C}$ (the complex numbers), the sheaf of holomorphic functions $\mathcal{F} = \mathcal{C}^\omega$, and $\varphi = d$, the differentiation operator on $\mathcal{F}$ (here, $\mathcal{G} = \mathcal{F}$). For any $x \in \mathbb{C}$, locally near $x$ a holomorphic function $f$ can be integrated as a holomorphic function $g$ such that $d/dz(g) = f$, but if $U$ is not simply connected there are holomorphic functions which cannot be expressed as $d/dz(g)$ for some holomorphic function $g$, for example $f = 1/z$ on $U = \{z \in \mathbb{C} \mid z \neq 0\}$.

**Definition 11.14.** A map of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *surjective* if any of the Conditions (i) and (ii) of Proposition 11.17 holds. A map of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *surjective* if any of the Conditions (i)–(iv) of Proposition 11.18 holds.

**Remark:** A presheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$ is said to an *epimorphism* if for every presheaf $\mathcal{H}$ any two presheaf maps $\psi_1, \psi_2\colon \mathcal{G} \to \mathcal{H}$, if $\psi_1 \circ \varphi = \psi_2 \circ \varphi$, then $\psi_1 = \psi_2$. Similarly, A sheaf map $\varphi\colon \mathcal{F} \to \mathcal{G}$ is said to an *epimorphism* if for every sheaf $\mathcal{H}$ any two sheaf maps $\psi_1, \psi_2\colon \mathcal{G} \to \mathcal{H}$, if $\psi_1 \circ \varphi = \psi_2 \circ \varphi$, then $\psi_1 = \psi_2$. It can be shown that being a presheaf epimorphism is equivalent to any of the conditions of Proposition 11.17, and being a sheaf epimorphism is equivalent to any of the conditions of Proposition 11.18; see Tennison [48] (Chapter III, Theorems 4.7 and 4.8). Technically, Definition 11.14 defines the notions of presheaf epimorphism and sheaf epimorphism. A presheaf morphism is surjective on sections (*i.e.* all $\varphi_U$ are surjective). The failure of a sheaf morphism to be a surjection on sections is closely related to sheaf cohomology.

We can combine Propositions 11.13, 11.17, and 11.18 to obtain the following criteria for a map of presheaves or a map of sheaves to be an isomorphism.

**Proposition 11.19.** *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a map of presheaves on a space $X$. Then the following conditions are equivalent:*

(i) *$\varphi$ is an isomorphism.*

(ii) *For every open subset $U$ of $X$, $\varphi_U$ is bijective.*

*If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then we have the further equivalent condition:*

(iii) *$\varphi_x$ is bijective for all $x \in X$.*

*Proof.* By definition $\varphi$ is a presheaf isomorphism iff there is some presheaf morphism $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\psi \circ \varphi = \mathrm{id}_\mathcal{F}$ and $\varphi \circ \psi = \mathrm{id}_\mathcal{G}$ iff there is some $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\psi_U \circ \varphi_U = \mathrm{id}_{\mathcal{F}(U)}$ and $\varphi_U \circ \psi_U = \mathrm{id}_{\mathcal{G}(U)}$ for all open subsets $U$ iff $\varphi_U$ is an isomorphism for all open subsets $U$. It remains to check that the inverses $\psi_U\colon \mathcal{G}(U) \to \mathcal{F}(U)$ are compatible with the restriction functions, which is easy to do. This proves that (i) and (ii) are equivalent.

It is clear that (i) implies (iii). Now, assume that $\mathcal{F}$ and $\mathcal{G}$ are sheaves and that the $\varphi_x$ are bijective. Since each $\varphi_x$ is injective, we know from Proposition 11.13 that $\varphi_U$ is injective for every open subset $U$. We now prove that because the $\varphi_x$ are surjective, each $\varphi_U$ is also surjective.

By Proposition 11.18(iv), for every open subset $U$ of $X$, for every $t \in \mathcal{G}(U)$, there is some open cover $(U_i)_{i \in I}$ of $U$ and a family $(s_i)_{i \in I}$ of sections $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = (\rho_\mathcal{G})^U_{U_i}(t)$ for all $i \in I$. By applying $\rho^{U_i}_{U_i \cap U_j}$ to both sides of the equation $\varphi_{U_i}(s_i) = (\rho_\mathcal{G})^U_{U_i}(t)$ and $\rho^{U_j}_{U_i \cap U_j}$ to both sides of the equation $\varphi_{U_j}(s_j) = (\rho_\mathcal{G})^U_{U_j}(t)$ and using the fact that

$$(\rho_\mathcal{G})^{U_i}_{U_i \cap U_j}(\varphi_{U_i}(s_i)) = \varphi_{U_i \cap U_j}((\rho_\mathcal{F})^{U_i}_{U_i \cap U_j}(s_i))$$
$$(\rho_\mathcal{G})^{U_j}_{U_i \cap U_j}(\varphi_{U_j}(s_j)) = \varphi_{U_i \cap U_j}((\rho_\mathcal{F})^{U_j}_{U_i \cap U_j}(s_j))$$

as shown by the commutativity of the diagrams

$$
\begin{array}{ccc}
\mathcal{F}(U_i) & \xrightarrow{\varphi_{U_i}} & \mathcal{G}(U_i) \\
{\scriptstyle (\rho_{\mathcal{F}})_{U_i \cap U_j}^{U_i}} \downarrow & & \downarrow {\scriptstyle (\rho_{\mathcal{G}})_{U_i \cap U_j}^{U_i}} \\
\mathcal{F}(U_i \cap U_j) & \xrightarrow{\varphi_{U_i \cap U_j}} & \mathcal{G}(U_i \cap U_j)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}(U_j) & \xrightarrow{\varphi_{U_j}} & \mathcal{G}(U_j) \\
{\scriptstyle (\rho_{\mathcal{F}})_{U_i \cap U_j}^{U_j}} \downarrow & & \downarrow {\scriptstyle (\rho_{\mathcal{G}})_{U_i \cap U_j}^{U_j}} \\
\mathcal{F}(U_i \cap U_j) & \xrightarrow{\varphi_{U_i \cap U_j}} & \mathcal{G}(U_i \cap U_j),
\end{array}
$$

we get

$$
\varphi_{U_i \cap U_j}((\rho_{\mathcal{F}})_{U_i \cap U_j}^{U_i}(s_i)) = \varphi_{U_i \cap U_j}((\rho_{\mathcal{F}})_{U_i \cap U_j}^{U_j}(s_j)) = (\rho_{\mathcal{G}})_{U_i \cap U_j}^{U}(t),
$$

and since $\varphi_{U_i \cap U_j}$ is injective, we conclude that

$$
(\rho_{\mathcal{F}})_{U_i \cap U_j}^{U_i}(s_i) = (\rho_{\mathcal{F}})_{U_i \cap U_j}^{U_j}(s_j)
$$

for all $i, j$. Since $\mathcal{F}$ is a sheaf, there is some $s \in \mathcal{F}(U)$ such that $(\rho_{\mathcal{F}})_{U_i}^{U}(s) = s_i$ for all $i$. We claim that $\varphi_U(s) = t$. For this, observe that

$$
(\rho_{\mathcal{G}})_{U_i}^{U}(\varphi_U(s)) = \varphi_{U_i}((\rho_{\mathcal{F}})_{U_i}^{U}(s)) = \varphi_{U_i}(s_i) = (\rho_{\mathcal{G}})_{U_i}^{U}(t)
$$

for all $i$, and since $\mathcal{G}$ is a sheaf, by Condition (M) we get

$$
\varphi_U(s) = t,
$$

as claimed. Therefore, $\varphi_U$ is surjective. $\qquad\square$

We also have the following result that we state without proof. The proof consists in unwinding the defintions; see Tennison [48] (Chapter III, Proposition 4.11).

**Proposition 11.20.** *Let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a map of presheaves on a space $X$. Then*

$$
(\text{PCoker }\varphi)_x = \text{Coker } \varphi_x = \mathcal{G}_x / \text{Im } \varphi_x
$$

*for all $x \in X$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then*

$$
(\text{SCoker }\varphi)_x = \text{Coker } \varphi_x
$$

*for all $x \in X$.*

In general, if $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a presheaf morphism, even if $\varphi$ is surjective and $\mathcal{F}$ is a sheaf $\mathcal{G}$ need not be a sheaf. However, it is under the following conditions.

**Proposition 11.21.** *Let $\mathcal{F}$ be a sheaf and $\mathcal{G}$ be a presheaf. If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a presheaf isomorphism, then $\mathcal{G}$ is a sheaf.*

*Proof.* Let $\psi \colon \mathcal{G} \to \mathcal{F}$ be the inverse of $\varphi$. For any open subset $U$ of $X$ and any open cover $(U_i)_{i \in I}$ of $U$, let $s, t \in \mathcal{G}(U)$ be such that $(\rho_{\mathcal{G}})^U_{U_i}(s) = (\rho_{\mathcal{G}})^U_{U_i}(t)$ for all $i$. Since

$$\psi_{U_i}((\rho_{\mathcal{G}})^U_{U_i}(s)) = (\rho_{\mathcal{F}})^U_{U_i}(\psi_U(s))$$
$$\psi_{U_i}((\rho_{\mathcal{G}})^U_{U_i}(t)) = (\rho_{\mathcal{F}})^U_{U_i}(\psi_U(t)),$$

we get

$$(\rho_{\mathcal{F}})^U_{U_i}(\psi_U(s)) = (\rho_{\mathcal{F}})^U_{U_i}(\psi_U(t))$$

for all $i$, and since $\mathcal{F}$ is a sheaf, we must have $\psi_U(s) = \psi_U(t)$. Since $\psi_U$ is injective, $s = t$; that is, $\mathcal{G}$ satisfies Condition (M).

Next, let $(t_i)_{\in I}$ be a family with $t_i \in \mathcal{G}(U_i)$ such that $(\rho_{\mathcal{G}})^{U_i}_{U_i \cap U_j}(t_i) = (\rho_{\mathcal{G}})^{U_j}_{U_i \cap U_j}(t_j)$ for all $i, j$. Since

$$\psi_{U_i \cap U_j}((\rho_{\mathcal{G}})^{U_i}_{U_i \cap U_j}(t_i)) = (\rho_{\mathcal{F}})^{U_i}_{U_i \cap U_j}(\psi_{U_i}(t_i))$$
$$\psi_{U_i \cap U_j}((\rho_{\mathcal{G}})^{U_j}_{U_i \cap U_j}(t_j)) = (\rho_{\mathcal{F}})^{U_j}_{U_i \cap U_j}(\psi_{U_j}(t_j)),$$

we get

$$(\rho_{\mathcal{F}})^{U_i}_{U_i \cap U_j}(\psi_{U_i}(t_i)) = (\rho_{\mathcal{F}})^{U_j}_{U_i \cap U_j}(\psi_{U_j}(t_j))$$

for all $i, j$. Since $\mathcal{F}$ is a sheaf, there is some $s \in \mathcal{F}(U)$ such that

$$(\rho_{\mathcal{F}})^U_{U_i}(s) = \psi_{U_i}(t_i)$$

for all $i \in I$. Now, since $\varphi_{U_i}$ and $\psi_{U_i}$ are mutual inverses, we get

$$(\rho_{\mathcal{G}})^U_{U_i}(\varphi_U(s)) = \varphi_{U_i}((\rho_{\mathcal{F}})^U_{U_i}(s)) = \varphi_{U_i}(\psi_{U_i}(t_i)) = t_i$$

for all $i \in I$, which shows that Condition (G) holds with $\varphi_U(s) \in \mathcal{G}(U)$. Therefore, $\mathcal{G}$ is a sheaf. $\qquad\square$

**Remark:** The notions of image and quotient of a map of stalk spaces do not present the difficulties encountered with sheaves. If $\varphi \colon (E_1, p_1) \to (E_2, p_2)$ is a map of stalk spaces, because $\varphi$ is a local homeomorphism (see Proposition 11.6(c)), the subspace $\varphi(E_1)$ is open in $E_2$, and so it is a substalk space of $(E_2, p_2)$. Similarly, if $(E_1, p_1)$ is a substalk space of $(E_2, p_2)$, then for every $x \in X$ we can form the quotient $H_x = p_2^{-1}(x)/p_1^{-1}(x)$ and make the disjoint union of the $H_x$ into a stalk space by giving it the quotient topology of the topology of $E_2$. This what Serre does in FAC [44] (Chapter 1, Section 7.1).

## 11.5    Exact Sequences of Presheaves and Sheaves

The key to the "correct" definition of an exact sequence of sheaves is the appropriate notion of image of a sheaf morphism.

**Definition 11.15.** If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is map of presheaves on a space $X$, then the *(presheaf) image* of $\varphi$, denoted $\operatorname{PIm}\varphi$, is the kernel $\operatorname{Ker}\operatorname{pcoker}(\varphi)$ of the cokernel map $\operatorname{pcoker}(\varphi) \colon \mathcal{G} \to \operatorname{PCoker}(\varphi)$. If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is map of sheaves on a space $X$, then the *(sheaf) image* of $\varphi$, denoted $\operatorname{SIm}\varphi$, is the kernel $\operatorname{Ker}\operatorname{scoker}(\varphi)$ of the cokernel map $\operatorname{scoker}(\varphi) \colon \mathcal{G} \to \operatorname{SCoker}(\varphi)$.

It is not hard to check that if $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves, then $(\operatorname{PIm}\varphi)(U) = \operatorname{Im}\varphi_U$, while if $\varphi \colon \mathcal{F} \to \mathcal{G}$ is map of sheaves, then $(\operatorname{SIm}\varphi)_x = \operatorname{Im}\varphi_x$ for all $x \in X$.

**Remark:** The image $\operatorname{Im}\varphi$ of a map of sheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ is often defined as the sheafification $\widetilde{\operatorname{PIm}\varphi}$ of the presheaf $\operatorname{PIm}\varphi$. The small problem with this approach is that this sheaf is not a subsheaf of $\mathcal{G}$. There is an injective morphism from $\widetilde{\operatorname{PIm}\varphi}$ into $\mathcal{G}$ so the image of $\varphi$ should really be the image of $\widetilde{\operatorname{PIm}\varphi}$ by that morphism. It seems to us that using $\operatorname{SIm}\varphi$ for the image of $\varphi$ is a cleaner approach (which agrees with the definition of image in an abelian category).

If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is map of sheaves and $\operatorname{PIm}\varphi$ is a sheaf, then $\operatorname{SIm}\varphi = \operatorname{PIm}\varphi$. Indeed, both are subsheaves of $\mathcal{G}$ and their stalks are equal to $\operatorname{Im}\varphi_x$ for all $x$, so by Proposition 11.16 they are equal. As a consequence, we obtain the following result.

**Proposition 11.22.** *If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is an injective map of sheaves, then* $\operatorname{SIm}\varphi = \operatorname{PIm}\varphi$.

*Proof.* Indeed, since $\varphi$ is injective there is a presheaf isomorphism from $\mathcal{F}$ to $\operatorname{PIm}\varphi$, and by Proposition 11.21 we conclude that $\operatorname{PIm}\varphi$ is sheaf, so by the fact stated just before this proposition $\operatorname{SIm}\varphi = \operatorname{PIm}\varphi$. $\qquad\square$

**Definition 11.16.** Let

$$\cdots \longrightarrow \mathcal{F} \xrightarrow{\ \varphi\ } \mathcal{G} \xrightarrow{\ \psi\ } \mathcal{H} \longrightarrow \cdots$$

be a sequence of maps of preshaves (over a space $X$). We say that the sequence is *exact at $\mathcal{G}$ as a sequence of presheaves* if

$$\operatorname{PIm}\varphi = \operatorname{Ker}\psi.$$

We say that it is an *exact sequence of presheaves* if it is exact at each point where it makes sense.

If the sequence consists of sheaves, then we say that it is *exact at $\mathcal{G}$ as a sequence of sheaves* if

$$\operatorname{SIm}\varphi = \operatorname{Ker}\psi.$$

It is an *exact sequence of sheaves* if it is exact at each point where it makes sense.

We have the following result stating more convenient conditions for checking that a sequence is an exact sequence of presheaves or an exact sequence of sheaves.

**Proposition 11.23.** *The following facts hold:*

(i) *If the sequence*

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

*is an exact sequence of presheaves, then for every open subset $U$ of $X$*

$$\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$$

*is an exact sequence of $R$-modules (or rings).*

(ii) *The sequence*

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

*is an exact sequence of sheaves iff the sequence*

$$\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

*is an exact sequence of $R$-modules (or rings) for all $x \in X$.*

(iii) *If the sequence of sheaves*

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

*is exact as a sequence of presheaves then it is exact as a sequence of sheaves.*

*Proof.* A complete proof is given in Tennison [48] (Chapter III, Theorem 6.5). We only give the proof of (ii). By definition, the sequence is exact iff $\operatorname{SIm} \varphi = \operatorname{Ker} \psi$ iff by Proposition 11.16

$$(\operatorname{SIm} \varphi)_x = (\operatorname{Ker} \psi)_x$$

for all $x \in X$. But, by definition

$$
\begin{aligned}
(\operatorname{SIm} \varphi)_x &= (\operatorname{Ker} (\operatorname{scoker} \varphi))_x \\
&= \operatorname{Ker} ((\operatorname{scoker} \varphi)_x \colon \mathcal{G}_x \longrightarrow (\operatorname{SCoker} \varphi)_x) \qquad \text{by Proposition 11.14} \\
&= \operatorname{Ker} ((\operatorname{scoker} \varphi)_x \colon \mathcal{G}_x \longrightarrow (\mathcal{G}_x/\operatorname{Im} \varphi_x)) \qquad \text{by Proposition 11.20} \\
&= \operatorname{Im} \varphi_x.
\end{aligned}
$$

Therefore, $\operatorname{SIm} \varphi = \operatorname{Ker} \psi$ iff $\operatorname{Im} \varphi_x = (\operatorname{Ker} \psi)_x = \operatorname{Ker} \psi_x$, as claimed. $\qquad\square$

As a corollary of Proposition 11.23, we have the following result.

**Proposition 11.24.** *The following facts hold as sequences of preseaves or sheaves:*

(i) *The sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$$

*is exact iff $\varphi$ is injective (a monomorphism).*

(ii) *The sequence*

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \longrightarrow 0$$

*is exact iff $\varphi$ is surjective (an epimorphism).*

(iii) *For any map $\varphi \colon \mathcal{F} \to \mathcal{G}$ of preshaves the sequence*

$$0 \longrightarrow \operatorname{Ker} \varphi \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \longrightarrow \operatorname{PCoker} \varphi \longrightarrow 0$$

*is exact, and for any map $\varphi \colon \mathcal{F} \to \mathcal{G}$ of sheaves the sequence*

$$0 \longrightarrow \operatorname{Ker} \varphi \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \longrightarrow \operatorname{SCoker} \varphi \longrightarrow 0$$

*is exact.*

We now discuss the preservation of exactness by various operations (functors). Some examples of these operations are:

1. The inclusion map $i \colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ which maps a sheaf to the corresponding presheaf, and a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ to the corresponding presheaf morphism.

2. The sheafification operation $\Gamma S \colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ which maps a presheaf $\mathcal{F}$ to its sheafification $\widetilde{\mathcal{F}}$, and a map of preshaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ to the map of sheaves $\widetilde{\varphi} \colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$.

3. For every open subset $U$ of $X$, for every presheaf $\mathcal{F} \in \mathbf{PSh}(X)$, we have the operation $\Gamma(U, -)$, "sections over $U$," given by

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U),$$

which yields an $R$-module (or a ring). Any presheaf morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ is mapped to the $R$-module (or ring) homomorphism $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$.

4. For every open subset $U$ of $X$, for every sheaf $\mathcal{F} \in \mathbf{Sh}(X)$, we have the operation $\Gamma(U, -)$, "sections over $U$," given by

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U),$$

which yields an $R$-module (or a ring). Any presheaf morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ is mapped to the $R$-module (or ring) homomorphism $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$.

The common thread between these examples is that we have two types of structures (categories) $\mathbf{C}$ and $\mathbf{D}$, and we have a transformation $T$ (a functor) which works as follows:

(i) Each object $A$ of $\mathbf{C}$ is mapped to some object $T(A)$ of $\mathbf{D}$.

(ii) Each map $A \xrightarrow{f} B$ between two objects $A$ and $B$ in $\mathbf{C}$ is mapped to some map $T(A) \xrightarrow{T(f)} T(B)$ between the objects $T(A)$ and $T(B)$ in $\mathbf{D}$ in such a way that the following properties hold:

(a) Given any two maps $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ between objects $A, B, C$ in $\mathbf{C}$ such that the composition $A \xrightarrow{g \circ f} C = A \xrightarrow{f} B \xrightarrow{g} C$ makes sense, the composition $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ makes sense in $\mathbf{D}$, and

$$T(g \circ f) = T(g) \circ T(f).$$

(b) If $A \xrightarrow{\mathrm{id}_A} A$ is the identity map of the object $A$ in $\mathbf{C}$, then $T(A) \xrightarrow{T(\mathrm{id}_A)} T(A)$ is the identity map of $T(A)$ in $\mathbf{D}$; that is,

$$T(\mathrm{id}_A) = \mathrm{id}_{T(A)}.$$

Whenever a transformation $T \colon \mathbf{C} \to \mathbf{D}$ satisfies the Properties (i), (ii) (a), (b), we call it a *(covariant) functor* from $\mathbf{C}$ to $\mathbf{D}$.

If $T \colon \mathbf{C} \to \mathbf{D}$ satisfies Properties (i), (b), and if Properties (ii) and (a) are replaced by the Properties (ii') and (a') below

(ii') Each map $A \xrightarrow{f} B$ between two objects $A$ and $B$ in $\mathbf{C}$ is mapped to some map $T(B) \xrightarrow{T(f)} T(A)$ between the objects $T(B)$ and $T(A)$ in $\mathbf{D}$ in such a way that the following properties hold:

(a') Given any two maps $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ between objects $A, B, C$ in $\mathbf{C}$ such that the composition $A \xrightarrow{g \circ f} C = A \xrightarrow{f} B \xrightarrow{g} C$ makes sense, the composition $T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A)$ makes sense in $\mathbf{D}$, and

$$T(g \circ f) = T(f) \circ T(g),$$

then $T$ is called a *contravariant functor* from $\mathbf{C}$ to $\mathbf{D}$.

An example of a (covariant) functor is the functor $\mathrm{Hom}(A, -)$ (for a fixed $R$-module $A$) from $R$-modules to $R$-modules (abelian groups if $R$ is not commutative) which maps a module $B$ to the module $\mathrm{Hom}(A, B)$ and a module homomorphism $f \colon B \to C$ to the module homomorphism $\mathrm{Hom}(A, f)$ from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(A, C)$ given by

$$\mathrm{Hom}(A, f)(\varphi) = f \circ \varphi \quad \text{for all } \varphi \in \mathrm{Hom}(A, B);$$

see Section 2.2.

An example of a contravariant functor is the functor $\text{Hom}(-, A)$ (for a fixed $R$-module $A$) from $R$-modules to $R$-modules (abelian groups if $R$ is not commutative) which maps a module $B$ to the module $\text{Hom}(B, A)$ and a module homomorphism $f\colon B \to C$ to the module homomorphism $\text{Hom}(f, A)$ from $\text{Hom}(C, A)$ to $\text{Hom}(B, A)$ given by

$$\text{Hom}(f, A)(\varphi) = \varphi \circ f \quad \text{for all } \varphi \in \text{Hom}(C, A);$$

see Section 2.2.

Given a type of structures (category) $\mathbf{C}$ let us denote the set of all maps from an object $A$ to an object $B$ by $\text{Hom}_{\mathbf{C}}(A, B)$. For all the types of structures $\mathbf{C}$ that we will dealing with, each set $\text{Hom}_{\mathbf{C}}(A, B)$ has some additional structure; namely it is an abelian group.

**Definition 11.17.** A type of structures (category) $\mathbf{C}$ is an **Ab**-*category* (or a *pre-additive category*) if for all $A, B \in \mathbf{C}$ the set of maps $\text{Hom}_{\mathbf{C}}(A, B)$ is an abelian group (with addition operation $+_{A,B}$ and a zero map $0_{A,B}$), and if the following distributivity axioms hold: for all $A, B, C, D \in \mathbf{C}$, for all maps $f \in \text{Hom}_{\mathbf{C}}(A, B), g_1, g_2 \in \text{Hom}_{\mathbf{C}}(B, C)$ and $h \in \text{Hom}_{\mathbf{C}}(C, D)$,

$$f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$$
$$(g_1 + g_2) \circ h = g_1 \circ h + g_2 \circ h.$$

If $\mathbf{C}$ and $\mathbf{D}$ are two **Ab**-categories, a functor $T\colon \mathbf{C} \to \mathbf{D}$ is *additive* if for all $A, B \in \mathbf{C}$ and all $f, g \in \text{Hom}_{\mathbf{C}}(A, B)$,
$$T(f + g) = T(f) + T(g).$$

Observe that if $T$ is an additive functor, then $T(0_{A,B}) = 0_{T(A),T(B)}$. For simplicity of notation we usually drop the subscript $A, B$ in $+_{A,B}$ and $0_{A,B}$.

The category of $R$-modules is an **Ab**-category. The category of sheaves (or presheaves) of $R$-modules or rings is also an **Ab**-category. The functors $\text{Hom}_R(A, -)$, $\text{Hom}_R(-, A)$, $- \otimes B$, and $\Gamma(U, -)$ are additive.

**Definition 11.18.** An **Ab**-category $\mathbf{C}$ is an *additive category* if there is a zero object $0$, with a unique map $0 \longrightarrow A$ and a unique map $A \longrightarrow 0$ for all $A \in \mathbf{C}$, and if the notion of direct sum makes sense for any two objects $A, B \in \mathbf{C}$.

Intuitively speaking an *abelian category* if an additive category in which the notion of kernel and cokernel of a map makes sense. Then we can define the notion of image of a map $f$ as the kernel of the cokernel of $f$, so the notion of exact sequence makes sense.

Technically, an *abelian category* $\mathbf{C}$ is an additive catgeory such that the following three properties hold:

1. Every map (that is, a map in $\mathbf{C}(A, B)$ for any $A, B \in \mathbf{C}$) has a kernel and a cokernel.

2. Every monomorphism is the kernel of its cokernel.

3. Every epimorphism is the cokernel of its kernel.

For precise definitions, see Weibel [51], MacLane [29], or Cartan–Eilenberg [7]. For our purposes it is enough to think of an abelian category as an additive category in which the notion of exact sequence makes sense. The categories of $R$-modules and the categories of of sheaves (or presheaves) are abelian categories.

**Definition 11.19.** Given two types of structures (categories) $\mathbf{C}$ and $\mathbf{D}$ in each of which the concept of exactness is defined (abelian categories), an additive functor $T\colon \mathbf{C} \to \mathbf{D}$ is said to be *exact* (resp. *left exact*, *right exact*) if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in $\mathbf{C}$, then the sequence

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow 0$$

is exact in $\mathbf{D}$ (left exact if the sequence

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C)$$

is exact, right exact if the sequence

$$T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow 0$$

is exact). If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant additive functor, then $T$ is said to be *exact* (resp. *left exact*, *right exact*) if whenever the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact in $\mathbf{C}$, then the sequence

$$0 \longrightarrow T(C) \longrightarrow T(B) \longrightarrow T(A) \longrightarrow 0$$

is exact in $\mathbf{D}$ (left exact if the sequence

$$0 \longrightarrow T(C) \longrightarrow T(B) \longrightarrow T(A)$$

is exact, right exact if the sequence

$$T(C) \longrightarrow T(B) \longrightarrow T(A) \longrightarrow 0$$

is exact).

For example, the functor $\mathrm{Hom}(-, A)$ is left-exact but not exact in general. The proof that $\mathrm{Hom}(-, A)$ is left-exact is identical to the proof of Proposition 2.5 except that $R$ is replaced by any $R$-module $A$ and $f^{\top}$ is replaced by $\mathrm{Hom}(f, A)$. Similarly, the functor $\mathrm{Hom}(A, -)$ is left-exact but not exact in general. Modules for which the functor $\mathrm{Hom}(A, -)$ is exact play an important role. They are called *projective module*. Similarly, modules for which the functor $\mathrm{Hom}(-, A)$ is exact are called *injective modules*.

Another important functor is given by the tensor product of modules. Given a fixed $R$-module $M$, we have a functor $T$ from $R$-modules to $R$-modules such that $T(A) = A \otimes_R M$ for any $R$-module $A$, and $T(f) = f \otimes_R \mathrm{id}_M$ for any $R$-linear map $f\colon A \to B$. This functor usually denoted $- \otimes_R M$ is right-exact; see Section 2.2. Modules $M$ for which the functor $- \otimes_R M$ is exact are called *flat*.

Here is a result giving us more exact or left exact functors.

**Proposition 11.25.** *The following results hold:*

(1) *The inclusion functor $i\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ is left-exact.*

(2) *The sheafification functor $\Gamma S\colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ which maps a presheaf $\mathcal{F}$ to its sheafification $\widetilde{\mathcal{F}}$, is exact.*

(3) *For every open subset $U$ of $X$, the functor $\Gamma(U, -)$ (sections over $U$) from $\mathbf{PSh}(X)$ to abelian groups is exact.*

(4) *For every open subset $U$ of $X$, the functor $\Gamma(U, -)$, (sections over $U$) from $\mathbf{Sh}(X)$ to abelian groups is left-exact.*

*Proof.* A proof of Proposition 11.25 can be found in Tennison [48] (Chapter III, Theorem 6.9). We simply indicate how to prove (1) and (4).

(1) if

$$0 \longrightarrow \mathcal{F} \overset{\varphi}{\longrightarrow} \mathcal{G} \overset{\psi}{\longrightarrow} \mathcal{H} \longrightarrow 0$$

is exact as sheaves, then by Proposition 11.24 $\varphi$ is injective. It follows from Proposition 11.22 that $\mathrm{PIm}\,\varphi = \mathrm{SIm}\,\varphi$, and then exactness at $\mathcal{G}$ (in the sense of sheaves) means that

$$\mathrm{PIm}\,\varphi = \mathrm{SIm}\,\varphi = \mathrm{Ker}\,\psi,$$

which is exactness in the sense of presheaves.

(4) By (1), if

$$0 \longrightarrow \mathcal{F} \overset{\varphi}{\longrightarrow} \mathcal{G} \overset{\psi}{\longrightarrow} \mathcal{H} \longrightarrow 0$$

is exact as sheaves, then

$$0 \longrightarrow \mathcal{F} \overset{\varphi}{\longrightarrow} \mathcal{G} \overset{\psi}{\longrightarrow} \mathcal{H}$$

is exact as presheaves. By Proposition 11.23 we deduce that the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$$

is exact for all open subsets of $X$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

One of the most useful applications of sheaves is that they can be used to generalize the notion of manifold. In the next section, we sketch this approach.

## 11.6   Ringed Spaces

The notion of a manifold $X$ captures the intuition that many spaces look locally like familiar spaces, such as $\mathbb{R}^n$ (which means that for every point $x \in X$ there is some open subset $U$ containing $x$ which "looks" like $\mathbb{R}^n$, more precisely $U$ is homeomorphic to $\mathbb{R}^n$), and that certain types of functions can be defined on them; for example continuous functions, smooth functions, analytic functions, *etc.* The notion of a ringed space provides an abstract way of specifying which are the "nice" functions on a space.

**Definition 11.20.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings called the *structure sheaf*.

The next step is to define the notion of map between two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. The basic idea is that such a map $f$ is a continuous map between the underlying spaces $X$ and $Y$ that pulls back the sheaf of functions on $Y$ to the sheaf of functions on $X$. For simplicity, let us first assume that $\mathcal{O}_X$ and $\mathcal{O}_Y$ are both sheaves of functions respectively on $X$ and $Y$. Let $f \colon X \to Y$ be a continuous function (where $V$ is some open subset of $Y$). Given any function $h \in \mathcal{O}_Y(V)$, denote the restriction of $h \circ f$ to $f^{-1}(V)$ by $f^*h$. Then $f$ should be a map of ringed spaces if the following condition holds: For every open subset $V$ of $Y$,

$$\text{if } h \in \mathcal{O}_Y(V) \text{ then } f^*h \in \mathcal{O}_X(f^{-1}(V)).$$

See Figure 11.7.

Observe that the assignment $h \mapsto f^*h$ defines a map

$$f_V^* \colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$$

which is a ring homomorphism. Thus, to define the notion of map of ringed spaces, it seems natural to require that there is a map of sheaves between $\mathcal{O}_Y$ and some sheaf over the base space $Y$ whose sections over any open subset $V$ of $Y$ come from sections of $\mathcal{O}_X$ over $f^{-1}(V)$. Such a sheaf corresponds to the notion of direct image of a sheaf.
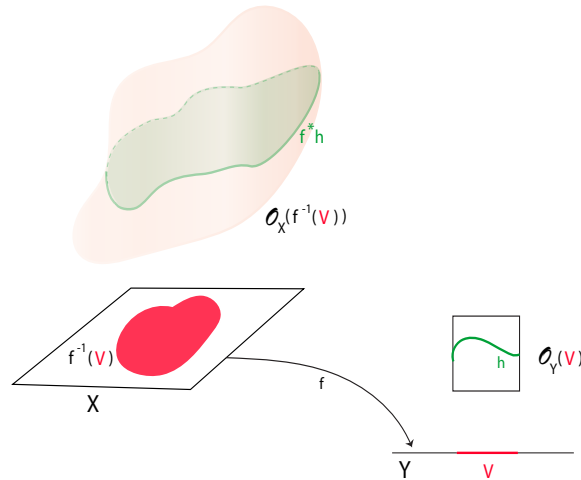
Figure 11.7: A schematic illustration of $f^*h$ where $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. The green plane in the peach balloon is the pull back of the section $h \in \mathcal{O}_Y(V)$.

**Definition 11.21.** Given any continuous function $f \colon X \to Y$ between two topological spaces $X$ and $Y$, for any sheaf $\mathcal{F}$ on $X$, define the presheaf $f_*\mathcal{F}$ on $Y$ by

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for all open subsets $V$ of $Y$. It is easily verified that $f_*\mathcal{F}$ is a sheaf on $Y$ called the *direct image* of $\mathcal{F}$ under $f$.

We can now define the notion of morphism of ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ even if $\mathcal{O}_X$ and $\mathcal{O}_Y$ are not sheaves of functions.

**Definition 11.22.** A *map* (or *morphism*) between two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair $(f, g)$, where $f \colon X \to Y$ is a continuous function and $g \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves, with each $g_V \colon \mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V)$ a ring homomorphisms for every open subset $V$ of $Y$.

Given two maps of ringed spaces $(f_1, g_1) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(f_2, g_2) \colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, their composition is the ring space map $(f_2, g_2) \circ (f_1, g_1) \colon (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ given by the pair of maps

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1).$$

A map of ringed spaces $(f, g) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an *isomorphism* iff there is some ring map $(f', g') \colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ such that $(f, g) \circ (f', g') = (\mathrm{id}, \mathrm{id})$ and $(f', g') \circ (f, g) = (\mathrm{id}, \mathrm{id})$. Given a ringed space $(X, \mathcal{O}_X)$, for every open subset $U$ ot $C$ it is clear that $(U, \mathcal{O}_X|U)$ is a ringed space on $U$.

We can now use the above notions to define a far reaching definition of the notion of a manifold. The idea is that a ringed space $(X, \mathcal{O}_X)$ is a certain type of manifold (also called a variety in the algebraic case) if it it is locally isomorphic to some other ringed space of the required type. The sheaf $\mathcal{O}_X$ specifies the "nice" functions on $X$.

**Definition 11.23.** Given two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, we say that $(X, \mathcal{O}_X)$ is *locally isomorphic to* $(Y, \mathcal{O}_Y)$ if for every $x \in X$ there is some open subset $U$ of $X$ containing $x$ and some open subset $V$ of $Y$ such that the ringed spaces $(U, \mathcal{O}_X|U)$ and $(V, \mathcal{O}_Y|V)$ are isomorphic.

Here are some examples illustrating that familiar types of manifolds can be cast in the framework of ringed spaces.

**Example 11.1.**

1. A *topological (or continuous) manifold* $M$ is a ringed space which is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$, where $\mathcal{C}(\mathbb{R}^n)$ is the sheaf of algebras of continuous (real-valued) functions on $\mathbb{R}^n$.

2. A *smooth manifold* $M$ is a ringed space which is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty(\mathbb{R}^n))$, where $\mathcal{C}^\infty(\mathbb{R}^n)$ is the sheaf of algebras of smooth (real-valued) functions on $\mathbb{R}^n$.

3. A *complex analytic manifold* $M$ is a ringed space which is locally isomorphic to $(\mathbb{C}^n, \mathrm{Hol}(\mathbb{C}^n))$, where $\mathrm{Hol}(\mathbb{C}^n)$ is the sheaf of smooth (complex-valued) functions on $\mathbb{C}^n$.

To illustrate the power of the notion of ringed space, if we had defined the notion of affine variety (where the functions are given by ratios of polynomials), then an *algebraic variety* is a ringed space which is locally isomorphic to an affine variety.

More generally, in algebraic geometry the central notion is that of a *scheme*, which is a ringed space locally isomorphic to an *affine scheme* (an affine scheme is a ringed space locally isomorphic to the "spectrum" of a ring, whatever that is). Ambitious readers are referred to Hartshorne [24] for an advanced treatment of algebraic geometry based on schemes.

# Chapter 12

# Derived Functors, $\delta$-Functors, and $\partial$-Functors

## 12.1 Projective, Injective, and Flat Modules

We saw in Section 2.2 that the functors $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, M)$ are left-exact but not exact in general, and that the functor $- \otimes M$ is right-exact but not exact in general. Thus it is natural to take a closer look at the modules for which these functors are exact.

**Definition 12.1.** An $R$-module $M$ is *projective* if the functor $\mathrm{Hom}(M, -)$ is exact, *injective* if the functor $\mathrm{Hom}(-, M)$ is exact, and *flat* if the functor $- \otimes M$ is exact.

Observe that the trivial module $(0)$ is injective, projective, and flat. The above definition does not tell us what kind of animals these modules are. The propositions of this section give somewhat more illuminating characterizations. Recall that for any linear map $h \colon A \to B$, we have $\mathrm{Hom}(M, h)(\varphi) = h \circ \varphi$ for all $\varphi \in \mathrm{Hom}(M, A)$; see Definition 2.6.

**Proposition 12.1.** *Let $P$ be an $R$-module. Then the following properties are equivalent:*

(1) *$P$ is projective.*

(2) *For any surjective linear map $h \colon A \to B$ and any linear map $f \colon P \to B$, there is some linear map $\widehat{f} \colon P \to A$ lifting $f \colon P \to B$ in the sense that $f = h \circ \widehat{f}$, as in the following commutative diagram:*

$$
\begin{array}{ccc}
 & & P \\
 & \overset{\widehat{f}}{\swarrow} & \downarrow f \\
A & \underset{h}{\longrightarrow} & B \longrightarrow 0.
\end{array}
$$

(3) *Any exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

*splits.*

(4)  *There is a free module $F$ and some other module $Q$ such that $F \cong P \oplus Q$.*

*Proof.* This is a standard result of commutative algebra. Proofs can be found in Dummit and Foote [11], Rotman [40], MacLane [29], Cartan–Eilenberg [7], and Weibel [51], among others. We only show that (1) is equivalent to (2) and that (2) implies (3).

Since $\mathrm{Hom}(P, -)$ is left exact, to say that it is exact means that if

$$A \xrightarrow{\ h\ } B \longrightarrow 0$$

is exact, then the sequence

$$\mathrm{Hom}(P, A) \xrightarrow{\ \mathrm{Hom}(P,h)\ } \mathrm{Hom}(P, B) \longrightarrow 0$$

is also exact. This is equivalent to saying that if $h\colon A \to B$ is surjective, then the map $\mathrm{Hom}(P, h)\colon \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$ is surjective, which by definition of $\mathrm{Hom}(P, h)$ means that for any linear map $f \in \mathrm{Hom}(P, B)$ there is some $\widehat{f} \in \mathrm{Hom}(P, A)$ such that $f = h \circ \widehat{f}$ as in

$$
\begin{array}{ccc}
 & & P \\
 & \overset{\widehat{f}}{\swarrow} & \big\downarrow{\scriptstyle f} \\
A & \xrightarrow{\ h\ } & B \longrightarrow 0,
\end{array}
$$

which is exactly (2).

Suppose

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } P \longrightarrow 0$$

is an exact sequence. We have the diagram

$$
\begin{array}{ccc}
 & & P \\
 & \overset{j}{\swarrow} & \big\| \\
B & \xrightarrow{\ g\ } & P \longrightarrow 0
\end{array}
$$

and since $P$ is projective the lifting property gives a map $j\colon P \to B$ such that $g \circ j = \mathrm{id}_P$, which show that (3) holds.  □

Proposition 12.1(4) shows that projective modules are almost free, in the sense that they are a summand of a free module. It also shows that free modules are projective, an invaluable fact. Another fact that we will need later is that every module is the image of some projective module.

**Proposition 12.2.** *For every $R$-module $M$, there is some projective (in fact, free) module $P$ and a surjective homomorphism $\rho\colon P \to M$.*

*Proof.* Pick any set $S$ of generators for $M$ (possibly $M$ itself) and let $P = R^{(S)}$ be the free $R$-module generated by $S$. The inclusion map $i\colon S \to M$ extends to a surjective linear map $\rho\colon P \to M$. $\qquad\square$

Injective modules are more elusive, although the diagram in 12.1(2) dualizes. Recall that for any linear map $h\colon A \to B$, we have $\mathrm{Hom}(h, M)(\varphi) = \varphi \circ h$ for all $\varphi \in \mathrm{Hom}(B, M)$; see Definition 2.5.

**Proposition 12.3.** *Let $I$ be an $R$-module. Then the following properties are equivalent:*

(1) *$I$ is injective.*

(2) *For any injective linear map $h\colon A \to B$ and any linear map $f\colon A \to I$, there is some linear map $\widehat{f}\colon B \to I$ extending $f\colon A \to I$ in the sense that $f = \widehat{f} \circ h$, as in the following commutative diagram:*

$$0 \longrightarrow A \xrightarrow{\ h\ } B$$

with $f\colon A \to I$ and $\widehat{f}$ the diagonal map to $I$.

(3) *Any exact sequence*

$$0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$$

*splits.*

*Proof.* This is also a standard result of commutative algebra. Proofs can be found in Dummit and Foote [11], Rotman [40], MacLane [29], Cartan–Eilenberg [7], and Weibel [51], among others. We only show that (1) is equivalent to (2) and that (2) implies (3). Since $\mathrm{Hom}(-, I)$ is left exact, to say that it is exact means that if

$$0 \longrightarrow A \xrightarrow{\ h\ } B$$

is exact, then the sequence

$$\mathrm{Hom}(B, I) \xrightarrow{\ \mathrm{Hom}(h,I)\ } \mathrm{Hom}(A, I) \longrightarrow 0$$

is also exact. This is equivalent to saying that if $h\colon A \to B$ is injective, then the map $\mathrm{Hom}(h, I)\colon \mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I)$ is surjective, which by definition of $\mathrm{Hom}(h, I)$ means that for any linear map $f \in \mathrm{Hom}(A, I)$ there is some $\widehat{f} \in \mathrm{Hom}(B, I)$ such that $f = \widehat{f} \circ h$ as in

$$0 \longrightarrow A \xrightarrow{\ h\ } B$$

with $f\colon A \to I$ and $\widehat{f}$ the diagonal map to $I$,

which is exactly (2).

Suppose

$$0 \longrightarrow I \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

is an exact sequence. We have the diagram

$$
\begin{array}{ccc}
0 \longrightarrow I & \xrightarrow{\ f\ } & B \\
\Big\| & \swarrow^{p} & \\
I & &
\end{array}
$$

and since $I$ is injective the lifting property gives a map $p \colon B \to I$ such that $p \circ f = \mathrm{id}_I$, which is (3).   □

The following theorem due to Baer shows that to test whether a module is injective it is enough to check the extension property (Proposition 12.3(2)) for sequences $0 \longrightarrow \mathfrak{A} \longrightarrow R$ for all ideals $\mathfrak{A}$ of the ring $R$.

**Theorem 12.4.** *(Baer Representation Theorem) An $R$-module $I$ is injective iff it has the extension property with respect to all sequences $0 \longrightarrow \mathfrak{A} \longrightarrow R$  where $\mathfrak{A}$ is an ideal of the ring $R$.*

A proof can be found in Dummit and Foote [11], Rotman [40], MacLane [29], Cartan–Eilenberg [7], and Weibel [51], among others.

As a corollary of Theorem 12.4, it is possible to characterize injective modules when the ring $R$ is a PID.

**Definition 12.2.** An $R$-module $M$ is divisible if for every nonzero $\lambda \in R$, the multiplication map given by $u \mapsto \lambda u$ for all $u \in M$ is surjective.

**Proposition 12.5.** *If the ring $R$ has no zero divisors then any injective module is divisible. Furthermore, if $R$ is a PID then a module is injective iff it is divisible.*

A proof can be found in Dummit and Foote [11], Rotman [40], MacLane [29], Cartan–Eilenberg [7].

The reader should check that the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective. A result dual to the statement of Proposition 12.2 holds for injective modules but is harder to prove.

**Theorem 12.6.** *(Baer Embedding Theorem) For every $R$-module $M$, there is some injective module $I$ and an injection $i \colon M \to I$.*

A particularly short poof of Theorem 12.6 can be found in Godement [18]. It uses the fact that if $M$ is a projective module, then $\mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ is an injective module.

Finally, we come to flat modules.

**Proposition 12.7.** *Let $M$ an $N$ be any two $R$-modules. If $M \oplus N$ is flat, then $M$ and $N$ are flat. Every projective module is flat. Direct sums of flat modules are flat.*

A proof of Proposition 12.7 can be found in Rotman [41]. The following result gives us a precise idea of what a flat module is when the ring $R$ is a PID.

**Proposition 12.8.** *If the ring $R$ has nonzero divisors, then any flat module is torsion-free. Furthermore, if $R$ is a PID then a module is a flat module iff it is torsion-free.*

A proof of Proposition 12.8 can be found in Weibel [51] (Chapter 3, Section 3.2), Bourbaki [3] (Chapter I, §2, Section 4, Proposition 3), and as a exercise in Dummit and Foote [11]. In particular, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

More generally, if $R$ is an integral domain and if $K$ is its fraction field, then $K$ is a flat $R$-module; see Atiyah and MacDonald [1] (Chapter 3, Corollary 3.6) or Bourbaki [3] (Chapter II, §2, Section 4, Theorem 1). This last result has an interesting application.

If $M$ is a finitely generated $R$-module where $R$ is an integral ring, recall that the *rank* $\text{rank}\,M$ of $M$ is the largest number of linearly independent vectors in $M$. Since the fraction field $K$ of $R$ is a field, the tensor product $M \otimes_R K$ is a vector space, and it is easy to see that the dimension of the vector space $M \otimes_R K$ is equal to the rank of $M$; see Matsumura [33] (Chapter 4, Section 11, page 84).

**Proposition 12.9.** *Let $R$ be an integral ring. For any finitely generated $R$-module $A, B, C$, if there is a short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

*then*

$$\text{rank}\,B = \text{rank}\,A + \text{rank}\,C.$$

*Proof.* Since the fraction field $K$ of $R$ is a flat $R$-module, if we tensor with $K$ we get the short exact sequence

$$0 \longrightarrow A \otimes_R K \longrightarrow B \otimes_R K \longrightarrow C \otimes_R K \longrightarrow 0,$$

in which all the modules involved are vector spaces over $K$. But then this is a split exact sequence and we have

$$\dim B \otimes_R K = \dim A \otimes_R K + \dim C \otimes_R K.$$

By a previous remark, $\text{rank}\,A = \dim A \otimes_R K$ and similarly with $B$ and $C$, so we obtain

$$\text{rank}\,B = \text{rank}\,A + \text{rank}\,C,$$

as claimed. $\square$

In the special case where $R = \mathbb{Z}$ and $A, B, C$ are finitely generated abelian groups, the equation of Proposition 12.9 is obtained by tensoring with $\mathbb{Q}$. Another proof of this formula (for abelian groups) is given in Greenberg and Harper [19] (Chapter 20, Lemma 20.7 and Lemma 20.8).

This is an equation which is used in proving the Euler–Poincaré formula; see Theorem 6.14.

It can be shown that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module which is not flat and the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}$ is flat but neither projective nor injective.

We are now ready to discuss (projective and injective) resolutions, one of the most important technical tools in homological algebra.

## 12.2    Projective and Injective Resolutions

We saw in Section 12.1 that in general there are modules that are not projective or not injective (or neither). Then it is natural to ask whether it is possible to quantify how much a module deviates from being projective or injective. Let us first consider the projective case.

We know from Proposition 12.2 that given any module $M$, there is some projective (in fact, free) module $P_0$ and a surjection $p_0 \colon P_0 \to M$. It follows that $M$ is isomorphic to $P_0/\mathrm{Ker}\, p_0$, but the module $K_0 = \mathrm{Ker}\, p_0$ may not be projective, so we repeat the process. There is some projective module $P_1$ and a surjection $p_1 \colon P_1 \to K_0$. Again $K_0$ is isomorphic to $P_1/\mathrm{Ker}\, p_1$, but $K_1 = \mathrm{Ker}\, p_1$ may not be projective. We repeat the process.

By induction, we we obtain exact sequences

$$0 \longrightarrow K_n \xrightarrow{\ i_n\ } P_n \xrightarrow{\ p_n\ } K_{n-1} \longrightarrow 0$$

with $P_n$ projective, $K_n = \mathrm{Ker}\, p_n$, and $i_n$ the inclusion map for all $n \geq 1$, and the starting sequence

$$0 \longrightarrow K_0 \xrightarrow{\ i_0\ } P_0 \xrightarrow{\ p_0\ } M \longrightarrow 0,$$

as illustrated by the following diagram:



If we define $d_n \colon P_n \to P_{n-1}$ by

$$d_n = i_{n-1} \circ p_n \quad (n \geq 1),$$

then since $i_{n-1}$ is injective we have

$$\mathrm{Ker}\, d_n = \mathrm{Ker}\, p_n = K_n,$$

and since $p_n$ is surjective we have

$$\mathrm{Im}\, d_n = \mathrm{Im}\, i_{n-1} = K_{n-1}.$$

Therefore, $\mathrm{Im}\, d_{n+1} = \mathrm{Ker}\, d_n$ for all $n \geq 1$. We also have $\mathrm{Im}\, d_1 = K_0 = \mathrm{Ker}\, p_0$ and $p_0$ is surjective, therefore the top row is an exact sequence. In summary, we proved the following result:

**Proposition 12.10.** *For every $R$-module $M$, there is some exact sequence*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

*in which every $P_n$ is a projective module. Furthermore, we may assume that the $P_n$ are free.*

Exact sequences of the above from are called resolutions.

**Definition 12.3.** Given any $R$-module $M$, a *projective* (resp. *free*, resp. *flat*) *resolution* of $M$ is any exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} M \longrightarrow 0 \qquad (*)$$

in which every $P_n$ is a projective (resp. free, resp. flat) module. The exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0$$

obtained by truncating the projective resolution of $M$ after $P_0$ is denoted by $\mathbf{P}^M$ or $\mathbf{P}_\bullet$, and the projective resolution $(*)$ is denoted by

$$\mathbf{P}^M \xrightarrow{p_0} M \longrightarrow 0.$$

An exact sequence $(*)$ where the $P_i$ are not necessarily projective (nor free, nor flat) is called a *left acyclic resolution* of $M$.

**Remark:** Following the convention for writing complexes with lower indices discussed in Section 2.3, the exact sequence $(*)$ of Definition 12.3 can also be written as

$$0 \longleftarrow M \xleftarrow{p_0} P_0 \xleftarrow{d_1} P_1 \longleftarrow \cdots \xleftarrow{d_{n-1}} P_{n-1} \xleftarrow{d_n} P_n \longleftarrow \cdots \qquad (**)$$

and the truncated sequence

$$P_0 \xleftarrow{d_1} P_1 \longleftarrow \cdots \xleftarrow{d_{n-1}} P_{n-1} \xleftarrow{d_n} P_n \longleftarrow \cdots$$

is still denoted by $\mathbf{P}^M$ or $\mathbf{P}_\bullet$. The projective resolution $(**)$ is denoted by

$$0 \longleftarrow M \xleftarrow{\;p_0\;} \mathbf{P}^M.$$

Proposition 12.10 shows that every module has some projective (resp. free, resp. flat) resolution. A projective resolution may stop after finitely many steps, which means that there is some $m$ such that $P_n = (0)$ for all $n \geq m$. For example, if the ring $R$ is a PID, since every submodule of a free module is free, every $R$-module has a free resolution with two steps:

$$0 \longrightarrow P_1 \xrightarrow{\;d_1\;} P_0 \xrightarrow{\;p_0\;} M \longrightarrow 0.$$

If we apply the functor $\mathrm{Hom}(-, B)$ to the exact sequence $\mathbf{P}^A$

$$P_0 \xleftarrow{\;d_1\;} P_1 \longleftarrow \quad \cdots \quad \xleftarrow{\;d_{n-1}\;} P_{n-1} \xleftarrow{\;d_n\;} P_n \longleftarrow \quad \cdots$$

obtained from a projective resolution of a module $A$ by dropping the term $A$, exactness is usually lost but we still obtain the chain complex $\mathrm{Hom}(\mathbf{P}^A, B)$ given by

$$0 \longrightarrow \mathrm{Hom}(P_0, B) \longrightarrow \quad \cdots \longrightarrow \mathrm{Hom}(P_{n-1}, B) \longrightarrow \mathrm{Hom}(P_n, B) \longrightarrow \quad \cdots \;,$$

with the maps $\mathrm{Hom}(P_{n-1}, B) \xrightarrow{\;\mathrm{Hom}(d_n, B)\;} \mathrm{Hom}(P_n, B)$.

Consequently, we have the cohomology groups $H^p(\mathrm{Hom}(\mathbf{P}^A, B))$ of the cohomology complex $\mathrm{Hom}(\mathbf{P}^A, B)$.

These cohomology modules seem to depend of the choice of the projective resolution $\mathbf{P}^A$. However, the remarkable fact about projective resolutions is that these cohomology groups *are independent* of the projective resolution chosen. This is what makes projective resolutions so special. In our case where we applied the functor $\mathrm{Hom}(-, B)$, the cohomology groups are denoted by $\mathrm{Ext}_R^n(A, B)$ (the "Ext" groups). Since $\mathrm{Hom}(-, B)$, is left exact, the exact sequence

$$P_1 \xrightarrow{\;d_1\;} P_0 \xrightarrow{\;p_0\;} A \longrightarrow 0$$

yields the exact sequences

$$0 \longrightarrow \mathrm{Hom}(A, B) \xrightarrow{\;\mathrm{Hom}(p_0, B)\;} \mathrm{Hom}(P_0, B) \xrightarrow{\;\mathrm{Hom}(d_1, B)\;} \mathrm{Hom}(P_1, B).$$

This implies that $\mathrm{Hom}(A, B)$ is isomorphic to $\mathrm{Ker}\,\mathrm{Hom}(d_1, B) = H^0(\mathrm{Hom}(\mathbf{P}^A, B))$ that is,

$$\mathrm{Ext}_R^0(A, B) \cong \mathrm{Hom}(A, B).$$

If $A$ is a projective module, then we have the trivial resolution $0 \longrightarrow A \xrightarrow{\;\mathrm{id}\;} A \longrightarrow 0$ , and $\mathrm{Ext}_R^n(A, B) = (0)$ for all $n \geq 1$.

If the ring $R$ is a PID, then every module $A$ has a free resolution

$$0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} A \longrightarrow 0 \,,$$

so $\operatorname{Ext}_R^n(A, B) = (0)$ for all $n \geq 2$. The group $\operatorname{Ext}_R^1(A, B)$ plays a a crucial role in the universal coefficient theorem for cohomology which expresses the cohomology groups of a complex in terms of its cohomology. The cohomology complex is obtain from the homology complex by appying the functor $\operatorname{Hom}(-, R)$.

If we apply the functor $- \otimes B$ to the exact sequence $\mathbf{P}^A$

$$P_0 \xleftarrow{d_1} P_1 \longleftarrow \quad \cdots \quad \xleftarrow{d_{n-1}} P_{n-1} \xleftarrow{d_n} P_n \longleftarrow \quad \cdots$$

obtained from a projective resolution of a module $A$ by dropping the term $A$, exactness is usually lost but we still obtain the chain complex $\mathbf{P}^A \otimes B$ given by

$$0 \longleftarrow P_0 \otimes B \longleftarrow \quad \cdots \longleftarrow P_{n-1} \otimes B \longleftarrow P_n \otimes B \longleftarrow \quad \cdots$$

with maps $P_n \otimes B \xrightarrow{d_n \otimes \operatorname{id}_B} P_{n-1} \otimes B$.

This time, we have the homology groups $H_p(\mathbf{P}^A \otimes B)$ of the homology complex $\mathbf{P}^A \otimes B$.

As before, these homology groups *are independent* of the resolution chosen. These homology groups are denoted by $\operatorname{Tor}_n^R(A, B)$ (the "Tor" groups). Because $- \otimes B$ is right-exact, we have an isomorphism

$$\operatorname{Tor}_0^R(A, B) \cong A \otimes B.$$

If the ring $R$ is a PID, then $\operatorname{Tor}_0^R(A, B) = (0)$ for all $n \geq 2$. The group $\operatorname{Tor}_1^R(A, B)$ plays a crucial role in the universal coefficient theorem that expresses the homology groups with coefficients in an $R$-module $B$ in terms of the homology groups with coefficients in $R$.

Using Theorem 12.6, we can dualize the construction of Proposition 12.10 to show that every module has an injective resolution, a notion defined below.

**Definition 12.4.** Given any $R$-module $M$, an *injective resolution* of $M$ is any exact sequence

$$0 \longrightarrow M \xrightarrow{i_0} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \quad \cdots \longrightarrow I^n \xrightarrow{d^n} I^{n+1} \longrightarrow \cdots \qquad (*)$$

in which every $I^n$ is an injective module. The exact sequence

$$I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \quad \cdots \longrightarrow I^n \xrightarrow{d^n} I^{n+1} \longrightarrow \cdots$$

obtained by truncating the injective resolution of $M$ before $I^0$ is denoted by $\mathbf{I}_M$ or $\mathbf{I}^\bullet$, and the injective resolution $(*)$ is denoted by

$$0 \longrightarrow M \xrightarrow{i_0} \mathbf{I}_M.$$

An exact sequence $(*)$ where the $I^i$ are not necessarily injective is called a *right acyclic resolution* of $M$.

**Proposition 12.11.** *Every R-module $M$ has some injective resolution.*

*Proof.* Using Theorem 12.6 we can find an injective module $I^0$ and an injection $i^0 \colon M \to I^0$. Let $C^1 = \operatorname{Coker} i^0$ be the cokernel of $i^0$. If $C^1$ is not injective then by Theorem 12.6 we can find an injective module $I^1$ and an injection $i^1 \colon C^1 \to I^1$. Let $C^2 = \operatorname{Coker} i^1$. If $C^2$ is not injective we repeat the process. By induction we obtain exact sequences

$$0 \longrightarrow C^n \xrightarrow{\ i^n\ } I^n \xrightarrow{\ p^n\ } C^{n+1} \longrightarrow 0,$$

where $C^{n+1} = \operatorname{Coker} i^n = I^n/\operatorname{Im} i^n$ and $p^n$ is the projection map for all $n \geq 0$, starting with

$$0 \longrightarrow M \xrightarrow{\ i^0\ } I^0 \xrightarrow{\ p^0\ } C^1 \longrightarrow 0,$$

as illustrated by the following diagram:



If we define $d^n \colon I^n \to I^{n+1}$ by

$$d^n = i^{n+1} \circ p^n \quad (n \geq 0),$$

then we immediately check $\operatorname{Ker} d^n = \operatorname{Ker} p^n = \operatorname{Im} i^n$ and $\operatorname{Im} d^n = \operatorname{Im} i^{n+1}$, so the top row is exact; that is, it is an injective resolution of $M$. $\qquad\square$

If we apply the functor $\operatorname{Hom}(A, -)$ to the exact sequence

$$I^0 \xrightarrow{\ d^0\ } I^1 \xrightarrow{\ d^1\ } \cdots \longrightarrow I^n \xrightarrow{\ d^n\ } I^{n+1} \longrightarrow \cdots$$

obtained by truncating the injective resolution of $B$ before $I^0$ we obtain the complex

$$\operatorname{Hom}(A, I^0) \longrightarrow \operatorname{Hom}(A, I^1) \longrightarrow \cdots \longrightarrow \operatorname{Hom}(A, I^n) \longrightarrow \operatorname{Hom}(A, I^{n+1}) \longrightarrow \cdots$$

with maps $\operatorname{Hom}(A, I^n) \xrightarrow{\ \operatorname{Hom}(A, d^n)\ } \operatorname{Hom}(A, I^{n+1})$ .

We have the cohomology groups $H^p(\operatorname{Hom}(A, \mathbf{I}_B))$ of the complex $\operatorname{Hom}(A, \mathbf{I}_B)$. Remarkably, as in the case of projective resolutions, these cohomology groups are *independent* of the injective resolution chosen. This is what makes injective resolutions so special. In our case where we applied the functor $\operatorname{Hom}(A, -)$ we obtain some cohomology modules $\operatorname{Ext}_R'^p(A, B)$.

It is natural to ask whether the modules $\operatorname{Ext}_R'^p(A, B)$ are related to the cohomology modules $\operatorname{Ext}_R^p(A, B)$ induced by the functor $\operatorname{Hom}(-, B)$ and defined in terms of projective resolutions. The answer is that they are *isomorphic*; see Rotman [41] or Weibel [51] for a thorough exposition.

We now return to the fundamental property of projective and injective resolutions, a kind of quasi-uniqueness. To be more precise, there is a chain homotopy equivalence between the complexes $\mathbf{P}^A$ and $\mathbf{P}'^A$ arising from any two projective resolutions of a module $A$ (a similar result holds for injective resolutions). To understand this, let us review the notions of chain map and chain homotopy from Section 2.4 in the context of projective and injective resolutions.

**Definition 12.5.** Let $A$ and $B$ be two $R$-modules, let

$$\mathbf{P}^A \xrightarrow{\epsilon} A \longrightarrow 0 \tag{$*$}$$

and

$$\mathbf{P}'^B \xrightarrow{\epsilon'} B \longrightarrow 0 \tag{$**$}$$

be two complexes, and let $f \colon A \to B$ be a map of $R$-modules. A *map* (or *morphism*) *from* $\mathbf{P}^A$ *to* $\mathbf{P}'^B$ *over* $f$ (or *lifting $f$*) is a family $g = (g_n)_{n \geq 0}$ of maps $g_n \colon P_n \to P'_n$ such that the following diagrams commute for all $n \geq 1$:

$$
\begin{array}{ccc}
P_n & \xrightarrow{d_n^P} & P_{n-1} \\
g_n \downarrow & & \downarrow g_{n-1} \\
P'_n & \xrightarrow{d_n^{P'}} & P'_{n-1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
P_0 & \xrightarrow{\epsilon} & A \\
g_0 \downarrow & & \downarrow f \\
P'_0 & \xrightarrow{\epsilon'} & B.
\end{array}
$$

Given two morphisms $g$ and $h$ from $\mathbf{P}^A$ to $\mathbf{P}'^B$ over $f$, a *chain homotopy* between $g$ and $h$ is a family $s = (s_n)_{n \geq 0}$ of maps $s_n \colon P_n \to P'_{n+1}$ for $n \geq 0$, such that

$$g_n - h_n = s_{n-1} \circ d_n^P + d_{n+1}^{P'} \circ s_n, \qquad n \geq 1$$

and

$$g_0 - h_0 = d_1^{P'} \circ s_0,$$

as illustrated in the diagrams

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}^P} & P_n & \xrightarrow{d_n^P} & P_{n-1} & \xrightarrow{d_{n-1}^P} \cdots \\
& \Delta^{n+1} \downarrow & \diagup s_n & \Delta_n \downarrow & \diagup s_{n-1} & \downarrow \Delta_{n-1} & \\
\cdots \longrightarrow & P'_{n+1} & \xrightarrow[d_{n+1}^{P'}]{} & P'_n & \xrightarrow[d_n^{P'}]{} & P'_{n-1} & \xrightarrow[d_{n-1}^{P'}]{} \cdots
\end{array}
$$

and

$$
\begin{array}{ccc}
\cdots \longrightarrow P_1 & \xrightarrow{\ d_1^P\ } & P_0 \\
\Big\downarrow{\scriptstyle \Delta^1} \quad\diagup{\scriptstyle s_0} & & \Big\downarrow{\scriptstyle \Delta_0} \\
\cdots \longrightarrow P_1' & \xrightarrow[\ d_1^{P'}\ ]{} & P_0'
\end{array}
$$

where $\Delta_n = g_n - h_n$.

In particular, a special case of Definition 12.5 is the case where $(*)$ and $(**)$ are projective resolutions. Dually, we have a definition that specializes to injective resolutions.

**Definition 12.6.** Let $A$ and $B$ be two $R$-modules, let

$$0 \longrightarrow A \xrightarrow{\ \epsilon\ } \mathbf{I}_A \tag{$*$}$$

and

$$0 \longrightarrow B \xrightarrow{\ \epsilon'\ } \mathbf{I}'_B \tag{$**$}$$

be two complexes, and let $f \colon A \to B$ be a map of $R$-modules. A *map* (or *morphism*) *from* $\mathbf{I}_A$ *to* $\mathbf{I}'_B$ *over* $f$ (or *lifting* $f$) is a family $g = (g^n)_{n \geq 0}$ of maps $g^n \colon I^n \to I'^n$ such that the following diagrams commute for all $n \geq 0$:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \epsilon\ } & I^0 \\
{\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle g^0} \\
B & \xrightarrow[\ \epsilon'\ ]{} & I'^0
\end{array}
\qquad\qquad
\begin{array}{ccc}
I^n & \xrightarrow{\ d_P^n\ } & I^{n+1} \\
{\scriptstyle g^n}\Big\downarrow & & \Big\downarrow{\scriptstyle g^{n+1}} \\
I'^n & \xrightarrow[\ d_{P'}^n\ ]{} & I'^{n+1}
\end{array}
$$

Given two morphisms $g$ and $h$ from $\mathbf{I}_A$ to $\mathbf{I}'_B$ over $f$, a *chain homotopy* between $g$ and $h$ is a family $s = (s^n)_{n \geq 1}$ of maps $s^n \colon I^n \to I'^{n-1}$ for $n \geq 1$, such that

$$g^n - h^n = s^{n+1} \circ d_I^n + d_{I'}^{n-1} \circ s^n, \qquad n \geq 1$$

and

$$g^0 - h^0 = s^1 \circ d_I^0,$$

as illustrated in the diagrams

$$
\begin{array}{ccc}
I^0 & \xrightarrow{\ d_I^0\ } & I^1 \longrightarrow \cdots \\
\Big\downarrow{\scriptstyle \Delta^0} \quad\diagup{\scriptstyle s^1} & & \Big\downarrow{\scriptstyle \Delta^1} \\
I'^0 & \xrightarrow[\ d_{I'}^0\ ]{} & I'^1 \longrightarrow \cdots
\end{array}
$$

and

$$\cdots \longrightarrow I_{n-1} \xrightarrow{d_I^{n-1}} I^n \xrightarrow{d_I^n} I^{n+1} \xrightarrow{d_I^{n+1}} \cdots$$

where $\Delta^n = g^n - h^n$.

We now come to the small miracle about projective resolutions.

**Theorem 12.12.** *(Comparison Theorem, Projective Case) Let $A$ and $B$ be $R$-modules. If $\mathbf{P}^A \xrightarrow{\epsilon} A \longrightarrow 0$ is a chain complex with all $P_n$ in $\mathbf{P}^A$ projective and if $\mathbf{X}^B \xrightarrow{\epsilon'} B \longrightarrow 0$ is an exact sequence (a left resolution of $B$), then any $R$-linear map $f : A \to B$ lifts to a morphism $g$ from $\mathbf{P}^A$ to $\mathbf{X}^B$ as illustrated by the following commutative diagram:*

$$\cdots \longrightarrow P_2 \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$
$$\cdots \longrightarrow X_2 \xrightarrow{d_2^X} X_1 \xrightarrow{d_1^X} X_0 \xrightarrow{\epsilon'} B \longrightarrow 0.$$

*Any two morphisms from $\mathbf{P}^A$ to $\mathbf{X}^B$ lifting $f$ are chain homotopic.*

*Proof.* Here is a slightly expanded version of the classical proof from Cartan–Eilenberg [7] (Chapter V, Proposition 1.1). We begin with a crucial observation.

If we have a diagram

$$\begin{array}{ccc} & & P \\ & \theta \nearrow & \downarrow f \\ A \xrightarrow{\varphi} & B \xrightarrow{\psi} & C \end{array}$$

in which

(1) $P$ is projective.

(2) The lower sequence is exact (i.e., Im $\varphi = \mathrm{Ker}\,\psi$).

(3) $\psi \circ f = 0$,

then there is a map $\theta : P \to A$ lifting $f$ (as shown by the dotted arrow above).

*Proof.* Indeed, $\psi \circ f = 0$ implies that $\operatorname{Im} f \subseteq \operatorname{Ker} \psi = \operatorname{Im} \varphi$; so, we have $\operatorname{Im} f \subseteq \operatorname{Im} \varphi$, and we are reduced to the usual diagram

$$
\begin{array}{ccc}
 & & P \\
 & {\scriptstyle \theta} \nearrow & \downarrow {\scriptstyle f} \\
A & \xrightarrow[\varphi]{} & \operatorname{Im} \varphi \longrightarrow 0
\end{array}
$$

where $\varphi$ is surjective.    □

We now prove our theorem.

We begin by proving the existence of the lift, stepwise, by induction. Since we have morphisms $\epsilon \colon P_0 \to A$ and $f \colon A \to B$, we get a morphism $f \circ \epsilon \colon P_0 \to B$ and we have the diagram

$$
\begin{array}{ccc}
 & & P_0 \\
 & {\scriptstyle g_0} \nearrow & \downarrow {\scriptstyle f \circ \epsilon} \\
X_0 & \longrightarrow & B \longrightarrow 0.
\end{array}
$$

As $P_0$ is projective, the map $g_0 \colon P_0 \to X_0$ exists and makes the diagram commute. Assume the lift exists up to level $n$. We have the diagram

$$
\begin{array}{ccccccc}
P_{n+1} & \xrightarrow{d^P_{n+1}} & P_n & \xrightarrow{d^P_n} & P_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow {\scriptstyle g_n} & & \downarrow {\scriptstyle g_{n-1}} & & \\
X_{n+1} & \xrightarrow[d^X_{n+1}]{} & X_n & \xrightarrow[d^X_n]{} & X_{n-1} & \longrightarrow & \cdots ,
\end{array} \tag{$\dagger$}
$$

so we get a map $g_n \circ d^P_{n+1} \colon P_{n+1} \to X_n$ and a diagram

$$
\begin{array}{ccc}
 & & P_{n+1} \\
 & {\scriptstyle g_{n+1}} \nearrow & \downarrow {\scriptstyle g_n \circ d^P_{n+1}} \\
X_{n+1} & \longrightarrow & X_n \xrightarrow[d^X_n]{} X_{n-1}.
\end{array}
$$

But, by commutativity in ($\dagger$), we get

$$
d^X_n \circ g_n \circ d^P_{n+1} = g_{n-1} \circ d^P_n \circ d^P_{n+1} = 0.
$$

Observe that in the above step we only use the fact that the first sequence is a chain complex. Now, $P_{n+1}$ is projective and the lower row in the above diagram is exact, so there is a lifting $g_{n+1} \colon P_{n+1} \to X_{n+1}$, as required.

Say we have two lifts $g = (g_n)$ and $h = (h_n)$. We construct the chain homotopy $(s_n)$, by induction on $n \geq 0$.

For the base case, we have the diagram

$$
\begin{array}{ccccc}
 & P_0 & \xrightarrow{\ \epsilon\ } & A & \longrightarrow & 0 \\
\small{s_0}\ \ & \small{g_0}\Big\downarrow\ \Big\downarrow\small{h_0} & & \Big\downarrow\small{f} & & \\
X_1 & \xrightarrow[\ d_1^X\ ]{} & X_0 & \xrightarrow[\ \epsilon'\ ]{} & B & \longrightarrow & 0.
\end{array}
$$

As $\epsilon'(g_0 - h_0) = (f - f)\epsilon = 0$, the lower row is exact and $P_0$ is projective, we get our lifting $s_0 \colon P_0 \to X_1$ with $g_0 - h_0 = d_1^X \circ s_0$.

Assume, for the induction step, that we already have $s_0, \ldots, s_{n-1}$. Write $\Delta_n = g_n - h_n$, then we get the diagram

$$
\begin{array}{ccccccc}
P_n & \xrightarrow{\ d_n^P\ } & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots \\
\small{\Delta_n}\Big\downarrow\ \ \small{s_{n-1}}\!\!\!\nearrow & & \Big\downarrow\small{\Delta_{n-1}} & & \Big\downarrow\small{\Delta_{n-2}} & & \\
X_{n+1} \longrightarrow & X_n & \xrightarrow[\ d_n^X\ ]{} & X_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & \cdots
\end{array}
\qquad (\dagger\!\dagger)
$$

There results a map $\Delta_n - s_{n-1} \circ d_n^P \colon P_n \longrightarrow X_n$ and a diagram

$$
\begin{array}{ccc}
 & P_n & \\
 & \Big\downarrow \small{\Delta_n - s_{n-1}\circ d_n^P} & \\
X_{n+1} \xrightarrow[\ d_{n+1}^X\ ]{} & X_n & \xrightarrow[\ d_n^X\ ]{} X_{n-1}.
\end{array}
$$

As usual, if we show that $d_n^X \circ (\Delta_n - s_{n-1} \circ d_n^P) = 0$, then there will be a lift $s_n \colon P_n \to X_{n+1}$ making the diagram

$$
\begin{array}{ccc}
 & P_n & \\
\small{s_n}\ \ \nearrow & \Big\downarrow \small{\Delta_n - s_{n-1}\circ d_n^P} & \\
X_{n+1} \xrightarrow[\ d_{n+1}^X\ ]{} & X_n & \xrightarrow[\ d_n^X\ ]{} X_{n-1}
\end{array}
$$

commute. Now, by the commutativity of $(\dagger\!\dagger)$, we have $d_n^X \circ \Delta_n = \Delta_{n-1} \circ d_n^P$; so

$$
d_n^X \circ (\Delta_n - s_{n-1} \circ d_n^P) = \Delta_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P.
$$

By the induction hypothesis, $\Delta_{n-1} = g_{n-1} - h_{n-1} = s_{n-2} \circ d_{n-1}^P + d_n^X \circ s_{n-1}$, and therefore

$$
\Delta_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P = s_{n-2} \circ d_{n-1}^P \circ d_n^P + d_n^X \circ s_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P = 0.
$$

Hence, $s_n$ exists and we are done. $\qquad\qquad\square$

Note that Theorem 12.12 holds under hypotheses weaker than the assumption that both If $\mathbf{P}^A \xrightarrow{\epsilon} A \longrightarrow 0$ and $\mathbf{X}^B \xrightarrow{\epsilon'} B \longrightarrow 0$ are projective resolutions. It suffices that the first sequence is a chain complex with all $P_n$ projective and that the second sequence is exact (with arbitrary $X_n$).

There are two important corollaries of the Comparison Theorem.

**Proposition 12.13.** *Given any R-linear map $f\colon A \to B$ between some R-modules A and B, if $\mathbf{P}^A \xrightarrow{\epsilon} A \longrightarrow 0$ and $\mathbf{P}'^B \xrightarrow{\epsilon'} B \longrightarrow 0$ are any two projective resolutions of A and B, then f has a lift g from $\mathbf{P}^A$ to $\mathbf{P}'^B$. Furthermore, any two lifts of f are chain homotopic.*

Recall that a *homotopy equivalence* between two chain complexes $C$ and $D$ consists of a pair $(g, h)$ of chain maps $g\colon C \to D$ and $h\colon D \to C$ such that $h \circ g$ is chain homotopic to $\mathrm{id}_C$ and $g \circ h$ is chain homotopic to $\mathrm{id}_D$.

We have the following important result which plays a key role in showing that the notion of derived functor does not depend on the choice of a projective resolution.

**Theorem 12.14.** *Given any R-module A, if $\mathbf{P}^A \xrightarrow{\epsilon} A \longrightarrow 0$ and $\mathbf{P}'^A \xrightarrow{\epsilon'} A \longrightarrow 0$ are any two projective resolutions of A, then $\mathbf{P}^A$ and $\mathbf{P}'^A$ are homotopy equivalent.*

*Proof.* By Proposition 12.13, the identity map $\mathrm{id}\colon A \to A$ has a lift $g$ from $\mathbf{P}^A$ and $\mathbf{P}'^A$ and a lift $h$ from $\mathbf{P}'^A$ and $\mathbf{P}^A$. Then $h \circ g$ is a lift of $\mathrm{id}_A$ from $\mathbf{P}^A$ to $\mathbf{P}^A$, and since the identity map $\mathrm{id}_{\mathbf{P}}$ of the complex $\mathbf{P}^A$ is also a lift of $\mathrm{id}_A$, by Proposition 12.13 there f is a chain homotopy from $h \circ g$ to $\mathrm{id}_{\mathbf{P}^A}$. Similarly, $g \circ h$ is a lift of $\mathrm{id}_A$ from $\mathbf{P}'^A$ to $\mathbf{P}'^A$, and since the identity map $\mathrm{id}_{\mathbf{P}'}$ of the complex $\mathbf{P}'^A$ is also a lift of $\mathrm{id}_A$, by Proposition 12.13 there is a chain homotopy from $g \circ h$ to $\mathrm{id}_{\mathbf{P}'^A}$. Therefore, $g$ and $h$ define a homotopy equivalence between $\mathbf{P}^A$ and $\mathbf{P}'^A$.                                   $\square$

Since the definition of an injective module is obtained from the definition of a projective module by changing the direction of the arrows it is not unreasonable to expect that a version of Theorem 12.12 holds. The proof is basically obtained by changing the direction of the arrows, but it takes a little more than that. Indeed, some quotients show up in the proof. Paraphrasing Lang [28]: "The books on homological algebra that I know of in fact carry out the projective case, and leave the injective case to the reader."

**Theorem 12.15.** *(Comparison Theorem, Injective Case) Let A and B be R-modules. If $0 \longrightarrow A \xrightarrow{\epsilon'} \mathbf{X}_A$ is an exact sequence (a right resolution of A) and if $0 \longrightarrow B \xrightarrow{\epsilon'} \mathbf{I}_B$ is a chain complex with all $I^n$ in $\mathbf{I}_B$ injective, then any R-linear map $f\colon A \to B$ lifts to a morphism g from $\mathbf{X}_A$ to $\mathbf{I}_B$ as illustrated by the following commutative diagram:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ \epsilon\ } & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^1} & & \downarrow{\scriptstyle g^2} & & \\
0 & \longrightarrow & B & \xrightarrow{\ \epsilon'\ } & I^0 & \xrightarrow{d_I^0} & I^1 & \xrightarrow{d_I^1} & I^2 & \xrightarrow{d_I^2} & \cdots .
\end{array}
$$

*Any two morphisms from $\mathbf{X}_A$ to $\mathbf{I}_B$ lifting $f$ are chain homotopic.*

*Proof.* We begin with a crucial observation dual to the crucial observation of the proof of Theorem 12.12.

If we have a diagram

$$A \xrightarrow{\psi} B \xrightarrow{\varphi} C$$

$$f \downarrow \quad \swarrow \theta$$

$$I$$

in which

(1) $I$ is injective.

(2) The upper sequence is exact (i.e., $\mathrm{Im}\,\psi = \mathrm{Ker}\,\varphi$).

(3) $f \circ \psi = 0$,

then there is a map $\theta \colon C \to I$ lifting $f$ (as shown by the dotted arrow above).

*Proof.* Indeed, $f \circ \psi = 0$ implies that $\mathrm{Im}\,\psi \subseteq \mathrm{Ker}\,f$; so we have $\mathrm{Ker}\,\varphi = \mathrm{Im}\,\psi \subseteq \mathrm{Ker}\,f$, that is $\mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\,f$. It follows that there is a unique map $\overline{f} \colon B/\mathrm{Ker}\,\varphi \to I$ such that the following diagram commutes:

$$B \xrightarrow{\pi} B/\mathrm{Ker}\,\varphi$$

$$f \searrow \quad \downarrow \overline{f}$$

$$I.$$

The map $\varphi \colon B \to C$ factors through the quotient map $\overline{\varphi} \colon B/\mathrm{Ker}\,\varphi \to C$ as $\varphi = \overline{\varphi} \circ \pi$ so we have the commutative diagram

$$
\begin{array}{c}
B \\
\pi \downarrow \quad \searrow \varphi \\
0 \longrightarrow B/\mathrm{Ker}\,\varphi \xrightarrow{\overline{\varphi}} C \\
\overline{f} \downarrow \quad \swarrow \theta \\
I
\end{array}
$$

and since $I$ is injective there is a map $\theta \colon C \to I$ lifting $\overline{f}$ as shown in the diagram above. Since $f = \overline{f} \circ \pi$, the commutativity of the above diagram yields $f = \overline{f} \circ \pi = \theta \circ \varphi$, which shows that $\theta$ lifts $f$, as claimed. $\qquad \square$

Using the above fact, the proof of the theorem proceeds by induction and is very similar to the proof of Theorem 12.12. Lang [28] gives most of the details. $\qquad \square$

Note that Theorem 12.15 holds under hypotheses weaker than the assumption that both
$0 \longrightarrow A \xrightarrow{\epsilon'} \mathbf{X}_A$ and $0 \longrightarrow B \xrightarrow{\epsilon'} \mathbf{I}_B$ are injective resolutions. It suffices that the first
sequence is exact (with arbitrary $X^n$) and that the second sequence is a chain complex with
all $I^n$ injective.

Analogously to the projective case we have the following important corollaries.

**Proposition 12.16.** *Given any $R$-linear map $f \colon A \to B$ between some $R$-modules $A$ and
$B$, if $0 \longrightarrow A \xrightarrow{\epsilon} \mathbf{I}_A$ and $0 \longrightarrow B \xrightarrow{\epsilon'} \mathbf{I}'_B$ are any two injective resolutions of $A$ and
$B$, then $f$ has a lift $g$ from $\mathbf{I}_A$ to $\mathbf{I}'_B$. Furthermore, any two lifts of $f$ are chain homotopic.*

The following result plays a key role in showing that the notion of derived functor does
not depend on the choice of an injective resolution.

**Theorem 12.17.** *Given any $R$-module $A$, if $0 \longrightarrow A \xrightarrow{\epsilon} \mathbf{I}_A$ and $0 \longrightarrow A \xrightarrow{\epsilon'} \mathbf{I}'_A$ are
any two injective resolutions of $A$, then $\mathbf{I}_A$ and $\mathbf{I}'_A$ are homotopy equivalent.*

At this stage we are ready to define the central concept of this chapter, the notion of
derived functor. A key observation is that the existence of projective resolutions or injective
resolutions depends only on the fact that for every object $A$ there is some projective object
$P$ and a surjection $\rho \colon P \to A$, and there is some injective object $I$ and an injection $\epsilon \colon A \to I$.

If $\mathbf{C}$ is an abelian category then the notions of projective and injective objects make sense
since they are defined purely in terms of conditions on maps.

**Definition 12.7.** Given an abelian category $\mathbf{C}$, we say that $\mathbf{C}$ has *enough injectives* if for
every object $A \in \mathbf{C}$ there is some injective object $I \in \mathbf{C}$ and a monomorphism $\epsilon \colon A \to I$
(which means that ker $\epsilon = 0$) (resp. *enough projectives* if for every $A \in \mathbf{C}$ there is some
projective object $P \in \mathbf{C}$ and an epimorphism $\rho \colon P \to A$ (which means that coker $\rho = 0$)).

If can be shown that if an abelian category $\mathbf{C}$ has enough projectives, then the results of
this section (in particular Proposition 12.13 and Theorem 12.14) hold. Similarly, if an abelian
category $\mathbf{C}$ has enough injectives, then the results of this section (in particular Proposition
12.16 and Theorem 12.17) hold.

As we saw, the category of $R$-modules has enough injectives and projecticves. Now, it
turns out that the category of sheaves (which is abelian) has enough injectives, but does not
have enough projectives (as we saw, cokernels and quotients are problematic).

Derived functors have the property that any short exact sequence yields a long coho-
mology (or homology) exact sequence, and that it is so naturally (as in Theorem 2.19 and
Proposition 2.20). To prove these facts requires some rather technical propositions involving
projective and injective resolutions. We content ourselves with stating these results. Futher-
more, since our ultimate goal is to apply derived functors to the category of sheaves to obtain
sheaf cohomology, and since the category of sheaves does not have enough projectives but
has enough injectives, we will focus our attention on results involving injectives.

We need to define what we mean by an exact sequence of chain complexes. If $\mathfrak{A} = (A, d_A)$, $\mathfrak{B} = (B, d_B)$ and $\mathfrak{C} = (C, d_C)$ are three chain complexes and $f\colon \mathfrak{A} \to \mathfrak{B}$ and $g\colon \mathfrak{B} \to \mathfrak{C}$ are two chain maps with $f = (f^n)$ and $g = (g^n)$, we say that the the sequence of complexes

$$0 \longrightarrow \mathfrak{A} \xrightarrow{\ f\ } \mathfrak{B} \xrightarrow{\ g\ } \mathfrak{C} \longrightarrow 0$$

is *exact* iff the sequence

$$0 \longrightarrow A^n \xrightarrow{\ f^n\ } B^n \xrightarrow{\ g^n\ } C^n \longrightarrow 0$$

is exact for every $n$.

**Proposition 12.18.** *(Horseshoe Lemma, Projective Case) Consider the diagram (in some abelian category* $\mathbf{C}$*)*



*where the left column is a projective resolution* $\mathfrak{P}'\colon \ \mathbf{P}^{A'} \xrightarrow{\ \epsilon'\ } A' \longrightarrow 0$ *of* $A'$*, the right column* $\mathfrak{P}''\colon \ \mathbf{P}^{A''} \xrightarrow{\ \epsilon''\ } A'' \longrightarrow 0$ *is a projective resolution of* $A''$*, and the row is an exact sequence. Then there is a projective resolution* $\mathfrak{P}\colon \ \mathbf{P}^A \xrightarrow{\ \epsilon\ } A \longrightarrow 0$ *of* $A$ *and chain maps* $f\colon \mathfrak{P}' \to \mathfrak{P}$ *and* $g\colon \mathfrak{P} \to \mathfrak{P}''$ *such that the sequence*

$$0 \longrightarrow \mathfrak{P}' \xrightarrow{\ f\ } \mathfrak{P} \xrightarrow{\ g\ } \mathfrak{P}'' \longrightarrow 0$$

*is exact.*

A proof of Proposition 12.18 can be found in Rotman [40] (Chapter 6, Lemma 6.20).

**Proposition 12.19.** *(Horseshoe Lemma, Injective Case) Consider the diagram (in some*

*abelian category* **C***)*

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
I'^1 & & I''^1 \\
\uparrow & & \uparrow \\
I'^0 & & I''^0 \\
\epsilon'\uparrow & & \uparrow\epsilon'' \\
0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0 \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

*where the left column is an injective resolution* $\mathfrak{I}'\colon\; 0 \longrightarrow A' \xrightarrow{\epsilon'} \mathbf{I}_{A'}$ *of* $A'$*, the right column*
$\mathfrak{I}''\colon\; 0 \longrightarrow A'' \xrightarrow{\epsilon''} \mathbf{I}_{A''}$ *is an injective resolution of* $A''$*, and the row is an exact sequence.*
*Then there is an injective resolution* $\mathfrak{I}\colon\; 0 \longrightarrow A \xrightarrow{\epsilon} \mathbf{I}_A$ *of* $A$ *and chain maps* $f\colon \mathfrak{I}' \to \mathfrak{I}$
*and* $g\colon \mathfrak{I} \to \mathfrak{I}''$ *such that the sequence*

$$
0 \longrightarrow \mathfrak{I}' \xrightarrow{f} \mathfrak{I} \xrightarrow{g} \mathfrak{I}'' \longrightarrow 0
$$

*is exact.*

We will also need a generalization of the Horseshoe Lemma for chain maps of exact sequences.

**Proposition 12.20.** *Suppose we have a map of exact sequences (in some abelian category*
**C***)*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{\varphi} & A & \xrightarrow{\psi} & A'' & \longrightarrow & 0 \\
& & \downarrow{f'} & & \downarrow{f} & & \downarrow{f''} & & \\
0 & \longrightarrow & B' & \xrightarrow{\varphi'} & B & \xrightarrow{\psi'} & B'' & \longrightarrow & 0
\end{array}
$$

*and that we have some injective resolutions* $0 \longrightarrow A' \xrightarrow{\epsilon^{A'}} \mathbf{I}_{A'}$ , $0 \longrightarrow A'' \xrightarrow{\epsilon^{A''}} \mathbf{I}_{A''}$ ,

$0 \longrightarrow B' \xrightarrow{\epsilon^{B'}} \mathbf{I}_{B'}$ *and* $0 \longrightarrow B'' \xrightarrow{\epsilon^{B''}} \mathbf{I}_{B''}$ *of the corners* $A', A'', B', B''$*, and chain maps*
$F'\colon \mathbf{I}_{A'} \to \mathbf{I}_{B'}$ *over* $f'$ *and* $F''\colon \mathbf{I}_{A''} \to \mathbf{I}_{B''}$ *over* $f''$*. Then there exist injective resolutions*
$0 \longrightarrow A \xrightarrow{\epsilon^A} \mathbf{I}_A$ *of* $A$ *and* $0 \longrightarrow B \xrightarrow{\epsilon^B} \mathbf{I}_B$ *of* $B$ *and a chain map* $F\colon \mathbf{I}_A \to \mathbf{I}_B$ *over* $f$

*such that the following diagram commmutes*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{I}_{A'} & \longrightarrow & \mathbf{I}_A & \longrightarrow & \mathbf{I}_{A''} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F''} & & \\
0 & \longrightarrow & \mathbf{I}_{B'} & \longrightarrow & \mathbf{I}_B & \longrightarrow & \mathbf{I}_{B''} & \longrightarrow & 0
\end{array}
$$

*and has exact rows.*

There is also a version of Proposition 12.20 for projective resolutions; see Rotman [40] (Chapter 6, Lemma 6.24). The reader should enjoy the use of three-dimensional diagrams involving cubes.

# 12.3 Left and Right Derived Functors

Let $\mathbf{C}$ and $\mathbf{D}$ be two abelian categories, and let $T\colon \mathbf{C} \to \mathbf{D}$ be an additive functor. Actually, in all our examples $\mathbf{C}$ is either the category of $R$-modules, the category of presheaves, or the category or sheaves, and $\mathbf{D}$ is either the category of $R$-modules or the catgeory of abelian groups, so the reader may assume this if the abstract nature of abelian categories makes her/him uncomfortable.

Assume that $\mathbf{C}$ has enough injectives. For any $A \in \mathbf{C}$, if $0 \longrightarrow A \overset{\epsilon}{\longrightarrow} \mathbf{I}_A$ is an injective resolution of $A$, then if we apply $T$ to $\mathbf{I}_A$ we obtain the cochain complex

$$
0 \longrightarrow T(I^0) \xrightarrow{T(d^0)} T(I^1) \xrightarrow{T(d^1)} \cdots \longrightarrow T(I^n) \xrightarrow{T(d^n)} T(I^{n+1}) \longrightarrow \cdots , \tag{Ri}
$$

denoted $T(\mathbf{I}_A)$. If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor and if we apply $T$ to $\mathbf{I}_A$ we obtain the chain complex

$$
0 \longleftarrow T(I^0) \xleftarrow{T(d^0)} T(I^1) \xleftarrow{T(d^1)} \cdots \longleftarrow T(I^n) \xleftarrow{T(d^n)} T(I^{n+1}) \longleftarrow \cdots , \tag{Li}
$$

also denoted $T(\mathbf{I}_A)$.

Now assume that $\mathbf{C}$ has enough projectives. For any $A \in \mathbf{C}$, if $\mathbf{P}^A \overset{\epsilon}{\longrightarrow} A \longrightarrow 0$ is a projective resolution of $A$, then if we apply $T$ to $\mathbf{P}^A$ we obtain the chain complex

$$
0 \longleftarrow T(P_0) \xleftarrow{T(d_1)} T(P_1) \xleftarrow{T(d_2)} \cdots \longleftarrow T(P_{n-1}) \xleftarrow{T(d_n)} T(P_n) \longleftarrow \cdots , \tag{Lp}
$$

denoted $T(\mathbf{P}^A)$. If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor and if we apply $T$ to $\mathbf{P}^A$ we obtain the cochain complex

$$
0 \longrightarrow T(P_0) \xrightarrow{T(d_1)} T(P_1) \xrightarrow{T(d_2)} \cdots \longrightarrow T(P_{n-1}) \xrightarrow{T(d_n)} T(P_n) \longrightarrow \cdots , \tag{Rp}
$$

also denoted $T(\mathbf{P}^A)$. The above four complexes have (co)homology that defines the left and right derived functors of $T$.

**Definition 12.8.** Let $\mathbf{C}$ and $\mathbf{D}$ be two abelian categories, and let $T\colon \mathbf{C} \to \mathbf{D}$ be an additive functor.

(Ri) Assume that $\mathbf{C}$ has enough injectives. For any $A \in \mathbf{C}$, if $0 \longrightarrow A \overset{\epsilon}{\longrightarrow} \mathbf{I}_A$ is an injective resolution of $A$, then the cohomology groups of the complex $T(\mathbf{I}_A)$ are denoted by

$$R^n T(\mathbf{I}_A) = H^n(T(\mathbf{I}_A)), \qquad n \geq 0.$$

(Li) If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor, then the homology groups of the complex $T(\mathbf{I}_A)$ are denoted by

$$L_n T(\mathbf{I}_A) = H_n(T(\mathbf{I}_A)), \qquad n \geq 0.$$

(Lp) Now assume that $\mathbf{C}$ has enough projectives. For any $A \in \mathbf{C}$, if $\mathbf{P}^A \overset{\epsilon}{\longrightarrow} A \longrightarrow 0$ is a projective resolution of $A$, then the homology groups of the complex $T(\mathbf{P}^A)$ are denoted by

$$L_n T(\mathbf{P}^A) = H_n(T(\mathbf{P}^A)), \qquad n \geq 0.$$

(Rp) If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor, then the cohomology groups of the complex $T(\mathbf{P}^A)$ are denoted by

$$R^n T(\mathbf{P}^A) = H^n(T(\mathbf{P}^A)), \qquad n \geq 0.$$

The reason for using $R^n T$ or $L_n T$ is that the chain complexes $T(\mathbf{I}_A)$ in (Ri) and $T(\mathbf{P}^A)$ in (Rp) have arrows going to the right since they are cohomology complexes so the corresponding functors are $R^n T$, and the chain complexes $T(\mathbf{I}_A)$ in (Li) and $T(\mathbf{P}^A)$ in (Lp) have arrows going to the left since they are homology complexes so the corresponding functors are $L_n T$.

In the rest of this chapter we always assume that $\mathbf{C}$ and $\mathbf{D}$ are abelian categories and that $\mathbf{C}$ has enough injectives or projectives, as needed.

All the operators introduced in Definition 12.8 are actually functors so let us clarify what are the categories invoved. In Cases (Li) and (Ri) the domain category is the set of all injective resolutions $0 \longrightarrow A \overset{\epsilon}{\longrightarrow} \mathbf{I}_A$ for all $A \in \mathbf{C}$, and a morphism from $0 \longrightarrow A \overset{\epsilon}{\longrightarrow} \mathbf{I}_A$ to $0 \longrightarrow B \overset{\epsilon'}{\longrightarrow} \mathbf{I}'_B$ is simply a map $f\colon A \to B$. To be absolutely precise $R^n T(\mathbf{I}_A)$ and $L_n T(\mathbf{I}_A)$ should be denoted $R^n T(\, 0 \longrightarrow A \overset{\epsilon}{\longrightarrow} \mathbf{I}_A\, )$ and $L_n T(\, 0 \longrightarrow A \overset{\epsilon}{\longrightarrow} \mathbf{I}_A\, )$ but for the sake of notational simplicity we use the former notation.

In Cases (Lp) and (Rp) the domain category is the set of all projective resolutions $\mathbf{P}^A \overset{\epsilon}{\longrightarrow} A \longrightarrow 0$ $(A \in \mathbf{C})$, and a morphism from $\mathbf{P}^A \overset{\epsilon}{\longrightarrow} A \longrightarrow 0$ to $\mathbf{P}'^B \overset{\epsilon'}{\longrightarrow} B \longrightarrow 0$ is simply a map $f\colon A \to B$. Again, to be absolutely precise $L_n T(\mathbf{P}^A)$ and $R^n T(\mathbf{P}^A)$ should be denoted $L_n T(\, \mathbf{P}^A \overset{\epsilon}{\longrightarrow} A \longrightarrow 0\, )$ $R^n T(\, \mathbf{P}^A \overset{\epsilon}{\longrightarrow} A \longrightarrow 0\, )$ but we use the simpler notation.

In both cases the codomain category is $\mathbf{D}$. Definition 12.8 describes how $R^n T$ and $L_n T$ act on objects. We also have to explain how they act on maps $f\colon A \to B$. First, consider Case (Ri).

If $0 \longrightarrow A \xrightarrow{\epsilon} \mathbf{I}_A$ is any injective resolution of $A$ and $0 \longrightarrow B \xrightarrow{\epsilon'} \mathbf{I}'_B$ is any injective resolution of $B$, then by Proposition 12.16 the map $f$ has a lift $g$ from $\mathbf{I}_A$ to $\mathbf{I}'_B$ as illustrated by the following commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{\epsilon} & I^0 & \xrightarrow{d_I^0} & I^1 & \xrightarrow{d_I^1} & I^2 & \xrightarrow{d_I^2} & \cdots \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^1} & & \downarrow{\scriptstyle g^2} & & \\
0 & \longrightarrow & B & \xrightarrow{\epsilon'} & I'^0 & \xrightarrow{d_{I'}^0} & I'^1 & \xrightarrow{d_{I'}^1} & I'^2 & \xrightarrow{d_{I'}^2} & \cdots \ .
\end{array}
$$

Since $T$ is a functor, $T(g)$ is a chain map from $T(\mathbf{I}_A)$ to $T(\mathbf{I}'_B)$ lifting $T(f)$ as illustrated by the following commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & T(A) & \xrightarrow{T(\epsilon)} & T(I^0) & \xrightarrow{T(d_I^0)} & T(I^1) & \xrightarrow{T(d_I^1)} & T(I^2) & \xrightarrow{T(d_I^2)} & \cdots \\
& & \downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle T(g^0)} & & \downarrow{\scriptstyle T(g^1)} & & \downarrow{\scriptstyle T(g^2)} & & \\
0 & \longrightarrow & T(B) & \xrightarrow{T(\epsilon')} & T(I'^0) & \xrightarrow{T(d_{I'}^0)} & T(I'^1) & \xrightarrow{T(d_{I'}^1)} & T(I'^2) & \xrightarrow{T(d_{I'}^2)} & \cdots \ .
\end{array}
$$

By Proposition 2.16, $T(g)$ induces a homomorphism of cohomology $T(g^n)^*\colon H^n(T(\mathbf{I}_A)) \to H^n(T(\mathbf{I}'_B))$ for all $n \geq 0$. Furthermore, if $h$ is another lift of $f$, since by Proposition 12.16 any two lifts of $f$ are chain homotopic say by the chain homotopy $(s^n)_{n \geq 0}$, since $T$ is additive by applying $T$ to the equations

$$g^n - h^n = s^{n+1} \circ d_I^n + d_{I'}^{n-1} \circ s^n$$

we obtain

$$T(g^n) - T(h^n) = T(s^{n+1}) \circ T(d_I^n) + T(d_{I'}^{n-1}) \circ T(s^n),$$

which shows that $(T(s^n))_{n \geq 0}$ is a chain homotopy between $T(g)$ and $T(h)$, and by Proposition 2.17 we have $T(g^n)^* = T(h^n)^*$. Therefore, the homomorphism $T(g^n)^*\colon H^n(T(\mathbf{I}_A)) \to H^n(T(\mathbf{I}'_B))$ is independent of the lift $g$ of $f$, and we define $R^n T(\mathbf{I}_A, \mathbf{I}'_B)(f)\colon R^n T(\mathbf{I}_A) \to R^n T(\mathbf{I}'_B)$ by

$$R^n T(\mathbf{I}_A, \mathbf{I}'_B)(f) = T(g^n)^*.$$

In Case (Li), a lift $g$ of $f$ induces a chain map $T(g)$ between the homology complexes $T(\mathbf{I}'_B)$ and $T(\mathbf{I}_A)$. The map $T(g_n)_*\colon H_n(T(\mathbf{I}'_B)) \to H_n(T(\mathbf{I}_A))$ is a homomorphism of homology and we obtain a well-defined map $L_n T(\mathbf{I}'_B, \mathbf{I}_A)(f)\colon L_n T(\mathbf{I}'_B) \to L_n T(\mathbf{I}_A)$ (independent of the lifting $g$) given by

$$L_n T(\mathbf{I}'_B, \mathbf{I}_A)(f) = T(g_n)_*.$$

In Case (Lp) we use projective resolutions $\mathbf{P}^A \xrightarrow{\epsilon} A \longrightarrow 0$ and $\mathbf{P}'^B \xrightarrow{\epsilon'} B \longrightarrow 0$ and Proposition 12.13. In this case $T(g)$ is a chain map of homology from $T(\mathbf{P}^A)$ to $T(\mathbf{P}'^B)$, and $T(g_n)_*\colon H_n(T(\mathbf{P}^A)) \to H_n(T(\mathbf{P}'^B))$ is the induced map of homology. We obtain a well-defined map of homology (independent of the lifting $g$) $L_nT(\mathbf{P}^A, \mathbf{P}'^B)(f)\colon L_nT(\mathbf{P}^A) \to L_nT(\mathbf{P}'^B)$ given by

$$L_nT(\mathbf{P}^A, \mathbf{P}'^B)(f) = T(g_n)_*.$$

In Case (Rp), we use projective resolutions and Proposition 12.13. This time $T(g)$ is a chain map of cohomology from $T(\mathbf{P}'^B)$ to $T(\mathbf{P}^A)$ and $T(g^n)^*\colon H^n(T(\mathbf{P}'^B)) \to H^n(T(\mathbf{P}^A))$ is the induced map of cohomology. We obtain a well-defined map of cohomology (independent of the lifting $g$) $R^nT(\mathbf{P}'^B, \mathbf{P}^A)(f)\colon R^nT(\mathbf{P}'^B) \to R^nT(\mathbf{P}^A)$ given by

$$R^nT(\mathbf{P}'^B, \mathbf{P}^A)(f) = T(g^n)^*.$$

In summary we make the following definition.

**Definition 12.9.** Let $A, B \in \mathbf{C}$ be objects in $\mathbf{C}$ and let $f\colon A \to B$ be any map.

(Ri) If $0 \longrightarrow A \xrightarrow{\epsilon} \mathbf{I}_A$ is any injective resolution of $A$ and $0 \longrightarrow B \xrightarrow{\epsilon'} \mathbf{I}'_B$ is any injective resolution of $B$, then we define $R^nT(\mathbf{I}_A, \mathbf{I}'_B)(f)\colon R^nT(\mathbf{I}_A) \to R^nT(\mathbf{I}'_B)$ by

$$R^nT(\mathbf{I}_A, \mathbf{I}'_B)(f) = T(g^n)^*$$

for any lift $g$ of $f$. The map $T(g^n)^*\colon H^n(T(\mathbf{I}_A)) \to H^n(T(\mathbf{I}'_B))$ is independent of the lift $g$.

(Li) We define $L_nT(\mathbf{I}'_B, \mathbf{I}_A)(f)\colon L_nT(\mathbf{I}'_B) \to L_nT(\mathbf{I}_A)$ by

$$L_nT(\mathbf{I}'_B, \mathbf{I}_A)(f) = T(g_n)_*$$

for any lift $g$ of $f$. The map $T(g_n)_*\colon H_n(T(\mathbf{I}'_B)) \to H_n(T(\mathbf{I}_A))$ is independent of the lift $g$.

(Lp) If $\mathbf{P}^A \xrightarrow{\epsilon} A \longrightarrow 0$ is any projective resolution of $A$ and $\mathbf{P}'^B \xrightarrow{\epsilon'} B \longrightarrow 0$ is any projective resolution of $B$, then we define $L_nT(\mathbf{P}^A, \mathbf{P}'^B)(f)\colon L_nT(\mathbf{P}^A) \to L_nT(\mathbf{P}'^B)$ by

$$L_nT(\mathbf{P}^A, \mathbf{P}'^B)(f) = T(g_n)_*$$

for any lift $g$ of $f$. The map $T(g_n)_*\colon H_n(T(\mathbf{P}^A)) \to H_n(T(\mathbf{P}'^B))$ is independent of the lift $g$.

(Rp) We define $R^nT(\mathbf{P}'^B, \mathbf{P}^A)(f)\colon R^nT(\mathbf{P}'^B) \to R^nT(\mathbf{P}^A)$ by

$$R^nT(\mathbf{P}'^B, \mathbf{P}^A)(f) = T(g^n)^*$$

for any lift $g$ of $f$. The map $T(g^n)^*\colon H^n(T(\mathbf{P}'^B)) \to H^n(T(\mathbf{P}^A))$ is independent of the lift $g$.

It is an easy exercise to check that $R^n T$ and $L_n T$ are additive functors, contravariant in Cases (Li) and (Rp).

The next two theorems are absolutely crucial results. Indeed, they show that even though the objects $R^n T(\mathbf{I}_A)$ (and $L_n T(\mathbf{I}_A)$) depend on the injective resolution $\mathbf{I}_A$ chosen or $A$, this dependency is inessential because any other resolution $\mathbf{I}'_A$ for $A$ yields an object $R^n T(\mathbf{I}'_A)$ *isomorphic* to $R^n T(\mathbf{I}_A)$. Similarly if $\mathbf{P}^A$ and $\mathbf{P}'^A$ are two different resolutions for $A$ then $L_n T(\mathbf{P}^A)$ and $L_n T(\mathbf{P}'^A)$ are isomorphic. The key to these isomorphisms are the Comparison theorems. These isomorphisms are actually isomorphisms of functors known as natural transformations that we now define. A natural transformation is a simple generalization of the notion of morphism of presheaves.

**Definition 12.10.** Given two categories $\mathbf{C}$ and $\mathbf{D}$ and two functors $F, G \colon \mathbf{C} \to \mathbf{D}$ betwen them, a *natural transformation* $\eta \colon F \to G$ is a family $\eta = (\eta_A)_{A \in \mathbf{C}}$ of maps $\eta_A \colon F(A) \to G(A)$ in $\mathbf{D}$ such that the following diagrams commute for all maps $f \colon A \to B$ between objects $A, B \in \mathbf{C}$:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\eta_A} & G(A) \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\
F(B) & \xrightarrow[\eta_B]{} & G(B).
\end{array}
$$

We are now ready to state and prove our crucial theorems.

**Theorem 12.21.** *Let* $0 \longrightarrow A \xrightarrow{\epsilon_A} \mathbf{I}_A$ *and* $0 \longrightarrow A \xrightarrow{\epsilon'_A} \mathbf{I}'_A$ *be any two injective resolutions for any* $A \in \mathbf{C}$. *If* $T \colon \mathbf{C} \to \mathbf{D}$ *is any additive functor, then there are isomorphisms*

$$\eta_A^n \colon R^n T(\mathbf{I}_A) \to R^n T(\mathbf{I}'_A)$$

*for all* $n \geq 0$ *that depend only on* $A$ *and* $T$. *Furthermore, for any map* $f \colon A \to B$, *for any injective resolutions* $0 \longrightarrow B \xrightarrow{\epsilon_B} \mathbf{I}_B$ *and* $0 \longrightarrow B \xrightarrow{\epsilon'_B} \mathbf{I}'_B$ *of* $B$ *the following diagrams*

$$
\begin{array}{ccc}
R^n T(\mathbf{I}_A) & \xrightarrow{\eta_A^n} & R^n T(\mathbf{I}'_A) \\
{\scriptstyle R^n T(\mathbf{I}_A, \mathbf{I}_B)(f)}\downarrow & & \downarrow{\scriptstyle R^n T(\mathbf{I}'_A, \mathbf{I}'_B)(f)} \\
R^n T(\mathbf{I}_B) & \xrightarrow[\eta_B^n]{} & R^n T(\mathbf{I}'_B)
\end{array}
$$

*commute for all* $n \geq 0$.

If $T \colon \mathbf{C} \to \mathbf{D}$ *is a contravariant additive functor, then there are isomorphisms*

$$\eta_n^A \colon L_n T(\mathbf{I}_A) \to L_n T(\mathbf{I}'_A)$$

*for all $n \geq 0$ that depend only on $A$ and $T$. Furthermore, the following diagrams*

$$
\begin{array}{ccc}
L_n T(\mathbf{I}_B) & \xrightarrow{\ \eta_n^B\ } & L_n T(\mathbf{I}'_B) \\
\scriptstyle{L_n T(\mathbf{I}_B, \mathbf{I}_A)(f)} \downarrow & & \downarrow \scriptstyle{L_n T(\mathbf{I}'_B, \mathbf{I}'_A)(f)} \\
L_n T(\mathbf{I}_A) & \xrightarrow[\ \eta_n^A\ ]{} & L_n T(\mathbf{I}'_A)
\end{array}
$$

*commute for all $n \geq 0$.*

*Proof.* By Theorem 12.17 the complexes $\mathbf{I}_A$ and $\mathbf{I}'_A$ are homotopy equivalent, which means that there are chain maps $g \colon \mathbf{I}_A \to \mathbf{I}'_A$ and $h \colon \mathbf{I}'_A \to \mathbf{I}_A$ both lifting $\mathrm{id}_A$ such that $h \circ g$ is chain homotopic to $\mathrm{id}_{\mathbf{I}_A}$ and $g \circ h$ is chain homotopic to $\mathrm{id}_{\mathbf{I}'_A}$. Since $T$ is additive, $T(h) \circ T(g)$ is chain homotopic to $\mathrm{id}_{T(\mathbf{I}_A)}$ and $T(g) \circ T(h)$ is chain homotopic to $\mathrm{id}_{T(\mathbf{I}'_A)}$. These chain maps induce cohomology homomorphisms for all $n \geq 0$ and by Proposition 2.17, we obtain

$$
T(h^n)^* \circ T(g^n)^* = \mathrm{id}_{T(\mathbf{I}_A)}
$$
$$
T(g^n)^* \circ T(h^n)^* = \mathrm{id}_{T(\mathbf{I}'_A)}.
$$

Therefore, $T(g^n)^* \colon H^n(T(\mathbf{I}_A)) \to H^n(T(\mathbf{I}'_A))$ is an isomorphism of cohomology.

We still have to show that this map depends only on $T$ and $A$. This is because by Proposition 12.16, any two lifts $g$ and $g'$ of $\mathrm{id}_A$ are chain homotopic, so $T(g)$ and $T(g')$ are chain homotopic, and by Proposition 2.17 we have $T(g^n)^* = T(g'^n)^*$. As a consequence, it is legitimate to set $\eta_A^n = T(g^n)^*$, a well-defined isomorphism $\eta_A^n \colon R^n T(\mathbf{I}_A) \to R^n T(\mathbf{I}'_A)$.

Finally, we need to check that the $\eta_A^n$ yield a natural transformation. For any map $f \colon A \to B$ we need to show that the following diagram commutes:

$$
\begin{array}{ccc}
R^n T(\mathbf{I}_A) & \xrightarrow{\ \eta_A^n\ } & R^n T(\mathbf{I}'_A) \\
\scriptstyle{R^n T(\mathbf{I}_A, \mathbf{I}_B)(f)} \downarrow & & \downarrow \scriptstyle{R^n T(\mathbf{I}'_A, \mathbf{I}'_B)(f)} \\
R^n T(\mathbf{I}_B) & \xrightarrow[\ \eta_B^n\ ]{} & R^n T(\mathbf{I}'_B).
\end{array}
$$

The map $\eta_A^n$ is given by a lifting $g_A$ of $\mathrm{id}_A$ from $\mathbf{I}_A$ to $\mathbf{I}'_A$, and the map $R^n T(\mathbf{I}'_A, \mathbf{I}'_B)(f)$ is given by a lifting $h'$ of $f$ from $\mathbf{I}'_A$ to $\mathbf{I}'_B$. Thus $h' \circ g_A$ is a lifting of $f \circ \mathrm{id}_A = f$ from $\mathbf{I}_A$ to $\mathbf{I}'_B$. Similarly the map $\eta_B^n$ is given by a lifting $g_B$ of $\mathrm{id}_B$ from $\mathbf{I}_B$ to $\mathbf{I}'_B$, and the map $R^n T(\mathbf{I}_A, \mathbf{I}_B)(f)$ is given by a lifting $h$ of $f$ from $\mathbf{I}_A$ to $\mathbf{I}_B$. Thus $g_B \circ h$ is a lifting of $\mathrm{id}_A \circ f = f$ from $\mathbf{I}_A$ to $\mathbf{I}'_B$. Since $T$ is a functor, $T(h') \circ T(g_A)$ and $T(g_B) \circ T(h)$ both lift $T(f)$, and by Proposition 12.16 they are chain homotopic, so

$$
T(h'^n)^* \circ T(g_A^n)^* = T(g_B^n)^* \circ T(h^n)^*
$$

or equivalently

$$
R^n T(\mathbf{I}'_A, \mathbf{I}'_B)(f) \circ \eta_A^n = \eta_B^n \circ R^n T(\mathbf{I}_A, \mathbf{I}_B)(f)
$$

as desired. The proof in the case of a contravariant functor is similar. $\qquad \square$

We have a similar theorem for projective resolutions using Proposition 12.13 and Theorem 12.14 instead of Proposition 12.16 and Theorem 12.17.

**Theorem 12.22.** *Let* $\mathbf{P}^A \xrightarrow{\epsilon^A} A \longrightarrow 0$ *and* $\mathbf{P}'^A \xrightarrow{\epsilon'^A} A \longrightarrow 0$ *be any two projective resolutions for any* $A \in \mathbf{C}$. *If* $T\colon \mathbf{C} \to \mathbf{D}$ *is any additive functor, then there are isomorphisms*

$$\eta_n^A\colon L_nT(\mathbf{P}^A) \to L_nT(\mathbf{P}'^A)$$

*for all* $n \geq 0$ *that depend only on* $A$ *and* $T$. *Furthermore, for any map* $f\colon A \to B$, *for any projective resolutions* $\mathbf{P}^B \xrightarrow{\epsilon^B} B \longrightarrow 0$ *and* $\mathbf{P}'^B \xrightarrow{\epsilon'^B} B \longrightarrow 0$ *of* $B$, *the following diagrams*

$$
\begin{array}{ccc}
L_nT(\mathbf{P}^A) & \xrightarrow{\eta_n^A} & L_nT(\mathbf{P}'^A) \\
{\scriptstyle L_nT(\mathbf{P}^A,\mathbf{P}^B)(f)} \downarrow & & \downarrow {\scriptstyle L_nT(\mathbf{P}'^A,\mathbf{P}'^B)(f)} \\
L_nT(\mathbf{P}^B) & \xrightarrow[\eta_n^B]{} & L_nT(\mathbf{P}'^B)
\end{array}
$$

*commute for all* $n \geq 0$.

*If* $T\colon \mathbf{C} \to \mathbf{D}$ *is a contravariant additive functor, then there are isomorphisms*

$$\eta_A^n\colon R^nT(\mathbf{P}^A) \to R^nT(\mathbf{P}'^A)$$

*for all* $n \geq 0$ *that depend only on* $A$ *and* $T$. *Furthermore, the following diagrams*

$$
\begin{array}{ccc}
R^nT(\mathbf{P}^B) & \xrightarrow{\eta_B^n} & R^nT(\mathbf{P}'^B) \\
{\scriptstyle R^nT(\mathbf{P}^B,\mathbf{P}^A)(f)} \downarrow & & \downarrow {\scriptstyle R^nT(\mathbf{P}'^B,\mathbf{P}'^A)(f)} \\
R^nT(\mathbf{P}^A) & \xrightarrow[\eta_A^n]{} & R^nT(\mathbf{P}'^A)
\end{array}
$$

*commute for all* $n \geq 0$.

Theorem 12.21 and Theorem 12.22 suggest defining $R^nT$ and $L_nT$ as functors with domain $\mathbf{C}$ rather than projective or injective resolutions.

**Definition 12.11.** Let $\mathbf{C}$ and $\mathbf{D}$ be two abelian categories, and let $T\colon \mathbf{C} \to \mathbf{D}$ be an additive functor.

(Ri) Assume that $\mathbf{C}$ has enough injectives and for every object $A$ in $\mathbf{C}$ choose (once and for all) some injective resolution $0 \longrightarrow A \xrightarrow{\epsilon} \mathbf{I}_A$. The *right derived functors* $R^nT$ of $T$ are defined for every $A \in \mathbf{C}$ by

$$R^nT(A) = R^nT(\mathbf{I}_A) = H^n(T(\mathbf{I}_A)), \qquad n \geq 0,$$

and for every map $f\colon A \to B$, by

$$R^nT(f) = R^nT(\mathbf{I}_A, \mathbf{I}'_B)(f), \qquad n \geq 0.$$

(Li) If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor, then the *left derived functors* $L_nT$ of $T$ are defined for every $A \in \mathbf{C}$ by

$$L_nT(A) = L_nT(\mathbf{I}_A) = H_n(T(\mathbf{I}_A)), \qquad n \geq 0,$$

and for every map $f\colon A \to B$, by

$$L_nT(f) = L_nT(\mathbf{I}'_B, \mathbf{I}_A)(f), \qquad n \geq 0.$$

(Lp) Now assume that $\mathbf{C}$ has enough projectives and for every object $A$ in $\mathbf{C}$ choose (once and for all) some projective resolution $\mathbf{P}^A \xrightarrow{\ \epsilon\ } A \longrightarrow 0$ . The *left derived functors* $L_nT$ of $T$ are defined for every $A \in \mathbf{C}$ by

$$L_nT(A) = L_nT(\mathbf{P}^A) = H_n(T(\mathbf{P}^A)), \qquad n \geq 0,$$

and for every map $f\colon A \to B$, by

$$L_nT(f) = L_nT(\mathbf{P}^A, \mathbf{P}'^B)(f), \qquad n \geq 0.$$

(Rp) If $T\colon \mathbf{C} \to \mathbf{D}$ is a contravariant functor, then the *right derived functors* $R^nT$ of $T$ are defined for every $A \in \mathbf{C}$ by

$$R^nT(A) = R^nT(\mathbf{P}^A) = H^n(T(\mathbf{P}^A)), \qquad n \geq 0,$$

and for every map $f\colon A \to B$, by

$$R^nT(f) = R^nT(\mathbf{P}'^B, \mathbf{P}^A)(f), \qquad n \geq 0.$$

Observe that in (Li) and (Rp) the derived functors are contravariant. Any other choice of injective resolutions or projective resolutions yields derived functors $(\widehat{R}^nT)_{n\geq 0}$ and $(\widehat{L}_nT)_{n\geq 0}$ that are naturally isomorphic to the derived functors $(R^nT)_{n\geq 0}$ and $(L_nT)_{n\geq 0}$ associated to the original fixed choice of resolutions (in the sense that the $(\eta_A^n)_{A\in\mathbf{C}}$ and $(\eta_n^A)_{A\in\mathbf{C}}$ in Theorems 12.21 and 12.22 are natural transformations with all $\eta_A^n$ and all $\eta_n^A$ isomorphisms. For example, in Case (Ri), for all maps $f\colon A \to B$ we have the commutative diagrams

$$
\begin{array}{ccc}
R^nT(A) & \xrightarrow{\ \eta_A^n\ } & \widehat{R}^nT(A) \\
\Big\downarrow{\scriptstyle R^nT(f)} & & \Big\downarrow{\scriptstyle \widehat{R}^nT(f)} \\
R^nT(B) & \xrightarrow[\ \eta_B^n\ ]{} & \widehat{R}^nT(B)
\end{array}
$$

for all $n \geq 0$).

One of the main reasons for defining the derived functors $(R^n T)_{n \geq 0}$ and $(L_n T)_{n \geq 0}$ is to investigate properties of $T$, in particular how much does $T$ preserve exactness. For $T$ fixed, the objects $R^n T(A)$ (or $L_n T(A)$) (groups if $\mathbf{D}$ is the category of abelian goups) are important invariants of the object $A$.

It turns out that more useful information is obtained if either $R^0 T$ is isomorphic to $T$ or $L_0 T$ is isomorphic to $T$. The following proposition gives sufficient conditions for this to happen.

**Proposition 12.23.** *Let $\mathbf{C}$ and $\mathbf{D}$ be two abelian categories, and let $T \colon \mathbf{C} \to \mathbf{D}$ be an additive functor.*

(1) *If $T$ is left-exact then $R^0 T$ is naturally isomorphic to $T$. If $T$ is right-exact and contravariant then $L_0 T$ is naturally isomorphic to $T$.*

(2) *If $T$ is right-exact then $L_0 T$ is naturally isomorphic to $T$. If $T$ is left-exact and contravariant then $R^0 T$ is naturally isomorphic to $T$.*

*Proof.* (1) Let $0 \longrightarrow A \overset{\epsilon}{\longrightarrow} \mathbf{I}_A$ be an injective resolution of $A$. Since $T$ is left-exact we have the exact sequence

$$0 \longrightarrow T(A) \xrightarrow{T(\epsilon)} T(I_0) \xrightarrow{T(d^0)} T(I_1).$$

Since $T(\epsilon)$ is injective It follows that $T(A)$ is isomorphic to $\operatorname{Im} T(\epsilon) = \operatorname{Ker} T(d^0)$. The chain complex $T(\mathbf{I}_A)$ given by

$$0 \longrightarrow T(I_0) \xrightarrow{T(d^0)} T(I_1) \xrightarrow{T(d^1)} T(I^2) \longrightarrow \cdots$$

yields $R^0 T(A) = H^0(T(\mathbf{I}_A)) = \operatorname{Ker} T(d^0)$, so $T(A)$ is isomorphic to $R^0 T(A)$. We leave it as an exercise to show that these isomorphisms constitute a natural transformation. The case where $T$ is right-exact and contravariant is left as an exercise.

(2) Let $\mathbf{P}^A \overset{\epsilon}{\longrightarrow} A \longrightarrow 0$ be a projective resolution of $A$. Since $T$ is right-exact we have the exact sequence
$$0 \longleftarrow T(A) \xleftarrow{T(\epsilon)} T(P^0) \xleftarrow{T(d_1)} T(P^1).$$

Since $T(\epsilon)$ is surjective $T(A)$ is isomorphic to $T(P^0)/\operatorname{Ker} T(\epsilon) = T(P^0)/\operatorname{Im} T(d_1)$. The chain complex $T(\mathbf{P}^A)$ given by

$$0 \longleftarrow T(P^0) \xleftarrow{T(d_1)} T(P^1) \xleftarrow{T(d_2)} T(P^2) \longleftarrow \cdots$$

yields $L_0 T(A) = H_0(T(\mathbf{P}^A)) = T(P^0)/\operatorname{Im} T(d_1)$, so $T(A)$ is isomorphic to $L_0 T(A)$. We leave it as an exercise to show that these isomorphisms constitute a natural transformation. The case where $T$ is left-exact and contravariant is also left as an exercise. $\qquad \square$

**Remark:** We will show later that in Case (Ri) $R^0T$ is left-exact, in Case (Li) $L_0T$ is right-exact, in Case (Lp) $L_0T$ is right-exact, and in Case (Rp) $R^0T$ is left-exact. These properties also proved in Rotman [40]. As a consequence, the conditions of Proposition 12.23 are necessary and sufficient.

**Example 12.1.** We know that the contravariant functor $T_B(A) = \mathrm{Hom}(A, B)$ with $B$ fixed is left-exact. Its right derived functors are the "Ext" functors

$$\mathrm{Ext}_R^n(A, B) = (R^nT_B)(A),$$

with

$$\mathrm{Ext}_R^0(A, B) = \mathrm{Hom}(A, B).$$

We also know that the functor $T_A'(B) = \mathrm{Hom}(A, B)$ with $A$ fixed is left-exact. Its right derived functors are also "Ext" functors

$$\mathrm{Ext}_R^{'n}(A, B) = (R^nT_A')(B),$$

with

$$\mathrm{Ext}_R^{'0}(A, B) = \mathrm{Hom}(A, B).$$

It turns out that $\mathrm{Ext}_R^n(A, B)$ and $\mathrm{Ext}_R^{'n}(A, B)$ are isomorphic; see Rotman [40] (Chapter 7, Theorem 7.8).

The functor $T_B(A) = A \otimes B$ with $B$ fixed is right-exact. Its left derived functors are the "Tor" functors

$$\mathrm{Tor}_n^R(A, B) = (L_nT_B)(A),$$

with

$$\mathrm{Tor}_0^R(A, B) = A \otimes B.$$

Similarly the functor $T_A(B) = A \otimes B$ with $A$ fixed is right-exact. Its left derived functors are also the "Tor" functors

$$\mathrm{Tor}_n^{'R}(A, B) = (L_nT_A)(B),$$

with

$$\mathrm{Tor}_0^{'R}(A, B) = A \otimes B.$$

It turns out that $\mathrm{Tor}_n^R(A, B)$ and $\mathrm{Tor}_n^{'R}(A, B)$ are isomorphic; see Rotman [40] (Chapter 7, Theorem 7.9). It can be shown that for all $R$-modules $A$ and $B$, the $R$-module $\mathrm{Tor}_n^R(A, B)$ is a torsion module for all $n \geq 1$; see Rotman [40] (Chapter 8, Theorem 8.21).

Since Hom is not right-exact, its left derived functors convey no obvious information about Hom. Similarly, since $\otimes$ is not left-exact, its right derived functors convey no obvious information about it.

Although quite trivial the following proposition has significant implications, namely that the family of right derived functors $(R^nT)_{n \geq 0}$ are universal $\delta$-functors, and that the family of left derived functors $(L_nT)_{n \geq 0}$ are universal $\partial$-functors; See Section 12.4.

**Proposition 12.24.** *Let* **C** *and* **D** *be two abelian categories, and let* $T \colon \mathbf{C} \to \mathbf{D}$ *be an additive functor.*

(1) *For every injective object* $I$, *we have* $R^n T(I) = (0)$ *for all* $n \geq 1$, *and* $T(I)$ *is isomorphic to* $R^0 T(I)$. *If* $T$ *is contravariant we have* $L_n T(I) = (0)$ *for all* $n \geq 1$, *and* $T(I)$ *is isomorphic to* $L^0 T(I)$.

(2) *For every projective object* $P$, *we have* $L_n T(P) = (0)$ *for all* $n \geq 1$, *and* $T(P)$ *is isomorphic to* $L_0 T(P)$. *If* $T$ *is contravariant we have* $R^n T(P) = (0)$ *for all* $n \geq 1$, *and* $T(P)$ *is isomorphic to* $R^0 T(P)$.

*Proof.* (1) if $I$ is injective we can pick the resolution

$$0 \longrightarrow I \xrightarrow{\mathrm{id}} I \longrightarrow 0,$$

which yields the complex $T(\mathbf{I})$ given by

$$0 \longrightarrow T(I) \longrightarrow 0,$$

and obviously $R^0 T(I) = H^0(T(\mathbf{I})) = T(I)$ and $H^n(T(\mathbf{I})) = (0)$ for all $n \geq 1$. The proof for the other cases is similar and left as an exercise. $\qquad\square$

It should also be noted that if $T$ is an exact functor then $R^n T = (0)$ and $L_n T = (0)$ for all $n \geq 1$.

Proposition 12.24 implies that if $A$ or $B$ is a projective $R$-module (in particular, a free module), then

$$\mathrm{Tor}_n^R(A, B) = (0) \quad \text{for all } n \geq 1.$$

It can also be shown that the above property holds if $A$ or $B$ is a flat $R$-module; see Rotman [40] (Chapter 8, Theorem 8.7). Proposition 12.24 also implies that if $A$ is a projective $R$-module (in particular, a free module) or if $B$ is an injective $R$-module then

$$\mathrm{Ext}_R^n(A, B) = (0) \quad \text{for all } n \geq 1.$$

We now come to the most important properties of derived functors, that short-exact sequences yield long exact sequences of cohomology or homology.

**Theorem 12.25.** *(Long exact sequence, Case (Ri)) Assume the abelian category* **C** *has enough injectives, let* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *be an exact sequence in* **C**, *and let* $T \colon \mathbf{C} \to \mathbf{D}$ *be an additive left-exact functor.*

(1) *Then for every* $n \geq 0$, *there is a map*

$$(R^n T)(A'') \xrightarrow{\delta^n} (R^{n+1} T)(A'),$$

*and the sequence*

$$0 \longrightarrow T(A') \longrightarrow T(A) \xrightarrow{\delta^0} T(A'') \longrightarrow$$

$$\longrightarrow (R^1T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow (R^nT)(A') \longrightarrow (R^nT)(A) \xrightarrow{\delta^n} (R^nT)(A'') \longrightarrow$$

$$\longrightarrow (R^{n+1}T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots$$

*is exact.*

(2) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ *is another exact sequence in* **C**, *and if there is a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0,
\end{array}
$$

*then the induced diagram beginning with*

$$
\begin{array}{ccccccc}
0 & \longrightarrow & T(A') & \longrightarrow & T(A) & \longrightarrow & T(A'') \xrightarrow{\delta_A^0} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T(B') & \longrightarrow & T(B) & \longrightarrow & T(B'') \xrightarrow[\delta_B^0]{} 
\end{array}
$$

*and continuing with*

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & R^nT(A') & \longrightarrow & R^nT(A) & \longrightarrow & R^nT(A'') & \xrightarrow{\delta_A^n} & (R^{n+1}T)(A') & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & R^nT(B') & \longrightarrow & R^nT(B) & \longrightarrow & R^nT(B'') & \xrightarrow[\delta_B^n]{} & (R^{n+1}T)(B') & \longrightarrow \cdots
\end{array}
$$

*is also commutative.*

*Proof.* We have injective resolutions (from the collection of resolutions picked once and for all) $0 \longrightarrow A' \xrightarrow{\epsilon'} \mathbf{I}_{A'}$ and $0 \longrightarrow A'' \xrightarrow{\epsilon''} \mathbf{I}_{A''}$ for $A'$ and $A''$. We are in the situation where we can apply the Horseshose Lemma (Proposition 12.19) to obtain an injective resolution $0 \longrightarrow A \xrightarrow{\epsilon} \widehat{\mathbf{I}}_A$ for $A$ as illustrated in the following diagram in which all rows and columns

are exact:

$$
\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^1 & \longrightarrow & \widehat{I}^1 & \longrightarrow & I''^1 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^0 & \longrightarrow & \widehat{I}^0 & \longrightarrow & I''^0 & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle\epsilon'} & & \uparrow{\scriptstyle\epsilon} & & \uparrow{\scriptstyle\epsilon''} & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & & .
\end{array}
$$

Since all the rows are exact we obtain an exact sequence of complexes

$$
\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^2 & \longrightarrow & \widehat{I}^2 & \longrightarrow & I''^2 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^1 & \longrightarrow & \widehat{I}^1 & \longrightarrow & I''^1 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^0 & \longrightarrow & \widehat{I}^0 & \longrightarrow & I''^0 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & & .
\end{array}
$$

denoted by

$$
0 \longrightarrow \mathbf{I}_{A'} \longrightarrow \widehat{\mathbf{I}}_A \longrightarrow \mathbf{I}_{A''} \longrightarrow 0
$$

Observe that the injective resolution $\widehat{\mathbf{I}}_A$ for $A$ given by the Horseshoe Lemma may not be the original resolution that was picked originally and this is why it is denoted with hats. In the end, we will see that Theorem 12.21 implies that this does not matter.

If we apply $T$ to this complex we obtain another sequence of complexes

$$
0 \longrightarrow T(\mathbf{I}_{A'}) \longrightarrow T(\widehat{\mathbf{I}}_A) \longrightarrow T(\mathbf{I}_{A''}) \longrightarrow 0
$$

as illustrated below

$$
\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & T(I'^2) & \longrightarrow & T(\widehat{I}^2) & \longrightarrow & T(I''^2) & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & T(I'^1) & \longrightarrow & T(\widehat{I}^1) & \longrightarrow & T(I''^1) & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & T(I'^0) & \longrightarrow & T(\widehat{I}^0) & \longrightarrow & T(I''^0) & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
& 0 & & 0 & & 0 & .
\end{array}
$$

Because the $I'^n$ are injective and the rows

$$
0 \longrightarrow I'^n \longrightarrow \widehat{I}^n \longrightarrow I''^n \longrightarrow 0
$$

are exact, by Proposition 12.3 these sequence split and since $T$ is an additive functor the sequences

$$
0 \longrightarrow T(I'^n) \longrightarrow T(\widehat{I}^n) \longrightarrow T(I''^n) \longrightarrow 0
$$

also split and thus are exact. Therefore the sequence

$$
0 \longrightarrow T(\mathbf{I}_{A'}) \longrightarrow T(\widehat{\mathbf{I}}_A) \longrightarrow T(\mathbf{I}_{A''}) \longrightarrow 0
$$

is a short exact sequence, so our fundamental theorem applies (Theorem 2.19) and we obtain a long exact sequence of cohomology

$$
\begin{array}{l}
0 \longrightarrow H^0(T(\mathbf{I}_{A'})) \longrightarrow H^0(T(\widehat{\mathbf{I}}_A)) \longrightarrow H^0(T(\mathbf{I}_{A''})) \longrightarrow \\
\qquad \longrightarrow H^1(T(\mathbf{I}_{A'})) \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \\
\qquad \longrightarrow H^n(T(\mathbf{I}_{A'})) \longrightarrow H^n(T(\widehat{\mathbf{I}}_A)) \longrightarrow H^n(T(\mathbf{I}_{A''})) \longrightarrow \\
\qquad \longrightarrow H^{n+1}(T(\mathbf{I}_{A'})) \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots
\end{array}
$$

namely the following long exact sequence:

$$
\begin{array}{l}
0 \longrightarrow R^0 T(A') \longrightarrow \widehat{R}^0 T(A) \longrightarrow R^0 T(A'') \longrightarrow \\
\longrightarrow (R^1 T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \\
\longrightarrow (R^n T)(A') \longrightarrow (\widehat{R}^n T)(A) \longrightarrow (R^n T)(A'') \longrightarrow \\
\longrightarrow (R^{n+1} T)(A') \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots
\end{array}
$$

The right derived functors $\widehat{R}^n T$ may not be those corresponding to the original choice of injective resolutions but we can use Theorem 12.21 to replace it by the isomorphic derived functors $R^n T$ corresponding to the original choice of injective resolutions and adjust the isomorphisms. Since $T$ is left-exact, by Proposition 12.23 we may also replace the $R^0 T$ terms (as well as the $\widehat{R}^0 T$ terms) by $T$ and adjust the isomorphisms. After all this, we do obtain the promised long exact sequence.

To prove naturality we use Proposition 12.20. Assume we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0
\end{array}
$$

with exact rows. We have injective resolutions $0 \longrightarrow A' \xrightarrow{\epsilon^{A'}} \mathbf{I}_{A'}$, $0 \longrightarrow A'' \xrightarrow{\epsilon^{A''}} \mathbf{I}_{A''}$, $0 \longrightarrow B' \xrightarrow{\epsilon^{B'}} \mathbf{I}_{B'}$ and $0 \longrightarrow B'' \xrightarrow{\epsilon^{B''}} \mathbf{I}_{B''}$ of the corners $A', A'', B', B''$, and chain maps $A' \colon \mathbf{I}_{A'} \to \mathbf{I}_{B'}$ over $f'$ and $A'' \colon \mathbf{I}_{B''} \to \mathbf{I}_{B''}$ over $f''$. Then there exist injective resolutions $0 \longrightarrow A \xrightarrow{\epsilon^A} \widehat{\mathbf{I}}_A$ of $A$ and $0 \longrightarrow B \xrightarrow{\epsilon^B} \widehat{\mathbf{I}}_B$ of $B$ and a chain map $A \colon \widehat{\mathbf{I}}_A \to \widehat{\mathbf{I}}_B$ over $f$ such that the following diagram commmutes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{I}_{A'} & \longrightarrow & \widehat{\mathbf{I}}_A & \longrightarrow & \mathbf{I}_{A''} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle A'} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A''} & & \\
0 & \longrightarrow & \mathbf{I}_{B'} & \longrightarrow & \widehat{\mathbf{I}}_B & \longrightarrow & \mathbf{I}_{B''} & \longrightarrow & 0
\end{array}
$$

Since the $I_{A'}^n$ and the $I_{B'}^n$ are injective, every row of the diagram above splits, thus after

applying $T$ we obtain a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T(\mathbf{I}_{A'}) & \longrightarrow & T(\widehat{\mathbf{I}}_A) & \longrightarrow & T(\mathbf{I}_{A'}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle T(A')} & & \downarrow{\scriptstyle T(A)} & & \downarrow{\scriptstyle T(A'')} & & \\
0 & \longrightarrow & T(\mathbf{I}_{B'}) & \longrightarrow & T(\widehat{\mathbf{I}}_B) & \longrightarrow & T(\mathbf{I}_{B''}) & \longrightarrow & 0.
\end{array}
$$

We now conclude by applying Proposition 2.20 and replacing the terms $\widehat{R}^n T$ by $R^n T$ as we did before. $\qquad\square$

**Remark:** If $T$ is not left-exact, the proof of Theorem 12.25 shows that $R^0 T$ is left-exact.

A similar theorem holds for the left derived functors $L_n T$ of a (right-exact) functor; we obtain a long exact sequence of homology type involving the $L_n T$ applied to $A', A, A''$, and $L_0 T$ is right-exact.

**Theorem 12.26.** *(Long exact sequence, Case (Lp)) Assume the abelian category* $\mathbf{C}$ *has enough projectives, let* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *be an exact sequence in* $\mathbf{C}$, *and let* $T\colon \mathbf{C} \to \mathbf{D}$ *be an additive right-exact functor.*

*(1) Then for every $n \geq 1$, there is a map*

$$
(L_n T)(A'') \xrightarrow{\partial_n} (L_{n-1} T)(A'),
$$

*and the sequence*

$$
\begin{array}{l}
\cdots \longrightarrow L_n T(A') \longrightarrow L_n T(A) \longrightarrow L_n T(A'') \\
\qquad\qquad\qquad\qquad\qquad\quad {\scriptstyle \partial_n} \\
\qquad L_{n-1} T(A') \longrightarrow \cdots \qquad\qquad \cdots \\
\\
\qquad\qquad \cdots \qquad\qquad \cdots \longrightarrow L_1 T(A'') \\
\qquad\qquad\qquad\qquad\qquad {\scriptstyle \partial_1} \\
\qquad T(A') \longrightarrow T(A) \longrightarrow T(A'') \longrightarrow 0
\end{array}
$$

*is exact.*

*(2) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is another exact sequence in $\mathbf{C}$, and if there is a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0,
\end{array}
$$

*then the induced diagram*

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & (L_nT)(A') & \longrightarrow & (L_nT)(A) & \longrightarrow & (L_nT)(A'') & \xrightarrow{\partial_n^A} & (L_{n-1}T)(A') & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & (L_nT)(B') & \longrightarrow & (L_nT)(B) & \longrightarrow & (L_nT)(B'') & \xrightarrow[\partial_n^B]{} & (L_{n-1}T)(B') & \longrightarrow \cdots
\end{array}
$$

*and ending with*

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & L_1T(A'') & \xrightarrow{\partial_1^A} & T(A') & \longrightarrow & T(A) & \longrightarrow & T(A'') & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & L_1T(B'') & \xrightarrow[\partial_1^B]{} & T(B') & \longrightarrow & T(B) & \longrightarrow & T(B'') & \longrightarrow 0
\end{array}
$$

*is also commutative.*

**Remark:** If $T$ is not right-exact, the proof of Theorem 12.25 shows that $L_0T$ is right-exact.

If $\mathbf{C}$ has enough injectives and $T$ is a contravariant (right-exact) functor, we have a version of Theorem 12.26 showing that there is a long-exact sequence of homology type involving the $L_nT$ applied to $A', A, A''$, with the terms $A', A, A''$ appearing in reverse order (Case (Li)). As a consequence, $L_0T$ is right-exact. This case does not seem to arise in practice.

If $\mathbf{C}$ has enough projectives and $T$ is a contravariant (left-exact) functor, we have a version of Theorem 12.25 showing that there is a long-exact sequence of cohomology type involving the $R^nT$ applied to $A', A, A''$ with the terms $A', A, A''$ appearing in reverse order (Case (Rp)). As a consequence, $R_0T$ is left-exact.

Remember: Right derived functors go with left-exact functors; left derived functors go with right-exact functors.

There are situations (for example, when dealing with sheaves) where it is useful to know that right derived functors can be computed by resolutions involving objects that are not necessarily injective, but $T$-acyclic, as defined below.

**Definition 12.12.** Given a left-exact functor $T\colon \mathbf{C} \to \mathbf{D}$, an object $J \in \mathbf{C}$ is *T-acyclic* if $R^nT(J) = (0)$ for all $n \geq 1$.

The following proposition shows that right derived functors can be computed using $T$-acyclic resolutions.

**Proposition 12.27.** *Given an additive left-exact functor $T\colon \mathbf{C} \to \mathbf{D}$, for any $A \in \mathbf{C}$ suppose there is an exact sequence*

$$ 0 \longrightarrow A \xrightarrow{\;\epsilon\;} J^0 \xrightarrow{\;d^0\;} J^1 \xrightarrow{\;d^1\;} J^2 \xrightarrow{\;d^2\;} \cdots \tag{\dagger} $$

*in which every $J^n$ is T-acyclic (a right T-acyclic resolution $\mathbf{J}^A$). Then for every $n \geq 0$ we have a natural isomorphism between $R^nT(A)$ and $H^n(T(\mathbf{J}_A))$.*

*Proof.* The proof is a good illustration of the use of the long exact sequence given by Theorem 12.25. First, observe that if

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \longrightarrow 0$$

is a short exact sequence and if $T$ is left-exact, then $\operatorname{Ker} T(g) \cong T(\operatorname{Ker} g)$.

*Proof.* Since the above is a short exact sequence

$$A \cong \operatorname{Im} f = \operatorname{Ker} g,$$

and as $T$ is a functor

$$T(A) \cong T(\operatorname{Ker} g).$$

Since $T$ is left-exact we obtain the exact sequence

$$0 \longrightarrow T(A) \xrightarrow{\;T(f)\;} T(B) \xrightarrow{\;T(g)\;} T(C),$$

so

$$T(A) \cong \operatorname{Im} T(f) = \operatorname{Ker} T(g),$$

and thus

$$\operatorname{Ker} T(g) \cong T(A) \cong T(\operatorname{Ker} g),$$

as claimed. $\qquad\square$

Since ($\dagger$) is exact and $T$ is left-exact we obtain the exact sequence

$$0 \longrightarrow T(A) \xrightarrow{\;T(\epsilon)\;} T(J^0) \xrightarrow{\;T(d^0)\;} T(J^1),$$

which implies that

$$R^0 T(A) \cong T(A) \cong \operatorname{Ker} T(d^0) = H^0(T(\mathbf{J}_A)).$$

Let $K^n = \operatorname{Ker} d^n$ for all $n \geq 1$. The exact sequence ($\dagger$) implies that $\operatorname{Im} d^n = \operatorname{Ker} d^{n+1} = K^{n+1}$ and the surjection $p^n \colon J^n \to K^{n+1}$ has kernel $K^n$ so we have the short exact sequence

$$0 \longrightarrow K^n \longrightarrow J^n \xrightarrow{\;p^n\;} K^{n+1} \longrightarrow 0 \qquad\qquad (*)$$

for all $n \geq 1$. We also have the short exact sequence

$$0 \longrightarrow A \longrightarrow J^0 \xrightarrow{\;p^0\;} K^1 \longrightarrow 0. \qquad\qquad (**)$$

If we denote the injection of $K^{n+1}$ into $J^{n+1}$ by $\epsilon^{n+1}$, then we can factor $d^n$ as

$$d^n = \epsilon^{n+1} \circ p^n.$$

We have the following commutative diagram



If we apply $T$ we get

$$T(d^n) = T(\epsilon^{n+1}) \circ T(p^n).$$

Since $\epsilon^{n+1}$ is injective, the sequence $0 \longrightarrow K^{n+1} \xrightarrow{\epsilon^{n+1}} J^{n+1}$ is exact, and since $T$ is left exact we see that $0 \longrightarrow T(K^{n+1}) \xrightarrow{T(\epsilon^{n+1})} T(J^{n+1})$ is also exact, so $T(\epsilon^{n+1})$ is injective. It follows that the restriction of $T(\epsilon^{n+1})$ to $\operatorname{Im} T(p^n)$ is an isomorphism onto the image of $T(d^n)$, which implies that

$$\operatorname{Im} T(d^n) \cong \operatorname{Im} T(p^n), \qquad n \geq 0.$$

If we apply Theorem 12.25 to $(**)$, the long exact sequence begins with

$$0 \longrightarrow T(A) \longrightarrow T(J^0) \xrightarrow{T(p^0)} T(K^1) \longrightarrow R^1 T(A) \longrightarrow R^1 T(J^0) = (0),$$

which yields

$$R^1 T(A) \cong T(K^1)/\operatorname{Im} T(p^0) = T(\operatorname{Ker} d^1)/\operatorname{Im} T(p^0) \cong \operatorname{Ker} T(d^1)/\operatorname{Im} T(d^0) = H^1(T(\mathbf{J}_A)).$$

So far, we proved that $R^0 T(A) \cong H^0(T(\mathbf{J}_A))$ and $R^1 T(A) \cong H^1(T(\mathbf{J}_A))$. To prove that $R^n T(A) \cong H^n(T(\mathbf{J}_A))$ for $n \geq 2$ again we use the long exact sequence applied to $(**)$, which gives

$$R^{n-1} T(J^0) \longrightarrow R^{n-1} T(K^1) \longrightarrow R^n T(A) \longrightarrow R^n T(J^0),$$

and since $J^0$ is $T$-acyclic $R^{n-1} T(J^0) = R^n T(J^0) = (0)$ for $n \geq 2$, so we obtain isomorphisms

$$R^{n-1} T(K^1) \cong R^n T(A), \qquad n \geq 2.$$

The long exact sequence applied to $(*)$ yields

$$R^{n-i-1} T(J^i) \longrightarrow R^{n-i-1} T(K^{i+1}) \longrightarrow R^{n-i} T(K^i) \longrightarrow R^{n-i} T(J^i),$$

and since $J^i$ is $T$-acyclic $R^{n-i-1} T(J^i) = R^{n-i} T(J^i) = (0)$ so we have the isomorphisms

$$R^{n-i-1} T(K^{i+1}) \cong R^{n-i} T(K^i), \qquad 1 \leq i \leq n - 2.$$

By induction we obtain

$$R^{n-1}T(K^1) \cong R^1T(K^{n-1}), \qquad n \geq 2.$$

However, we showed that $R^{n-1}T(K^1) \cong R^nT(A)$, so we obtain

$$R^nT(A) \cong R^{n-1}T(K^1) \cong R^1T(K^{n-1}).$$

The long exact sequence applied to $(*)$ yields

$$T(J^{n-1}) \xrightarrow{T(p^{n-1})} T(K^n) \longrightarrow R^1T(K^{n-1}) \longrightarrow R^1T(J^{n-1}) = (0)$$

which implies that

$$
\begin{aligned}
R^nT(A) &\cong R^1T(K^{n-1}) \\
&\cong T(K^n)/\operatorname{Im} T(p^{n-1}) \\
&= T(\operatorname{Ker} d^n)/\operatorname{Im} T(p^{n-1}) \\
&\cong \operatorname{Ker} T(d^n)/\operatorname{Im} T(d^{n-1}) = H^n(T(\mathbf{J}_A)).
\end{aligned}
$$

Therefore we proved that $R^nT(A) \cong H^n(T(\mathbf{J}_A))$ for all $n \geq 0$, as claimed.  $\square$

Another proof of Proposition 12.27 can be found in Lang [28] (Chapter XX, §6, Theorem 6.2).

A similar proposition holds for left $T$-acyclic resolutions and the left derived functors $L_nT$.

Proposition 12.27 has an interesting application to de Rham cohomology. Say $M$ is a smooth manifold. Recall that for every $p \geq 0$ we have the sheaf $\mathcal{A}_M^p$ of differential forms on $M$ (where for every open subset $U$ of $M$, $\mathcal{A}_M^p(U) = \mathcal{A}^p(U)$ is the vector space of smooth $p$-forms on $U$).

**Proposition 12.28.** *If $\widetilde{\mathbb{R}}_M$ denotes the sheaf of locally constant real-valued functions on a smooth manifold $M$, then*

$$0 \longrightarrow \widetilde{\mathbb{R}}_M \xrightarrow{\epsilon} \mathcal{A}_M^0 \xrightarrow{d} \mathcal{A}_M^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_M^p \xrightarrow{d} \mathcal{A}_M^{p+1} \xrightarrow{d} \cdots$$

*is a resolution of $\widetilde{\mathbb{R}}_M$, where $\epsilon$ is the inclusion map.*

*Proof.* The above fact is proved using Proposition 11.23(ii) by showing that for every $x \in M$, the stalk complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}_{M,x}^0 \longrightarrow \mathcal{A}_{M,x}^1 \longrightarrow \cdots \longrightarrow \mathcal{A}_{M,x}^p \longrightarrow \mathcal{A}_{M,x}^{p+1} \longrightarrow \cdots$$

is exact. Since $M$ is a smooth manifold, we may assume that $M$ is an open subset of $\mathbb{R}^n$, and use a fundamental system of convex open neighborhoods of $x$ to compute the direct limit $\mathcal{A}^p_{M,x} = \varinjlim (\mathcal{A}^p(U))_{U \ni x}$. If $U$ is convex, the complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}^0(U) \longrightarrow \mathcal{A}^1(U) \longrightarrow \cdots \longrightarrow \mathcal{A}^p(U) \longrightarrow \mathcal{A}^{p+1}(U) \longrightarrow \cdots$$

is exact by the Poincaré lemma (Proposition 3.1). Since a direct limit of exact sequences is exact, we conclude that

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}^0_{M,x} \longrightarrow \mathcal{A}^1_{M,x} \longrightarrow \cdots \longrightarrow \mathcal{A}^p_{M,x} \longrightarrow \mathcal{A}^{p+1}_{M,x} \longrightarrow \cdots$$

is exact. For details, see Brylinski [6] (Section 1.4, Proposition 1.4.3). $\qquad\square$

If $\Gamma(M, -)$ is the global section functor with $\Gamma(M, \mathcal{A}^p_M) = \mathcal{A}^p(M)$, then it can also be shown that the sheaves $\mathcal{A}^p_M$ are $\Gamma(M, -)$-acyclic. This is because the sheaves $\mathcal{A}^p_M$ are soft, and soft sheaves on a paracompact space are $\Gamma(M, -)$-acyclic; see Godement [18] (Chapter 3, Section 3.9), or Brylinski [6] (Section 1.4, Theorem 1.4.6 and Proposition 1.4.9), or Section 13.5.

Now, it is also true that sheaves have enough injectives (we will see this in the next chapter). Therefore, we conclude that the cohomology groups $R^p\Gamma(M, -)(\widetilde{\mathbb{R}}_M)$ and the de Rham cohomology groups $H^p_{\mathrm{dR}}(M)$ are isomorphic. The groups $R^p\Gamma(M, -)(\widetilde{\mathbb{R}}_M)$ are called the *sheaf cohomology groups* of the sheaf $\widetilde{\mathbb{R}}_M$ and are denoted by $H^p(M, \widetilde{\mathbb{R}}_M)$. We will also show in the next chapter that for a paracompact space $M$, the Čech cohomology groups $\check{H}^p(M, \mathcal{F})$ and the sheaf cohomology groups $H^p(M, \mathcal{F}) = R^p\Gamma(M, -)(\mathcal{F})$ are isomorphic (where $\Gamma(M, -)$ is the global section functor, $\Gamma(M, \mathcal{F}) = \mathcal{F}(M)$); thus, for smooth manifolds we have isomorphisms

$$H^p(M, \widetilde{\mathbb{R}}_M) \cong \check{H}^p(M, \widetilde{\mathbb{R}}_M) \cong H^p_{\mathrm{dR}}(M),$$

proving part of Theorem 10.4.

Theorems 12.25 and 12.26 suggest the definition of families of functors originally proposed by Cartan and Eilenberg [7] and then investigated by Grothendieck in his legendary "Tohoku" paper [21] (1957).

## 12.4 Universal $\delta$-Functors and $\partial$-Functors

In his famous Tohoku paper [21] Grothendieck introduced the terminology "$\partial$-functor" and "$\partial^*$-functor;" see Chapter II, Section 2.1. The notion of $\partial$-functor is a slight generalization of the notion of "connected sequence of functors" introduced earlier by Cartan and Eilenberg [7] (Chapter 3). Since $\partial$-functor have a cohomological flavor and $\partial^*$-functor have a homological flavor, everybody now appears to use the terminology $\delta$-functor instead of $\partial$-functor and $\partial$-functor for $\partial^*$-functor.

**Definition 12.13.** Given two abelian categories $\mathbf{C}$ and $\mathbf{D}$, a $\delta$-*functor* consists of a countable family $T = (T^n)_{n \geq 0}$ of additive functors $T^n \colon \mathbf{C} \to \mathbf{D}$, and for every short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in the abelian category $\mathbf{C}$ and every $n \geq 0$ of a map

$$T^n(A'') \xrightarrow{\ \delta^n\ } T^{n+1}(A')$$

such that the following two properties hold:

(i) The sequence



is exact (a long exact sequence).

(ii) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is another exact sequence in $\mathbf{C}$, and if there is a commutative diagram



then the induced diagram beginning with



and continuing with



is also commutative.

In particular, $T^0$ is left-exact.

The notion of morphism of $\delta$-functors is defined as follows.

**Definition 12.14.** Given two $\delta$-functors $S = (S^n)_{n \geq 0}$ and $T = (T^n)_{n \geq 0}$ a *morphism* $\eta \colon S \to T$ between $S$ and $T$ is a family $\eta = (\eta^n)_{n \geq 0}$ of natural transformations $\eta^n \colon S^n \to T^n$ such that the following diagrams commute

$$
\begin{array}{ccc}
S^n(A'') & \xrightarrow{\ \delta_S^n\ } & S^{n+1}(A') \\
{\scriptstyle (\eta^n)_{A''}}\Big\downarrow & & \Big\downarrow{\scriptstyle (\eta^{n+1})_{A'}} \\
T^n(A'') & \xrightarrow[\ \delta_T^n\ ]{} & T^{n+1}(A')
\end{array}
$$

for all $n \geq 0$ and for every short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$.

Morphisms of $\delta$-functors are composed in the obvious way. The notion of isomorphism is also obvious (each $\eta^n$ is an an isomorphism). Grothendieck introduced the important notion of universal $\delta$-functor; see Grothendieck [21] (Chapter II, Section 2.2).

**Definition 12.15.** A $\delta$-functor $T = (T^n)_{n \geq 0}$ is *universal* if for every $\delta$-functor $S = (S^n)_{n \geq 0}$ and every natural transformation $\varphi \colon T^0 \to S^0$ there is a *unique* morphism $\eta \colon T \to S$ such that $\eta^0 = \varphi$; we say that $\eta$ *lifts* $\varphi$.

**Proposition 12.29.** *Suppose $S = (S^n)_{n \geq 0}$ and $T = (T^n)_{n \geq 0}$ are both universal $\delta$-functors and there is an isomorphism $\varphi \colon S^0 \to T^0$ (a natural transformation $\varphi$ which is an isomorphism). Then, there is a unique isomorphism $\eta \colon S \to T$ lifting $\varphi$.*

*Proof.* Since $\varphi$ is an isomorphism, it has an inverse $\psi \colon T^0 \to S^0$, that is, we have $\psi \circ \varphi = \mathrm{id}_{S^0}$ and $\varphi \circ \psi = id_{T^0}$. Since $S$ is universal there is a unique lift $\eta \colon S \to T$ of $\varphi$ and since $T$ is universal there is a unique lift $\theta \colon T \to S$ of $\psi$. But $\theta \circ \eta$ lifts $\psi \circ \varphi = \mathrm{id}_{S^0}$ and $\eta \circ \theta$ lifts $\varphi \circ \psi = \mathrm{id}_{T^0}$. However, $\mathrm{id}_S$ is a lift of $\mathrm{id}_{S^0}$ and $\mathrm{id}_T$ is a lift of $\mathrm{id}_{T^0}$, so by uniqueness of lifts we must have $\theta \circ \eta = \mathrm{id}_S$ and $\eta \circ \theta = \mathrm{id}_T$, which shows that $\eta$ is an isomorphism. $\quad\square$

Proposition 12.29 shows a significant property of a universal $\delta$-functor $T$: it is completely determined by the component $T^0$.

One might wonder whether (universal) $\delta$-functors exist. Indeed there are plenty of them.

**Theorem 12.30.** *Assume the abelian category $\mathbf{C}$ has enough injectives. For every additive left-exact functor $T \colon \mathbf{C} \to \mathbf{D}$, the family $(R^n T)_{n \geq 0}$ of right derived functors of $T$ is a $\delta$-functor. Furthermore $T$ is isomorphic to $R^0 T$.*

*Proof.* Now that we have done all the hard work the proof is short: apply Theorem 12.25. The second property follows from Proposition 12.23. $\quad\square$

In fact, the $\delta$-functors $(R^n T)_{n \geq 0}$ are universal. Before explaining the technique due to Grothendieck for proving this fact, let us take a quick look at $\partial$-functors.

**Definition 12.16.** Given two abelian categories $\mathbf{C}$ and $\mathbf{D}$, a $\partial$-*functor* consists of a countable family $T = (T_n)_{n \geq 0}$ of additive functors $T_n \colon \mathbf{C} \to \mathbf{D}$, and for every short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in the abelian category $\mathbf{C}$ and every $n \geq 1$ of a map

$$T_n(A'') \xrightarrow{\partial_n} T_{n-1}(A')$$

such that the following two properties hold:

(i) The sequence



is exact.

(ii) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is another exact sequence in $\mathbf{C}$, and if there is a commutative diagram



then the induced diagram



and ending with



is also commutative.

In particular, $T_0$ is right-exact.

**Definition 12.17.** Given two $\partial$-functors $S = (S_n)_{n \geq 0}$ and $T = (T_n)_{n \geq 0}$ a *morphism* $\eta \colon S \to T$ between $S$ and $T$ is a family $\eta = (\eta_n)_{n \geq 0}$ of natural transformations $\eta_n \colon S_n \to T_n$ such that the following diagrams commute

$$
\begin{array}{ccc}
S_n(A'') & \xrightarrow{\ \partial_n^S\ } & S_{n-1}(A') \\
{\scriptstyle (\eta_n)_{A''}}\Big\downarrow & & \Big\downarrow{\scriptstyle (\eta_{n-1})_{A'}} \\
T_n(A'') & \xrightarrow[\ \partial_n^T\ ]{} & T_{n-1}(A')
\end{array}
$$

for all $n \geq 1$ and for every short exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$.

Morphisms of $\partial$-functors are composed in the obvious way. The notion of isomorphism is clear (each $\eta_n$ is an an isomorphism). Grothendieck also introduced the important notion of universal $\partial$-functor; see Grothendieck [21] (Chapter II, Section 2.2).

**Definition 12.18.** A $\partial$-functor $T = (T_n)_{n \geq 0}$ is *universal* if for every $\partial$-functor $S = (S_n)_{n \geq 0}$ and every natural transformation $\varphi \colon S_0 \to T_0$ there is a *unique* morphism $\eta \colon S \to T$ such that $\eta_0 = \varphi$; we say that $\eta$ *lifts* $\varphi$.

**Proposition 12.31.** *Suppose $S = (S_n)_{n \geq 0}$ and $T = (T_n)_{n \geq 0}$ are both universal $\partial$-functors and there is an isomorphism $\varphi \colon S_0 \to T_0$ (a natural transformation $\varphi$ which is an isomorphism). Then, there is a unique isomorphism $\eta \colon S \to T$ lifting $\varphi$.*

The proof of Proposition 12.31 is the same as the proof of Proposition 12.29. Proposition 12.31 shows a significant property of a universal $\partial$-functor $T$: it is completely determined by the component $T_0$.

There are plenty of (universal) $\partial$-functors.

**Theorem 12.32.** *Assume the abelian category $\mathbf{C}$ has enough projectives. For every additive right-exact functor $T \colon \mathbf{C} \to \mathbf{D}$, the family $(L_n T)_{n \geq 0}$ of left derived functors of $T$ is a $\partial$-functor. Furthermore $T$ is isomorphic to $L_0 T$.*

*Proof.* Now that we have done all the hard work the proof is short: apply Theorem 12.26. The second property follows from Proposition 12.23.  □

Grothendieck came up with an ingenious sufficient condition for a $\delta$-functor to be universal: the notion of an *erasable* functor. Since Grothendieck's paper is written in French, this notion defined in Section 2.2 (page 141) of [21] is called *effaçable*, and many books and paper use it. Since the English translation of "effaçable" is "erasable," as advocated by Lang we will use the the English word.

**Definition 12.19.** An additive functor $T \colon \mathbf{C} \to \mathbf{D}$ is *erasable* (or *effaçable*) if for every object $A \in \mathbf{C}$ there is some object $M_A$ and an injection $u \colon A \to M_A$ such that $T(u) = 0$. In particular this will be the case if $T(M_A)$ is the zero object of $\mathbf{D}$. We say that $T$ *coerasable* (or *coeffaçable*) if for every object $A \in \mathbf{C}$ there is some object $M_A$ and a surjection $u \colon M_A \to A$ such that $T(u) = 0$.

In many cases if $T$ is erasable by injectives (which means that $M_A$ can be chosen to be injective) and $T$ is coerasable by projectives (which means that $M_A$ can be chosen to be projective). However, this is not always desirable.

The following proposition shows that our favorite functors, namely right derived functors, are erasable functors (and left derived functors are coerasable by projectives).

**Proposition 12.33.** *Assume the abelian category* $\mathbf{C}$ *has enough injectives. For every additive (left-exact) functor* $T \colon \mathbf{C} \to \mathbf{D}$ *the right derived functors* $R^n T$ *are erasable by injectives for all* $n \geq 1$. *Assume the abelian category* $\mathbf{C}$ *has enough projectives. For every additive (right-exact) functor* $T \colon \mathbf{C} \to \mathbf{D}$ *the left derived functors* $L_n T$ *are coerasable by projectives for all* $n \geq 1$.

*Proof.* For every $A \in \mathbf{C}$ there is an injection $u \colon A \to I$ into some injective $I$. Applying $R^n T$ we get a map $R^n T(u) \colon R^n T(A) \to R^n T(I)$, but by Proposition 12.24 we have $R^n T(I) = (0)$ for all $n \geq 1$. The proof in the projective case is similar and left as an exercise. □

The following theorem shows the significance of the seemingly strange notion of erasable functor.

**Theorem 12.34.** *(Grothendieck) Let* $T = (T^n)_{n \geq 0}$ *be a δ-functor between two abelian categories* $\mathbf{C}$ *and* $\mathbf{D}$. *If* $T^n$ *is erasable for all* $n \geq 1$, *then* $T$ *is a universal δ-functor.*

*Proof Idea.* Theorem 12.34 is Proposition 2.2.1 on page 141 of Grothendieck's Tohoku [21]. The proof takes two third of a page. Even if you read French, you are likely to be frustrated. All the pieces are there but as Grothendieck says

"Des raisonnements standarts montrent que le morphisme ainsi défini ne dépend pas du choix particulier de la suite exacte $0 \longrightarrow A \longrightarrow M \longrightarrow A' \longrightarrow 0$, puis le fait que ce morphisme est fonctoriel, et "permute à $\partial$.""

Roughly translated, the above says that the details constitute "standard reasoning." No doubt that experts in the field will have no trouble supplying the details but for the rest of us, where is a complete proof?

Let us explain the beginning of the proof. The proof is by induction on $n$; we shall treat only the case $n = 1$; the other cases are very similar. Let $S = (S^n)_{n \geq 0}$ be another δ-functor and let $u_0 \colon T^0 \to S^0$ be a given map of functors. If $A$ is an object of $\mathbf{C}$, the erasability of $T^1$ shows that there is an exact sequence

$$0 \longrightarrow A \overset{v}{\longrightarrow} M_A \overset{p}{\longrightarrow} A'' \longrightarrow 0 \tag{†}$$

such that the map $\delta_T$ in the induced sequence

$$T^0(M_A) \xrightarrow{T(p)} T^0(A'') \xrightarrow{\delta_T} T^1(A) \xrightarrow{0} T^1(M_A)$$

is surjective. Since $T$ is a $\delta$-functor we have the commutative diagram

$$
\begin{array}{ccccccc}
T^0(M_A) & \xrightarrow{T(p)} & T^0(A'') & \xrightarrow{\delta_T} & T^1(A) & \xrightarrow{0} & T^1(M_A) \\
\downarrow{\scriptstyle u_0(M_A)} & & \downarrow{\scriptstyle u_0(A'')} & & \downarrow{\scriptstyle u_1} & & \\
S^0(M_A) & \xrightarrow[S(p)]{} & S^0(A'') & \xrightarrow[\delta_S]{} & S^1(A), & &
\end{array}
$$

Since $\operatorname{Ker}\delta_T = \operatorname{Im}T(p)$, since the left square commutes

$$u_0(A'') \circ T(p) = S(p) \circ u_0(M_A),$$

and since the bottom row is exact, we get

$$\delta_S \circ u_0(A'') \circ T(p) = \delta_S \circ S(p) \circ u_0(M_A) = 0,$$

which proves that

$$\operatorname{Ker}\delta_T \subseteq \operatorname{Ker}\delta_S \circ u_0(A'').$$

Therefore there is a unique map $u_1 \colon T^1(A) \to S^1(A)$ making the second square commute. It remains to check that $u_1$ has the required properties and that it does not depend on the choice of the exact sequence (†). Lang [28] actually spells out most of the details but leaves out the verification that the argument does not depend on choice of the short exact sequence; see Chapter XX, §7, Theorem 7.1. □

**Remark:** There is a version of Theorem 12.34 for a contravariant $\partial$-functor which is erasable.

Combining Theorem 12.34 and Theorem 12.30 we obtain the most important result of this chapter.

**Theorem 12.35.** *Assume the abelian category* **C** *has enough injectives. For every additive left-exact functor* $T \colon \mathbf{C} \to \mathbf{D}$, *the right derived functors* $(R^n T)_{n \geq 0}$ *form a universal $\delta$-functor such that $T$ is isomorphic to $R^0 T$. Conversely, every universal $\delta$-functor $T = (T^n)_{n \geq 0}$ is isomorphic to the right derived $\delta$-functor* $(R^n T^0)_{n \geq 0}$.

*Proof.* The first statement is obtained by Combining Theorem 12.34 and Theorem 12.30. Conversely, if $T = (T^n)_{n \geq 0}$ is a universal $\delta$-functor, then $T^0$ is left-exact, so by the first part of the theorem applied to $T^0$, $(R^n T^0)_{n \geq 0}$ is a universal $\delta$-functor with $R^0 T^0$ isomorphic to $T^0$, thus $T$ and $(R^n T^0)_{n \geq 0}$ are isomorphic by Proposition 12.29. □

After all, the mysterious universal $\delta$-functors are just the right derived functors of left-exact functors. As an example, the functors $\operatorname{Ext}_R^n(A, -)$ constitute a universal $\delta$-functor (for any fixed $R$-module $A$).

Of course there is a version of Theorem 12.34 for coerasable $\partial$-functors.

**Theorem 12.36.** *(Grothendieck) Let $T = (T_n)_{n \geq 0}$ be a $\partial$-functor between two abelian categories $\mathbf{C}$ and $\mathbf{D}$. If $T_n$ is coerasable for all $n \geq 1$, then $T$ is a universal $\partial$-functor.*

**Remark:** There is a version of Theorem 12.36 for a contravariant $\delta$-functor which is coerasable.

Combining Theorem 12.36 and Theorem 12.32 we obtain the other most important result of this section.

**Theorem 12.37.** *Assume the abelian category $\mathbf{C}$ has enough projectives. For every additive right-exact functor $T \colon \mathbf{C} \to \mathbf{D}$ the left derived functors $(L_n T)_{n \geq 0}$ form a universal $\partial$-functor such that $T$ is isomorphic to $L_0 T$. Conversely, every universal $\partial$-functor $T = (T_n)_{n \geq 0}$ is isomorphic to the left derived $\partial$-functor $(L_n T_0)_{n \geq 0}$.*

After all, the mysterious universal $\partial$-functors are just the left derived functors of right-exact functors. For example, the functors $\operatorname{Tor}_n^R(A, -)$ and $\operatorname{Tor}_n^R(-, B)$ constitute universal $\partial$-functors.

**Remark:** Theorem 12.35 corresponds to Case (Ri). If $\mathbf{C}$ has enough injectives there is also a version of Theorem 12.35 for a contravariant right-exact functor $T$ saying that $(L_n T)_{n \geq 0}$ is a contravariant universal $\partial$-functor (Case (Li)). There doesn't seem to be any practical example of this case.

Theorem 12.37 corresponds to Case (Lp). If $\mathbf{C}$ has enough projectives there is a version of Theorem 12.37 for a contravariant left-exact functor $T$ saying that $(R^n T)_{n \geq 0}$ is a contravariant universal $\delta$-functor (Case (Rp)). As an example, the functors $\operatorname{Ext}_R^n(-, B)$ constitute a contravariant universal $\delta$-functor (for any fixed $R$-module $B$).

## 12.5   Universal Coefficient Theorems

Suppose we have a homology chain complex

$$0 \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \longleftarrow \cdots \xleftarrow{d_{p-1}} C_{p-1} \xleftarrow{d_p} C_p \xleftarrow{d_{p+1}} C_{p+1} \longleftarrow \cdots \,,$$

where the $C_i$ are $R$-modules over some commutative ring $R$ with a multiplicative identity element (recall that $d_i \circ d_{i+1} = 0$ for all $i \geq 0$). Given another $R$-module $G$ we can form the homology complex

$$0 \xleftarrow{d_0 \otimes \operatorname{id}} C_0 \otimes_R G \xleftarrow{d_1 \otimes \operatorname{id}} C_1 \otimes_R G \longleftarrow \cdots \xleftarrow{d_p \otimes \operatorname{id}} C_p \otimes_R G \longleftarrow \cdots \,,$$

obtained by tensoring with $G$, denoted $C \otimes_R G$, and the cohomology complex

$$0 \xrightarrow{\mathrm{Hom}_R(d_0, G)} \mathrm{Hom}_R(C_0, G) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(C_p, G) \xrightarrow{\mathrm{Hom}_R(d_{p+1}, G)} \mathrm{Hom}_R(C_{p+1}, G) \longrightarrow \cdots$$

obtained by applying $\mathrm{Hom}_R(-, G)$, and denoted $\mathrm{Hom}_R(C, G)$.

The question is: what is the relationship between the homology groups $H_p(C \otimes_R G)$ and the original homology groups $H_p(C)$ in the first case, and what the relationship between the cohomology groups $H^p(\mathrm{Hom}_R(C, G))$ and the original homology groups $H_p(C)$ in the second case?

The ideal situation would be that

$$H_p(C \otimes_R G) \cong H_p(C) \otimes_R G \quad \text{and} \quad H^p(\mathrm{Hom}_R(C, G)) \cong \mathrm{Hom}_R(H_p(C), G)),$$

but this is generally not the case. If the ring $R$ is nice enough if the modules $C_p$ are nice enough, then $H_p(C \otimes_R G)$ can be expressed in terms of $H_p(C) \otimes_R G$ and $\mathrm{Tor}_1^R(H_{p-1}(C), G)$, where $\mathrm{Tor}_1^R(-, G)$ is a one of the left-derived functors of $- \otimes_R G$, and $H^p(\mathrm{Hom}_R(C, G))$ can be expressed in terms of $\mathrm{Hom}_R(H_p(C), G))$ and $\mathrm{Ext}_R^1(H_{p-1}(C), G)$, where $\mathrm{Ext}_R^1(-, G)$ is one of the right-derived functors of $\mathrm{Hom}_R(-, G)$; both derived functors are defined in Section 12.3. These formulae are known as universal coefficient theorems.

Following Rotman [41] (Chapter 8), we give universal coefficients formulae that are general enough to cover all the cases of interest in singular homology and singular cohomology, for (commutative) rings that are hereditary and modules that are projective.

**Definition 12.20.** A commutative ring $R$ (with an identity element) is *hereditary* if every ideal in $R$ is a projective module.

Every PID is hereditary (and every semisimple ring is hereditary). The reason why hereditary rings are interesting is that if $R$ is hereditary, then every submodule of a projective $R$-module is also projective. In fact, a theorem of Cartan and Eilenberg states that *a ring is hereditary iff every submodule of a projective R-module is also projective*; see Rotman [41] (Chapter 4, Theorem 4.23). The next theorem is a universal coefficient theorem for homology.

**Theorem 12.38.** *(Universal Coefficient Theorem for Homology) Let $R$ be a commutative hereditary ring, $G$ be any $R$-module, and let $C$ be a chain complex of projective $R$-modules. Then there is a split exact sequence*

$$0 \longrightarrow H_n(C) \otimes_R G \xrightarrow{\mu} H_n(C \otimes_R G) \xrightarrow{p} \mathrm{Tor}_1^R(H_{n-1}(C), G) \longrightarrow 0$$

*for all $n \geq 0$. (It is assumed that $H_n(C) = (0)$ for all $n < 0$.) Thus, we have an isomorphism*

$$H_n(C \otimes_R G) \cong (H_n(C) \otimes_R G) \oplus \mathrm{Tor}_1^R(H_{n-1}(C), G)$$

*for all $n \geq 0$. Furthermore, the maps involved in the exact sequence of the theorem are natural, which means that for any chain map $\varphi \colon C \to C'$ between two chain complexes $C$ and $C'$ the following diagram commutes:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(C) \otimes_R G & \xrightarrow{\ \mu\ } & H_n(C \otimes_R G) & \xrightarrow{\ p\ } & \mathrm{Tor}_1^R(H_{n-1}(C), G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle (\varphi \otimes \mathrm{id})_*} & & \downarrow{\scriptstyle \mathrm{Tor}_1^R(\varphi_*)} & & \\
0 & \longrightarrow & H_n(C') \otimes_R G & \xrightarrow[\ \mu'\ ]{} & H_n(C' \otimes_R G) & \xrightarrow[\ p'\ ]{} & \mathrm{Tor}_1^R(H_{n-1}(C'), G) & \longrightarrow & 0.
\end{array}
\tag{$\dagger$}
$$

*Proof.* Theorem 12.38 is proved in Rotman [41] and we follow this proof (Chapter 8, Theorem 8.22). We warn the reader that in all the proofs that we are aware of (including Rotman's proof), the details involved in verifying that the maps $\mu$ and $p$ are natural are omitted (or sketched). We decided to provide complete details, which makes the proof quite long. The reader is advised to skip such details upon first reading.

We begin by observing that we have some exact sequences

$$
0 \longrightarrow Z_n(C) \xrightarrow{\ i_n\ } C_n \xrightarrow{\ d_n^B\ } B_{n-1}(C) \longrightarrow 0
\tag{$*$}
$$

and

$$
0 \longrightarrow B_{n-1}(C) \longrightarrow Z_{n-1}(C) \longrightarrow H_{n-1}(C) \longrightarrow 0.
$$

The first sequence $(*)$ is exact by definition of $Z_n(C)$ as $Z_n(C) = \mathrm{Ker}\, d_n$ and $B_{n-1}(C)$ as $B_{n-1}(C) = \mathrm{Im}\, d_n$, where the map $d_n^B \colon C_n \to B_{n-1}(C)$ is the corestriction of $d_n \colon C_n \to C_{n-1}$ to $B_{n-1}(C)$. The second sequence is exact by definition of $H_{n-1}(C)$, as $H_{n-1}(C) = Z_{n-1}(C)/B_{n-1}(C) = \mathrm{Ker}\, d_{n-1}/\mathrm{Im}\, d_n$. From now on, to simplify notation we drop the argument $(C)$ in $Z_n(C), B_n(C), H_n(C)$. These can be spliced using the diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n & \xrightarrow{\ i_n\ } & C_n & \xrightarrow{\ \tilde{d}_n\ } & Z_{n-1} & \longrightarrow & H_{n-1} & \longrightarrow & 0 \\
 & & & & & \searrow{\scriptstyle d_n^B} \quad \nearrow{\scriptstyle \iota_{n-1}} & & & & \\
 & & & & & B_{n-1} & & & & \\
 & & & & \nearrow \qquad \searrow & & & & \\
 & & & & 0 \qquad\qquad 0 & & & & &
\end{array}
$$

to form an exact sequence

$$
0 \longrightarrow Z_n \xrightarrow{\ i_n\ } C_n \xrightarrow{\ \tilde{d}_n\ } Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0.
\tag{$**$}
$$

Here $\iota_{n-1}$ is the inclusion map of $B_{n-1}$ into $Z_{n-1}$ and $\tilde{d}_n \colon C_n \to Z_{n-1}$ is the corestriction of $d_n \colon C_n \to C_{n-1}$ to $Z_{n-1}$. Since every $C_n$ is projective and $R$ is hereditary, the submodules $Z_{n-1}$ and $B_{n-1}$ of $C_{n-1}$ are also projective. This implies that the short exact sequence $(*)$ splits (by Proposition 12.1 (3)) and that the exact sequence $(**)$ is a projective resolution

of $H_{n-1}$. If we tensor $(**)$ with $G$ and drop the term $H_{n-1}$ we obtain the homology chain complex

$$0 \longrightarrow Z_n \otimes G \xrightarrow{\; i_n \otimes \mathrm{id} \;} C_n \otimes G \xrightarrow{\; \widetilde{d}_n \otimes \mathrm{id} \;} Z_{n-1} \otimes G \longrightarrow 0$$

denoted $L$, and by definition of $\mathrm{Tor}^R(-, G)$, we have

$$\mathrm{Tor}_j^R(H_{n-1}, G) = H_j(L), \quad j \geq 0.$$

Because $(*)$ is a split exact sequence, the sequence obtained by tensoring $(*)$ with $G$ is also exact, so $i_n \otimes \mathrm{id}$ is injective. This implies that $\mathrm{Tor}_2^R(H_{n-1}, G) = (0)$. We can compute $\mathrm{Tor}_j^R(H_{n-1}, G)$ for $j = 0, 1$ as follows:

$$\mathrm{Tor}_1^R(H_{n-1}, G) = H_1(L) = \mathrm{Ker}\,(\widetilde{d}_n \otimes \mathrm{id})/\mathrm{Im}\,(i_n \otimes \mathrm{id}) \cong \mathrm{Ker}\,(\widetilde{d}_n \otimes \mathrm{id})/(Z_n \otimes G)$$
$$H_{n-1} \otimes G = \mathrm{Tor}_0^R(H_{n-1}, G) = H_0(L) = (Z_{n-1} \otimes G)/\mathrm{Im}(\widetilde{d}_n \otimes \mathrm{id}).$$

Since $d_n = i_{n-1} \circ \widetilde{d}_n$, we have

$$d_n \otimes \mathrm{id} = (i_{n-1} \circ \widetilde{d}_n) \otimes \mathrm{id} = (i_{n-1} \otimes \mathrm{id}) \circ (\widetilde{d}_n \otimes \mathrm{id}),$$

and since $i_{n-1} \otimes \mathrm{id}$ is injective, $\mathrm{Ker}\,(d_n \otimes \mathrm{id}) = \mathrm{Ker}\,(\widetilde{d}_n \otimes \mathrm{id})$, which implies that

$$\mathrm{Tor}_1^R(H_{n-1}, G) \cong \mathrm{Ker}\,(d_n \otimes \mathrm{id})/(Z_n \otimes G).$$

Now look at the sequence

$$C_{n+1} \xrightarrow{\; d_{n+1} \;} C_n \xrightarrow{\; d_n \;} C_{n-1}$$

and tensor it with $G$ to obtain the sequence

$$C_{n+1} \otimes G \xrightarrow{\; d_{n+1} \otimes \mathrm{id} \;} C_n \otimes G \xrightarrow{\; d_n \otimes \mathrm{id} \;} C_{n-1} \otimes G.$$

One verifies that

$$\mathrm{Im}(d_{n+1} \otimes \mathrm{id}) \subseteq Z_n \otimes G \subseteq \mathrm{Ker}\,(d_n \otimes \mathrm{id}) \subseteq C_n \otimes G.$$

By the Third Isomorphism Theorem, we have

$$(\mathrm{Ker}\,(d_n \otimes \mathrm{id})/\mathrm{Im}(d_{n+1} \otimes \mathrm{id}))/[(Z_n \otimes G)/\mathrm{Im}(d_{n+1} \otimes \mathrm{id})] \cong \mathrm{Ker}\,(d_n \otimes \mathrm{id})/(Z_n \otimes G),$$

which may be rewritten as an exact sequence

$$0 \longrightarrow (Z_n \otimes G)/\mathrm{Im}(d_{n+1} \otimes \mathrm{id}) \longrightarrow (\mathrm{Ker}\,(d_n \otimes \mathrm{id})/\mathrm{Im}(d_{n+1} \otimes \mathrm{id})) \longrightarrow$$
$$\mathrm{Ker}\,(d_n \otimes \mathrm{id})/(Z_n \otimes G) \longrightarrow 0.$$

The middle term is just $H_n(C \otimes G)$, while the first term is isomorphic to $H_n(C) \otimes G$ because

$$\mathrm{Im}(d_{n+1} \otimes \mathrm{id}) = \{d_{n+1}(c) \otimes g \in C_n \otimes G \mid c \in C_{n+1}, g \in G\} = B_n \otimes G,$$

and the third term is isomorphic to $\mathrm{Tor}_1^R(H_{n-1}, G)$, so we obtain the exact sequence of the theorem.

It remains to prove that this sequence splits. Since $(*)$ splits, we have an isomorphism

$$C_n \cong Z_n \oplus B_{n-1}$$

and by tensoring with $G$ we obtain

$$C_n \otimes G \cong (Z_n \otimes G) \oplus (B_{n-1} \otimes G).$$

The reader should check that this implies that $Z_n \otimes G$ is a summand of $\mathrm{Ker}\,(d_n \otimes \mathrm{id})$. It follows from this that $Z_n \otimes G/(B_n \otimes G)$ is a summand of $\mathrm{Ker}\,(d_n \otimes \mathrm{id})/(B_n \otimes G)$, and the sequence of the theorem splits.

Suppose we have a chain map $\varphi \colon C \to C'$ between two chain complexes $C$ and $C'$. First we prove that the left square of the diagram (†) commutes, that is the following diagram commutes:

$$
\begin{array}{ccc}
H_n(C) \otimes_R G & \xrightarrow{\;\mu\;} & H_n(C \otimes_R G) \\
{\scriptstyle \varphi_* \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle (\varphi \otimes \mathrm{id})_*} \\
H_n(C') \otimes_R G & \xrightarrow[\;\mu'\;]{} & H_n(C' \otimes_R G).
\end{array}
\qquad (\dagger_1)
$$

Since

$$H_n(C) \otimes G = (Z_n/B_n) \otimes G \cong (Z_n \otimes G)/(B_n \otimes G),$$

the linear map $\varphi_* \otimes \mathrm{id} \colon H_n \otimes G \to H_n' \otimes G$ is given by

$$(\varphi_* \otimes \mathrm{id})([c \otimes g]) = [\varphi(c) \otimes g]', \qquad\qquad (*_1)$$

where $[c \otimes g]$ is the equivalence class of $c \otimes g \in Z_n \otimes G$ modulo $B_n \otimes G$ and $[\varphi(c) \otimes g]'$ is the equivalence class of $\varphi(c) \otimes g \in Z_n' \otimes G$ modulo $B_n' \otimes G$. Since $\varphi$ is a chain map, $\varphi(B_n) \subseteq B_n'$ and $\varphi(Z_n) \subseteq Z_n'$, so for any $d \otimes g' \in B_n \otimes G$ we have

$$(\varphi_* \otimes \mathrm{id})([c \otimes g + d \otimes g']) = [\varphi(c) \otimes g]' + [\varphi(d) \otimes g']' = [\varphi(c) \otimes g]'$$

since $\varphi(d) \otimes g' \in B_n' \otimes G$, and $\varphi(c) \otimes g \in Z_n' \otimes G$. Thus, the map $\varphi_* \otimes \mathrm{id}$ is well defined.

Since

$$H_n(C \otimes_R G) = \mathrm{Ker}\,(d_n \otimes \mathrm{id})/(B_n \otimes G),$$

the linear map $\mu \colon H_n(C) \otimes_R G \to H_n(C \otimes_R G)$ is given by

$$\mu([c \otimes g]) = [c \otimes g], \qquad\qquad (*_2)$$

where $c \in Z_n$ is a cycle and $g$ is any element in $G$, and where equivalence classes are taken modulo $B_n \otimes G$. If $c \in Z_n$ is a cycle then $d_n(c) = 0$ so $(d_n \otimes \mathrm{id})(c \otimes g) = d_n(c) \otimes g = 0$, which implies that $c \otimes g \in \mathrm{Ker}\,(d_n \otimes \mathrm{id})$. If $d \otimes g' \in B_n \otimes G$, then

$$\mu([c \otimes g + d \otimes g']) = [c \otimes g] + [d \otimes g'] = [c \otimes g]$$

because $d \otimes g' \in B_n \otimes G$, so the map $\mu$ is well defined. The map $\mu' \colon H_n(C') \otimes_R G \to H_n(C' \otimes_R G)$ is given by

$$\mu'([c' \otimes g]') = [c' \otimes g]', \qquad (*_3)$$

where $c' \in Z'_n$ is a cycle and $g$ is any element in $G$, and where the equivalence classes are taken modulo $B'_n \otimes G$.

The linear map $(\varphi \otimes \mathrm{id})_* \colon H_n(C \otimes G) \to H_n(C' \otimes G)$ is given by

$$(\varphi \otimes \mathrm{id})_*([c \otimes g]) = [\varphi(c) \otimes g]' \qquad (*_4)$$

where $[c \otimes g]$ is the equivalence class of $c \otimes g \in \mathrm{Ker}\,(d_n \otimes \mathrm{id})$ modulo $B_n \otimes G$ and $[\varphi(c) \otimes g] \in \mathrm{Ker}\,(d'_n \otimes \mathrm{id})$ is the equivalence class of $\varphi(c) \otimes g \in \mathrm{Ker}\,(d'_n \otimes \mathrm{id})$ modulo $B'_n \otimes G$. Since $\varphi$ is a chain map, we have $\varphi \circ d_n = d'_n \circ \varphi$, so

$$(d'_n \otimes \mathrm{id})(\varphi(c) \otimes g) = d'_n(\varphi(c)) \otimes g = \varphi(d_n(c)) \otimes g = (\varphi \otimes \mathrm{id})((d_n \otimes \mathrm{id})(c \otimes g)) = 0$$

so $\varphi(c) \otimes g \in \mathrm{Ker}\,(d'_n \otimes \mathrm{id})$. Since $\varphi$ is a chain map $\varphi(B_n) \subseteq B'_n$, and for any $d \otimes g \in B_n \otimes G$

$$(\varphi \otimes \mathrm{id})_*([c \otimes g + d \otimes g']) = [\varphi(c) \otimes g]' + [\varphi(d) \otimes g']' = [c \otimes g]'$$

since $\varphi(d) \otimes g' \in B'_n \otimes G$. Therefore, $(\varphi \otimes \mathrm{id})_*$ is well defined. Then we have

$$
\begin{aligned}
(\varphi \otimes \mathrm{id})_*(\mu([c \otimes g])) &= (\varphi \otimes \mathrm{id})_*([c \otimes g]), && \text{by } (*_2) \\
&= [\varphi(c) \otimes g]', && \text{by } (*_4) \\
&= \mu'([\varphi(c) \otimes g]'), && \text{by } (*_3) \\
&= \mu'([\varphi_*([c]) \otimes g]') \\
&= \mu'((\varphi_* \otimes \mathrm{id})([c \otimes g])), && \text{by } (*_1),
\end{aligned}
$$

which shows that

$$(\varphi \otimes \mathrm{id})_* \circ \mu = \mu' \circ (\varphi_* \otimes \mathrm{id}),$$

so the left square of the diagram (†) commutes.

Next we prove that the right square of the diagram (†) commutes, that is the following diagram commutes:

$$
\begin{array}{ccc}
H_n(C \otimes_R G) & \xrightarrow{\ p\ } & \mathrm{Tor}_1^R(H_{n-1}(C), G) \\
{\scriptstyle (\varphi \otimes \mathrm{id})_*}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Tor}_1^R(\varphi_*)} \\
H_n(C' \otimes_R G) & \xrightarrow[\ p'\ ]{} & \mathrm{Tor}_1^R(H_{n-1}(C'), G).
\end{array} \qquad (\dagger_2)
$$

To figure out what $\mathrm{Tor}_1(\varphi_*)$ is we go back to the projective resolution $(**)$ of $H_{n-1}$

$$0 \longrightarrow Z_n \xrightarrow{\ i_n\ } C_n \xrightarrow{\ \widetilde{d}_n\ } Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0. \qquad (**)$$

If $\varphi\colon C_n \to C_n'$ is a chain map, we claim that the following diagram commutes:

$$
\begin{array}{ccccccc}
Z_n & \xrightarrow{\ i_n\ } & C_n & \xrightarrow{\ \widetilde{d}_n\ } & Z_{n-1} & \longrightarrow & H_{n-1} \\
\downarrow{\scriptstyle \varphi|Z_n} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi|Z_{n-1}} & & \downarrow{\scriptstyle \varphi_*} \\
Z_n' & \xrightarrow[\ i_n'\ ]{} & C_n' & \xrightarrow[\ \widetilde{d}_n'\ ]{} & Z_{n-1}' & \longrightarrow & H_{n-1}'.
\end{array}
\qquad (**_1)
$$

The leftmost square commutes because $i_n$ and $i_n'$ are inclusions, the middle square commutes because $\varphi$ is a chain map, and the rightmost square commutes because $H_{n-1} = Z_{n-1}/B_{n-1}$ and $H_{n-1}' = Z_{n-1}'/B_{n-1}'$ and by the definition of $\varphi_*\colon H_{n-1} \to H_{n-1}'$, namely $\varphi_*([c]) = [\varphi(c)]$, for any $c \in Z_n$. Therefore we obtain a lifting of $\varphi_*$ between two projective resolutions of $H_{n-1}$ and $H_{n-1}'$ so by applying $- \otimes G$ we obtain

$$
\begin{array}{ccccc}
Z_n \otimes G & \xrightarrow{\ i_n \otimes \mathrm{id}\ } & C_n \otimes G & \xrightarrow{\ \widetilde{d}_n \otimes \mathrm{id}\ } & Z_{n-1} \otimes G \\
{\scriptstyle (\varphi|Z_n)\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi \otimes \mathrm{id}} & & \downarrow{\scriptstyle (\varphi|Z_{n-1})\otimes \mathrm{id}} \\
Z_n' \otimes G & \xrightarrow[\ i_n' \otimes \mathrm{id}\ ]{} & C_n' \otimes G & \xrightarrow[\ \widetilde{d}_n' \otimes \mathrm{id}\ ]{} & Z_{n-1}' \otimes G,
\end{array}
\qquad (**_2)
$$

and if we denote the upper row by $\mathcal{C}$ and the lower row by $\mathcal{C}'$, as explained just after Definition 12.8, the maps $\mathrm{Tor}_j^R(\varphi_*)\colon \mathrm{Tor}_j^R(H_{n-1}, G) \to \mathrm{Tor}_j^R(H_{n-1}', G)$ are the maps of homology $\mathrm{Tor}_j^R(\varphi_*)\colon H_j(\mathcal{C}) \to H_j(\mathcal{C}')$ induced by the chain map of the diagram $(**_2)$ and are independent of the lifting of $\varphi_*$ in $(**_1)$. Since

$$\mathrm{Tor}_1^R(H_{n-1}(C), G) \cong \mathrm{Ker}\,(\widetilde{d}_n \otimes \mathrm{id})/(Z_n \otimes G) = \mathrm{Ker}\,(d_n \otimes \mathrm{id})/(Z_n \otimes G)$$

and

$$\mathrm{Tor}_1^R(H_{n-1}'(C), G) \cong \mathrm{Ker}\,(\widetilde{d}_n' \otimes \mathrm{id})/(Z_n' \otimes G) = \mathrm{Ker}\,(d_n' \otimes \mathrm{id})/(Z_n' \otimes G),$$

the map $\mathrm{Tor}_1^R(\varphi_*)\colon \mathrm{Tor}_1^R(H_{n-1}(C), G) \to \mathrm{Tor}_1^R(H_{n-1}(C'), G)$ is the unique linear map given by

$$\mathrm{Tor}_1^R(\varphi_*)([c \otimes g]_{Z_n \otimes G}) = [\varphi(c) \otimes g]_{Z_n' \otimes G} \qquad (*_5)$$

for any $c \in C_n$ and any $g \in G$ such that $c \otimes g \in \mathrm{Ker}\,(d_n \otimes \mathrm{id})$. If $(d_n \otimes \mathrm{id})(c \otimes g) = 0$, that is, $d_n(c) \otimes g = 0$, since $\varphi$ is a chain map

$$(d_n' \otimes \mathrm{id})(\varphi(c) \otimes \mathrm{id}) = d_n'(\varphi(c)) \otimes g = \varphi(d_n(c)) \otimes g = (\varphi \otimes \mathrm{id})(d_n(c) \otimes g) = 0.$$

Also, for any $d \otimes g' \in Z_n \otimes G$, since $\varphi$ is a chain map $\varphi(Z_n) \subseteq Z_n'$, and we have

$$\mathrm{Tor}_1^R(\varphi_*)([c \otimes g + d \otimes g']_{Z_n \otimes G}) = [\varphi(c) \otimes g]_{Z_n' \otimes G} + [\varphi(d) \otimes g']_{Z_n' \otimes G} = [\varphi(c) \otimes g]_{Z_n' \otimes G},$$

so $\mathrm{Tor}_1^R(\varphi_*)$ is well defined. Since

$$H_n(C \otimes_R G) = \mathrm{Ker}\,(d_n \otimes \mathrm{id})/(B_n \otimes G)$$

the map $p \colon H_n(C \otimes_R G) \to \mathrm{Tor}_1^R(H_{n-1}(C), G)$ is given by

$$p([c \otimes g]_{B_n \otimes G}) = [c \otimes g]_{Z_n \otimes G} \qquad (*_6)$$

for any $c \otimes g \in \mathrm{Ker}\,(d_n \otimes \mathrm{id})$. Since $B_n \otimes G \subseteq Z_n \otimes G$, this map is well defined. Similarly, the map $p' \colon H_n(C' \otimes_R G) \to \mathrm{Tor}_1^R(H_{n-1}(C'), G)$ is given by

$$p'([c' \otimes g]_{B'_n \otimes G}) = [c' \otimes g]_{Z'_n \otimes G} \qquad (*_7)$$

for any $c' \otimes g \in \mathrm{Ker}\,(d'_n \otimes \mathrm{id})$. Then we have

$$
\begin{aligned}
\mathrm{Tor}_1^R(\varphi_*)(p([c \otimes g]_{B_n \otimes G})) &= \mathrm{Tor}_1^R(\varphi_*)([c \otimes g]_{Z_n \otimes G}), && \text{by } (*_6), \\
&= [\varphi(c) \otimes g]_{Z'_n \otimes G}, && \text{by } (*_5),
\end{aligned}
$$

and

$$
\begin{aligned}
p'((\varphi \otimes \mathrm{id})_*([c \otimes g]_{B_n \otimes G})) &= p'([\varphi(c) \otimes g]_{B'_n \otimes G}), && \text{by } (*_1), \\
&= [\varphi(c) \otimes g]_{Z'_n \otimes G}, && \text{by } (*_7).
\end{aligned}
$$

Therefore

$$\mathrm{Tor}_1^R(\varphi_*) \circ p = p' \circ (\varphi \otimes \mathrm{id})_*,$$

which proves that the second square of the diagram (†) commutes. $\qquad \square$

However, the splitting is not natural. This means that a splitting of the upper row may not map to a splitting of the lower row. Also, the theorem holds if the $C_n$ are flat; what is needed is that if $R$ is hereditary, then any submodule of a flat $R$-module is flat (see Rotman [41], Theorem 9.25 and Theorem 11.31).

A weaker version of Theorem 12.38 is proved in Munkres for $R = \mathbb{Z}$ and where the $C_n$ are free abelian groups; see Munkres [38] (Chapter 7, Theorem 55.1). This version of Theorem 12.38 is also proved in Hatcher; see Hatcher [25] (Chapter 3, Appendix 3.A, Theorem 3.A.3). Theorem 12.38 is proved in Spanier for free modules over a PID; see Spanier [47] (Chapter 5, Section 2, Theorem 8).

**Remark:** The injective map $\mu \colon H_n(C) \otimes G \to H_n(C \otimes G)$ is given by $\mu([c \otimes g]) = [c \otimes g]$ if we view $H_n(C)$ as isomorphic to $(Z_n \otimes G)/(B_n \otimes G)$, or by $\mu([c] \otimes g) = [c \otimes g]$ if we don't use this isomorphism; see Spanier [47] (Chapter 5, Section 1, page 214).

Whenever $\mathrm{Tor}_1^R(H_{n-1}(C), G)$ vanishes we obtain the "ideal result." This happens in the following two cases.

**Proposition 12.39.** *If $C$ is a complex of vector spaces and if $V$ is a vector space over the same field $K$, then we have*

$$H_n(C \otimes_K V) \cong H_n(C) \otimes_K V$$

*for all $n \geq 0$.*

**Proposition 12.40.** *If $C$ is a complex of free abelian groups, $G$ is an abelian group, and if either $H_{n-1}(C)$ or $G$ is torsion-free, then we have*

$$H_n(C \otimes_{\mathbb{Z}} G) \cong H_n(C) \otimes_{\mathbb{Z}} G$$

*for all $n \geq 0$.*

As a corollary of Theorem 12.38, we obtain the following result about singular homology, since $\mathbb{Z}$ is a PID, and the abelian groups in the complex $S_*(X, A; \mathbb{Z})$ are free.

**Theorem 12.41.** *If $X$ is a topological space, $A$ is a subset of $X$, and $G$ is any abelian group, then we have the following isomorphism of relative singular homology:*

$$H_n(X, A; G) \cong (H_n(X, A; \mathbb{Z}) \otimes_{\mathbb{Z}} G) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A; \mathbb{Z}), G)$$

*for all $n \geq 0$.*

*Proof.* By definition $H_n(X, A; \mathbb{Z}) = H_n(S_*(X, A; \mathbb{Z}))$ and $H_n(X, A; G) = H_n(S_*(X, A; G))$. But by definition $S_*(X, A; G) \cong S_*(X, A; \mathbb{Z}) \otimes_{\mathbb{Z}} G$, and the $S_n(X, A; \mathbb{Z})$ are free abelian groups, and thus projective. $\qquad\square$

Theorem 12.41 shows that the singular homology groups with coefficients in an abelian group $G$ are determined by the singular homology groups with integer coefficients.

Since the modules in the relative chain complex $S_*(X, A; R)$ are free, and thus projective, and a PID is hereditary, Theorem 12.38 has the following corollary.

**Theorem 12.42.** *If $X$ is a topological space, $A$ is a subset of $X$, $R$ is a PID, and $G$ is any $R$-module, then we have the following isomorphism of relative singular homology:*

$$H_n(X, A; G) \cong (H_n(X, A; R) \otimes_R G) \oplus \mathrm{Tor}_1^R(H_{n-1}(X, A; R), G)$$

*for all $n \geq 0$.*

Theorem 12.42 is also proved in Spanier [47] (Chapter 5, Section 2, Theorem 8). The reader should be warned that the assumption that $R$ is a PID is missing in the statement of his Theorem 8. This is because Spanier reminds the reader earlier on page 220 that $R$ is a PID. Spanier also proves a more general theorem similar to Theorem 12.38 but applying to

a chain complex $C$ such that $C \otimes G$ is acyclic and with $R$ a PID; see Theorem 14 in Spanier [47] (Chapter 5, Section 2).

If $G$ is a finitely generated abelian group and $A$ is any abelian group, then $\operatorname{Tor}_1^{\mathbb{Z}}(A, G)$ can be computed recursively using some simple rules. It is customary to drop the subscript 1 in $\operatorname{Tor}_1^{\mathbb{Z}}(-, -)$.

The main rules that allow us to use a recursive method are

$$\operatorname{Tor}^R\left(\bigoplus_{i \in I} A_i, B\right) \cong \bigoplus_{i \in I} \operatorname{Tor}^R(A_i, B)$$

$$\operatorname{Tor}^R\left(A, \bigoplus_{i \in I} B_i\right) \cong \bigoplus_{i \in I} \operatorname{Tor}^R(A, B_i)$$

$$\operatorname{Tor}^R(A, B) \cong \operatorname{Tor}^R(B, A)$$

$$\operatorname{Tor}^R(A, B) \cong (0) \quad \text{if } A \text{ or } B \text{ is flat (in particular, projective, or free),}$$

which hold for any commutative ring $R$ (with an identity element) and any $R$-modules. When $R = \mathbb{Z}$, we also have

$$\operatorname{Tor}^{\mathbb{Z}}(\mathbb{Z}, A) = (0)$$

and

$$\operatorname{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \cong \operatorname{Ker}(A \xrightarrow{m} A),$$

where $A$ is an abelian group and the map $A \xrightarrow{m} A$ is multiplication by $m$. The proof of this last equation involves a clever use of a free resolution.

*Proof.* It is immediately checked that the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

is exact, and since $\mathbb{Z}$ is a free abelian group, the above sequence is a free resolution of $\mathbb{Z}/m\mathbb{Z}$. Then, since $\operatorname{Tor}^{\mathbb{Z}}(-, A)$ is the left derived functor of $- \otimes A$, we deduce that $\operatorname{Tor}_j^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) = (0)$ for all $j \geq 2$, and the long exact sequence given by Theorem 12.26 yields the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} A \xrightarrow{m \otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} A \longrightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow 0$$

But $\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A$, so we obtain an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \xrightarrow{j} A \xrightarrow{m} A \longrightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow 0,$$

and since $j$ is injective and $\operatorname{Im} j = \operatorname{Ker} m$, we get $\operatorname{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \cong \operatorname{Ker}(A \xrightarrow{m} A)$, as claimed. $\square$

We also use the following identities about tensor products:

$$\left(\bigoplus_{i\in I} A_i\right) \otimes_R B \cong \bigoplus_{i\in I} A_i \otimes_R B$$

$$A \otimes_R B \cong B \otimes_R A$$

$$R \otimes_R A \cong A,$$

which hold for any commutative ring $R$ (with an identity element) and any $R$-modules. When $R = \mathbb{Z}$, we also have

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/mA$$

where $A$ is an abelian group. These rules imply that

$$\mathrm{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = (0)$$

and

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathrm{Tor}^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

For details, see Munkres [38] (Chapter 7, Section 54) and Hatcher [25] (Chapter 3, Appendix 3.A, Proposition 3.A.5).

Regarding the cohomology complex obtained by using $\mathrm{Hom}_R(-, G)$, we have the following theorem.

**Theorem 12.43.** *(Universal Coefficient Theorem for Cohomology) Let $R$ be a commutative hereditary ring, $G$ be any $R$-module, and let $C$ be a chain complex of projective $R$-modules. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_R(H_{n-1}(C), G) \xrightarrow{j} H^n(\mathrm{Hom}_R(C, G)) \xrightarrow{h} \mathrm{Hom}_R(H_n(C), G) \longrightarrow 0$$

*for all $n \geq 0$. (It is assumed that $H_n(C) = (0)$ for all $n < 0$.) Thus, we have an isomorphism*

$$H^n(\mathrm{Hom}_R(C, G)) \cong \mathrm{Hom}_R(H_n(C), G) \oplus \mathrm{Ext}^1_R(H_{n-1}(C), G)$$

*for all $n \geq 0$. Furthermore, the maps in the exact sequence of the theorem are natural, which means that for any chain map $\theta \colon C \to C'$ between two chain complexes $C$ and $C'$ we have the following commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1_R(H_{n-1}(C'), G) & \xrightarrow{j'} & H^n(\mathrm{Hom}_R(C', G)) & \xrightarrow{h'} & \mathrm{Hom}_R(H_n(C'), G) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{Ext}^1_R(\theta_*)} & & \downarrow{\scriptstyle (\mathrm{Hom}_R(\theta,\mathrm{id}))^*} & & \downarrow{\scriptstyle \mathrm{Hom}_R(\theta_*,\mathrm{id})} & & \\
0 & \longrightarrow & \mathrm{Ext}^1_R(H_{n-1}(C), G) & \xrightarrow{j} & H^n(\mathrm{Hom}_R(C, G)) & \xrightarrow{h} & \mathrm{Hom}_R(H_n(C), G) & \longrightarrow & 0.
\end{array}
$$

*Proof.* Theorem 12.43 is proved by modifying the proof of Theorem 12.38 by replacing the functor $- \otimes_R G$ by the functor $\mathrm{Hom}_R(-, G)$. Again, we warn the reader that in all the proofs that we are aware of (Rotman leaves the entire proof to the reader), the details involved in verifying that the maps $j$ and $h$ are natural are omitted (or sketched). The dualization process (applying $\mathrm{Hom}(-, G)$) also causes technical complications that do not come up when tensoring with $G$. In particular it is no longer obvious how to identify $\mathrm{Hom}(H_n(C), G)$, and some auxiliary proposition is needed (Proposition 2.7). We decided to provide complete details, which makes the proof quite long. The reader is advised to skip such details upon first reading.

Recall from the beginning of the proof of Theorem 12.38 that we have the split short exact sequence

$$0 \longrightarrow Z_n(C) \xrightarrow{\ i_n\ } C_n \xrightarrow{\ d_n^B\ } B_{n-1}(C) \longrightarrow 0 \tag{$*$}$$

and the exact sequence

$$0 \longrightarrow Z_n \xrightarrow{\ i_n\ } C_n \xrightarrow{\ \widetilde{d}_n\ } Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0 \tag{$**$}$$

where $\widetilde{d}_n \colon C_n \to Z_{n-1}$ is the corestriction of $d_n \colon C_n \to C_{n-1}$ to $Z_{n-1}$ and $d_n^B \colon C_n \to B_{n-1}(C)$ is the corestriction of $d_n \colon C_n \to C_{n-1}$ to $B_{n-1}(C)$. Since every $C_n$ is projective and $R$ is hereditary, the exact sequence $(**)$ is a projective resolution of $H_{n-1}$. If we apply $\mathrm{Hom}(-, G)$ to $(**)$ and drop the term $H_{n-1}$ we obtain the cohomology chain complex

$$0 \longrightarrow \mathrm{Hom}(Z_{n-1}, G) \xrightarrow{\ \mathrm{Hom}(\widetilde{d}_n, \mathrm{id})\ } \mathrm{Hom}(C_n, G) \xrightarrow{\ \mathrm{Hom}(i_n, \mathrm{id})\ } \mathrm{Hom}(Z_n, G) \longrightarrow 0$$

denoted $\mathcal{C}$, and by definition of $\mathrm{Ext}_R^j(-, G)$, we have

$$\mathrm{Ext}_R^j(H_{n-1}, G) = H^j(\mathcal{C}).$$

Since the sequence $(*)$ is a split exact sequence and $i_n$ is injective, $\mathrm{Hom}(i_n, \mathrm{id})$ is surjective, and this implies that

$$\mathrm{Ext}_R^2(H_{n-1}, G) = H^2(\mathcal{C}) = \mathrm{Hom}(Z_n, G)/\mathrm{Im}\,\mathrm{Hom}(i_n, \mathrm{id}) = \mathrm{Hom}(Z_n, G)/\mathrm{Hom}(Z_n, G) = (0).$$

We also have

$$\mathrm{Ext}_R^1(H_{n-1}, G) = H^1(\mathcal{C}) = \mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(\widetilde{d}_n, \mathrm{id}).$$

From the original chain complex

$$0 \xleftarrow{\ d_0\ } C_0 \xleftarrow{\ d_1\ } C_1 \xleftarrow{\quad} \cdots \xleftarrow{\ d_{n-1}\ } C_{n-1} \xleftarrow{\ d_n\ } C_n \xleftarrow{\ d_{n+1}\ } C_{n+1} \xleftarrow{\quad} \cdots$$

we have

$$H_n = \mathrm{Ker}\,d_n/\mathrm{Im}\,d_{n+1} = Z_n/B_n, \tag{$*_1$}$$

and from the complex

$$0 \xrightarrow{\mathrm{Hom}_R(d_0,\mathrm{id})} \mathrm{Hom}_R(C_0, G) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(C_{n-1}, G) \xrightarrow{\mathrm{Hom}_R(d_n,\mathrm{id})} \mathrm{Hom}_R(C_n, G) \longrightarrow \cdots$$

we have

$$H^n(\mathrm{Hom}(C, G)) = \mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}). \tag{$*_2$}$$

Since $d_n = i_{n-1} \circ \widetilde{d}_n$, with $d_n \colon C_n \to C_{n-1}$, $\widetilde{d}_n \colon C_n \to Z_{n-1}$, and $i_{n-1} \colon Z_{n-1} \to C_{n-1}$ we have

$$\mathrm{Hom}(d_n, \mathrm{id}) = \mathrm{Hom}(\widetilde{d}_n, \mathrm{id}) \circ \mathrm{Hom}(i_{n-1}, \mathrm{id}).$$

Since $\mathrm{Hom}(C_{n-1}, G) \xrightarrow{\mathrm{Hom}(i_{n-1},G)} \mathrm{Hom}(Z_{n-1}, G)$ is a surjection, we have

$$\mathrm{Im}\,\mathrm{Hom}(\widetilde{d}_n, \mathrm{id}) = \mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}). \tag{$*_3$}$$

Consequently

$$\mathrm{Ext}^1_R(H_{n-1}, G) = \mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}). \tag{$*_4$}$$

We claim that

$$\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}) \subseteq \mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id}) \subseteq \mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id}). \tag{$*_5$}$$

Since $\mathrm{Hom}(d_n, \mathrm{id}) \colon \mathrm{Hom}(C_{n-1}, G) \to \mathrm{Hom}(C_n, G)$ is given by $\varphi \mapsto \varphi \circ d_n$ for all $\varphi \in \mathrm{Hom}(C_{n-1}, G)$, we have

$$\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}) = \{\psi \circ d_n \in \mathrm{Hom}(C_n, G) \mid \psi \in \mathrm{Hom}(C_{n-1}, G)\}.$$

Also, since $\mathrm{Hom}(d_{n+1}, \mathrm{id}) \colon \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G)$ is given by $\varphi \mapsto \varphi \circ d_{n+1}$ for all $\varphi \in \mathrm{Hom}(C_n, G)$, and $\mathrm{Hom}(i_n, \mathrm{id}) \colon \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(Z_n, G)$ is given by $\varphi \mapsto \varphi \circ i_n$ for all $\varphi \in \mathrm{Hom}(C_n, G)$, we see that $\varphi \in \mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id})$ iff $\varphi \circ d_{n+1} = 0$ iff $\varphi$ vanishes on $B_n = \mathrm{Im}\,d_{n+1}$, and $\varphi \in \mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id})$ iff $\varphi \circ i_n = 0$ iff $\varphi$ vanishes on $Z_n = \mathrm{Im}\,i_n$. Therefore

$$\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}) = \{\psi \circ d_n \in \mathrm{Hom}(C_n, G) \mid \psi \in \mathrm{Hom}(C_{n-1}, G)\}$$
$$\mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id}) = \{\varphi \in \mathrm{Hom}(C_n, G) \mid \varphi(c) = 0 \quad \text{for all } c \in Z_n\}$$
$$\mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id}) = \{\varphi \in \mathrm{Hom}(C_n, G) \mid \varphi(c) = 0 \quad \text{for all } c \in B_n\}.$$

Since $Z_n = \mathrm{Ker}\,d_n$, any function $\psi \circ d_n \in \mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id})$ vanishes on $Z_n$, so $\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}) \subseteq \mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id})$, and since $B_n \subseteq Z_n$, any function $\varphi \in \mathrm{Hom}(C_n, G)$ that vanishes on $Z_n$ also vanishes on $B_n$, so $\mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id}) \subseteq \mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id})$.

Then we can apply the third isomorphism theorem and we get

$$\big(\mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id})\big)/\big(\mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id})\big)$$
$$\cong \mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id})/\mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id}),$$

and this can be rewritten as the exact sequence

$$0 \longrightarrow \operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id})/\operatorname{Im} \operatorname{Hom}(d_n, \mathrm{id}) \longrightarrow \operatorname{Ker} \operatorname{Hom}(d_{n+1}, \mathrm{id})/\operatorname{Im} \operatorname{Hom}(d_n, \mathrm{id}) \longrightarrow$$
$$\operatorname{Ker} \operatorname{Hom}(d_{n+1}, \mathrm{id})/\operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id}) \longrightarrow 0.$$

Since

$$\operatorname{Ext}^1_R(H_{n-1}, G) = \operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id})/\operatorname{Im} \operatorname{Hom}(d_n, \mathrm{id})$$

the first term in the exact sequence is $\operatorname{Ext}^1_R(H_{n-1}, G)$, and the second term is $H^n(\operatorname{Hom}(C, G))$, so our exact sequence can be written as

$$0 \longrightarrow \operatorname{Ext}^1_R(H_{n-1}, G) \longrightarrow H^n(\operatorname{Hom}(C, G)) \longrightarrow \operatorname{Ker} \operatorname{Hom}(d_{n+1}, \mathrm{id})/\operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id}) \longrightarrow 0. \quad (\dagger)$$

It remains to figure out what is $\operatorname{Ker} \operatorname{Hom}(d_{n+1}, \mathrm{id})/\operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id})$. We will show that this term is isomorphic to $\operatorname{Hom}(H_n, G)$.

We proved earlier that

$$\operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id}) = \{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi(c) = 0 \quad \text{for all } c \in Z_n\}$$
$$\operatorname{Ker} \operatorname{Hom}(d_{n+1}, \mathrm{id}) = \{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi(c) = 0 \quad \text{for all } c \in B_n\},$$

so

$$\operatorname{Ker} \operatorname{Hom}(d_{n+1}, \mathrm{id})/\operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id}) =$$
$$\{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi|B_n \equiv 0\}/\{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi|Z_n \equiv 0\}.$$

We use Proposition 2.7 to conclude that

$$\operatorname{Ker} \operatorname{Hom}(d_{n+1}, \mathrm{id})/\operatorname{Ker} \operatorname{Hom}(i_n, \mathrm{id}) =$$
$$\{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi|B_n \equiv 0\}/\{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi|Z_n \equiv 0\}$$
$$= B^0_n/Z^0_n \cong \operatorname{Hom}(Z_n/B_n, G) = \operatorname{Hom}(H_n, G),$$

where

$$B^0_n = \{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi(a) = 0 \quad \text{for all } b \in B_n\}$$
$$Z^0_n = \{\varphi \in \operatorname{Hom}(C_n, G) \mid \varphi(z) = 0 \quad \text{for all } z \in Z_n\}.$$

Since the exact sequence $(*)$ splits, we have $C_n = Z_n \oplus Z'_n$ for some submodule $Z'_n$ of $C_n$, and we can apply Proposition 2.7 to $M = C_n$, $Z = Z_n$, and $B = B_n$. Therefore, the exact sequence $(\dagger)$ yields

$$0 \longrightarrow \operatorname{Ext}^1_R(H_{n-1}, G) \longrightarrow H^n(\operatorname{Hom}(C, G)) \longrightarrow \operatorname{Hom}(H_n, G) \longrightarrow 0. \quad (\dagger\dagger)$$

We now prove that the exact sequence $(\dagger\dagger)$ splits. For this, we use the fact that since the exact sequence $(*)$ splits we have an isomorphim

$$C_n \cong Z_n \oplus B_{n-1}.$$

Applying $\mathrm{Hom}(-, G)$, we get

$$\mathrm{Hom}(C_n, G) \cong \mathrm{Hom}(Z_n, G) \oplus \mathrm{Hom}(B_{n-1}, G). \qquad (*_6)$$

Recall that

$$\mathrm{Ker}\, \mathrm{Hom}(i_n, \mathrm{id}) = \{\varphi \in \mathrm{Hom}(C_n, G) \mid \varphi|Z_n \equiv 0\}$$
$$\mathrm{Ker}\, \mathrm{Hom}(d_{n+1}, \mathrm{id}) = \{\varphi \in \mathrm{Hom}(C_n, G) \mid \varphi|B_n \equiv 0\}.$$

We deduce from the above that

$$\mathrm{Ker}\, \mathrm{Hom}(i_n, \mathrm{id}) \cong \mathrm{Hom}(B_{n-1}, G), \qquad (*_7)$$

so by $(*_4)$ we obtain

$$\mathrm{Ext}^1_R(H_{n-1}, G) = \mathrm{Hom}(B_{n-1}, G)/\mathrm{Im}\, \mathrm{Hom}(d_n, \mathrm{id}). \qquad (*_8)$$

Since $(*_5)$ implies that $\mathrm{Ker}\, \mathrm{Hom}(i_n, \mathrm{id}) \subseteq \mathrm{Ker}\, \mathrm{Hom}(d_{n+1}, \mathrm{id})$, we have

$$\mathrm{Ker}\, \mathrm{Hom}(d_{n+1}, \mathrm{id}) \cong \{\varphi \in \mathrm{Hom}(Z_n, G) \mid \varphi|B_n \equiv 0\} \oplus \mathrm{Hom}(B_{n-1}, G).$$

Now by Proposition 2.8 there is an isomorphism

$$\kappa \colon \{\varphi \in \mathrm{Hom}(Z_n, G) \mid \varphi|B_n \equiv 0\} \to \mathrm{Hom}(Z_n/B_n, G), \qquad (*_9)$$

where $\kappa$ is given by

$$(\kappa(\varphi))([z]) = \varphi(z) \quad \text{for all } [z] \in Z_n/B_n. \qquad (*_\kappa)$$

Since $Z_n/B_n = H_n$, we obtain

$$\mathrm{Ker}\, \mathrm{Hom}(d_{n+1}, \mathrm{id}) \cong \mathrm{Hom}(H_n, G) \oplus \mathrm{Hom}(B_{n-1}, G). \qquad (*_{10})$$

We now take the quotient modulo $\mathrm{Im}\, \mathrm{Hom}(d_n, \mathrm{id})$. Since we showed that $\mathrm{Im}\, \mathrm{Hom}(d_n, \mathrm{id}) \subseteq \mathrm{Ker}\, \mathrm{Hom}(i_n, \mathrm{id}) \cong \mathrm{Hom}(B_{n-1}, G)$, we get

$$\mathrm{Ker}\, \mathrm{Hom}(d_{n+1}, \mathrm{id})/\mathrm{Im}\, \mathrm{Hom}(d_n, \mathrm{id}) \cong \mathrm{Hom}(H_n, G) \oplus \mathrm{Hom}(B_{n-1}, G)/\mathrm{Im}\, \mathrm{Hom}(d_n, \mathrm{id}),$$

and by $(*_8)$ this means that

$$H^n(\mathrm{Hom}(C, G)) \cong \mathrm{Hom}(H_n, G) \oplus \mathrm{Ext}^1_R(H_{n-1}, G),$$

which proves that the exact sequence (††) splits.

To prove naturality of the exact sequence (††) we first give another expression for $\mathrm{Hom}(Z_n/B_n, G) = \mathrm{Hom}(H_n, G)$ in terms of the inclusion map $\gamma_n \colon B_n \to Z_n$ as in Spanier [47] (Chapter 5, Section 5, Theorem 3). We claim that

$$\mathrm{Hom}(H_n, G) = \mathrm{Hom}(Z_n/B_n, G) \cong \mathrm{Ker}\, \mathrm{Hom}(\gamma_n, \mathrm{id}). \qquad (*_{11})$$

Indeed, since $\gamma_n \colon B_n \to Z_n$ we have $\mathrm{Hom}(\gamma_n, \mathrm{id}) \colon \mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G)$, and we have $\varphi \in \mathrm{Ker}\,\mathrm{Hom}(\gamma_n, \mathrm{id})$ iff $\varphi \circ \gamma_n = 0$ iff $\varphi$ vanishes on $B_n$, thus

$$\mathrm{Ker}\,\mathrm{Hom}(\gamma_n, \mathrm{id}) = \{\varphi \in \mathrm{Hom}(Z_n, G) \mid \varphi|B_n \equiv 0\},$$

but we know $(*_9)$ that this last term is isomorphic to $\mathrm{Hom}(Z_n/B_n, G) = \mathrm{Hom}(H_n, G)$. We now prove the naturality of $(\dagger\dagger)$.

Let $\theta \colon C \to C'$ be a chain map. First we prove that the diagram

$$
\begin{array}{ccc}
H^n(\mathrm{Hom}_R(C', G)) & \xrightarrow{\ h'\ } & \mathrm{Hom}_R(H_n(C'), G) \\
{\scriptstyle (\mathrm{Hom}_R(\theta,\mathrm{id}))^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Hom}_R(\theta_*,\mathrm{id})} \\
H^n(\mathrm{Hom}_R(C, G)) & \xrightarrow[\ h\ ]{} & \mathrm{Hom}_R(H_n(C), G)
\end{array}
\qquad (\dagger_1)
$$

commutes, which in view of $(*_2)$ and $(*_{11})$ is equivalent to the commutativity of the following diagram

$$
\begin{array}{ccc}
\mathrm{Ker}\,\mathrm{Hom}(d'_{n+1}, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d'_n, \mathrm{id}) & \xrightarrow{\ h'\ } & \mathrm{Ker}\,\mathrm{Hom}(\gamma'_n, \mathrm{id}) \\
{\scriptstyle (\mathrm{Hom}(\theta,\mathrm{id}))^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Hom}(\theta_*,\mathrm{id})} \\
\mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}) & \xrightarrow[\ h\ ]{} & \mathrm{Ker}\,\mathrm{Hom}(\gamma_n, \mathrm{id}),
\end{array}
\qquad (\dagger_2)
$$

where the various maps involved are defined below. Recall that

$$\mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id}) = \{\varphi \in \mathrm{Hom}(C_n, G) \mid \varphi|B_n \equiv 0\}$$
$$\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}) = \{\psi \circ d_n \in \mathrm{Hom}(C_n, G) \mid \psi \in \mathrm{Hom}(C_{n-1}, G)\}$$
$$\mathrm{Ker}\,\mathrm{Hom}(\gamma_n, \mathrm{id}) = \{\varphi \in \mathrm{Hom}(Z_n, G) \mid \varphi|B_n \equiv 0\}.$$

The map $(\mathrm{Hom}(\theta, \mathrm{id}))^*$ is given by

$$(\mathrm{Hom}(\theta, \mathrm{id}))^*([\varphi']) = [\varphi' \circ \theta] \qquad (*_{12})$$

for any $\varphi' \in \mathrm{Hom}(C'_n, G)$ such that $\varphi'|B'_n \equiv 0$, the map $\mathrm{Hom}(\theta_*, \mathrm{id})$ is given by

$$\mathrm{Hom}(\theta_*, \mathrm{id})(\varphi') = \varphi' \circ (\theta|Z_n) \qquad (*_{13})$$

for any $\varphi' \in \mathrm{Hom}(Z'_n, G)$ such that $\varphi'|B'_n \equiv 0$, the map $h$ is given by

$$h([\varphi]) = \varphi|Z_n \qquad (*_{14})$$

for any $\varphi \in \mathrm{Hom}(C_n, G)$ such that $\varphi|B_n \equiv 0$, and the map $h'$ is given by

$$h'([\varphi']) = \varphi'|Z'_n \qquad (*_{15})$$

for any $\varphi' \in \mathrm{Hom}(C'_n, G)$ such that $\varphi'|B'_n \equiv 0$. The map $(\mathrm{Hom}(\theta, \mathrm{id}))^*$ is well defined because $\theta$ is a chain map so for any $\psi' \circ d'_n \in \mathrm{Im}\,\mathrm{Hom}(d'_n, \mathrm{id})$ we have

$$(\mathrm{Hom}(\theta, \mathrm{id}))^*([\varphi' + \psi' \circ d'_n]) = [\varphi' \circ \theta + \psi' \circ d'_n \circ \theta]) = [\varphi' \circ \theta + \psi' \circ \theta \circ d_n]) = [\varphi' \circ \theta].$$

If $\varphi'|B_n' \equiv 0$, then because $\theta$ is a chain map, for any $c \in C_{n+1}$

$$(\varphi' \circ \theta)(d_{n+1}(c)) = \varphi'(d_{n+1}'(\theta(c))) = 0$$

so $(\varphi' \circ \theta)|B_n \equiv 0$. The map $\mathrm{Hom}(\theta_*, \mathrm{id})$ is well defined because $\theta(Z_n) \subseteq Z_n'$ since $\theta$ is a chain map, and if $\varphi'|B_n' \equiv 0$ for any $\varphi' \in \mathrm{Hom}(Z_n', G)$, then using the same reasoning as above $(\varphi' \circ \theta)|B_n \equiv 0$. The map $h$ is well defined because if $\varphi \in \mathrm{Hom}(C_n, G)$ with $\varphi|B_n \equiv 0$ then $\varphi|Z_n$ vanishes on $B_n$ since $B_n \subseteq Z_n$, and for any $\psi \circ d_n \in \mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id})$, we have

$$(\varphi + \psi \circ d_n)|Z_n = \varphi|Z_n + (\psi \circ d_n)|Z_n = \varphi|Z_n,$$

since $d_n|Z_n \equiv 0$ ($Z_n = \mathrm{Ker}\, d_n$). Similarly the map $h'$ is well defined.

Then by $(*_{15})$ an $(*_{13})$ we have

$$\mathrm{Hom}(\theta_*, \mathrm{id})(h'([\varphi'])) = \mathrm{Hom}(\theta_*, \mathrm{id})(\varphi'|Z_n') = (\varphi'|Z_n') \circ (\theta|Z_n),$$

and by $(*_{12})$ and $(*_{14})$

$$h((\mathrm{Hom}(\theta, \mathrm{id}))^*([\varphi'])) = h([\varphi' \circ \theta]) = (\varphi' \circ \theta)|Z_n.$$

Since $\theta(Z_n) \subseteq Z_n'$, we have

$$(\varphi'|Z_n') \circ (\theta|Z_n) = (\varphi' \circ \theta)|Z_n,$$

which proves that the diagram $(\dagger_2)$ commutes.

We now prove that the diagram

$$
\begin{array}{ccc}
\mathrm{Ext}_R^1(H_{n-1}(C'), G) & \xrightarrow{\ j'\ } & H^n(\mathrm{Hom}_R(C', G)) \\
{\scriptstyle \mathrm{Ext}_R^1(\theta_*)} \downarrow & & \downarrow {\scriptstyle (\mathrm{Hom}_R(\theta, \mathrm{id}))^*} \\
\mathrm{Ext}_R^1(H_{n-1}(C), G) & \xrightarrow{\ j\ } & H^n(\mathrm{Hom}_R(C, G))
\end{array}
\qquad (\dagger_3)
$$

commutes, which in view of $(*_2)$ and $(*_4)$ is equivalent to the commutativity of the following diagram

$$
\begin{array}{ccc}
\mathrm{Ker}\,\mathrm{Hom}(i_n', \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n', \mathrm{id}) & \xrightarrow{\ j'\ } & \mathrm{Ker}\,\mathrm{Hom}(d_{n+1}', \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n', \mathrm{id}) \\
{\scriptstyle \mathrm{Ext}^1(\theta_*)} \downarrow & & \downarrow {\scriptstyle (\mathrm{Hom}(\theta, \mathrm{id}))^*} \\
\mathrm{Ker}\,\mathrm{Hom}(i_n, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}) & \xrightarrow{\ j\ } & \mathrm{Ker}\,\mathrm{Hom}(d_{n+1}, \mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id}),
\end{array}
\qquad (\dagger_4)
$$

where the maps involved (besides the right vertical map) are defined below.

To figure out what $\mathrm{Ext}^1(\theta_*)$ is we go back to the projective resolution $(**)$ of $H_{n-1}$

$$0 \longrightarrow Z_n \xrightarrow{\ i_n\ } C_n \xrightarrow{\ \tilde{d}_n\ } Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0. \qquad (**)$$

If $\theta\colon C_n \to C'_n$ is a chain map, we showed during the proof of Theorem 12.38 that the following diagram commutes:

$$
\begin{array}{ccccccc}
Z_n & \xrightarrow{i_n} & C_n & \xrightarrow{\widetilde{d}_n} & Z_{n-1} & \longrightarrow & H_{n-1} \\
\downarrow{\scriptstyle \theta|Z_n} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta|Z_{n-1}} & & \downarrow{\scriptstyle \theta_*} \\
Z'_n & \xrightarrow{i'_n} & C'_n & \xrightarrow{\widetilde{d}'_n} & Z'_{n-1} & \longrightarrow & H'_{n-1}.
\end{array}
\qquad (**_1)
$$

Therefore we obtain a lifting of $\theta_*$ between two projective resolutions of $H_{n-1}$ and $H'_{n-1}$ so by applying $\mathrm{Hom}(-,G)$ we obtain the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(Z'_{n-1},G) & \xrightarrow{\mathrm{Hom}(\widetilde{d}'_n,G)} & \mathrm{Hom}(C'_{n-1},G) & \xrightarrow{\mathrm{Hom}(i'_n,\mathrm{id})} & \mathrm{Hom}(Z'_n,\mathrm{id}) \\
& & \downarrow{\scriptstyle \mathrm{Hom}(\theta|Z_{n-1},\mathrm{id})} & & \downarrow{\scriptstyle \mathrm{Hom}(\theta|C_n,\mathrm{id})} & & \downarrow{\scriptstyle \mathrm{Hom}(\theta|Z_n,\mathrm{id})} \\
0 & \longrightarrow & \mathrm{Hom}(Z_{n-1},G) & \xrightarrow{\mathrm{Hom}(\widetilde{d}_n,G)} & \mathrm{Hom}(C_{n-1},G) & \xrightarrow{\mathrm{Hom}(i_n,\mathrm{id})} & \mathrm{Hom}(Z_n,\mathrm{id}),
\end{array}
\qquad (**_2)
$$

and if we denote the upper row by $\mathcal{C}'$ and the lower row by $\mathcal{C}$, as explained just after Definition 12.8, the maps $\mathrm{Ext}_R^j(\theta_*)\colon \mathrm{Ext}_R^j(H'_{n-1},G) \to \mathrm{Ext}_R^j(H_{n-1},G)$ are the maps of cohomology $\mathrm{Ext}_R^j(\theta_*)\colon H^j(\mathcal{C}') \to H^j(\mathcal{C})$ induced by the chain map of the diagram $(**_2)$ and are independent of the lifting of $\theta_*$ in $(**_1)$.

Recall that

$$
\begin{aligned}
\mathrm{Ker}\,\mathrm{Hom}(d_{n+1},\mathrm{id}) &= \{\varphi \in \mathrm{Hom}(C_n,G) \mid \varphi|B_n \equiv 0\} \\
\mathrm{Im}\,\mathrm{Hom}(d_n,\mathrm{id}) &= \{\psi \circ d_n \in \mathrm{Hom}(C_n,G) \mid \psi \in \mathrm{Hom}(C_{n-1},G)\} \\
\mathrm{Ker}\,\mathrm{Hom}(i_n,\mathrm{id}) &= \{\varphi \in \mathrm{Hom}(C_n,G) \mid \varphi|Z_n \equiv 0\}.
\end{aligned}
$$

Since by $(*_4)$

$$
\mathrm{Ext}_R^1(H_{n-1},G) = \mathrm{Ker}\,\mathrm{Hom}(i_n,\mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(\widetilde{d}_n,\mathrm{id}) = \mathrm{Ker}\,\mathrm{Hom}(i_n,\mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n,\mathrm{id})
$$

and similarly for $\mathrm{Ext}_R^1(H'_{n-1},G)$, the cohomology map $\mathrm{Ext}_R^1(\theta_*)$ is given by

$$
\mathrm{Ext}_R^1(\theta_*)([\varphi']) = [\varphi' \circ \theta], \qquad (*_{16})
$$

for all $\varphi' \in \mathrm{Hom}(C'_n,G)$ such that $\varphi'|Z'_n \equiv 0$. It is well defined because $\theta$ is a a chain map and for any $\psi' \circ d'_n \in \mathrm{Im}\,\mathrm{Hom}(d'_n,\mathrm{id})$ we have

$$
\mathrm{Ext}_R^1(\theta_*)([\varphi' + \psi' \circ d'_n]) = [\varphi' \circ \theta + \psi' \circ d'_n \circ \theta] = [\varphi' \circ \theta + \psi' \circ \theta \circ d_n] = [\varphi' \circ \theta].
$$

The map $j\colon \mathrm{Ker}\,\mathrm{Hom}(i_n,\mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n,\mathrm{id}) \to \mathrm{Ker}\,\mathrm{Hom}(d_{n+1},\mathrm{id})/\mathrm{Im}\,\mathrm{Hom}(d_n,\mathrm{id})$ is the quotient of the inclusion map $\mathrm{Ker}\,\mathrm{Hom}(i_n,\mathrm{id}) \longrightarrow \mathrm{Ker}\,\mathrm{Hom}(d_{n+1},\mathrm{id})$ given by

$$
j([\varphi]) = [\varphi], \qquad (*_{17})
$$

for any $\varphi \in \mathrm{Hom}(C_n, G)$ such that $\varphi|Z_n \equiv 0$. This map is well defined because for any $\psi \circ d_n \in \mathrm{Im}\,\mathrm{Hom}(d_n, \mathrm{id})$ we have

$$j([\varphi + \psi \circ d_n]) = [\varphi + \psi \circ d_n] = [\varphi],$$

because $B_n \subseteq Z_n$ and $Z_n = \mathrm{Ker}\,d_n$ so $\psi \circ d_n$ vanishes on $B_n$. The map $j'$ is defined anagolously. By $(*_{12})$ and $(*_{17})$ we have

$$(\mathrm{Hom}(\theta, \mathrm{id}))^*(j'([\varphi'])) = (\mathrm{Hom}(\theta, \mathrm{id}))^*([\varphi']) = [\varphi' \circ \theta]$$

for any $\varphi' \in \mathrm{Hom}(C'_n, G)$ such that $\varphi'|Z'_n \equiv 0$, and by $(*_{16})$ and $(*_{17})$ we have

$$j(\mathrm{Ext}^1_R(\theta_*)([\varphi'])) = j([\varphi' \circ \theta]) = [\varphi' \circ \theta].$$

Therefore,

$$(\mathrm{Hom}(\theta, \mathrm{id}))^* \circ j' = j \circ \mathrm{Ext}^1(\theta_*),$$

which proves that $(\dagger_4)$ commutes, and finishes the proof of naturality. $\qquad\square$

As in the case of homology, the splitting is not natural.

Spanier proves a version of Theorem 12.43 for a chain complex $C$ such that $\mathrm{Ext}_R(C, G)$ is acyclic and with $R$ a PID; see Theorem 3 in Spanier [47] (Chapter 5, Section 5).

**Remarks:**

(1) Under the isomorphism $\kappa\colon \{\varphi \in \mathrm{Hom}(Z_n, G) \mid \varphi|B_n \equiv 0\} \to \mathrm{Hom}(Z_n/B_n, G)$, the map

$$h\colon H^n(\mathrm{Hom}(C, G)) \to \{\varphi \in \mathrm{Hom}(Z_n, G) \mid \varphi|B_n \equiv 0\}$$

is given by $h([\varphi]) = \varphi|Z_n$ for any $[\varphi] \in H^n(\mathrm{Hom}(C, G))$. Composing with the isomorphism $\kappa$, we obtain the surjection (also denoted $h$)

$$h\colon H^n(\mathrm{Hom}(C, G)) \to \mathrm{Hom}(H_n(C), G)$$

given by

$$(h([\varphi]))([z]) = \varphi(z),$$

for any $[\varphi] \in H^n(\mathrm{Hom}(C, G))$ and any $[z] \in H_n(C)$; this matches Spanier's definition; see Spanier [47] (Chapter 5, Section 5, page 242). In Munkres, the map $h\colon H^n(\mathrm{Hom}(C, G)) \to \mathrm{Hom}(H_n(C), G)$ is defined on page 276 ([38], Section 45), and called the *Kronecker map* (it is denoted by $\kappa$ rather than $h$).

(2) We can prove that

$$\mathrm{Ext}^1_R(H_{n-1}, G) \cong \mathrm{Coker}\,\mathrm{Hom}(\gamma_{n-1}, \mathrm{id}) = \mathrm{Hom}(B_{n-1}, G)/\mathrm{Im}\,\mathrm{Hom}(\gamma_{n-1}, \mathrm{id}). \qquad (*_{18})$$

This will establish a connection with Spanier's proof of the naturality of the exact sequence (††); see Spanier [47] (Chapter 5, Section 5).

Recall from $(*_4)$ that $\operatorname{Ext}^1_R(H_{n-1}, G) = \operatorname{Ker} \operatorname{Hom}(i_n, \operatorname{id})/\operatorname{Im} \operatorname{Hom}(d_n, \operatorname{id})$. We already showed in $(*_7)$ that $\operatorname{Ker} \operatorname{Hom}(i_n, \operatorname{id}) \cong \operatorname{Hom}(B_{n-1}, G)$ so we just have to prove that

$$\operatorname{Im} \operatorname{Hom}(d_n, \operatorname{id}) \cong \operatorname{Im} \operatorname{Hom}(\gamma_{n-1}, \operatorname{id}). \tag{$*_{19}$}$$

This is because

$$\operatorname{Im} \operatorname{Hom}(d_n, \operatorname{id}) = \{\psi \circ d_n \in \operatorname{Hom}(C_n, G) \mid \psi \in \operatorname{Hom}(C_{n-1}, G)\}$$
$$\operatorname{Im} \operatorname{Hom}(\gamma_{n-1}, \operatorname{id}) = \{\psi \circ \gamma_{n-1} \in \operatorname{Hom}(B_{n-1}, G) \mid \psi \in \operatorname{Hom}(Z_{n-1}, G)\}$$

and since $d_n \colon C_n \to B_{n-1}$ is a surjection and $\gamma_n \colon B_n \to Z_n$ is an injection,

$$\{\psi \circ d_n \in \operatorname{Hom}(C_n, G) \mid \psi \in \operatorname{Hom}(C_{n-1}, G)\}$$
$$\cong \{\psi|B_{n-1} \in \operatorname{Hom}(B_{n-1}, G) \mid \psi \in \operatorname{Hom}(C_{n-1}, G)\}$$

and

$$\{\psi \circ \gamma_{n-1} \in \operatorname{Hom}(B_{n-1}, G) \mid \psi \in \operatorname{Hom}(Z_{n-1}, G)\}$$
$$\cong \{\psi|B_{n-1} \in \operatorname{Hom}(B_{n-1}, G) \mid \psi \in \operatorname{Hom}(Z_{n-1}, G)\},$$

but since $B_{n-1} \subseteq Z_{n-1} \subseteq C_{n-1}$, the sets of the right-hand sides of the two equations above are identical.

Therefore, we proved that the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_R(H_{n-1}, G) \longrightarrow H^n(\operatorname{Hom}(C, G)) \longrightarrow \operatorname{Hom}(H_n, G) \longrightarrow 0. \tag{††}$$

is equivalent to the exact sequence

$$0 \longrightarrow \operatorname{Coker} \operatorname{Hom}(\gamma_{n-1}, \operatorname{id}) \longrightarrow H^n(\operatorname{Hom}(C, G)) \longrightarrow \operatorname{Ker} \operatorname{Hom}(\gamma_n, \operatorname{id}) \longrightarrow 0, \tag{††$_2$}$$

which is the exact sequence found in the middle of page 243 in Spanier (and others, such as Munkres and Hatcher); see Spanier [47] (Chapter 5, Section 5). We can now refer to Spanier's proof of naturality of this sequence.

Whenever $\operatorname{Ext}^1_R(H_{n-1}(C), G)$ vanishes, we obtain the "ideal result."

Recall form Definition 12.2 that a $R$-module $M$ is divisible if for every nonzero $\lambda \in R$, the multiplication map given by $u \mapsto \lambda u$ for all $u \in M$ is surjective. Here, we let $R = \mathbb{Z}$ and $M$ be an abelian group.

**Proposition 12.44.** *If $C$ is a complex of free abelian groups, $G$ is an abelian group, and if either $H_{n-1}(C)$ or $G$ is divisible, then we have an isomorphism*

$$H^n(\mathrm{Hom}_{\mathbb{Z}}(C, G)) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(C), G)$$

*for all $n \geq 0$.*

We also have the following generalization of Theorem 4.27 to $G$-coefficients.

**Proposition 12.45.** *If $R$ is a PID, $G$ is an $R$-module, $C$ is a complex of free $R$-modules, and if $H_{n-1}(C)$ is a free $R$-module or $(0)$, then we have an isomorphism*

$$H^n(\mathrm{Hom}_R(C, G)) \cong \mathrm{Hom}_R(H_n(C), G)$$

*for all $n \geq 0$.*

**Proposition 12.46.** *If $C$ is a complex of vector spaces and $V$ is a vector space, both over the same field $K$, then we have an isomorphism*

$$H^n(\mathrm{Hom}_K(C, V)) \cong \mathrm{Hom}_K(H_n(C), V)$$

*for all $n \geq 0$. In particular, for $V = K$, we have isomorphisms*

$$H^n(\mathrm{Hom}_K(C, K)) \cong \mathrm{Hom}_K(H_n(C), K) = H_n(C)^*,$$

*where $H_n(C)^*$ is the dual of the vector space $H_n(C)$, for all $n \geq 0$.*

Since the modules $S_*(X, A; \mathbb{Z})$ are free abelian groups, Theorem 12.43 yields the following result showing that the singular cohomology groups with coefficients in an abelian group $G$ are determined by the singular homology groups with coefficients in $\mathbb{Z}$.

**Theorem 12.47.** *If $X$ is a topological space, $A$ is a subset of $X$, and $G$ is any abelian group, then there is an isomorphism relative singular cohomology*

$$H^n(X, A; G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(X, A; \mathbb{Z}), G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X, A; \mathbb{Z}), G)$$

*for all $n \geq 0$.*

Theorem 12.47 is also proved in Munkres [38] (Chapter 7, Section 53, Theorem 53.1) and in Hatcher [25] (Chapter 3, Section 3.1, Theorem 3.2).

Since the modules $S_*(X, A; R)$ are free, Theorem 12.43 has the following corollary.

**Theorem 12.48.** *If $X$ is a topological space, $A$ is a subset of $X$, $R$ is any PID, and $G$ is any $R$-module, then there is an isomorphism of relative singular cohomology*

$$H^n(X, A; G) \cong \mathrm{Hom}_R(H_n(X, A; R), G) \oplus \mathrm{Ext}^1_R(H_{n-1}(X, A; R), G)$$

*for all $n \geq 0$.*

If $A$ is a finitely generated abelian group and $G$ is any abelian group, then $\operatorname{Ext}^1_{\mathbb{Z}}(A, G)$ can be computed recursively. It is customary to drop the superscript 1 in $\operatorname{Ext}^1_R(-, -)$. We have the identities

$$\operatorname{Ext}_R\left(\bigoplus_{i \in I} A_i, B\right) \cong \prod_{i \in I} \operatorname{Ext}_R(A_i, B)$$

$$\operatorname{Ext}_R\left(A, \prod_{i \in I} B_i\right) \cong \prod_{i \in I} \operatorname{Ext}_R(A, B_i)$$

$$\operatorname{Ext}_R(A, B) \cong (0) \quad \text{if } A \text{ is projective or } B \text{ is injective,}$$

for any commutative ring $R$ and any $R$-modules. If the index set $I$ is finite, we can replace $\prod$ by $\bigoplus$. When $R = \mathbb{Z}$ we also have

$$\operatorname{Ext}_R(\mathbb{Z}, G) \cong (0)$$

$$\operatorname{Ext}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \cong G/mG,$$

where $G$ is an abelian group. This last equation is proved as follows.

*Proof.* We know that the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ m\ } \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}/m\mathbb{Z}$. Since $\operatorname{Ext}_{\mathbb{Z}}(-, G)$ is the right derived functor of $\operatorname{Hom}_{\mathbb{Z}}(-, G)$, we deduce that $\operatorname{Ext}^j_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) = (0)$ for all $j \geq 2$, and the long exact sequence given by Theorem 12.25 yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \longrightarrow \operatorname{Hom}(\mathbb{Z}, G) \xrightarrow{\operatorname{Hom}(m, G)} \operatorname{Hom}(\mathbb{Z}, G) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \longrightarrow 0.$$

Since $\operatorname{Hom}(\mathbb{Z}, G) \cong G$, we obtain an exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \longrightarrow G \xrightarrow{\ m\ } G \xrightarrow{\ p\ } \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \longrightarrow 0,$$

and since $p$ is surjective and $\operatorname{Im} m = \operatorname{Ker} p$, we have

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \cong G/\operatorname{Ker} p \cong G/mG,$$

as claimed. $\qquad\square$

We also use the following rules for $\operatorname{Hom}_R(-, -)$:

$$\operatorname{Hom}_R\left(\bigoplus_{i \in I} A_i, B\right) \cong \prod_{i \in I} \operatorname{Hom}_R(A_i, B)$$

$$\operatorname{Hom}_R\left(A, \prod_{i \in I} B_i\right) \cong \prod_{i \in I} \operatorname{Hom}_R(A, B_i)$$

for any commutative ring and any $R$-modules. If the index set $I$ is finite, we can replace $\prod$ by $\bigoplus$. When $R = \mathbb{Z}$, we also have

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \cong G$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \cong \mathrm{Ker}\,(G \xrightarrow{m} G),$$

where $G$ is an abelian group. The above formula is proved as follows.

*Proof.* We have the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0.$$

Since $\mathrm{Hom}_{\mathbb{Z}}(-, G)$ is right-exact, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(m,G)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G).$$

Since $\mathrm{Hom}(\mathbb{Z}, G) \cong G$, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \longrightarrow G \xrightarrow{m} G,$$

which yields $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, G) \cong \mathrm{Ker}\,(G \xrightarrow{m} G)$, as claimed. $\qquad\square$

These rules imply that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \cong (0)$$

and

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}.$$

For details, see Munkres [38] (Chapter 7, Sections 52 and 54) and Hatcher [25] (Chapter 3, Section 3.1).

If $A$ is a finitely generated abelian group, we know that $A$ can be written (uniquely) as a direct sum

$$A = F \oplus T$$

where $A$ is a free abelian group and $F$ is a torsion abelian group. Then, the above rules imply the following useful result that allows to compute integral cohomology from integral homology.

**Proposition 12.49.** *Let $C$ be a chain complex of free abelian groups. If $H_{n-1}(C)$ and $H_n(C)$ are finitely generated and if we write $H_n(C) = F_n \oplus T_n$ where $F_n$ is the free part of $H_n(C)$ and $T_n$ is the torsion part of $H_n(C)$ (and similarly $H_{n-1}(C) = F_{n-1} \oplus T_{n-1}$), then we have an isomorphism*

$$H^n(\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})) \cong F_n \oplus T_{n-1}.$$

*In particular, the above holds for the singular homology groups $H_n(X; \mathbb{Z})$ and the singular cohomology groups $H^n(X; \mathbb{Z})$ of a topological space $X$; that is,*

$$H^n(X; \mathbb{Z}) \cong F_n \oplus T_{n-1}$$

*where $H_n(X; \mathbb{Z}) = F_n \oplus T_n$ with $F_n$ free and $T_n$ a torsion abelian group.*

*Proof.* Using the above rules, since $T_n$ is a finitely generated torsion abelian group it is a direct sum of abelian groups of the form $\mathbb{Z}/m\mathbb{Z}$, and since $F_n$ is a finitely generated free abelian group it is of the form $\mathbb{Z}^n$, so we have

$$\text{Hom}_{\mathbb{Z}}(H_n(C), \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(F_n \oplus T_n, \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(F_n, \mathbb{Z}) \oplus \text{Hom}_{\mathbb{Z}}(T_n, \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(F_n, \mathbb{Z}) \cong F_n,$$

and

$$\text{Ext}_{\mathbb{Z}}(H_{n-1}(C), \mathbb{Z}) = \text{Ext}_{\mathbb{Z}}(F_{n-1} \oplus T_{n-1}, \mathbb{Z}) \cong \text{Ext}_{\mathbb{Z}}(F_{n-1}, \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}(T_{n-1}, \mathbb{Z})$$
$$\cong \text{Ext}_{\mathbb{Z}}(T_{n-1}, \mathbb{Z}) \cong T_{n-1}.$$

By Theorem 12.43, we conclude that $H^n(\text{Hom}_{\mathbb{Z}}(C, \mathbb{Z})) \cong F_n \oplus T_{n-1}$. $\qquad\square$

Proposition 12.49 is found in Bott and Tu [2] (Chapter III, Corollary 15.14.1), Hatcher [25] (Chapter 3, Corollary 3.3), and Spanier [47] (Chapter 5, Section 5, Corollary 4). As an application of Proposition 12.49, we can compute the cohomology groups of the real projective spaces $\mathbb{RP}^n$ and of the complex projective space $\mathbb{CP}^n$. Recall from Section 4.2 that the homology groups of $\mathbb{CP}^n$ and $\mathbb{RP}^n$ are given by

$$H_p(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0, 2, 4, \ldots, 2n \\ (0) & \text{otherwise,} \end{cases}$$

and

$$H_p(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0 \text{ and for } p = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } p \text{ odd, } 0 < p < n \\ (0) & \text{otherwise.} \end{cases}$$

Using Proposition 12.49, we obtain

$$H^p(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0, 2, 4, \ldots, 2n \\ (0) & \text{otherwise,} \end{cases}$$

and

$$H^p(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } p = 0 \text{ and for } p = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } p \text{ even, } 0 < p \leq n \\ (0) & \text{otherwise.} \end{cases}$$

Spanier [47] (Chapter 5, Sections 2 and 5) and Munkres [38] (Chapter 7, Section 56) discuss other types of universal coefficient theorems.

There is also a notion of tensor product $C \otimes D$ of chain complexes $C$ and $D$, and there are formulae relating the homology of $C \otimes D$ to the homology of $C$ and the homology of $D$. There are also formulae relating the cohomology of $C \otimes D$ to the cohomology of $C$ and the cohomology of $D$. Such formulae are known as *Künneth Theorems* (or *Künneth Formulae*). We will not discuss these theorems and instead refer the reader to Munkres [38] (Chapter 7, Sections 58 and 60), Hatcher [25] (Chapter 3, Sections 3.2 and 3.B), Spanier [47] (Chapter 5), and Rotman [41] (Chapter 11).

# Chapter 13

# Cohomology of Sheaves

In this chapter we apply the results of Sections 12.3 and 12.4 to the case where $\mathbf{C}$ is the abelian category of sheaves of $R$-modules on a topological space $X$, $\mathbf{D}$ is the (abelian) category of abelian groups, and $T$ is the left-exact global section functor $\Gamma(X, -)$, with $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ for every sheaf $\mathcal{F}$ on $X$. It turns out that the category of sheaves has enough injectives, thus the right derived functors $R^p\Gamma(X, -)$ exist, and for every sheaf $\mathcal{F}$ on $X$, the cohomology groups $R^p\Gamma(X, -)(\mathcal{F})$ are defined. These groups denoted by $H^p(X, \mathcal{F})$ are called the *cohomology groups* of the sheaf $\mathcal{F}$ (or the *cohomology groups of $X$ with values in $\mathcal{F}$*).

In principle, computing the cohomology groups $H^p(X, \mathcal{F})$ requires finding injective resolutions of sheaves. However injective sheaves are very big and hard to deal with. Fortunately, there is a class of sheaves known as *flasque* sheaves (due to Godement) which are $\Gamma(X, -)$-acyclic, and every sheaf has a resolution by flasque sheaves. Therefore, by Proposition 12.27, the cohomology groups $H^p(X, \mathcal{F})$ can be computed using flasque resolutions.

If the space $X$ is paracompact, then it turns out that for any sheaf $\mathcal{F}$, the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ are isomorphic to the cohomology groups $H^p(X, \mathcal{F})$. Furthermore, if $\mathcal{F}$ is a presheaf, then the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ and $\check{H}^p(X, \widetilde{\mathcal{F}})$ are isomorphic, where $\widetilde{\mathcal{F}}$ is the sheafification of $\mathcal{F}$. Several other results (due to Leray and Henri Cartan) about the relationship between Čech cohomology and sheaf cohomology will be stated.

When $X$ is a topological manifold (thus paracompact), for every $R$-module $G$, we will show that the singular cohomology groups $H^p(X; G)$ are isomorphic to the cohomology groups $H^p(X, \widetilde{G}_X)$ of the constant sheaf $\widetilde{G}_X$. Technically, we will need to define *soft* and *fine* sheaves.

We will also define Alexander–Spanier cohomology and prove that it is equivalent to sheaf cohomology (and Čech cohomology) for paracompact spaces and for the constant sheaf $\widetilde{G}_X$.

# 13.1   Cohomology Groups of a Sheaf of Modules

It is convenient to use for a definition of an injective sheaf the condition of Proposition 12.3 which applies to abelian categories.

**Definition 13.1.** A sheaf $\mathcal{I}$ is *injective* if for any injective sheaf map $h \colon \mathcal{F} \to \mathcal{G}$ and any sheaf map $f \colon \mathcal{F} \to \mathcal{I}$, there is some sheaf map $\widehat{f} \colon \mathcal{G} \to \mathcal{I}$ extending $f \colon \mathcal{F} \to \mathcal{I}$ in the sense that $f = \widehat{f} \circ h$, as in the following commutative diagram:

$$
\begin{array}{ccc}
0 \longrightarrow \mathcal{F} & \overset{h}{\longrightarrow} & \mathcal{G} \\
\quad {\scriptstyle f}\downarrow & \swarrow\ {\scriptstyle \widehat{f}} & \\
\mathcal{I}. &  &
\end{array}
$$

We need to prove that the category of sheaves of $R$-modules has enough injectives.

**Proposition 13.1.** *For any sheaf $\mathcal{F}$ of $R$-modules, there is an injective sheaf $\mathcal{I}$ and an injective sheaf homomorphism $\varphi \colon \mathcal{F} \to \mathcal{I}$.*

*Proof.* We know that the category of $R$-modules has enough injectives (see Theorem 12.6). For every $x \in X$, pick some injection $\mathcal{F}_x \longrightarrow I^x$ with $I^x$ an injective $R$-module, which always exists by Theorem 12.6. Define the "skyscraper sheaf" $\mathcal{I}^x$ as the sheaf given by

$$
\mathcal{I}^x(U) = \begin{cases} I^x & \text{if } x \in U \\ (0) & \text{if } x \notin U \end{cases}
$$

for every open subset $U$ of $X$ (we use a superscript in $\mathcal{I}^X$ to avoid the potential confusion with the stalk at $x$). It is easy to check that there is an isomorphism

$$
\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, \mathcal{I}^x) \cong \mathrm{Hom}_R(\mathcal{F}_x, I^x)
$$

for any sheaf $\mathcal{F}$, and this implies that $\mathcal{I}^x$ is an injective sheaf. We also have a sheaf map from $\mathcal{F}$ to $\mathcal{I}^x$. Consequently we obtain an injective sheaf map

$$
\mathcal{F} \longrightarrow \prod_{x \in X} \mathcal{I}^x.
$$

Since a product of injective sheaves is injective, $\mathcal{F}$ is embedded into an injective sheaves.   $\square$

**Remark:** The category of sheaves does have enough projectives. This is the reason why projective resolutions of sheaves are of little interest.

As we explained in Section 12.2, since the category of sheaves is an abelian category and since it has enough injectives, Proposition 12.11 holds for sheaves; that is, every sheaf has some injective resolution. Since the global section functor on sheaves is left-exact (see Proposition 11.25(4)), as a corollary of Theorem 12.21 we make the following definition.

**Definition 13.2.** Let $X$ be a topological space, and let $\Gamma(X, -)$ be the global section functor from the abelian category $\mathbf{Sh}(X)$ of sheaves of $R$-modules to the category of abelian groups. The *cohomology groups* of the sheaf $\mathcal{F}$ (or the *cohomology groups of $X$ with values in $\mathcal{F}$*), denoted by $H^p(X, \mathcal{F})$, are the groups $R^p\Gamma(X, -)(\mathcal{F})$ induced by the right derived functor $R^p\Gamma(X, -)$ (with $p \geq 0$).

To compute the sheaf cohomology groups $H^p(X, \mathcal{F})$, pick any resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \cdots$$

by injective sheaves $\mathcal{I}^n$, apply the global section functor $\Gamma(X, -)$ to obtain the complex of $R$-modules

$$0 \xrightarrow{\delta^{-1}} \mathcal{I}^0(X) \xrightarrow{\delta^0} \mathcal{I}^1(X) \xrightarrow{\delta^1} \mathcal{I}^2(X) \xrightarrow{\delta^2} \cdots ,$$

and then

$$H^p(X, \mathcal{F}) = \operatorname{Ker} \delta^p / \operatorname{Im} \delta^{p-1}.$$

By Theorem 12.35 the right derived functors $R^p\Gamma(X, -)$ constitute a universal $\delta$-functor, so all the properties of $\delta$-functors apply.

In algebraic geometry it is useful to consider sheaves defined on a ringed space generalizing modules.

**Definition 13.3.** Given a ringed space $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-*module* (or *sheaf of modules over $X$*) is a sheaf $\mathcal{F}$ of abelian groups on $X$ such that for every open subset $U$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and the following conditions hold for all open subsets $V \subseteq U$:

$$
\begin{array}{ccc}
\mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\
{\scriptstyle (\rho\mathcal{O}_V^U, \rho\mathcal{F}_V^U)} \Big\downarrow & & \Big\downarrow {\scriptstyle \rho\mathcal{F}_V^U} \\
\mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V).
\end{array}
$$

Any sheaf of $R$-modules on $X$ can be viewed as an $\mathcal{O}_X$-module with respect to the constant sheaf $\widetilde{R}_X$. There is an obvious notion of morphism of $\mathcal{O}_X$-modules induced by the notion of morphism of sheaves. The category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ is denoted by $\mathfrak{Mod}(X, \mathcal{O}_X)$. Proposition 13.1 has the following generalization.

**Proposition 13.2.** *For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, there is an injective $\mathcal{O}_X$-module $\mathcal{I}$ and an injective morphism $\varphi \colon \mathcal{F} \to \mathcal{I}$.*

A proof of Proposition 13.2 can be found in Hartshorne [24] (Chapter III, Section 2, Proposition 2.2). As a consequence, we can define the cohomology groups $H^p(X, \mathcal{F})$ of the $\mathcal{O}_X$-module $\mathcal{F}$ over the ringed space $(X, \mathcal{O}_X)$ as the groups induced by the right derived functors $R^p\Gamma(X, -)$ of the functor $\Gamma(X, -)$ from the category $\mathfrak{Mod}(X, \mathcal{O}_X)$ of $\mathcal{O}_X$-modules to the category of abelian groups (with $p \geq 0$).

We now turn to flasque sheaves.

## 13.2    Flasque Sheaves

The notion of flasque sheaf is due to Godement (see [18], Chapter 3). The word *flasque* is French and it is hard to find an accurate English translation for it. The closest appoximations we can think of are *flabby*, *limp*, or *soggy*; a good example of a "flasque" object is a slab of jello, or a jellyfish. Most authors use the French word "flasque" so we will use it too.

**Definition 13.4.** A sheaf $\mathcal{F}$ on a topological space $X$ is *flasque* if for every open subset $U$ of $X$ the restriction map $\rho_U^X \colon \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

We will see shortly that injective sheaves are flasque. Although this is not obvious from the definition, the notion of being flasque is local.

**Proposition 13.3.** *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flasque, so is $\mathcal{F} \restriction U$ for every open subset $U$ of $X$. Conversely, if for every $x \in X$, there is a neighborhood $U$ such that $\mathcal{F} \restriction U$ is flasque, then $\mathcal{F}$ is flasque.*

*Proof.* The first statement is trivial, let us prove the converse. Given any open set $V$ of $X$, let $s$ be a section of $\mathcal{F}$ over $V$. Let $T$ be the set of all pairs $(U, \sigma)$, where $U$ is an open in $X$ containing $V$, and $\sigma$ is an extension of $s$ to $U$. Partially order $T$ by saying that $(U_1, \sigma_1) \leq (U_2, \sigma_2)$ if $U_1 \subseteq U_2$ and $\sigma_2$ extends $\sigma_1$, and observe that $T$ is inductive, which means that every chain has an upper bound. Zorn's lemma provides us with a maximal extension of $s$ to a section $\sigma$ over an open set $U_0$. Were $U_0$ not $X$, there would exist an open set $W$ in $X$ not contained in $U_0$ such that $\mathcal{F} \restriction W$ is flasque. Thus we could extend the section $\rho_{U_0 \cap W}^{U_0}(\sigma)$ to a section $\sigma'$ of $\mathcal{F}$. Since $\sigma$ and $\sigma'$ agree on $U_0 \cap W$ by construction, their common extension to $U_0 \cup W$ extends $s$, a contradiction. $\qquad\square$

**Proposition 13.4.** *Every $\mathcal{O}_X$-module may be embedded in a canonical functorial way into a flasque $\mathcal{O}_X$-module. Consequently, every $\mathcal{O}_X$-module has a canonical flasque resolution (i.e., a resolution by flasque $\mathcal{O}_X$-modules.)*

*Proof.* Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and define a presheaf $C^0(X, \mathcal{F})$ by

$$U \mapsto \prod_{x \in U} \mathcal{F}_x.$$

It is immediate that $C^0(X, \mathcal{F})$ is actually a sheaf and that we have an *injection* of $\mathcal{O}_X$-modules $\mathcal{F} \longrightarrow C^0(X, \mathcal{F})$. An element of $C^0(X, \mathcal{F})$ over any open set $U$ is a collection $(s_x)$ of elements indexed by $U$, each $s_x$ lying over the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. Such a sheaf is flasque because every $U$-indexed sequence $s_x$ can be extended to an $X$-indexed sequence by assigning any arbitrary element of $\mathcal{F}_x$ to any $x \in X - U$. hence $\mathfrak{Mod}(X, \mathcal{O}_X)$ possesses enough flasque sheaves.

If $\mathcal{Z}_1$ is the (sheaf) cokernel of the canonical injection $\mathcal{F} \longrightarrow C^0(X, \mathcal{F})$, we define $C^1(X, \mathcal{F})$ to be the flasque sheaf $C^0(X, \mathcal{Z}_1)$. In general, $\mathcal{Z}_n$ is the cokernel of the injection

$\mathcal{Z}_{n-1} \longrightarrow C^0(X, \mathcal{Z}_{n-1})$, and $C^n(X, \mathcal{F})$ is the flasque sheaf $C^0(X, \mathcal{Z}_n)$. Putting all this information together, we obtain the desired flasque resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0(X, \mathcal{F}) \longrightarrow C^1(X, \mathcal{F}) \longrightarrow C^2(X, \mathcal{F}) \longrightarrow \cdots$$

as claimed. $\square$

**Remark:** The resolution of $\mathcal{F}$ constructed in Proposition 13.4 will be called the *canonical flasque resolution of $\mathcal{F}$* or the *Godement resolution of $\mathcal{F}$*.

Given two sheaves of $R$-modules $\mathcal{F}'$ and $\mathcal{F}''$, we obtain a presheaf $\mathcal{F}' \oplus \mathcal{F}''$ by setting

$$\mathcal{F} = (\mathcal{F}' \oplus \mathcal{F}'')(U) = \mathcal{F}'(U) \oplus \mathcal{F}''(U)$$

for every open subset $U$ of $X$. Actually, $\mathcal{F}' \oplus \mathcal{F}''$ is a sheaf. We call $\mathcal{F}'$ and $\mathcal{F}''$ *direct factors* of $\mathcal{F}$.

Here is the principal property of flasque sheaves.

**Theorem 13.5.** *Let* $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ *be an exact sequence of $\mathcal{O}_X$-modules, and assume $\mathcal{F}'$ is flasque. Then this sequence is exact as a sequence of presheaves. If both $\mathcal{F}'$ and $\mathcal{F}$ are flasque, so is $\mathcal{F}''$. Finally, any direct factor of a flasque sheaf is flasque.*

*Proof.* Given any open set, $U$, we must prove that

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\varphi} \mathcal{F}(U) \xrightarrow{\psi} \mathcal{F}''(U) \longrightarrow 0$$

is exact. By Proposition 11.25(4), the sole problem is to prove that $\mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$ is surjective. By restricting attention to $U$, we may assume $U = X$; hence, we are going to prove that a global section of $\mathcal{F}''$ may be lifted to a global section of $\mathcal{F}$. Let $s''$ be a global section of $\mathcal{F}''$, then by Proposition 11.18(iv), locally $s''$ may be lifted to sections of $\mathcal{F}$. Let $T$ be the family of all pairs $(U, \sigma)$ where $U$ is an open in $X$, and $\sigma$ is a section of $\mathcal{F}$ over $U$ whose image, $\sigma''$, in $\mathcal{F}''(U)$ equal $\rho^X_{\mathcal{F}''(U)}(s'')$. Partially order $T$ as in the proof of Proposition 13.3 and observe that $T$ is inductive. Zorn's lemma provides us with a maximal lifting of $s''$ to a section $\sigma \in \mathcal{F}(U_0)$.

Were $U_0$ not $X$, there would exist $x \in X - U_0$, a neighborhood, $V$, of $x$, and a section $\tau$ of $\mathcal{F}$ over $V$ which is a local lifting of $\rho^X_V(s'')$. The sections $\rho^{U_0}_{U_0 \cap V}(\sigma)$, $\rho^V_{U_0 \cap V}(\tau)$ have the same image under $\psi$ in $\mathcal{F}''(U_0 \cap V)$ so their difference maps to 0. Since $\mathrm{SIm}\,\varphi = \mathrm{Ker}\,\psi$, there is a section $t$ of $\mathcal{F}'(U_0 \cap V)$ such that

$$\rho^{U_0}_{U_0 \cap V}(\sigma) = \rho^V_{U_0 \cap V}(\tau) + \varphi(t).$$

Since $\mathcal{F}'$ is flasque, the section $t$ is the restriction of a section $t' \in \mathcal{F}'(V)$. Upon replacing $\tau$ by $\tau + \varphi(t')$ (which does not affect the image in $\mathcal{F}''(V)$), we may assume that $\rho^{U_0}_{U_0 \cap V}(\sigma) = \rho^V_{U_0 \cap V}(\tau)$; that is, that $\sigma$ and $\tau$ agree on the overlap $U_0 \cap V$. Clearly, we may extend $\sigma$ (*via* $\tau$) to $U_0 \cup V$, contradicting the maximality of $(U_0, \sigma)$; hence, $U_0 = X$.

Now suppose that $\mathcal{F}'$ and $\mathcal{F}$ are flasque. If $s'' \in \mathcal{F}''(U)$, then by the above, there is a section $s \in \mathcal{F}(U)$ mapping onto $s''$. Since $\mathcal{F}$ is also flasque, we may lift $s$ to a global section, $t$, of $\mathcal{F}$. The image, $t''$, of $t$ in $\mathcal{F}''(X)$ is the required extension of $s''$ to a global section of $\mathcal{F}''$.

Finally, assume that $\mathcal{F}$ is flasque, and that $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ for some sheaf $\mathcal{F}''$. For any open subset $U$ of $X$ and any section $s \in \mathcal{F}'(U)$, we can make $s$ into a section $\widehat{s} \in \mathcal{F}(U)$ by setting the component of $\widetilde{s}(U)$ in $\mathcal{F}''(U)$ equal to the zero section. Since $\mathcal{F}$ is flasque, there is some section $t \in \mathcal{F}(X)$ such that $\rho_U^X(t) = \widetilde{s}$. But $t = t_1 + t_2$ for some unique $t_1 \in \mathcal{F}'(X)$ and $t_2 \in \mathcal{F}''(X)$, and since $\rho_U^X$ is linear,

$$s + 0 = \widetilde{s} = \rho_U^X(t) = \rho_U^X(t_1) + \rho_U^X(t_2)$$

with $\rho_U^X(t_1) \in \mathcal{F}'(U)$ and $\rho_U^X(t_2) \in \mathcal{F}''(U)$, so $s = \rho_U^X(t_1)$ with $t_1 \in \mathcal{F}'(X)$, which shows that $\mathcal{F}'$ is flasque.                                                                                       $\square$

The following general proposition from Tohoku ([21], Section 3.3) implies that flasque sheaves are $\Gamma(X, -)$-acyclic. It will also imply that soft sheaves are $\Gamma(X, -)$-acyclic (see Section 13.5). Since the only functor involved is the global section functor, it is customary to abbreviate $\Gamma(X, -)$-acyclic to acyclic.

**Proposition 13.6.** *Let $T$ be an additive functor from the abelian category $\mathbf{C}$ to the abelian category $\mathbf{C}'$, and suppose that $\mathbf{C}$ has enough injectives. Let $X$ be a class of objects in $\mathbf{C}$ which satisfies the following conditions:*

*(i)  $\mathbf{C}$ possesses enough $X$-objects,*

*(ii)  If $A$ is an object of $\mathbf{C}$ and $A$ is a direct factor of some object in $X$, then $A$ belongs to $X$,*

*(iii)  If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact and if $A'$ belongs to $X$, then $0 \longrightarrow T(A') \longrightarrow T(A) \longrightarrow T(A'') \longrightarrow 0$ is exact, and if $A$ also belongs to $X$, then $A''$ belongs to $X$.*

*Under these conditions, every injective object belongs to $X$, for each $M$ in $X$ we have $R^n T(M) = (0)$ for $n > 0$, and finally the functors $R^n T$ may be computed by taking $X$-resolutions.*

*Proof.* The following proof is due to Steve Shatz. Let $I$ be an injective of $\mathbf{C}$. By (i), $I$ admits a monomorphism into some object $M$ of the class $X$. We have an exact sequence

$$0 \longrightarrow I \xrightarrow{\varphi} M \longrightarrow \operatorname{Coker} \varphi \longrightarrow 0,$$

and as $I$ is injective and $\varphi \colon I \to M$ is an injective sheaf map, there is a map $p \colon M \to I$ such that $p \circ \varphi = \mathrm{id}$ as in the following diagram

$$\begin{array}{ccc} 0 \longrightarrow I & \xrightarrow{\varphi} & M \\ {\scriptstyle \mathrm{id}} \downarrow & \swarrow {\scriptstyle p} & \\ I, & & \end{array}$$

and by Proposition 2.1(2), the above sequence is split so $I$ is a direct factor of $M$; hence (ii) implies $I$ lies in $X$. Let us now show that $R^n T(M) = (0)$ for $n > 0$ if $M$ lies in $X$. Now, $\mathbf{C}$ possesses enough injectives, so if we set $C_0 = \mathrm{Coker}(M \longrightarrow I_0)$ and inductively $C_{i+1} = \mathrm{Coker}(C_i \longrightarrow I_{i+1})$ where the maps $M \longrightarrow I_0$ and $C_i \longrightarrow I_{i+1}$ are injections and the $I_i$ are injective, we have the exact sequences

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow C_0 \longrightarrow 0$$
$$0 \longrightarrow C_0 \longrightarrow I_1 \longrightarrow C_1 \longrightarrow 0$$
$$0 \longrightarrow C_1 \longrightarrow I_2 \longrightarrow C_2 \longrightarrow 0$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$0 \longrightarrow C_n \longrightarrow I_{n+1} \longrightarrow C_{n+1} \longrightarrow 0$$
$$\dots\dots\dots\dots\dots\dots\dots$$

Here, each $I_i$ is injective, so lies in $X$. As $M$ belongs to $X$, (iii) shows that $C_0$ lies in $X$. By induction, $C_i$ belongs to $X$ for every $i \geq 0$. Again, by (iii), the sequences

$$0 \longrightarrow T(M) \longrightarrow T(I_0) \longrightarrow T(C_0) \longrightarrow 0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$0 \longrightarrow T(C_n) \longrightarrow T(I_{n+1}) \longrightarrow T(C_{n+1}) \longrightarrow 0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

are exact. Then, as in the proof of Proposition 12.11 we obtain the exact sequence

$$0 \longrightarrow T(M) \longrightarrow T(I_0) \longrightarrow T(I_1) \longrightarrow T(I_2) \longrightarrow \cdots$$

and this proves that $R^n T(M) = (0)$ for positive $n$. Finally, by Proposition 12.27, the functors $R^n T$ may be computed from arbitrary $X$-resolutions (which exist by (i)). $\qquad\square$

Using Theorem 13.5, Proposition 13.6 applied to the family of flasque sheaves yields the following result.

**Proposition 13.7.** *Flasque sheaves are acyclic, that is $H^p(X, \mathcal{F}) = (0)$ for every flasque sheaf $\mathcal{F}$ and all $p \geq 1$, and the cohomology groups $H^p(X, \mathcal{F})$ of any arbitrary sheaf $\mathcal{F}$ can be computed using flasque resolutions.*

In view of Proposition 13.2, we also have the following result.

**Proposition 13.8.** *If $(X, \mathcal{O}_X)$ is a ringed space, then the right derived functors of the functor $\Gamma(X, -)$ from the category $\mathfrak{Mod}(X, \mathcal{O}_X)$ of $\mathcal{O}_X$-modules to the category of abelian groups coincide with the sheaf cohomology functors $H^p(X, -)$.*

*Proof.* The right derived functors of the functor $\Gamma(X, -)$ from the category $\mathfrak{Mod}(X, \mathcal{O}_X)$ of $\mathcal{O}_X$-modules to the category of abelian groups is computed using resolutions of injectives in the category $\mathfrak{Mod}(X, \mathcal{O}_X)$. But injective sheaves are flasque, and flasque sheaves are acyclic, so by Proposition 12.27 these resolutions compute sheaf cohomology. $\qquad\square$

Our next goal is to compare Čech cohomology and sheaf cohomology.

# 13.3    Comparison of Čech Cohomology and Sheaf Cohomology

We begin by proving that for every space $X$, every open cover $\mathcal{U}$ of $X$, every sheaf $\mathcal{F}$ of $R$-modules on $X$, and every $p \geq 0$, there is a homomorphism

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}).$$

For every open subset $U$ of $X$ let $\mathcal{U}_{/U}$ denote the induced covering of $U$ consisting of all open subsets of the form $U_i \cap U$, with $U_i \in \mathcal{U}$. Then it is immediately verified that the presheaf $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ defined by

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F})(U) = C^p(\mathcal{U}_{/U}, \mathcal{F})$$

for any open subset $U$ of $X$ is a sheaf. The crucial property of the sheaves $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is that the complex

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \cdots \longrightarrow \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \cdots$$

is a resolution of the sheaf $\mathcal{F}$.

**Proposition 13.9.** *For every open cover $\mathcal{U}$ of the space $X$, for every $\mathcal{F}$ of $R$-modules on $X$, the complex*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \cdots \longrightarrow \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \cdots$$

*is a resolution of the sheaf $\mathcal{F}$.*

*Sketch of proof.* We follow Brylinski [6] (Section 1.3, Proposition 1.3.3). By Proposition 11.23(ii) it suffices to show that the stalk sequence

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F})_x \stackrel{\delta}{\longrightarrow} \cdots \longrightarrow \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \stackrel{\delta}{\longrightarrow} \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})_x \stackrel{\delta}{\longrightarrow} \cdots$$

is exact for every $x \in X$, and since direct limits of exact sequences are still exact it suffice to show that for every $x \in X$, there is some open neighborhood $V$ of $x$ such that the sequence

$$0 \longrightarrow \mathcal{F}(W) \longrightarrow C^0(\mathcal{U}_W, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \cdots \longrightarrow C^p(\mathcal{U}_{/W}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} C^{p+1}(\mathcal{U}_{/W}, \mathcal{F}) \stackrel{\delta}{\longrightarrow} \cdots$$

is exact for every open subset $W$ of $V$. Pick $V = U_{i_0}$ for some open subset $U_{i_0}$ in  such that $x \in U_{i_0}$. Then for $W \subseteq V = U_{i_0}$, the open cover $\{U_i \cap W \mid U_i \in \mathcal{U}\}$ contains $W = W \cap U_{i_0}$. The map with domain $\mathcal{F}(W)$ is clearly injective and we conclude by using the following simple proposition which is proved in Brylinski [6] (Section 1.3, Lemma 1.3.2) and Bredon [5] (Chapter III, Lemma 4.8):

**Proposition 13.10.** *If $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of $X$ and if $U_i = X$ for some index $i$, then for any presheaf $\mathcal{F}$ of $R$-modules we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = (0)$ for all $p > 0$.*

It follows that the above sequence is exact. □

**Proposition 13.11.** *For every space $X$, every open cover $\mathcal{U}$ of $X$, every sheaf $\mathcal{F}$ of $R$-modules on $X$, and every $p \geq 0$, there is a homomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

*from Čech cohomology to sheaf cohomology. Consequently there is also a homomorphism*

$$\check{H}^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

*for every $p \geq 0$.*

*Proof.* By Proposition 13.9, we have a resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^*(\mathcal{U}, \mathcal{F})$ of the sheaf $\mathcal{F}$. For every injective resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathbf{I}$ of $\mathcal{F}$, by Theorem 12.15, there is a map of resolutions from $\mathcal{C}^*(\mathcal{U}, \mathcal{F})$ to $\mathbf{I}$ lifting the identity and unique up to homotopy. Thus, there is a homomorphism of cohomology $\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$. Since $\check{H}^p(X, \mathcal{F})$ is a direct limit of the $\check{H}^p(\mathcal{U}, \mathcal{F})$, we obtain the homomorphism $\check{H}^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ by passing to the limit. □

In general, the homomorphism $\check{H}^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ of Proposition 13.11 is neither injective nor surjective. We will seek conditions that imply that it is an isomorphism.

The strategy to prove that the maps $\check{H}^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ are isomorphisms is to prove that (under certain conditions) the family of functors $(\check{H}^p(X, \mathcal{F}))_{p \geq 0}$ is a universal $\delta$-functor. Since for a sheaf, $\check{H}^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$, by Proposition 12.31 we obtain the desired isomorphisms.

We begin by proving that the functors $\check{H}^p(\mathcal{U}, -)$ on sheaves are erasable. Next, we will show that the family $(\check{H}^p(\mathcal{U}, -))_{p \geq 0}$ is a $\delta$-functors on sheaves. To do this, we will first show that they constitute a $\delta$-functor on preshaves, and then use the fact that if $X$ is paracompact and if $\mathcal{F}$ is a presheaf, then $\check{H}^p(X, \mathcal{F}) \cong \check{H}^p(X, \widetilde{\mathcal{F}})$ for all $p \geq 0$.

**Proposition 13.12.** *For every space $X$, every open cover $\mathcal{U}$ of $X$, if the sheaf $\mathcal{F}$ is flasque then*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = (0) \quad p \geq 1.$$

*Consequently the functors $\check{H}^p(\mathcal{U}, -)$ on sheaves are erasable for all $p \geq 1$.*

*Proof.* Proposition 13.12 is proved in Godement [18] (Chapter 5, Theorem 5.2.3), Hartshorne [24] (Chapter III, Proposition 4.3), and Bredon [5] (Chapter III, Corollary 4.10).

Observe that since $\mathcal{F}$ is assumed to be flasque, the sheaves $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are also flasque because the restriction of $\mathcal{F}$ to any open subset $U_{i_0 \cdots i_p}$ is flasque and a product of flasque

sheaves is flasque. Thus by Proposition 13.9 $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^*(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$ by flasque sheaves. By Proposition 13.7 the cohomology groups $H^p(X, \mathcal{F})$ can be computed using this resolution, but by definition this resolution computes the cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$, so we get

$$H^p(X, \mathcal{F}) = \check{H}^p(\mathcal{U}, \mathcal{F}), \quad \text{for all } p \geq 0.$$

However since $\mathcal{F}$ is flasque, by Proposition 13.7 we have $H^p(X, \mathcal{F}) = (0)$ for all $p \geq 1$, so $\check{H}^p(\mathcal{U}, \mathcal{F}) = (0)$ for all $p \geq 1$. Since every sheaf can be embedded in a flasque sheaf (Proposition 13.4), the functors $\check{H}^p(\mathcal{U}, -)$ are erasable for all $p \geq 1$. $\qquad \square$

**Remark:** In fact, it can be shown that $\check{H}^p(\mathcal{U}, \mathcal{F}) = (0)$ for all $p \geq 1$ if $\mathcal{F}$ is an injective presheaf, but this is harder to prove. Thus, the functors $\check{H}^p(\mathcal{U}, -)$ on presheaves are erasable for all $p \geq 1$.

The next important fact is that, on *presheaves*, the functors $C^p(\mathcal{U}, -)$ are exact.

**Proposition 13.13.** *For every space $X$ and every open cover $\mathcal{U}$ of $X$, the functor $C^p(\mathcal{U}, -)$ from presheaves to abelian groups is exact for all $p \geq 0$.*

*Proof.* If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \qquad\qquad (*)$$

is an exact sequence of presheaves, then the sequence

$$0 \longrightarrow C^p(\mathcal{U}, \mathcal{F}') \longrightarrow C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{F}'') \longrightarrow 0$$

is of the form

$$0 \longrightarrow \prod_{(i_0,\ldots,i_p)} \mathcal{F}'(U_{i_0\cdots i_p}) \longrightarrow \prod_{(i_0,\ldots,i_p)} \mathcal{F}(U_{i_0\cdots i_p}) \longrightarrow \prod_{(i_0,\ldots,i_p)} \mathcal{F}''(U_{i_0\cdots i_p}) \longrightarrow 0.$$

But since $(*)$ is an exact sequence of presheaves, every sequence

$$0 \longrightarrow \mathcal{F}'(U_{i_0\cdots i_p}) \longrightarrow \mathcal{F}(U_{i_0\cdots i_p}) \longrightarrow \mathcal{F}''(U_{i_0\cdots i_p}) \longrightarrow 0$$

is exact, and since exactness is preserved under direct products, the sequence

$$0 \longrightarrow \prod_{(i_0,\ldots,i_p)} \mathcal{F}'(U_{i_0\cdots i_p}) \longrightarrow \prod_{(i_0,\ldots,i_p)} \mathcal{F}(U_{i_0\cdots i_p}) \longrightarrow \prod_{(i_0,\ldots,i_p)} \mathcal{F}''(U_{i_0\cdots i_p}) \longrightarrow 0.$$

is exact. $\qquad \square$

As a corollary of Proposition 13.13, every exact sequence of presheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

yields an exact sequence of Čech complexes

$$0 \longrightarrow C^*(\mathcal{U}, \mathcal{F}') \longrightarrow C^*(\mathcal{U}, \mathcal{F}) \longrightarrow C^*(\mathcal{U}, \mathcal{F}'') \longrightarrow 0,$$

and thus, by Theorem 2.19, a long exact sequence of Čech cohomology groups over the cover $\mathcal{U}$. By passing to the limit over covers, we obtain the short exact sequence

$$0 \longrightarrow C^*(X, \mathcal{F}') \longrightarrow C^*(X, \mathcal{F}) \longrightarrow C^*(X, \mathcal{F}'') \longrightarrow 0,$$

which yields a long exact sequence of Čech cohomology groups. Thus, for *presheaves*, the family of functors $(\check{H}^p(X, \mathcal{F}))_{p \geq 0}$ is a $\delta$-functor (and even a universal $\delta$-functor, in view of a previous remark). The difficulty is that for sheaves, in general, it fails to be a $\delta$-functor.

Fortunately, the functors $\check{H}^p(X, \mathcal{F})$ are still *left-exact on sheaves*. Thus, given an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0 \qquad (*)$$

we can consider the exact sequence of presheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G} = \mathrm{PCoker}(\varphi)$, and we obtain a long exact sequence of cohomology whose rows

$$\longrightarrow \check{H}^p(X, \mathcal{F}') \longrightarrow \check{H}^p(X, \mathcal{F}) \longrightarrow \check{H}^p(X, \mathcal{G}) \longrightarrow \qquad (**)$$

involve the Čech cohomology groups $\check{H}^p(X, \mathcal{F}'), \check{H}^p(X, \mathcal{F})$, and $\check{H}^p(X, \mathcal{G})$. The exactness of $(*)$ means that $\mathcal{F}'' = \mathrm{SCoker}(\varphi)$, with $\mathrm{SCoker}(\varphi) = \widetilde{\mathrm{PCoker}(\varphi)}$, the sheafification of $\mathrm{PCoker}(\varphi) = \mathcal{G}$, so

$$\mathcal{F}'' = \widetilde{\mathcal{G}}.$$

Thus, if we can show that

$$\check{H}^p(X, \mathcal{G}) \cong \check{H}^p(X, \widetilde{\mathcal{G}}) \qquad (\dagger)$$

for every presheaf $\mathcal{G}$, by replacing $\check{H}^p(X, \mathcal{G})$ by $\check{H}^p(X, \widetilde{\mathcal{G}}) = \check{H}^p(X, \mathcal{F}'')$ in $(**)$ we obtain a long exact sequence with rows

$$\longrightarrow \check{H}^p(X, \mathcal{F}') \longrightarrow \check{H}^p(X, \mathcal{F}) \longrightarrow \check{H}^p(X, \mathcal{F}'') \longrightarrow$$

which constitutes a long exact sequence of cohomology associated with $(*)$, and the family $(\check{H}^p(X, \mathcal{F}))_{p \geq 0}$ is a $\delta$-functor. This is where the paracompactness condition comes in to save the day.

A space $X$ is *paracompact* if it is Hausdorff and if every open cover has an open, locally finite, refinement. An open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$ is *locally finite* if for every point $x \in X$, there is some open subset $V$ containing $x$ such that $V \cap U_i \neq \emptyset$ for only finitely many $i \in I$. Every metric space is paracompact, and so is every locally compact and second-countable space.

Assume that $X$ is paracompact. The key fact due to Godement is the following somewhat bizarre result which implies the crucial fact $(\dagger)$.

**Proposition 13.14.** *Assume the space $X$ is paracompact. For any presheaf $\mathcal{F}$ on $X$, if $\widetilde{\mathcal{F}} = (0)$ (the sheafification of $\mathcal{F}$ is the zero sheaf), then*

$$\check{H}^p(X, \mathcal{F}) = (0), \quad \text{for all } p \geq 0.$$

Proposition 13.14 is proved Godement [18], Chapter 5, Theorem 5.10.2. Another proof can be found in Bredon [5] (Chapter III, Theorem 4.4. See also Spanier [47] (Chapter 6, Theorem 16). None of these proofs are particularly illuminating. The significance of Proposition 13.14 is that it implies (†).

**Proposition 13.15.** *Assume the space $X$ is paracompact. For any presheaf $\mathcal{F}$ on $X$, we have isomorphisms*

$$\check{H}^p(X, \mathcal{F}) \cong \check{H}^p(X, \widetilde{\mathcal{F}}) \quad \text{for all } p \geq 0.$$

*Proof.* We follow Godement [18] (Chapter 5, page 230). Let $\eta \colon \mathcal{F} \to \widetilde{\mathcal{F}}$ be the morphism from $\mathcal{F}$ to its sheafification $\widetilde{\mathcal{F}}$, and let $\mathcal{K} = \operatorname{Ker} \eta$ and $\mathcal{I} = \operatorname{PIm} \eta$, as presheaves. We have the exact sequences of presheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{I} \longrightarrow \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{F}}/\mathcal{I} \longrightarrow 0.$$

Furthermore, we have

$$\widetilde{\mathcal{K}} = (0) \quad \text{and} \quad \widetilde{\widetilde{\mathcal{F}}/\mathcal{I}} = (0).$$

Is suffices to prove that $\mathcal{K}_x = (0)$ and $(\widetilde{\mathcal{F}}/\mathcal{I})_x = (0)$ for all $x \in X$. In the first case, by definition of $\eta$, for every open subset $U$ of $X$ and every $s \in \mathcal{F}(U)$ we have $\eta_U(s) = \widetilde{s}$, with $\widetilde{s}(x) = s_x$ for all $x \in U$, so $s \in \operatorname{Ker} \eta_U = \mathcal{K}(U)$ iff $s_x = 0$ for all $x \in U$, which implies that $\mathcal{K}_x = (0)$.

To prove that $(\widetilde{\mathcal{F}}/\mathcal{I})_x = (0)$ we use the fact (which is not hard to prove) that for any two presheaves $\mathcal{F}$ and $\mathcal{G}$, we have $(\mathcal{F}/\mathcal{G})_x = \mathcal{F}_x/\mathcal{G}_x$. Then $(\widetilde{\mathcal{F}}/\mathcal{I})_x = \widetilde{\mathcal{F}}_x/\mathcal{I}_x$, but it is easily shown that $\mathcal{I}_x = \widetilde{\mathcal{F}}_x$ since any continuous section in $\widetilde{\mathcal{F}}(U)$ agrees locally with some section of the form $\widetilde{s} \in \mathcal{I}(V)$ for some $V \subseteq U$.

By taking the long cohomology sequence associated with the first exact sequence we obtain exact sequences

$$\check{H}^p(X, \mathcal{K}) \longrightarrow \check{H}^p(X, \mathcal{F}) \longrightarrow \check{H}^p(X, \mathcal{I}) \longrightarrow \check{H}^{p+1}(X, \mathcal{K})$$

for all $p \geq 0$, and since $\widetilde{\mathcal{K}} = (0)$, by Proposition 13.14, we have

$$\check{H}^p(X, \mathcal{K}) = \check{H}^{p+1}(X, \mathcal{K}) = (0),$$

which yields isomorphisms

$$\check{H}^p(X, \mathcal{F}) \cong \check{H}^p(X, \mathcal{I}), \quad p \geq 0.$$

Similarly, by taking the long cohomology sequence associated with the second exact sequence we obtain exact sequences

$$0 \longrightarrow \check{H}^0(X, \mathcal{I}) \longrightarrow \check{H}^0(X, \widetilde{\mathcal{F}}) \longrightarrow \check{H}^0(X, \widetilde{\mathcal{F}}/\mathcal{I})$$

and

$$\check{H}^p(X, \widetilde{\mathcal{F}}/\mathcal{I}) \longrightarrow \check{H}^{p+1}(X, \mathcal{I}) \longrightarrow \check{H}^{p+1}(X, \widetilde{\mathcal{F}}) \longrightarrow \check{H}^{p+1}(X, \widetilde{\mathcal{F}}/\mathcal{I})$$

for all $p \geq 0$, and since $\widetilde{\widetilde{\mathcal{F}}/\mathcal{I}} = (0)$, by Proposition 13.14, we have

$$\check{H}^p(X, \widetilde{\mathcal{F}}/\mathcal{I}) = \check{H}^{p+1}(X, \widetilde{\mathcal{F}}/\mathcal{I}) = (0),$$

so we obtain isomorphisms

$$\check{H}^p(X, \mathcal{I}) \cong \check{H}^p(X, \widetilde{\mathcal{F}}), \quad p \geq 0.$$

It follows that

$$\check{H}^p(X, \mathcal{F}) \cong \check{H}^p(X, \widetilde{\mathcal{F}}), \quad p \geq 0,$$

as claimed.  $\square$

By putting the previous results together, we proved the following important theorem.

**Theorem 13.16.** *Assume the space $X$ is paracompact. For any sheaf $\mathcal{F}$ on $X$, we have isomorphisms*

$$\check{H}^p(X, \mathcal{F}) \cong H^p(X, \mathcal{F}) \quad \text{for all } p \geq 0$$

*between Čech cohomology and sheaf cohomology. Furthermore, for every presheaf $\mathcal{F}$, we have isomorphisms*

$$\check{H}^p(X, \mathcal{F}) \cong H^p(X, \widetilde{\mathcal{F}}) \quad \text{for all } p \geq 0.$$

**Remark:** The fact that for a paracompact space, every short exact sequence of sheaves yields a long exact sequence of cohomology is already proved in Serre's FAC [44] (Chapter 1, Section 25, Proposition 7).

Observe that all that is needed to prove Proposition 13.15 is the fact that for any presheaf $\mathcal{F}$, if $\widetilde{\mathcal{F}} = (0)$, then

$$\check{H}^p(X, \mathcal{F}) = (0), \quad \text{for all } p \geq 0.$$

This condition holds if $X$ paracompact (this is the content of Proposition 13.14), but there are other situations where it holds (perhaps for specific values of $p$). For example, for any space $X$ (not necessarily paracompact), it is shown in Godement ([18] Chapter 5, Lemma

on page 227) that for any presheaf $\mathcal{F}$, if $\widetilde{\mathcal{F}} = (0)$, then $\check{H}^0(X, \mathcal{F}) = (0)$. As a consequence, for any space $X$, for any sheaf $\mathcal{F}$ on $X$, we have isomorphisms

$$\check{H}^p(X, \mathcal{F}) \cong H^p(X, \mathcal{F}), \quad p = 0, 1;$$

see Godement ([18] Chapter 5, Corollary of Theorem 5.9.1 on page 227).

Grothendieck shows that the map $\check{H}^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F})$ is injective and gives an example where is it not an isomorphism; see Tohoku [21] (Section 3.8, Example, pages 177–179).

We now briefly discuss conditions not involving the space $X$ but instead the cover $\mathcal{U}$ that yield isomorphisms between the Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ and the sheaf cohomology groups $H^p(X, \mathcal{F})$.

First we state a result due to Leray involving the vanishing of certain sheaf cohomology groups on various open sets.

**Theorem 13.17.** *(Leray) For any topological space $X$ and any sheaf $\mathcal{F}$ on $X$, for any open cover $\mathcal{U}$, if $H^p(U_{i_0 \cdots i_p}, \mathcal{F}) = (0)$ for all $p > 0$ and all $(i_0, \ldots, i_p)$, then*

$$\check{H}^p(X, \mathcal{F}) \cong H^p(X, \mathcal{F}), \quad \text{for all} \ \ p \geq 0.$$

A proof of Theorem 13.17 can be found in Bredon [5] (Chapter III, Theorem 4.13). The proof involves a double complex. Leray's Theorem is used in algebraic geometry where $X$ is a scheme and $\mathcal{F}$ is a quasi-coherent sheaf; see Hartshorne [24] (Chapter III, Section 4, Theorem 4.5), and EGA III [22] (1.4.1).

Next, we state a result due to Henri Cartan involving the vanishing of certain Čech cohomology groups on various open sets.

**Theorem 13.18.** *(H. Cartan) For any topological space $X$ and any sheaf $\mathcal{F}$ on $X$, for any open cover $\mathcal{U}$, if $\mathcal{U}$ is a basis for the topology of $X$ closed under finite intersections and if $\check{H}^p(U_{i_0 \cdots i_p}, \mathcal{F}) = (0)$ for all $p > 0$ and all $(i_0, \ldots, i_p)$, then*

$$\check{H}^p(X, \mathcal{F}) \cong H^p(X, \mathcal{F}), \quad \text{for all} \ \ p \geq 0.$$

A proof of Theorem 13.18 is given in Grothendieck [21] (Section 3.8, Corollary 4), and in more details in Godement [18] (Chapter 5, Theorem 5.92).

We now compare singular cohomology and sheaf cohomology (for constant sheaves). To do so, we will need to introduce soft sheaves and fine sheaves.

# 13.4 Singular Cohomology and Sheaf Cohomology

If $R$ is a commutative ring with an an identity element and $G$ is an $R$-module, how can we relate the singular cohomology groups $H^p(X; G)$ to some sheaf cohomology groups? The answer is to consider the cohomology groups $H^p(X, \widetilde{G}_X)$ of the constant sheaf $\widetilde{G}_X$ (the sheafification of the constant sheaf $G$). The key idea is to consider some suitable resolution of $\widetilde{G}_X$ by acyclic sheaves such that the complex obtained by applying the global section functor to this resolution yields the singular cohomology groups, and to apply Proposition 12.27 to conclude that we have isomorphisms $H^p(X; G) \cong H^p(X, \widetilde{G}_X)$, provided some mild assumptions on $X$.

The natural candidate for the sheaves involved in a resolution of $\widetilde{G}_X$ are the presheaves $S^p(-; G)$ given by

$$U \mapsto S^p(U; G),$$

where $S^p(U; G)$ is the $R$-module of singular cochains on the open subset $U$, as defined in Definition 4.17, replacing $X$ by $U$.

The first problem is that the preshaves $S^p(-; G)$ satisfy axiom (G), but in general fail to satisfy axiom (M). To fix this problem we consider the sheafification $\mathcal{S}^p(-; G)$ of $S^p(-, G)$. The coboundary maps $\delta^p \colon S^p(U; G) \to S^{p+1}(U; G)$ induce maps $\delta^p \colon \mathcal{S}^p(-; G) \to \mathcal{S}^{p+1}(-; G)$, where we wrote $\delta$ instead of $\widetilde{\delta}$ to simplify the notation. Then, we obtain a complex

$$0 \longrightarrow \widetilde{G}_X \longrightarrow \mathcal{S}^0(-; G) \xrightarrow{\delta} \mathcal{S}^1(-; G) \xrightarrow{\delta} \mathcal{S}^2(-; G) \xrightarrow{\delta} \cdots . \qquad (*)$$

When is this a resolution of $\widetilde{G}_X$ and when are the sheaves $\mathcal{S}^p(-; G)$ acyclic?

It turns out that if $X$ is locally Euclidean, then the complex $(*)$ is exact; that is, a resolution. There is a more general condition implying that the complex $(*)$ is a resolution, namely that $X$ is an $HLC$-space ($X$ is homologically locally connected). Any locally contractible space, any manifold, or any CW-complex is $HLC$; for details, see Bredon [5] (Chapter II, Section 1). For our purposes, it suffices to assume that $X$ is a topological manifold. The proof that the complex $(*)$ is a resolution if $M$ is a topological manifold can be found in Warner [50] (Chapter V, Section 5.31). It is very technical.

Furthermore, if $X$ is paracompact, then the sheaves $\mathcal{S}^p(-; G)$ are acyclic. These sheaves are generally not flasque but they are soft sheaves, in fact, fines sheaves, and soft sheaves are acyclic; we will see this in the next section. By Proposition 12.27, if we apply the global section functor $\Gamma(X, -)$ to the resolution $(*)$, we obtain the complex $\mathcal{S}^*(X; G)$ (of modules)

$$0 \longrightarrow \mathcal{S}^0(X; G) \xrightarrow{\delta} \mathcal{S}^1(X; G) \xrightarrow{\delta} \mathcal{S}^2(X; G) \xrightarrow{\delta} \cdots$$

whose cohomology is isomorphic to the sheaf cohomology $H^*(X, \widetilde{G}_X)$.

However, there is a new problem: the cohomology groups of the complex $\mathcal{S}^*(X; G)$ involve the modules $\mathcal{S}^p(X; G)$, but the singular cohomology groups involve the modules $S^p(X; G)$;

how do we know that these groups are isomorphic? They are indeed isomorphic if $X$ is paracompact.

Let us settle this point before dealing with soft sheaves. Assume that $X$ is paracompact. If $\mathcal{F}$ is a presheaf on $X$ and if $\widetilde{\mathcal{F}}$ is its sheafification, the natural map $\eta \colon \mathcal{F} \to \widetilde{\mathcal{F}}$ induces the map $\eta \colon \mathcal{F}(X) \to \widetilde{\mathcal{F}}(X)$ given by $\eta = \eta_X$, as in Definition 11.3; that is, for every $s \in \mathcal{F}(X)$,

$$\eta(s) = \widetilde{s}$$

with $\widetilde{s}(x) = s_x$ for all $x \in X$. Define the presheaf $\mathcal{F}(X)_0$ by

$$\mathcal{F}(X)_0 = \{s \in \mathcal{F}(X) \mid \eta(s) = 0\} = \operatorname{Ker} \eta.$$

Then we have the following result.

**Proposition 13.19.** *Assume the space $X$ is paracompact. For every presheaf $\mathcal{F}$, if $\mathcal{F}$ satisfies condition (G) then the sequence*

$$0 \longrightarrow \mathcal{F}(X)_0 \longrightarrow \mathcal{F}(X) \overset{\theta}{\longrightarrow} \widetilde{\mathcal{F}}(X) \longrightarrow 0$$

*is exact.*

The only thing that needs to be proved is that $\theta$ is surjective. This is proved in Warner [50] (Chapter V, Proposition 5.27) and in Bredon [5] (Chapter I, Theorem 6.2). As a consequence of Proposition 13.19, we have an exact sequence of cochain complexes

$$0 \longrightarrow S^*(X;G)_0 \longrightarrow S^*(X;G) \longrightarrow \mathcal{S}^*(X;G) \longrightarrow 0. \tag{$\dagger$}$$

We claim that if we can prove that

$$H^p(S^*(X;G)_0) = (0) \quad \text{for all } p \geq 0,$$

then we have isomorphisms

$$H^p(X;G) = H^p(S^*(X;G)) \cong H^p(\mathcal{S}^*(X;G)), \quad \text{for all } p \geq 0.$$

*Proof.* This follows easily by taking the long exact sequence of cohomology associated with the exact sequence ($\dagger$). We have exact sequences

$$H^p(S^*(X;G)_0) \longrightarrow H^p(X;G) \longrightarrow H^p(\mathcal{S}^*(X;G)) \longrightarrow H^{p+1}(S^*(X;G)_0)$$

for all $p \geq 0$, and since by hypothesis $H^p(S^*(X;G)_0) = H^{p+1}(S^*(X;G)_0) = (0)$, we obtain the isomorphisms

$$H^p(X;G) = H^p(S^*(X;G)) \cong H^p(\mathcal{S}^*(X;G)), \quad \text{for all } p \geq 0,$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, it is shown in Warner [50] (Chapter 5, Section 5.32) that indeed

$$H^p(S^*(X;G)_0) = (0) \quad \text{for all } p \geq 0.$$

This is a very technical argument involving barycentric subdivision and a bit of topology.

In summary, we have shown that if $X$ is paracompact and a topological manifold, provided that the sheaves $\mathcal{S}^p(-;G)$ are acyclic, then we have isomorphisms

$$H^p(X;G) \cong H^p(X, \widetilde{G}_X), \quad \text{for all } p \geq 0$$

between singular cohomology and sheaf cohomology of the constant sheaf $\widetilde{G}_X$.

The sheaves $\mathcal{S}^p(-;G)$ are indeed acyclic because they are soft, and soft sheaves over a paracompact space are acyclic; this will be proved in Section 13.5. Assuming that this result has been proved, we have the following theorem showing the equivalence of singular cohomology and sheaf cohomology for the constant sheaf $\widetilde{G}_X$ and a (paracompact) topological manifold $X$.

**Theorem 13.20.** *Assume $X$ is a paracompact topological manifold. For any $R$-module $G$, there are isomorphisms*

$$H^p(X;G) \cong H^p(X, \widetilde{G}_X), \quad \text{for all } p \geq 0$$

*between singular cohomology and sheaf cohomology of the constant sheaf $\widetilde{G}_X$.*

**Remark:** There is a variant of singular cohomology that uses differentiable singular simplices instead of singular simplices as defined in Definition 4.2. Given a topological space $X$, if $p \geq 1$, a *differentiable singular $p$-simplex* is any map $\sigma \colon \Delta^p \to X$ that can be extended to a smooth map of a neighborhood of $\Delta^p$. Then, $S^p_\infty(U;G)$ denotes the $R$-module of functions which assign to each differentiable singular $p$-simplex an element of $G$ (for $p \geq 1$), and $S^0_\infty(U;G) = S^0(X;G)$. Elements of $S^p_\infty(U;G)$ are called *differentiable singular $p$-cochains*. Then, we obtain the cochain complex $S^*_\infty(X;G)$ and its cohomology groups denoted $H^p_{\Delta^\infty}(X;G)$ are called the *differentiable singular cohomology groups* of $X$ with coefficients in $G$. Each $S^p_\infty(-;G)$ is a presheaf satisfying condition (M), and we let $\mathcal{S}^p_\infty(-;G)$ be its sheafification. As in the continuous case, we obtain a version of Theorem 13.20.

**Theorem 13.21.** *Assume $X$ is a paracompact topological manifold. For any $R$-module $G$, there are isomorphisms*

$$H^p_{\Delta^\infty}(X;G) \cong H^p(X, \widetilde{G}_X), \quad \text{for all } p \geq 0$$

*between differentiable singular cohomology and sheaf cohomology of the constant sheaf $\widetilde{G}_X$.*

Details can be found in Warner [50] (Chapter 5, Sections 5.31, 5.32). The significance of differentiable singular cohomology is that it yields a stronger version of the equivalence with de Rham cohomology when $G = \mathbb{R}$ and $X$ is a smooth manifold; see Section 13.6.

## 13.5    Soft Sheaves and Fine Sheaves

Roughly speaking a sheaf is soft if it satisfies the condition for being flasque for closed subsets of $X$; that is, for every closed subset $A$ of $X$, the restriction map from $\mathcal{F}(X)$ to $\mathcal{F}(A)$ is surjective. The problem is that sheaves are only defined over open subsets!

The remedy is to work with stalk spaces $(E, p)$. Since every sheaf $\mathcal{F}$ is isomorphic to the sheaf of sections $\widetilde{\mathcal{F}}$ associated with the stalk space $(S\mathcal{F}, \pi)$, this is not a problem, although at times it is a little awkward.

If $(E, p)$ is a stalk space of $R$-modules on $X$ with $p\colon E \to X$, and $\Gamma[E, p]$ is the sheaf of continuous sections associated with $(E, p)$ (see Example 9.2 (1)), following Godement [18] (Chapter 1, bottom of page 110), for every subset $Y$ of $X$ (not necessarily open) we define

$$\Gamma(Y, \Gamma[E, p]) = \{s\colon Y \to E \mid p \circ s = \mathrm{id} \text{ and } s \text{ is continuous}\}$$

as the set of all continuous sections from $Y$ viewed as a subspace of $X$. We usually abuse notation a little and denote the sheaf $\Gamma[E, p]$ associated with the stalk space $(E, p)$ by $\mathcal{F}$. We write $\Gamma(Y, \mathcal{F})$ for $\Gamma(Y, \Gamma[E, p])$. Then we can make the following definition.

**Definition 13.5.** If $\mathcal{F}$ is the sheaf induced by a stalk space $(E, p)$ of $R$-modules on $X$, we say that the sheaf $\mathcal{F}$ is *soft* if the restriction map from $\Gamma(X, \mathcal{F})$ to $\Gamma(A, \mathcal{F})$ is surjective for every closed subset $A$ of $X$.

In order to prove that soft sheaves are acyclic, which is one of our main goals, we need to assume that $X$ is paracompact. Then, we will see that every flasque sheaf is soft.

Given a sheaf $\mathcal{F}$ and its sheafification $\widetilde{\mathcal{F}}$, the sheaf isomorphism $\eta\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ ensures that $\mathcal{F}$ is flasque iff $\widetilde{\mathcal{F}}$ is flasque, so there is no problem.

In this section, we will content ourselves with stating the properties of soft sheaves that are needed to finish the proof of the equivalence of singular cohomology and sheaf cohomology (for the constant sheaves $\widetilde{G}_X$), and the proof of the equivalence of de Rham cohomology and sheaf cohomology (for the constant sheaves $\widetilde{\mathbb{R}}_X$). Details and proofs can be found in Bredon [5] (Chapter II, Section 9) and Godement [18] (Chapters 3, 4, 5). Soft sheaves are also discussed in Brylinski [6] (Chapter I, Section 1,4), but a different definition is used.

**Proposition 13.22.** *Let $\mathcal{F}$ be the sheaf induced by a stalk space $(E, p)$ of $R$-modules over a space $X$, let $Y$ be any subset of $X$ and let $s \in \Gamma(Y, \mathcal{F})$ be any section over $Y$. If $Y$ admits a fundamental system of paracompact neighborhoods,[1] then $s$ has an extension to some open neighboorhood of $Y$ in $X$.*

Proposition 13.22 is proved in Godement [18] (Chapter III, Theorem 3.3.1) and Bredon [5] (Chapter I, Theorem 9.5). As an immediate corollary we obtain the following result.

---

[1]This means that there is a family $\mathcal{N}$ of paracompact neighborhoods of $A$ such that for every neighborhood $V$ of $A$ there is some $W$ in $\mathcal{N}$ such that $W \subseteq V$.

**Proposition 13.23.** *Let $\mathcal{F}$ be the sheaf induced by a stalk space $(E, p)$ of $R$-modules over a space $X$. If $X$ is paracompact and $\mathcal{F}$ is flasque, then $\mathcal{F}$ is soft.*

To prove that soft sheaves on a paracompact space are acyclic, we need the following two propositions.

**Proposition 13.24.** *If $X$ is paracompact, for any exact sequence of sheaves (induced by stalk spaces)*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

*if $\mathcal{F}'$ is soft then the sequence*

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0,$$

*is exact.*

A proof of Proposition 13.24 is given in Bredon [5] (Chapter II, Theorem 9.9); see also Godement [18] (Chapter 3, Theorem 3.5.2).

**Proposition 13.25.** *If $X$ is paracompact, for any exact sequence of sheaves (induced by stalk spaces)*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

*if $\mathcal{F}'$ and $\mathcal{F}$ are soft then $\mathcal{F}'$ is also soft.*

A proof of Proposition 13.25 is given in Bredon [5] (Chapter II, Theorem 9.10); see also Godement [18] (Chapter 3, Theorem 3.5.3).

It is also easy to see that every direct factor of a soft sheaf is soft; the proof given for flasque sheaves in Theorem 13.5 applies. But now (as in the case of flasque sheaves) the assumptions of Proposition 13.6 apply, and we imediately get the following result.

**Proposition 13.26.** *For any sheaf $\mathcal{F}$ induced by a stalk space $(E, p)$, if $X$ is paracompact and $\mathcal{F}$ is soft, then $\mathcal{F}$ is ayclic, that is*

$$H^p(X, \mathcal{F}) = (0) \quad \text{for all } p \geq 1.$$

Neither Godement nor Bredon have Proposition 13.6 from Tohoku at their disposal, so they need to prove Proposition 13.26; see Godement [18] (Chapter 4, Theorem 4.4.3) and Bredon [5] (Chapter, Theorem 9.11).

Going back to singular cohomology, it remains to prove that the sheaves $\mathcal{S}^p(X; G)$ are soft.

**Proposition 13.27.** *If the space $X$ is paracompact, then the sheaves (of singular cochains) $\mathcal{S}^p(X; G)$ are soft.*

A proof of Proposition 13.27 is given in Godement [18] (Chapter 3, Section 3.9, Example 3.9.1).

Propositions 13.26 and 13.27 conclude the proof of Theorem 13.20.

Another way to prove Proposition 13.27 is to prove that the sheaves $\mathcal{S}^p(X; G)$ are fine and that fine sheaves are soft.

If $\mathcal{F}$ is the sheaf induced by a stalk space $(E, p)$ where $p \colon E \to X$ is a continuous surjection, for any subset $Y$ of $X$, the sheaf $\mathcal{F}|Y$ is the sheaf of continuous sections of the stalk space $(p^{-1}(Y), p|p^{-1}(Y))$.

Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ induced by stalk spaces over the same space $X$ we have a definition of the presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ analogous to Definition 9.5:

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|U, \mathcal{G}|U)$$

for every open subset $U$ of $X$, where $\mathrm{Hom}(\mathcal{F}|U, \mathcal{G}|U)$ denotes the set of maps between the sheaves $\mathcal{F}|U$ and $\mathcal{G}|U$. Even though $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf if $\mathcal{F}$ and $\mathcal{G}$ are sheaves induced by stalk spaces, because we need to work with stalk spaces when dealing with soft sheaves, with some abuse of notation, we also denote the sheafification of the above presheaf by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. Then, we have the following definition due to Godement [18] (Chapter 3, Section 3.7).

**Definition 13.6.** For any sheaf $\mathcal{F}$ on $X$ induced by the stalk space $(E, p)$, we say that $\mathcal{F}$ is *fine* if $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ is soft.

The following results about fine and soft sheaves are proved in Godement [18] (Chapter 3, Section 3.7) and in Bredon [5] (Chapter II, Section 9).

**Proposition 13.28.** *Assume the space $X$ is paracompact. If $\mathcal{O}_X$ is any sheaf of rings with unit induced by a stalk space and if $\mathcal{O}_X$ is soft, then any $\mathcal{O}_X$-module is soft.*

This is Theorem 3.7.1 in Godement [18].

**Proposition 13.29.** *Assume the space $X$ is paracompact. If $\mathcal{O}_X$ is sheaf of rings with unit induced by a stalk space, then $\mathcal{O}_X$ is soft iff every $x \in X$ has some open neighborhood $U$ such that for any two disjoint open subsets $S, T$ contained in $U$, there is some section $s \in \mathcal{O}_X(U)$ such that $s \equiv 1$ on $S$ and $s \equiv 0$ on $T$.*

This is Theorem 3.7.2 in Godement [18].

**Proposition 13.30.** *Assume the space $X$ is paracompact. A sheaf $\mathcal{F}$ induced by a stalk space $(E, p)$ is fine iff for any two disjoint open subsets $S, T$ in $X$, there is a sheaf homomorphism $\varphi \colon \mathcal{F} \to \mathcal{F}$ such that $\varphi \equiv 1$ in a neighborhood of $S$ and $\varphi \equiv 0$ in a neighborhood of $T$. Every fine sheaf is soft.*

See Godement [18] (Section 3.7, page 157) and Bredon [5] (Chapter II,Theorem 9.16). Since every soft sheaf is acyclic, so is every fine sheaf (over a paracompact space).

**Remark:** If $X$ is paracompact, then any injective sheaf on $X$ is fine; see Bredon [5] (Chapter II, Exercise 17). The following diagram summarizes the relationships between injective, flasque, fine, and soft sheaves (assuming that $X$ is paracompact):

$$
\begin{array}{ccc}
\text{injective} & \Longrightarrow & \text{flasque} \\
\Big\Downarrow & & \Big\Downarrow \\
\text{fine} & \Longrightarrow & \text{soft.}
\end{array}
$$

Godement proves that the sheaves $\mathcal{S}^p(-; G)$ are fine (Godement, Example 3.7.1, page 161); see also Bredon [5] (Chapter III, page 180).

Besides being acyclic, fine sheaves behave well with respect to tensor products, which, historically motivated their introduction.

Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ of $R$-modules, the presheaf $\mathcal{F} \otimes \mathcal{G}$ is defined by

$$(\mathcal{F} \otimes \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U)$$

for any open subset $U$ of $X$. Actually, the presheaf $\mathcal{F} \otimes \mathcal{G}$ is a sheaf. If $\mathcal{F}$ and $\mathcal{G}$ are induced by stalk spaces of $R$-modules, with a minor abuse of notation we let $\mathcal{F} \otimes \mathcal{G}$ be the sheafification of the above sheaf.

**Proposition 13.31.** *Assume the space $X$ is paracompact. For any fine sheaf $\mathcal{F}$ and any sheaf $\mathcal{G}$ induced by stalk spaces on $X$, the sheaf $\mathcal{F} \otimes \mathcal{G}$ is fine.*

Proposition 13.31 is proved in Godement [18] (Chapter 3, Theorem 3.7.3), Bredon [5] (Chapter II, Corollary 9.18), and Warner [50] (Chapter V, Section 5.10).

Proposition 13.31 can used to create resolutions. Indeed, suppose that we have a resolution

$$0 \longrightarrow \widetilde{R}_X \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \mathcal{C}^2 \longrightarrow \cdots$$

of the locally constant sheaf $\widetilde{R}_X$ by fine and torsion-free sheaves $\mathcal{C}^p$ (which means that each stalk $\mathcal{C}^p_x$ is a torsion-free $R$-module, where by stalk we mean the fibre over $x \in X$ in the stalk space defining $\mathcal{C}^p$). Then it can be shown that for any sheaf $\mathcal{F}$ of $R$-modules, the complex

$$0 \longrightarrow \widetilde{R}_X \otimes \mathcal{F} \longrightarrow \mathcal{C}^0 \otimes \mathcal{F} \longrightarrow \mathcal{C}^1 \otimes \mathcal{F} \longrightarrow \mathcal{C}^2 \otimes \mathcal{F} \longrightarrow \cdots \qquad (*)$$

is a resolution of $\mathcal{F} \cong \widetilde{R}_X \otimes \mathcal{F}$ by fine sheaves; See Warner [50] (Chapter V, Section 5.10), Theorem 5.15). Furhermore, if $X$ is paracompact and if the $R$ is a PID, resolutions of $\widetilde{R}_X$ by

fine and torsion-free sheaves do exist; for example, the sheaves $\mathcal{S}^p(X; R)$ of singular cochains are fine and torsion-free; see Warner [50] (Chapter V, Section 5.31).

Thus, if $X$ is paracompact and if $R$ is a PID, we can define the sheaf cohomology groups $H^p(X, \mathcal{F})$ in terms of the resolution $(*)$ as

$$H^p(X, \mathcal{F}) = H^p(\Gamma(\mathcal{C}^* \otimes \mathcal{F})).$$

Since fine sheaves are acyclic, it follows that these groups are independent of the fine and torsion-free resolution of $\widetilde{R}_X$ chosen.

This method to define sheaf cohomology in terms of resolutions of fine sheaves is due to Henri Cartan and is presented in Chapter V of Warner [50]. It is also the approach used by Bredon [5].

The advantage of this method is that it does not require the machinery of derived functors. The disadvantage is that it relies on fine sheaves, and thus on paracompactness, and assumes that the ring $R$ is a PID. This makes it unsuitable for more general spaces and sheaves that arise naturally in algebraic geometry.

Fine sheaves are often defined in terms of partitions of unity, as in Warner [50] (Chapter V, Definition 5.10) or Spanier [47] (Chapter 6, Section 8). Given a sheaf $\mathcal{F}$ induced by a stalk space $(E, p)$, the *support* of a map $\varphi \colon \mathcal{F} \to \mathcal{F}$, denoted by $\mathrm{supp}(\varphi)$, is the closure of the set of elements $x \in E$ such that $\varphi(x) \neq 0$.

**Definition 13.7.** Given a sheaf $\mathcal{F}$ induced by a stalk space of rings $(E, p)$ over $X$, we say that $\mathcal{F}$ is *p-fine* if for each locally finite open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$, for each $i \in I$ there is some map $\varphi_i \colon \mathcal{F} \to \mathcal{F}$ such that

(a) $\mathrm{supp}(\varphi_i) \subseteq U_i$.

(b) $\sum \varphi_i = \mathrm{id}$.

   This sum makes sense because $\mathcal{U}$ is locally finite.

The family $(\varphi_i)_{i \in I}$ is called a *partition of unity for $\mathcal{F}$ subordinate to the cover $\mathcal{U}$*.

Then, if $X$ is paracompact, using a partition of unity, it is not hard to show to the sheaves $\mathcal{S}^p(-; G)$ and $\mathcal{S}^p_\infty(-; G)$ are $p$-fine; see Warner [50] (Chapter V, Sections 5.31 and 5.32, pages 193–196).

It is not obvious that on a paracompact space, a sheaf is fine iff it is $p$-fine. It is shown in Brylinski [6] (Chapter 1, Proposition 1.4.9) that a $p$-fine sheaf is soft. It is shown in Warner that a $p$-fine sheaf is acyclic; see [50] (Chapter V, Section 5.20, page 179). Therefore, *both fine sheaves and p-fine sheaves are acyclic*. It is also claimed in Exercise 13 in Bredon ([5], Chapter II, page 170) that Definition 13.6 is equivalent to Definition 13.7 for a paracompact space; thus, a sheaf is soft iff it is $p$-soft.

**Remark:** There is a slight generalization of the various cohomology theories involving "families of support." A *family of support* on $X$ is a family $\Phi$ of closed subsets of $X$ satisfying certain closure properties. Interesting families of support also *paracompactifying*. Then, given a sheaf $\mathcal{F}$ induced by a stalk space, for any section $s \in \Gamma(X, \mathcal{F})$, the *support* $|s|$ of $s$ is the closed set of $x \in X$ such that $s(x) \neq 0$. We define $\Gamma_\Phi$ by

$$\Gamma_\Phi(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) \mid |s| \in \Phi\}.$$

Then, we can define the cohomology groups $H_\Phi^p(X, \mathcal{F})$ by considering the (left-exact) functor $\Gamma_\Phi$ instead of $\Gamma$. We can also define $\Phi$-soft and $\Phi$-fine sheaves, and the results that we have presented generalize to paracompactifying families of support $\Phi$. For details on this approach, see Godement [18] and Bredon [5].

Another example of a $p$-fine sheaf is the sheaf $\mathcal{A}_X^p$ of differential forms on a smooth manifold $X$. Here, since we have to use stalk spaces, we are really dealing with the sheafification of the sheaf of differential forms, but we will use the same notation. This will allow us to finish the discussion of the comparison between the de Rham cohomology and sheaf cohomology started with Proposition 12.28.

# 13.6 de Rham Cohomology and Sheaf Cohomology

Let $X$ be a smooth manifold. Recall that we proved in Proposition 12.28 that the sequence

$$0 \longrightarrow \widetilde{\mathbb{R}}_X \xrightarrow{\epsilon} \mathcal{A}_X^0 \xrightarrow{d} \mathcal{A}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_X^p \xrightarrow{d} \mathcal{A}_X^{p+1} \xrightarrow{d} \cdots$$

is a resolution of the locally constant sheaf $\widetilde{\mathbb{R}}_M$. As we stated in the previous section, we have the following result.

**Proposition 13.32.** *For any (paracompact) smooth manifold $X$, the sheaves $\mathcal{A}_X^p$ (actually, the sheafifications of the sheaves $\mathcal{A}_X^p$) are $p$-fine and fine sheaves.*

That the $\mathcal{A}_X^p$ are fine sheaves is proved in Godement [18] (Chapter 3, Example 3.7.1, page 158). That the $\mathcal{A}_X^p$ are $p$-fine sheaves is proved in Warner [50] (Chapter V, Section 5.28) and Brylinski [6] (Section 1.4, page 139). Since fine sheaves and $p$-fine sheaves are equivalent and thus acyclic, by Proposition 12.27 the sheaf cohomology groups of the sheaf $\widetilde{\mathbb{R}}_X$ are computed by the resolution of fine (and $p$-fine) sheaves

$$0 \longrightarrow \widetilde{\mathbb{R}}_X \xrightarrow{\epsilon} \mathcal{A}_X^0 \xrightarrow{d} \mathcal{A}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_X^p \xrightarrow{d} \mathcal{A}_X^{p+1} \xrightarrow{d} \cdots$$

Thus, in view of Theorem 13.16 and Theorem 13.20, we obtain the following version of the *de Rham theorem*:

**Theorem 13.33.** *Let $X$ be a (paracompact) smooth manifold. There are isomorphisms*

$$H_{\mathrm{dR}}^p(X) \cong H^p(X, \widetilde{\mathbb{R}}_X) \cong \check{H}^p(X, \widetilde{\mathbb{R}}_X) \cong H^p(X; \mathbb{R})$$

*between de Rham cohomology, the sheaf cohomology of the locally constant sheaf $\widetilde{\mathbb{R}}_X$, Čech cohomology of $\widetilde{\mathbb{R}}_X$, and singular cohomology over $\mathbb{R}$.*

Theorem 13.21 also yields an isomorphism

$$H_{\mathrm{dR}}^p(X) \cong H_{\Delta^\infty}^p(X; \mathbb{R})$$

between de Rham cohomology and differentiable singular cohomology with coefficients in $\mathbb{R}$. It is possible to give a more explicit definition of the above isomorphism using integration.

For any $p \geq 1$, define the map $k_p \colon \mathcal{A}^p(X) \to S_\infty^p(X; \mathbb{R})$ by

$$k_p(\omega)(\sigma) = \int_\sigma \omega,$$

for any $p$-form $\omega \in \mathcal{A}^p(X)$ and any differentiable singular $p$-simplex $\sigma$ in $X$. Using Stokes' theorem, it can be shown that the $k_p$ induce a cochain map

$$k \colon \mathcal{A}^*(X) \to S_\infty^*(X; \mathbb{R}).$$

The above map induces a map of cohomology, and a strong version of the de Rham theorem is this:

**Theorem 13.34.** *For any smooth manifold $X$, the cochain map $k \colon \mathcal{A}^*(X) \to S_\infty^*(X; \mathbb{R})$ induces an isomorphism*

$$k_p^* \colon H_{\mathrm{dR}}^p(X) \to H_{\Delta^\infty}^p(X; \mathbb{R})$$

*for every $p \geq 0$, between de Rham cohomology and differentiable singular cohomology.*

For details, see Warner [50] (Chapter 5, Sections 5.35–5.37). Chapter 5 of Warner also contains a treatment of the multiplicative structure of cohomology.

There is yet another cohomology theory, *Alexander–Spanier cohomology*. It turns out to be equivalent to Čech cohomology, but it occurs naturally in a version of duality called Alexander–Lefschetz duality.

Alexander–Spanier cohomology is discussed extensively in Warner [50] (Chapter V), Bredon [5] (Chapters I, II, III), and Spanier [47] (Chapter 6).

## 13.7   Alexander–Spanier Cohomology and Sheaf Cohomology

Let $X$ be a paracompact space, and let $G$ be an $R$-module.

**Definition 13.8.** For any open subset $U$ of $X$, for any $p \geq 0$, let $A^p(U; G)$ denote the $G$-module of all functions $f \colon U^{p+1} \to G$. The homomorphism

$$d^p \colon A^p(U; G) \to A^{p+1}(U; G)$$

is defined by

$$d^p f(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(x_1, \ldots, \widehat{x_i}, \ldots, x_{p+1}),$$

for all $f \in A^p(U;G)$ and all $(x_0, \ldots, x_{p+1}) \in U^{p+2}$.

It is easily checked that $d^{p+1} \circ d^p = 0$ for all $p \geq 0$, so we obtain a cochain complex

$$0 \longrightarrow A^0(U;G) \xrightarrow{d^0} A^1(U;G) \xrightarrow{d^1} A^2(U;G) \xrightarrow{d^2} \cdots$$

denoted by $A^*(U;G)$. If $V \subseteq U$ then there is a restriction homomorphism

$$\rho_V^U \colon A^p(U;G) \to A^p(V;G),$$

so we obtain a presheaf $A^p(-;G)$ of $R$-modules called the *presheaf of Alexander–Spanier p-cochains*. The presheaf $A^p(-;G)$ satisfies Condition (G) for $p \geq 1$ but not Condition (M).

Let $\mathcal{A}_{\text{A-S}}^p(-;G)$ be the sheafification of $A^p(-;G)$. As in the case of singular cohomology we obtain a complex

$$0 \longrightarrow \widetilde{G}_X \longrightarrow \mathcal{A}_{\text{A-S}}^0(-;G) \xrightarrow{\delta} \mathcal{A}_{\text{A-S}}^1(-;G) \xrightarrow{\delta} \mathcal{A}_{\text{A-S}}^2(-;G) \xrightarrow{\delta} \cdots . \qquad (*)$$

The following result is proved in Warner [50] (Chapter 5, Section 5.26).

**Proposition 13.35.** *The sheaves $\mathcal{A}_{A\text{-}S}^p(-;G)$ are fine and the complex $(*)$ is a resolution of $\widetilde{G}_X$.*

By Proposition 12.27, if we apply the global section functor $\Gamma(X, -)$ to the resolution $(*)$, we obtain the complex $\mathcal{A}_{\text{A-S}}^*(X;G)$ (of modules)

$$0 \longrightarrow \mathcal{A}_{\text{A-S}}^0(X;G) \xrightarrow{\delta^0} \mathcal{A}_{\text{A-S}}^1(X;G) \xrightarrow{\delta^1} \mathcal{A}_{\text{A-S}}^2(X;G) \xrightarrow{\delta^2} \cdots$$

whose cohomology is isomorphic to the sheaf cohomology $H^*(X, \widetilde{G}_X)$.

Now, since $X$ is paracompact and since the presheaves $A^p(-;G)$ satisfy Condition (G), Proposition 13.19 implies that the sequence of cochain complexes

$$0 \longrightarrow A_0^*(X;G) \longrightarrow A^*(X;G) \longrightarrow \mathcal{A}_{\text{A-S}}^*(X;G) \longrightarrow 0$$

is exact, with

$$A_0^p(X;G) = \{f \in A^p(X;G) \mid f_x = 0 \text{ for all } x \in X\}.$$

Then, we have isomorphisms

$$A^p(X;G)/A_0^p(X;G) \cong \mathcal{A}_{\text{A-S}}^p(X;G)$$

for all $p \geq 0$, and the sheaf cohomology groups $H^p(X; \widetilde{G}_X)$ are the cohomology groups of the complex

$$0 \longrightarrow A^0(X;G)/A_0^0(X;G) \xrightarrow{\delta^0} A^1(X;G)/A_0^1(X;G) \xrightarrow{\delta^1} A^2(X;G)/A_0^2(X;G) \xrightarrow{\delta^2} \cdots$$

Now, the elements of $A_0^p(X;G)$ can be described as functions $f \in A^p(X;G)$ that are *locally zero*.

**Definition 13.9.** A function $f \in A^p(X;G)$ is *locally zero* if there is some open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$ such that $f(x_0, \ldots x_p) = 0$ for all $(x_0, \ldots, x_p) \in U_i^{p+1}$ in any $U_i \in \mathcal{U}$.

Equivalently, if we write

$$\mathcal{U}^{p+1} = \bigcup_{i \in I} U_i^{p+1} \subseteq X^{p+1},$$

then $f \in A^p(X;G)$ is locally zero if there is some open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$ such that $f$ vanishes on $\mathcal{U}^{p+1}$.

It follows that the restriction of $\delta$ to $A_0^p(X;G)$ has its image in $A_0^{p+1}(X;G)$, because if $f$ vanishes on $\mathcal{U}^{p+1}$, then $\delta f$ vanishes on $\mathcal{U}^{p+2}$. It follows that we obtain the quotient complex

$$0 \longrightarrow A^0(X;G)/A_0^0(X;G) \xrightarrow{\delta^0} A^1(X;G)/A_0^1(X;G) \xrightarrow{\delta^1} A^2(X;G)/A_0^2(X;G) \xrightarrow{\delta^2} \cdots$$

as above. By definition, its cohomology groups are the Alexander–Spanier cohomology groups.

**Definition 13.10.** For any topological space $X$, the *Alexander–Spanier* complex is the complex

$$0 \longrightarrow A^0(X;G)/A_0^0(X;G) \xrightarrow{\delta^0} A^1(X;G)/A_0^1(X;G) \xrightarrow{\delta^1} A^2(X;G)/A_0^2(X;G) \xrightarrow{\delta^2} \cdots$$

where the $A^p(-;G)$ are the Alexander–Spanier presheaves and $A_0^p(X;G)$ consists of the functions in $A^p(X;G)$ that are locally zero. The cohomology groups of the above complex are the *Alexander–Spanier* cohomology groups and are denoted by $H_{\text{A-S}}^p(X;G)$.

Observe that the Alexander–Spanier cohomology groups are defined for all topological spaces, not necessarily paracompact. However, we proved that if $X$ is paracompact, then they agree with the sheaf cohomology groups of the sheaf $\widetilde{G}_X$.

**Theorem 13.36.** *If the space $X$ is paracompact, then we have isomorphisms*

$$H_{A\text{-}S}^p(X;G) \cong H^p(X; \widetilde{G}_X) \quad \text{for all } p \geq 0$$

*between Alexander–Spanier cohomology and the sheaf cohomology of the constant sheaf $\widetilde{G}_X$.*

In view of Theorem 13.16, we also have the following theorem (proved in full in Warner [50], Chapter 5, Section 5.26, pages 187- 188).

**Theorem 13.37.** *If the space $X$ is paracompact, then we have isomorphisms*

$$H^p_{A\text{-}S}(X;G) \cong \check{H}^p(X;\widetilde{G}_X) \quad \text{for all } p \geq 0$$

*between Alexander–Spanier cohomology and the Čech cohomology of the constant sheaf $\widetilde{G}_X$ (classical Čech cohomology).*

Theorem 13.37 is also proved in Spanier [47] (Chapter 6, Section 8, Corollary 8). In fact, the above isomorphisms hold even if $X$ is not paracompact, a theorem due to Dowker; see Theorem 14.5, and also Spanier [47] (Chapter 6, exercise 6.D.3).

**Remark:** The cohomology of the complex

$$0 \longrightarrow A^0(X;G) \xrightarrow{d^0} A^1(X;G) \xrightarrow{d^1} A^2(X;G) \xrightarrow{d^2} \cdots$$

is trivial; that is, its cohomology group are all equal to $G$; see Spanier [47] (Chapter 6, Section 4, Lemma 1).

# Chapter 14

# Alexander and Alexander–Lefschetz Duality

Our goal is to present various generalizations of Poincaré duality. These versions of duality involve taking direct limits of direct mapping families of singular cohomology groups which, in general, are not singular cohomology groups. However, such limits are isomorphic to Alexander–Spanier cohomology groups, and thus to Čech cohomology groups. These duality results also require relative versions of homology and cohomology. Thus, in preparation for Alexander–Lefschetz duality we need to define relative Alexander–Spanier cohomology and relative Čech Cohomology.

## 14.1 Relative Alexander–Spanier Cohomology

Given a topological space $X$, let us denote by $A_{\text{A-S}}^p(X; G)$ the Alexander–Spanier cochain modules

$$A_{\text{A-S}}^p(X; G) = A^p(X; G)/A_0^p(X; G),$$

where $A_0^p(X; G)$ is the set of functions in $A^p(X; G)$ that are locally zero (which means that there is some open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$ such that $f(x_0, \ldots x_p) = 0$ for all $(x_0, \ldots, x_p) \in U_i^{p+1}$ in any $U_i \in \mathcal{U}$). Recall that if we write

$$\mathcal{U}^{p+1} = \bigcup_{i \in I} U_i^{p+1} \subseteq X^{p+1},$$

then $f \in A^p(X; G)$ is locally zero if there is some open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X$ such that $f$ vanishes on $\mathcal{U}^{p+1}$.

If $h \colon X \to Y$ is a continuous map, then we have an induced cochain maps

$$h^{p\sharp} \colon A^p(Y; G) \to A^p(X; G)$$

given by

$$h^{p\sharp}(\varphi)(x_0, \ldots, x_p) = \varphi(h(x_0), \ldots, h(x_p))$$

for all $(x_0, \ldots, x_p) \in X^{p+1}$ and all $\varphi \in A^p(Y; G)$.

If $\varphi$ vanishes on $\mathcal{V}^{p+1}$, where $\mathcal{V}$ is some open cover of $Y$, since $h$ is continuous we see that $h^{-1}(\mathcal{V})$ is an open cover of $X$ and then $h^{p\sharp}$ vanishes on $(h^{-1}(\mathcal{V}))^{p+1}$. It follows that $h^{p\sharp}$ maps $A_0^p(Y; G)$ into $A_0^p(X; G)$, so there is an induced map

$$h^{p\sharp} \colon A_{\text{A-S}}^p(Y; G) \to A_{\text{A-S}}^p(X; G),$$

and thus a module homomorphism

$$h^{p*} \colon H_{\text{A-S}}^p(Y; G) \to H_{\text{A-S}}^p(X; G).$$

If $A$ is a subspace of $X$ and $i \colon A \to X$ is the inclusion map, then the homomorphisms $i^{p\sharp} \colon A_{\text{A-S}}^p(X; G) \to A_{\text{A-S}}^p(A; G)$ are surjective. Therefore

$$A_{\text{A-S}}^p(X, A; G) = \operatorname{Ker} i^{p\sharp}$$

is a submodule of $A_{\text{A-S}}^p(X; G)$ called the module of *relative Alexander–Spanier p-cochains*, and by restriction we obtain a cochain complex

$$0 \longrightarrow A_{\text{A-S}}^0(X, A; G) \xrightarrow{d^0} A_{\text{A-S}}^1(X, A; G) \xrightarrow{d^1} A_{\text{A-S}}^2(X, A; G) \xrightarrow{d^2} \cdots . \qquad (*)$$

**Definition 14.1.** If $X$ is a topological space and if $A$ is a subspace of $X$, the *relative Alexander–Spanier* cohomology groups $H_{\text{A-S}}^p(X, A; G)$ are the cohomology groups of the complex $(*)$.

Observe that by definition the sequence

$$0 \longrightarrow A_{\text{A-S}}^*(X, A; G) \longrightarrow A_{\text{A-S}}^*(X; G) \longrightarrow A_{\text{A-S}}^*(A; G) \longrightarrow 0$$

is an exact sequence of cochain complexes. Therefore by Theorem 2.19 we have the following long exact sequence of cohomology:

$$
\begin{aligned}
&\cdots \longrightarrow H_{\text{A-S}}^{p-1}(A; G) \\
&\xrightarrow{\delta_{p-1}^*} H_{\text{A-S}}^p(X, A; G) \longrightarrow H_{\text{A-S}}^p(X; G) \longrightarrow H_{\text{A-S}}^p(A; G) \\
&\xrightarrow{\delta_p^*} H_{\text{A-S}}^{p+1}(X, A; G) \longrightarrow H_{\text{A-S}}^{p+1}(X; G) \longrightarrow H_{\text{A-S}}^{p+1}(A; G) \\
&\xrightarrow{\delta_{p+1}^*} H_{\text{A-S}}^{p+2}(X, A; G) \longrightarrow \cdots
\end{aligned}
$$

A continuous map $h\colon (X, A) \to (Y, B)$ (with $h(A) \subseteq B$) also yields the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A^*_{\text{A-S}}(Y, B; G) & \longrightarrow & A^*_{\text{A-S}}(Y; G) & \longrightarrow & A^*_{\text{A-S}}(B; G) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle h^\sharp} & & \Big\downarrow{\scriptstyle (h|X)^\sharp} & & \Big\downarrow{\scriptstyle (h|A)^\sharp} & & \\
0 & \longrightarrow & A^*_{\text{A-S}}(X, A; G) & \longrightarrow & A^*_{\text{A-S}}(X; G) & \longrightarrow & A^*_{\text{A-S}}(A; G) & \longrightarrow & 0.
\end{array}
$$

in which the rows are exact, and a diagram chasing argument proves the existence of a map $h^\sharp$ making the left square commute. We define the homomorphism

$$h^*\colon H^*_{\text{A-S}}(Y, B; G) \to H^*_{\text{A-S}}(X, A; G)$$

induced by $h\colon (X, A) \to (Y, B)$ as the homomorphism induced by the cochain homomorphism

$$h^\sharp\colon A^*_{\text{A-S}}(Y, B; G) \to A^*_{\text{A-S}}(X, A; G)$$

given by the above commutative diagram.

The Alexander–Spanier relative cohomology modules are also limits of certain cohomology groups defined in terms of open covers. This characterization is needed to prove that relative Alexander–Spanier cohomology satisfies the homotopy axiom, and also to prove later on its equivalence with relative classical Čech cohomology defined in Section 14.2. We now sketch this development

The first step is to give another characterization of $A^*_{\text{A-S}}(X, A; G)$ in terms of $A^*_0(X; G)$ and a certain submodule of $A^*(X; G)$.

**Definition 14.2.** For any space $X$ and any subspace $A$ of $X$, we define $A^p(X, A; G)$ as the submodule of $A^p(X; G)$ consisting of all functions in $A^p(X; G)$ which are locally zero on $A$. More precisely, there is some open cover $\mathcal{U}$ of $X$ such that $f \in A^p(X; G)$ vanishes on $\mathcal{U}^{p+1} \cap A^{p+1}$.

It is immediate that $d\colon A^*(X; G) \to A^*(X; G)$ restricts to $A^*(X, A; G)$ so $A^*(X, A; G)$ is a cochain complex. Observe that $A^*(X, \emptyset; G) = A^*(X; G)$

**Proposition 14.1.** *Let $(X, A)$ be a pair of spaces with $A \subseteq X$. There is an isomorphism*

$$A^*_{A\text{-}S}(X, A; G) \cong A^*(X, A; G)/A^*_0(X; G).$$

*Proof.* The surjective homomorphism $i^{p\sharp}\colon A^p_{\text{A-S}}(X; G) \to A^p_{\text{A-S}}(A; G)$ induced by the inclusion $i\colon A \to X$ is defined by

$$i^\sharp([f]) = [f|A],$$

where on the left-hand side $[f]$ is the equivalence class of $f \in A^p(X; G)$ modulo $A^p_0(X; G)$, and on the right-hand side $[f|A]$ is the equivalence modulo $A^p_0(A; G)$ of the restriction of $f$

to $A^p$. If $f' = f + g$ where $g$ is locally zero on $X$, there is some open cover $\mathcal{U}$ of $X$ such that $g$ vanishes on $\mathcal{U}^{p+1}$, and $g|A$ vanishes on $\mathcal{U}^{p+1} \cap A^{p+1}$. Since $f'|A = f|A + g|A$ this shows that $[f'|A] = [f|A]$ and the above map is well defined. This reasoning also shows that the map $\varphi$ given by the composition

$$A^*(X;G) \xrightarrow{\pi} A^*(X;G)/A_0^*(X;G) \xrightarrow{i^\sharp} A^*(A;G)/A_0^*(A;G) = A_{\text{A-S}}^*(A;G)$$

is given by

$$\varphi(f) = [f|A],$$

and that the kernel of $\varphi$ is equal to $A^*(X, A; G)$, so we have an exact sequence

$$0 \longrightarrow A^*(X, A; G) \xrightarrow{\iota} A^*(X;G) \xrightarrow{\varphi} A_{\text{A-S}}^*(A;G) \longrightarrow 0,$$

and $A_0^*(X;G) \subseteq A^*(X, A; G)$. Since $A_0^*(X;G) \subseteq A^*(X, A; G)$, $\operatorname{Ker} \varphi = A^*(X, A; G)$, and the following diagram commutes

$$A^*(X;G) \xrightarrow{\pi} A^*(X;G)/A_0^*(X;G)$$

$$\varphi \searrow \qquad \downarrow i^\sharp$$

$$A_{\text{A-S}}^*(A;G),$$

we have $\operatorname{Ker} i^\sharp \cong A^*(X, A; G)/A_0^*(X;G)$, and we conclude that we have the isomorphism

$$A_{\text{A-S}}^*(X, A; G) \cong A^*(X, A; G)/A_0^*(X;G),$$

as claimed.   $\square$

Observe that $A_{\text{A-S}}^*(X, \emptyset; G) = A_{\text{A-S}}^*(X;G)$.

The next step is to define some cohomology groups based on open covers of $(X, A)$, and for this we need a few facts about open covers.

**Definition 14.3.** Given a pair of topological spaces $(X, A)$ where $A$ is a subset of $X$, a pair $(\mathcal{U}, \mathcal{U}^A)$ is an *open cover of* $(X, A)$ if $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of $X$ and $\mathcal{U}^A = (U_i)_{i \in I^A}$ is a subcover of $\mathcal{U}$ which is a cover of $A$; that is, $I^A \subseteq I$ and $A \subseteq \bigcup_{i \in I^A} U_i$.

Recall from Definition 10.6 that given two covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ of a space $X$, we say that $\mathcal{V}$ *is a refinement of* $\mathcal{U}$, denoted $\mathcal{U} \prec \mathcal{V}$, if there is a function $\tau \colon J \to I$ (sometimes called a *projection*) such that

$$V_j \subseteq U_{\tau(j)} \quad \text{for all } j \in J.$$

**Definition 14.4.** Given a pair of topological spaces $(X, A)$ where $A$ is a subset of $X$, for any two open covers $(\mathcal{U}, \mathcal{U}^A)$ and $(\mathcal{V}, \mathcal{V}^A)$ of $(X, A)$, with $\mathcal{U} = (U_i)_{i \in I}$, $I^A \subseteq I$, $\mathcal{V} = (V_j)_{j \in J}$, $J^A \subseteq J$, we say that $(\mathcal{V}, \mathcal{V}^A)$ is a *refinement* of $(\mathcal{U}, \mathcal{U}^A)$, written $(\mathcal{U}, \mathcal{U}^A) \prec (\mathcal{V}, \mathcal{V}^A)$, if there is a function $\tau \colon J \to I$ (sometimes called a *projection*) such that $\tau(J^A) \subseteq I^A$

$$V_j \subseteq U_{\tau(j)} \quad \text{for all } j \in J.$$

Let $\mathrm{Cov}(X, A)$ be the preorder of open covers $(\mathcal{U}, \mathcal{U}^A)$ of $(X, A)$ under refinement. If $(\mathcal{U}, \mathcal{U}^A)$ and $(\mathcal{V}, \mathcal{V}^A)$ are two open covers of $(X, A)$, if we let

$$\mathcal{W} = \{U_i \cap V_j \mid (i, j) \in I \times J\}$$

and

$$\mathcal{W}^A = \{U_i \cap V_j \mid (i, j) \in I^A \times J^A\},$$

we see that $(\mathcal{W}, \mathcal{W}^A)$ is an open cover of $(X, A)$ that refines both $(\mathcal{U}, \mathcal{U}^A)$ and $(\mathcal{V}, \mathcal{V}^A)$. Therefore, $\mathrm{Cov}(X, A)$ is a directed preorder.

We also define $\mathrm{Cov}(X)$ as the preorder of open covers of $X$ under refinement; it is a directed preorder. However, observe that $\mathrm{Cov}(X)$ is not equal to $\mathrm{Cov}(X, \emptyset)$, because even if $A = \emptyset$, a cover of $(X, \emptyset)$ consists of a pair $(\mathcal{U}, \mathcal{U}^A)$ where $\mathcal{U}^A$ is a subcover of $\mathcal{U}$ associated with some index set $I^A \subseteq I$ which is not necessarily empty. Covers in $\mathrm{Cov}(X)$ correspond to those covers $(\mathcal{U}, \emptyset)$ in $\mathrm{Cov}(X, \emptyset)$ for which $I^A = \emptyset$. In the end this will not matter but this a subtle point that should not be overlooked.

We are ready to show that $A^*_{\mathrm{A\text{-}S}}(X, A; G)$ is the limit of cochain complexes associated with covers $(\mathcal{U}, \mathcal{U}^A)$ of $(X, A)$.

**Definition 14.5.** Let $(X, A)$ be a pair of topological spaces with $A \subseteq X$. For any open cover $(\mathcal{U}, \mathcal{U}^A)$ of $(X, A)$, let $A^p(\mathcal{U}, \mathcal{U}^A; G)$ be the submodule of $A^p(X; G)$ given by

$$A^p(\mathcal{U}, \mathcal{U}^A; G) = \{f \colon \mathcal{U}^{p+1} \to G \mid f(x_0, \ldots, x_p) = 0 \text{ if } (x_0, \ldots, x_p) \in (\mathcal{U}^A)^{p+1} \cap A^{p+1}\}.$$

The homomorphism

$$d^p \colon A^p(\mathcal{U}, \mathcal{U}^A; G) \to A^{p+1}(\mathcal{U}, \mathcal{U}^A; G)$$

is defined as in Definition 13.8 by

$$d^p f(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(x_1, \ldots, \widehat{x_i}, \ldots, x_{p+1}).$$

It is easily checked that $d^{p+1} \circ d^p = 0$ for all $p \geq 0$, so the modules $A^p(\mathcal{U}, \mathcal{U}^A; G)$ form a cochain complex.

**Remark:** The module $A^p(\mathcal{U}, \mathcal{U}^A; G)$ can be viewed as an ordered simplicial cochain complex; see Spanier [47] (Chapter 6, Section 4).

If $(\mathcal{V}, \mathcal{V}^A)$ is a refinement of $(\mathcal{U}, \mathcal{U}^A)$, then the restriction map is a cochain map

$$\rho^{\mathcal{U}, \mathcal{U}^A}_{\mathcal{V}, \mathcal{V}^A} \colon A^p(\mathcal{U}, \mathcal{U}^A; G) \to A^p(\mathcal{V}, \mathcal{V}^A; G),$$

so the directed family $(A^p(\mathcal{U},\mathcal{U}^A; G))_{(\mathcal{U},\mathcal{U}^A)\in\mathrm{Cov}(X,A)}$ together with the family of maps $\rho_{\mathcal{V},\mathcal{V}^A}^{\mathcal{U},\mathcal{U}^A}$ with $(\mathcal{U},\mathcal{U}^A)\prec(\mathcal{V},\mathcal{V}^A)$ is a direct mapping family.

**Remark:** As usual, one has to exercise some care because the set of all covers of $(X,A)$ is not a set. This can be delt with as in Serre's FAC [44] or as in Eilenberg and Steenrod [12] (Chapter IX, page 238).

The remarkable fact is that if $A\neq\emptyset$ then we have an isomorphism

$$A^*_{\text{A-S}}(X,A;G)\cong \varinjlim_{(\mathcal{U},\mathcal{U}^A)\in\mathrm{Cov}(X,A)} A^p(\mathcal{U},\mathcal{U}^A;G),$$

and if $A=\emptyset$ we have an isomorphism

$$A^*_{\text{A-S}}(X;G)\cong \varinjlim_{\mathcal{U}\in\mathrm{Cov}(X)} A^p(\mathcal{U},\emptyset;G).$$

To prove the above isomorphism, first if $A\neq\emptyset$ we define a map

$$\lambda\colon A^*(X,A;G)\to \varinjlim_{(\mathcal{U},\mathcal{U}^A)\in\mathrm{Cov}(X,A)} A^p(\mathcal{U},\mathcal{U}^A;G),$$

where $A^*(X,A;G)$ is the module defined in Definition 14.2, and if $A=\emptyset$ we define a map

$$\lambda\colon A^*(X;G)\to \varinjlim_{\mathcal{U}\in\mathrm{Cov}(X)} A^p(\mathcal{U},\emptyset;G).$$

Assume $A\neq\emptyset$. For any $f\in A^p(X,A;G)$, there is some open cover $\mathcal{U}^A$ of $A$ consisting of open subsets of $X$ such that $f$ vanishes on $(\mathcal{U}^A)^{p+1}\cap A^{p+1}$, and we let $\mathcal{U}$ be the open cover of $X$ obtained by adding $X$ itself to the cover $\mathcal{U}^A$ and giving it some new index, say $k$. Then $(\mathcal{U},\mathcal{U}^A)$ is an open cover of $(X,A)$ and by restriction $f$ determines an element $f|(\mathcal{U},\mathcal{U}^A)\in A^p(\mathcal{U},\mathcal{U}^A;G)$. Passing to the limit, we obtain a homomorphism

$$\lambda^p\colon A^p(X,A;G)\to \varinjlim_{(\mathcal{U},\mathcal{U}^A)\in\mathrm{Cov}(X,A)} A^p(\mathcal{U},\mathcal{U}^A;G).$$

**Theorem 14.2.** *If $A\neq\emptyset$ then the map*

$$\lambda\colon A^*(X,A;G)\to \varinjlim_{(\mathcal{U},\mathcal{U}^A)\in\mathrm{Cov}(X,A)} A^*(\mathcal{U},\mathcal{U}^A;G)$$

*is surjective and its kernel is given by* $\mathrm{Ker}\,\lambda=A^*_0(X;G)$. *Consequently, we have an isomorphism*

$$A^*_{A\text{-}S}(X,A;G)\cong \varinjlim_{(\mathcal{U},\mathcal{U}^A)\in\mathrm{Cov}(X,A)} A^*(\mathcal{U},\mathcal{U}^A;G).$$

*If $A = \emptyset$ then the map*

$$\lambda \colon A^*(X;G) \to \varinjlim_{\mathcal{U} \in \mathrm{Cov}(X)} A^*(\mathcal{U}, \emptyset; G)$$

*is surjective and its kernel is given by $\mathrm{Ker}\,\lambda = A_0^*(X;G)$. Consequently, we have an isomorphism*

$$A_{A\text{-}S}^*(X;G) \cong \varinjlim_{\mathcal{U} \in \mathrm{Cov}(X)} A^*(\mathcal{U}, \emptyset; G).$$

*Proof.* We follow Spanier's proof, see Spanier [47] (Chapter 6, Section 4, Theorem 1). Assume that $A \neq \emptyset$. First we prove that $\lambda$ is surjective. Pick any $u \in A^p(\mathcal{U}, \mathcal{U}^A; G)$, and define $f_u$ by

$$f_u(x_0, \ldots, x_p) = \begin{cases} u(x_0, \ldots, x_p) & \text{if } (x_0, \ldots, x_p) \in \mathcal{U}^{p+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_u$ vanishes on $(\mathcal{U}^A)^{p+1} \cap A^{p+1}$, and therefore $f_u|(\mathcal{U}, \mathcal{U}^A) \in A^p(X, A; G)$. By definition, we have $f_u| = u$, so $\lambda$ is surjective.

Next we prove that $\mathrm{Ker}\,\lambda = A_0^*(X;G)$. A function $f \in A^p(X, A; G)$ is in the kernel of $\lambda$ iff there is some open cover $(\mathcal{U}, \mathcal{U}^A)$ such that $f|(\mathcal{U}, \mathcal{U}^A) = 0$. Thus, $\lambda(f) = 0$ iff there is some open covering $\mathcal{U}$ such that $f$ vanishes on $\mathcal{U}^{p+1}$. By the definition of $A_0^*(X;G)$, we have $\lambda(f) = 0$ iff $f \in A_0^*(X;G)$.

The case where $A = \emptyset$ is similar but slightly simpler. $\qquad\square$

An important corollary of Theorem 14.2 is the following characterization of the relative Alexander–Spanier cohomology groups as certain limits of simpler cohomology groups (in fact, simplicial cohomology).

**Theorem 14.3.** *Let $(X, A)$ be a pair of spaces with $A \subseteq X$. If $A \neq \emptyset$ then we have an isomorphism*

$$H_{A\text{-}S}^p(X, A; G) \cong \varinjlim_{(\mathcal{U}, \mathcal{U}^A) \in \mathrm{Cov}(X, A)} H^p(\mathcal{U}, \mathcal{U}^A; G), \quad \text{for all } p \geq 0.$$

*If $A = \emptyset$ then we have an isomorphism*

$$H_{A\text{-}S}^p(X; G) \cong \varinjlim_{\mathcal{U} \in \mathrm{Cov}(X)} H^p(\mathcal{U}, \emptyset; G), \quad \text{for all } p \geq 0.$$

*Proof.* It is shown in Spanier [47] (Chapter 4) that cohomology commutes with direct limits (this is a general categorical fact about direct limits). Using Theorem 14.2 we obtain our result. $\qquad\square$

Spanier uses Theorem 14.3 to prove that Alexander–Spanier cohomology satisfies the homotopy axiom; see Spanier [47] (Chapter 6, Section 5). Actually, Spanier proves that

Alexander–Spanier cohomology satisfies all of the Eilenberg–Steenrod axioms. A more detailed treatment of Alexander–Spanier cohomology is found in Spanier [47] (Chapter 6, Sections 4–9).

In order to state the most general version of Alexander–Lefschetz duality (not restricted to the compact case), it is necessary to introduce Alexander–Spanier cohomology with compact support

**Definition 14.6.** A subset $A$ of a topological space $X$ is said to be *bounded* if its closure $\overline{A}$ is compact. A subset $B \subseteq X$ is said to be *cobounded* if its complement $X - B$ is bounded. A function $h \colon X \to Y$ is *proper* if it is continuous and if $h^{-1}(A)$ is bounded in $X$ whenever $A$ is bounded in $Y$.

It is immediate to check that the composition of two proper maps is proper. A proper map $h$ between $(X, A)$ and $(Y, B)$ (where $A \subseteq X$ and $B \subseteq Y$) is a proper map from $X$ to $Y$ such that $h(A) \subseteq B$.

**Definition 14.7.** Let $(X, A)$ be a pair of spaces with $A \subseteq X$. The module $A_c^p(X, A; G)$ is the submodule of $A^p(X, A; G)$ consisting of all functions $f \in A^p(X, A; G)$ such that $f$ is locally zero on some cobounded subset $B$ of $X$. If $f \in A^p(X, A; G)$ is locally zero on $B$, so is $\delta f$, thus the family of modules $A_c^p(X, A; G)$ with the restrictions of the $\delta^p$ is a cochain complex which is a subcomplex of $A^*(X, A; G)$. Since $A_0^*(X; G) \subseteq A_c^*(X, A; G)$, we obtain the cochain complex $A_{\text{A-S},c}^* * (X, A; G)$, with

$$A_{\text{A-S},c}^*(X, A; G) = A_c^*(X, A; G)/A_0^*(X; G).$$

The *Alexander–Spanier cohomology modules of* $(X, A)$ *with compact support* $H_{\text{A-S},c}^p(X, A; G)$ are the cohomology modules of the cochain complex $A_{\text{A-S},c}^*(X, A; G)$.

If $h \colon (X, A) \to (Y, B)$ is a proper map, then $h^\sharp$ maps $A_{\text{A-S},c}^*(Y, B; G)$ to $A_{\text{A-S},c}^*(X, A; G)$ and induces a homomorphism

$$h^* \colon H_{\text{A-S},c}^p(Y, B; G) \to H_{\text{A-S},c}^p(X, A; G).$$

Properties of Alexander–Spanier cohomology with compact support are investigated in [47] (Chapter 6, Sections 6). We just mention the following result.

**Proposition 14.4.** *Let* $(X, A)$ *be a pair of spaces with* $A \subseteq X$. *If* $A$ *is a cobounded subset of* $X$, *then there is an isomorphism*

$$H_{A\text{-}S,c}^*(X, A; G) \cong H_{A\text{-}S}^*(X, A; G).$$

In particular, Proposition 14.4 applies to the situation where $(X, A)$ is a *compact pair*, which means that $X$ is compact and $A$ is a closed subset of $X$.

We conclude this section by mentioning that Alexander–Spanier cohomology enjoys a very simple definition of the cup product. Indeed, given $f_1 \in A^p(X; G)$ and $f_2 \in A^q(X; G)$ we define $f_1 \smile f_2 \in A^{p+q}(X; G)$ by

$$(f_1 \smile f_2)(x_0, \ldots, x_{p+q}) = f_1(x_0, \ldots, x_p) f_2(x_p, \ldots, x_{p+q}).$$

If $f_1$ is locally zero on $A_1$ then so is $f_1 \smile f_2$, and if $f_2$ is locally zero on $A_1$ then so is $f_1 \smile f_2$. Consequently $\smile$ induces a cup product

$$\smile : A^p_{\text{A-S}}(X; G) \times A^q_{\text{A-S}}(X; G) \to A^{p+q}_{\text{A-S}}(X; G).$$

One verifies that

$$\delta(f_1 \smile f_2) = \delta f_1 \smile f_2 + (-1)^p f_1 \smile \delta f_2,$$

so we obtain a cup product

$$\smile : H^p_{\text{A-S}}(X; G) \times H^q_{\text{A-S}}(X; G) \to H^{p+q}_{\text{A-S}}(X; G)$$

at the cohomology level.

It is also easy to deal with relative cohomology; see Spanier [47] (Chapter 6, Section 5).

## 14.2 Relative Classical Čech Cohomology

In this section we deal with classical Čech cohomology, which means that given an open cover $\mathcal{U} = (U_i)_{i \in I}$ of the space $X$ and given a $R$-module $G$, the module $C^p(\mathcal{U}, G)$ of Čech $p$-cochains can be defined as the $R$-module of functions $f \colon I^{p+1} \to G$ such that for all $(i_0, \ldots, i_p) \in I^{p+1}$,

$$f(i_0, \ldots, i_p) = 0 \quad \text{if } U_{i_0 \cdots i_p} = \emptyset,$$

where $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$.

Our first goal is to explain how a continuous map $h \colon X \to Y$ induces a homomorphism of Čech cohomology

$$h^{p*} \colon \check{H}^p(Y, G) \to \check{H}^p(X, G).$$

For this, it necessary to take a closer look at the behavior of open covers of $Y$ under $h^{-1}$.

If $\mathcal{V} = (V_i)_{i \in I}$ is an open cover of $Y$, then since $h$ is continuous $h^{-1}(\mathcal{V}) = (h^{-1}(V_i))_{i \in I}$ is an open cover of $X$, with the same index set $I$. We also denote $h^{-1}(V_i)$ by $h^{-1}(V)_i$ or $V_i'$.

If $\mathcal{W} = (W_j)_{j \in J}$ is a refinement of $\mathcal{V} = (V_i)_{i \in I}$ and if $\tau \colon J \to I$ is a function such that

$$W_j \subseteq V_{\tau(j)} \quad \text{for all } j \in J,$$

since

$$h^{-1}(W_j) \subseteq h^{-1}(V_{\tau(j)}),$$

if we write $W'_j = h^{-1}(W_j)$ and $V'_i = h^{-1}(V_i)$, then we have

$$W'_j \subseteq V'_{\tau(j)} \quad \text{for all } j \in J,$$

which means that $h^{-1}(\mathcal{W})$ is a refinement of $h^{-1}(\mathcal{V})$ (as open covers of $X$).

Let $\mathrm{Cov}(X)$ be the preorder of open covers $\mathcal{U}$ of $X$ under refinement and let $\mathrm{Cov}(Y)$ be the preorder of open covers $\mathcal{V}$ of $Y$ under refinement. Observe that what we just showed implies that the map $\mathcal{V} \mapsto h^{-1}(\mathcal{V})$ between $\mathrm{Cov}(Y)$ and $\mathrm{Cov}(X)$ is an order-preserving map.

For any tuple $(i_0, \ldots, i_p) \in I^{p+1}$, we have

$$h^{-1}(V_{i_0 \cdots i_p}) = h^{-1}(V_{i_0} \cap \cdots \cap V_{i_p}) = h^{-1}(V_{i_0}) \cap \cdots \cap h^{-1}(V_{i_p}),$$

and if we let $h^{-1}(V)_{i_0 \cdots i_p} = h^{-1}(V_{i_0}) \cap \cdots \cap h^{-1}(V_{i_p})$, then

$$h^{-1}(V_{i_0 \cdots i_p}) = h^{-1}(V)_{i_0 \cdots i_p}.$$

Note that it is possible that $V_{i_0 \cdots i_p} \neq \emptyset$ but $h^{-1}(V_{i_0 \cdots i_p}) = h^{-1}(V)_{i_0 \cdots i_p} = \emptyset$.

Given a continuous map $h \colon X \to Y$ and an open cover $\mathcal{V} = (V_i)_{i \in I}$ of $Y$, we define a homomorphism from $C^p(\mathcal{V}, G)$ to $C^p(h^{-1}(\mathcal{V}), G)$ (where $h^{-1}(\mathcal{V})$ is an open cover of $X$).

**Definition 14.8.** Let $h \colon X \to Y$ be a continuous map between two spaces $X$ and $Y$ and let $\mathcal{V} = (V_i)_{i \in I}$ be some open cover of $Y$. The $R$-module homomorphism

$$h_{\mathcal{V}}^{p\sharp} \colon C^p(\mathcal{V}, G) \to C^p(h^{-1}(\mathcal{V}), G)$$

if defined as follows: for any $f \in C^p(\mathcal{V}; G)$, for all $(i_0, \ldots, i_p) \in I^{p+1}$,

$$h_{\mathcal{V}}^{p\sharp}(f)(i_0, \ldots, i_p) = \begin{cases} f(i_0, \ldots, i_p) & \text{if } h^{-1}(V)_{i_0 \cdots i_p} \neq \emptyset \\ 0 & \text{if } h^{-1}(V)_{i_0 \cdots i_p} = \emptyset. \end{cases}$$

The module homomorphism $h_{\mathcal{V}}^{p\sharp} \colon C^p(\mathcal{V}, G) \to C^p(h^{-1}(\mathcal{V}), G)$ induces a module homomorphism of Čech cohomology groups

$$h_{\mathcal{V}}^{p*} \colon \check{H}^p(\mathcal{V}; G) \to \check{H}^p(h^{-1}(\mathcal{V}); G).$$

For every refinement $\mathcal{W}$ of $\mathcal{V}$ ($\mathcal{V} \prec \mathcal{W}$), we have a commutative diagram

$$
\begin{array}{ccc}
\check{H}^p(\mathcal{V}; G) & \xrightarrow{\; h_{\mathcal{V}}^{p*} \;} & \check{H}^p(h^{-1}(\mathcal{V}); G) \\
\rho_{\mathcal{W}}^{\mathcal{V}} \downarrow & & \downarrow \rho_{h^{-1}(\mathcal{W})}^{h^{-1}(\mathcal{V})} \\
\check{H}^p(\mathcal{W}; G) & \xrightarrow{\; h_{\mathcal{W}}^{p*} \;} & \check{H}^p(h^{-1}(\mathcal{W}); G),
\end{array}
$$

where the restriction map $\rho_{\mathcal{W}}^{\mathcal{V}} \colon \check{H}^p(\mathcal{V}; G) \to \check{H}^p(\mathcal{W}; G)$ is defined just after 10.3 (and similarly for $\rho_{h^{-1}(\mathcal{W})}^{h^{-1}(\mathcal{V})} \colon \check{H}^p(h^{-1}(\mathcal{V}); G) \to \check{H}^p(h^{-1}(\mathcal{W}); G)$). If we define the map $\tau_h \colon \mathrm{Cov}(Y) \to \mathrm{Cov}(X)$ by $\tau_h(\mathcal{V}) = h^{-1}(\mathcal{V})$, then we see that $\tau_h$ and the family of maps

$$h_{\mathcal{V}}^{p*} \colon \check{H}^p(\mathcal{V}; G) \to \check{H}^p(h^{-1}(\mathcal{V}); G)$$

define a map from the direct mapping family $(\check{H}^p(\mathcal{V}; G))_{\mathcal{V} \in \mathrm{Cov}(Y)}$ to the direct mapping family $(\check{H}^p(\mathcal{U}; G))_{\mathcal{U} \in \mathrm{Cov}(X)}$, and by the discussion just before Definition 9.12 we obtain a homomorphism between their direct limits, that is, a homomorphism

$$h^{p*} \colon \check{H}^p(Y; G) \to \check{H}^p(X; G).$$

In order to define the relative Čech cohomology groups we need to consider a few more properties of the open covers of a pair $(X, A)$. Let $h \colon (X, A) \to (Y, B)$ be a continuous map (recall that $h \colon X \to Y$ is continuous and $h(A) \subseteq B$). If $(\mathcal{V}, \mathcal{V}^B)$ is any open cover of $(Y, B)$ (with index sets $(I, I^B)$) then $(h^{-1}(\mathcal{V}), h^{-1}(\mathcal{V}^B))$ is an open cover of $(X, A)$ with the same index sets $I$ and $I^B$.

If $(\mathcal{W}, \mathcal{W}^B)$ (with index sets $(J, J^B)$) is a refinement of $(\mathcal{V}, \mathcal{V}^B)$ (with index set $(I, I^B)$) with projection function $\tau \colon J \to I$, it is immediate to check that $(h^{-1}(\mathcal{W}), h^{-1}(\mathcal{W}^B))$ is a refinement of $(h^{-1}(\mathcal{V}), h^{-1}(\mathcal{V}^B))$. It follows that the map

$$(\mathcal{V}, \mathcal{V}^B) \mapsto (h^{-1}(\mathcal{V}), h^{-1}(\mathcal{V}^B))$$

is an order preserving map between $\mathrm{Cov}(Y, B)$ and $\mathrm{Cov}(X, A)$. As before, for any tuple $(i_0, \ldots, i_p)$ in $I^{p+1}$ or in $(I^A)^{p+1}$ we write

$$h^{-1}(V)_{i_0 \cdots i_p} = h^{-1}(V_{i_0 \cdots i_p}) = h^{-1}(V_{i_0}) \cap \cdots \cap h^{-1}(V_{i_p}).$$

It is possible that $V_{i_0 \cdots i_p} \neq \emptyset$ but $h^{-1}(V_{i_0 \cdots i_p}) = h^{-1}(V)_{i_0 \cdots i_p} = \emptyset$.

**Definition 14.9.** Let $(X, A)$ be a pair of spaces with $A \subseteq X$. For every open cover $(\mathcal{U}, \mathcal{U}^A)$ of $(X, A)$, the module $C^p(\mathcal{U}, \mathcal{U}^A; G)$ is the submodule of $C^p(\mathcal{U}; G)$ defined as follows:

$$C^p(\mathcal{U}, \mathcal{U}^A; G) = \{f \colon I^{p+1} \to G \mid \text{for all } (i_0, \ldots, i_p) \in I^{p+1}$$
$$f(i_0, \ldots, i_p) = 0 \text{ if } U_{i_0 \cdots i_p} = \emptyset \text{ or if } (i_0, \ldots, i_p) \in (\mathcal{U}^A)^{p+1} \cap A^{p+1}\}.$$

Observe that if $A = \emptyset$, then $C^p(\mathcal{U}, \mathcal{U}^A; G) = C^p(\mathcal{U}; G)$ for any $\mathcal{U}^A$. In this case, we will restrict ourselves to covers for which $\mathcal{U}^A = \emptyset$, to ensure that direct limits are taken over $\mathrm{Cov}(X)$ in order to obtain the Čech cohomology groups of Definition 10.7.

The analogy between the above definition of $C^p(\mathcal{U}, \mathcal{U}^A; G)$ and the Alexander–Spanier modules

$$A^p(\mathcal{U}, \mathcal{U}^A; G) = \{f \colon \mathcal{U}^{p+1} \to G \mid f(x_0, \ldots, x_p) = 0 \text{ if } (x_0, \ldots, x_p) \in (\mathcal{U}^A)^{p+1} \cap A^{p+1}\}$$

of Definition 14.5 is striking. Indeed, it turns out that they induce isomorphic cohomology.

It is immediately checked that the coboundary maps $d^p \colon C^p(U; G) \to C^{p+1}(U; G)$ restrict to the $C^p(\mathcal{U}, \mathcal{U}^A; G)$ and we obtain a cochain complex $C^*(\mathcal{U}, \mathcal{U}^A; G)$.

**Definition 14.10.** Let $(X, A)$ be a pair of spaces with $A \subseteq X$. For every open cover $(\mathcal{U}, \mathcal{U}^A)$ of $(X, A)$, the *Čech cohomology modules* $\check{H}^p(\mathcal{U}, \mathcal{U}^A; G)$ are the cohomology modules of the complex $C^*(\mathcal{U}, \mathcal{U}^A; G)$.

Observe that if $A = \emptyset$ then $\check{H}^p(\mathcal{U}, \mathcal{U}^A; G) = \check{H}^p(\mathcal{U}; G)$ for any $\mathcal{U}^A$.

If $(\mathcal{V}, \mathcal{V}^A)$ is a refinement of $(\mathcal{U}, \mathcal{U}^A)$ then there is a cochain map

$$\rho_{\mathcal{V}, \mathcal{V}^A}^{\mathcal{U}, \mathcal{U}^A} \colon C^p(\mathcal{U}, \mathcal{U}^A; G) \to C^p(\mathcal{V}, \mathcal{V}^A; G),$$

One need to prove that $\rho_{\mathcal{V}, \mathcal{V}^A}^{\mathcal{U}, \mathcal{U}^A}$ does not depend on the projection map $\tau \colon J \to I$, but this can be done as in Serre's FAC [44] or as in Eilenberg and Steenrod [12] (Chapter IX, Theorem 2.13 and Corollary 2.14).

Therefore, the directed family $(C^p(\mathcal{U}, \mathcal{U}^A; G))_{(\mathcal{U}, \mathcal{U}^A) \in \mathrm{Cov}(X, A)}$ together with the family of maps $\rho_{\mathcal{V}, \mathcal{V}^A}^{\mathcal{U}, \mathcal{U}^A}$ with $(\mathcal{U}, \mathcal{U}^A) \prec (\mathcal{V}, \mathcal{V}^A)$ is a direct mapping family.

**Remark:** As usual, one has to exercise some care because the set of all covers of $(X, A)$ is not a set. This can be delt with as in Serre's FAC [44] or as in Eilenberg and Steenrod [12] (Chapter IX, page 238).

**Definition 14.11.** Let $(X, A)$ be a pair of spaces with $A \subseteq X$. If $A \neq \emptyset$ then the *relative Čech cohomology modules* $\check{H}^p(X, A; G)$ are defined as the direct limits

$$\check{H}^p(X, A; G) = \varinjlim_{(\mathcal{U}, \mathcal{U}^A) \in \mathrm{Cov}(X, A)} \check{H}^p(\mathcal{U}, \mathcal{U}^A; G).$$

If $A = \emptyset$, then the *(absolute) Čech cohomology modules* $\check{H}^p(X; G)$ are defined as the direct limits

$$\check{H}^p(X; G) = \varinjlim_{\mathcal{U} \in \mathrm{Cov}(X)} \check{H}^p(\mathcal{U}; G).$$

It is clear that the absolute Čech cohomology modules $\check{H}^p(X; G)$ are equal to the classical Čech cohomology modules $\check{H}^p(X; G_X)$ of the constant presheaf $G_X$ as defined in Definition 10.7, since direct limits are taken over $\mathrm{Cov}(X)$.

At this stage, we could proceed with a study of the properties of the relative Čech cohomology modules as in Eilenberg and Steenrod [12], but instead we will state a crucial result due to Dowker [10] which proves that the relative Čech cohomology modules and the relative Alexander–Spanier cohomology modules are isomorphic; this is also true in the absolute case. This way, we are reduced to a study of the properties of the Alexander–Spanier cohomology modules, which is often simpler. For example the proof of the existence of the long exact cohomology sequence in Čech cohomology is quite involved (see Eilenberg and Steenrod [12] (Chapter IX), but is is quite simple in Alexander–Spanier cohomology.

This does not mean that Čech cohomology is not interesting. On the contrary, it arises naturally whenever the notion of cover is involved, and it plays an important role in algebraic geometry. It also lends itself to generalizations by extending the notion of cover.

**Theorem 14.5.** *(Dowker) Let $(X, A)$ be a pair of spaces with $A \subseteq X$. If $A \neq \emptyset$ then the Alexander–Spanier cohomology modules $H_{A\text{-}S}^p(X, A; G)$ and the Čech cohomology modules $\check{H}^p(X, A; G)$ are isomorphic:*

$$H_{A\text{-}S}^p(X, A; G) \cong \check{H}^p(X, A; G) \text{ for all } p \geq 0.$$

*If $A = \emptyset$ then we have isomorphisms*

$$H_{A\text{-}S}^p(X; G) \cong \check{H}^p(X; G) \text{ for all } p \geq 0.$$

A complete proof of Theorem 14.5 is given in Dowker [10]; see Theorem 2. Dowker is careful to parametrize the Alexander–Spanier cohomology modules and the Čech cohomology modules with a directed preorder of covers $\Omega$, so that he does not run into problems when taking direct limits when $A = \emptyset$. The proof of Theorem 14.5 is also proposed as a sequence of problems in Spanier [47] (Chapter 6, Problems D1, D2, D3).

## 14.3  Alexander–Lefschetz Duality

Given a $R$-orientable manifold $M$, Alexander–Lefschetz duality is a generalization of Poincaré duality that asserts that the Alexander–Spanier cohomology group $H_{\text{A-S}}^p(K, L; G)$ and the singular homology group $H_{n-p}(M - L, M - K; G)$ are isomorphic, where $L \subseteq K \subseteq M$ and $L$ and $K$ are compact. Actually, the method for proving this duality yields an isomorphism between a certain direct limit $\overline{H}^p(K, L; G)$ of singular cohomology groups $H^p(U, V; G)$ where $U$ is any open subset of $M$ containing $K$ and $V$ is any any open subset of $M$ containing $L$, and the singular homology group $H_{n-p}(M - L, M - K; G)$.

Furthermore, it can be shown that $\overline{H}^p(K, L; G)$ and $H_{\text{A-S}}^p(K, L; G)$ are isomorphic, so Alexander–Lefschetz duality can indeed be stated as an isomorphism between $H_{\text{A-S}}^p(K, L; G)$ and $H_{n-p}(M - L, M - K; G)$. Since Alexander–Lefschetz cohomology and Čech cohomology are isomorphic, Alexander–Lefschetz duality can also be stated as an isomorphism between $\check{H}^p(K, L; G)$ and $H_{n-p}(M - L, M - K; G)$, and this is what certain authors do, including Bredon [4] (Chapter 8, Section 8).

**Definition 14.12.** Given any topological space $X$, for any pair $(A, B)$ of subsets of $X$, let $N(A, B)$ be the set of all pairs $(U, V)$ of open subsets of $X$ such that $A \subseteq U$ and $B \subseteq V$ ordered such that $(U_1, V_1) \leq (U_2, V_2)$ iff $U_2 \subseteq U_1$ and $V_2 \subseteq V_1$ (reverse inclusion).

Clearly $N(A, B)$ is a directed preorder, and if $(U_1, V_1) \leq (U_2, V_2)$ then there is an induced map of singular cohomology $\rho_{U_2, V_2}^{U_1, V_1} \colon H^p(U_1, V_1; G) \to H^p(U_2, V_2; G)$, so the family $(H^p(U, V; G))_{(U,V) \in N(A,B)}$ together with the maps $\rho_{U_2, V_2}^{U_1, V_1}$ is a direct mapping family.

**Definition 14.13.** Given any topological space $X$, for any pair $(A, B)$ of subsets of $X$, the modules $\overline{H}^p(A, B; G)$ are defined

$$\overline{H}^p(A, B; G) = \varinjlim_{(U,V) \in N(A,B)} H^p(U, V; G) \text{ for all } p \geq 0.$$

The restriction maps $H^p(U, V; G) \longrightarrow H^p(A, B; G)$ yield a natural homomorphism

$$i^p \colon \overline{H}^p(A, B; G) \to H^p(A, B; G)$$

between $\overline{H}^p(A, B; G)$ and the singular cohomology module $H^p(A, B; G)$. In general, $i^p$ neither injective nor surjective. Following Spanier [47] (Chapter 6, Section 1), we say that the pair $(A, B)$ is *tautly imbedded* in $X$ if every $i^p$ is an isomorphism.

**Remark:** The notation $\overline{H}^p(A, B; G)$ is borrowed from Spanier [47] (Chapter 6, Section 1). Bredon denotes the direct limit in Definition 14.13 by $\check{H}^p(A, B; G)$; see Bredon [4] (Chapter 8, Section 8). He then goes on to say that if $X$ is a manifold and $A$ and $B$ are closed then this group (which is really $\overline{H}^p(A, B; G)$) is naturally isomorphic to the Čech cohomology group. This is indeed true, but this is proved by showing that $\overline{H}^p(A, B; G)$ is isomorphic to the Alexander–Spanier cohomology module $H^p_{\text{A-S}}(A, B; G)$ and then using the isomorphism between the Alexander–Spanier cohomology modules and the Čech cohomology modules. Since these results are nontrivial, we find Bredon's notation somewhat confusing.

It is shown in Spanier ([47], Chapter 6, Section 1, Corollary 11) that if $A$, $B$ and $X$ are compact polyhedra then the pair $(A, B)$ is taut in $X$, which means that there are isomorphisms $\overline{H}^p(A, B; G) \cong H^p(A, B; G)$, so we can simply use singular cohomology. This is the set-up in which Lefschetz duality was originally proved. We also have the following useful result about manifolds; see Spanier ([47], Chapter 6, Section 9, Corollary 7).

**Proposition 14.6.** *If $X$ is a manifold then $\overline{H}^*(X; G) \cong H^*(X; G)$.*

The following result shows that when $X$ is a manifold and $(A, B)$ is a closed pair, the groups $\overline{H}^p(A, B; G)$ are just the Alexander–Spanier cohomology groups.

**Proposition 14.7.** *Let $X$ be a manifold. For any pair $(A, B)$ of closed subsets of $X$, there are isomorphisms*

$$H^p_{\text{A-S}}(A, B; G) \cong \overline{H}^p(A, B; G) \ \text{for all } p \geq 0.$$

Proposition 14.7 is proved in Spanier [47] (Chapter 6, Section 9, Corollary 9).

We are now ready state the main result of this chapter. Let $M$ be a $R$-orientable manifold. By Theorem 7.7, for any compact subset $K$ of $M$, there is a unique $R$-fundamental class $\mu_K \in H_n(M, M - K; R)$ of $M$ at $K$.

Asume that $L \subseteq K \subseteq M$, $V \subseteq U$, $K \subseteq U$, and $L \subseteq V$, with $K, L$ compact. Then $U - K \subseteq U - L$ and $\{V, U - L\}$ is an open cover of $U$. We know from Section 7.5 that there is a relative cap product

$$\frown \colon S^p(U, V; G) \times S_n(U, V \cup (U - K); R) \to S_{n-p}(U, U - K; G).$$

We claim that the above cap product induces a cap product

$$\frown \colon S^p(U, V; G) \times S_n(U, U - K; R) \to S_{n-p}(U - L, U - K; G).$$

Since $U - K \subseteq L \cup (U - K)$, we have a homomorphism

$$i \colon S_n(U, U - K; R) \to S_n(U, V \cup (U - K); R),$$

where the equivalence class of $a \in S_n(U; R)$ mod $S_n(U - K; R)$ is mapped to the equivalence class of $a$ mod $S_n(L \cup (U - K); R)$. Recall that a cochain $f \in S^p(U, V; G)$ is a cochain in $S^p(U; G)$ that vanishes on simplices in $V$. Also since $U = V \cup (U - L)$, any chain $\sigma$ in $S_n(U, V \cup (U - K); R) = S_n(V \cup (U - L), V \cup (U - K); R)$ is represented by a sum of the form

$$a + b + c,$$

with $a \in S_n(V; R)$, $b \in S_n(U - L; R)$ and $c \in S_n(V \cup (U - K); R)$. Since $S_n(V; R) \subseteq S_n(V \cup (U - K); R)$, we see that $a \in S_n(V \cup (U - K); R)$ and so $\sigma$ is also represented by some element $b + d$ with $b \in S_n(U - L; R)$ and $d \in S_n(V \cup (U - K); R)$. Then we have

$$f \frown (b + d) = f \frown b + f \frown d,$$

with $f \frown b \in S_{n-p}(U - L; G)$, and since $f$ vanishes on $V$ and $d \in S_n(V \cup (U - K); R)$ the term $f \frown d$ belongs to $S_{n-p}(U - K; G)$, so in the end $f \frown (b + d)$ represents a cycle in $S_{n-p}(U - L, U - K; G)$. Passing to cohomology and homology, since by excision

$$H_n(M, M - K; R) \cong H_n(U, U - K; R)$$
$$H_{n-p}(M - L, M - L; G) \cong H_{n-p}(U - L, U - K; G),$$

the cap product

$$\frown \colon S^p(U, V; G) \times S_n(U, U - K; R) \to S_{n-p}(U - L, U - K; G)$$

induces a cap product

$$\frown \colon H^p(U, V; G) \times H_n(M, M - K; R) \to H_{n-p}(M - L, M - K; G).$$

If $M$ is a $R$-orientable manifold, for any pair $(K, L)$ of compact subsets of $M$ such that $L \subseteq K$ and for any pair $(U, V) \in N(K, L)$, we obtain a map

$$\frown \mu_K \colon H^p(U, V; G) \to H_{n-p}(M - L, M - K; G),$$

and by a limit argument, we obtain a map

$$\frown \mu_K \colon \overline{H}^p(K, L; G) \to H_{n-p}(M - L, M - K; G);$$

for details see Bredon [4] (Chapter 8, Section 8).

**Theorem 14.8.** *(Alexander–Lefschetz duality) Let $M$ be a $R$-orientable manifold where $R$ is any commutative ring with an identity element. For any $R$-module $G$, for any pair $(K, L)$ of compact subsets of $M$ such that $L \subseteq K$, the map $\omega \mapsto \omega \frown \mu_K$ yields an isomorphism*

$$\overline{H}^p(K, L; G) \cong H_{n-p}(M - L, M - K; G) \text{ for all } p \geq 0.$$

*Thus we also have isomorphisms*

$$H^p_{A\text{-}S}(K, L; G) \cong \check{H}^p(K, L; G) \cong H_{n-p}(M - L, M - K; G) \text{ for all } p \geq 0.$$

Theorem 14.8 is proved in Bredon [4] where it is called the *Poincaré–Alexander–Lefschetz duality* (Chapter 8, Section 8, Theorem 8.3) by using the Bootstrap Lemma (Proposition 7.4). It is also proved in Spanier [47] (Chapter 6, Section 2, Theorem 17), except that the isomorphism goes in the opposite direction and does not use the fundamental class $\mu_K$.

If we let $K = M$ and $L = \emptyset$, since for a manifold we have $\overline{H}^p(M; G) \cong H^p(M; G)$, then Theorem 14.8 yields isomorphisms

$$H^p(M; G) \cong H_{n-p}(M; G),$$

which is Poincaré duality if $M$ is compact and $R$-orientable.

In the special case where $K = M$, we get a version of *Lefschetz duality* for $M$ compact:

**Theorem 14.9.** *(Lefschetz Duality, Version 1) Let $M$ be a compact $R$-orientable $n$-manifold where $R$ is any commutative ring with an identity element. For any $R$-module $G$, for any compact subset $L$ of $M$, we have isomorphisms*

$$H^p_{A\text{-}S}(M, L; G) \cong \check{H}^p(M, L; G) \cong H_{n-p}(M - L; G) \ \text{for all } p \geq 0.$$

A version of Lefschetz duality where $M$ and $L$ are compact and triangulable, in which case singular cohomology suffices, is proved in Munkres [38] (Chapter 8, Theorem 72.3).

Spanier proves a slightly more general version. A pair $(X, A)$ is called a *relative $n$-manifold* if $X$ is a Hausdorff space, $A$ is closed in $X$, and $X - A$ is an $n$-manifold.

**Theorem 14.10.** *(Lefschetz Duality, Version 2) Let $(X, A)$ be a compact relative $n$-manifold such that $X - A$ is $R$-orientable where $R$ is any commutative ring with an identity element. For any $R$-module $G$, there are isomorphisms*

$$H^p_{A\text{-}S}(X, A; G) \cong \check{H}^p(X, A; G) \cong H_{n-p}(X - A; G) \ \text{for all } p \geq 0.$$

Theorem 14.9 is proved in Spanier [47] (Chapter 8, Section 2, Theorem 18).

There are also version of Poincaré and Lefschetz duality for manifolds with boundary but we will omit this topic. The interested reader is referred to Spanier [47] (Chapter 8, especially Section 2).

We now turn to two versions of Alexander duality.

## 14.4   Alexander Duality

Alexander duality correspond to the special case of Alexander–Lefschetz duality in which $L = \emptyset$. We begin with a version of Alexander duality in the situation where $M = \mathbb{R}^n$.

**Theorem 14.11.** *(Alexander Pontrjagin duality) Let $A$ be a compact subset of $\mathbb{R}^n$. For any commutative ring $R$ with an identity element, for any $R$-module $G$, we have isomorphisms*

$$H_{A\text{-}S}^{n-p-1}(A; G) \cong \check{H}^{n-p-1}(A; G) \cong \widetilde{H}_p(\mathbb{R}^n - A; G) \text{ for all } p \leq n.$$

*Proof.* By Theorem 14.8 with $M = \mathbb{R}^n$, $K = A$ and $L = \emptyset$, there are isomorphisms

$$\check{H}^{n-p-1}(A; G) \cong H_{p+1}(\mathbb{R}^n, \mathbb{R}^n - A; G) \text{ for all } p \leq n - 1.$$

We also have the long exact sequence of reduced homology of the pair $(\mathbb{R}^n, \mathbb{R}^n - A)$, which yields exact sequences

$$\widetilde{H}_{p+1}(\mathbb{R}^n; G) \longrightarrow \widetilde{H}_{p+1}(\mathbb{R}^n, \mathbb{R}^n - A; G) \longrightarrow \widetilde{H}_p(\mathbb{R}^n - A; G) \longrightarrow \widetilde{H}_p(\mathbb{R}^n; G),$$

and since $\widetilde{H}_{p+1}(\mathbb{R}^n; G) \cong \widetilde{H}_p(\mathbb{R}^n; G) \cong (0)$, we conclude that

$$H_{p+1}(\mathbb{R}^n, \mathbb{R}^n - A; G) = \widetilde{H}_{p+1}(\mathbb{R}^n, \mathbb{R}^n - A; G) \cong \widetilde{H}_p(\mathbb{R}^n - A; G),$$

which proves our result. $\square$

Here is another version of Alexander duality in which $M = S^n$. Recall from Section 4.7 that the relationship between the cohomology and the reduced cohomology of a space $X$ is

$$H^0(X; G) \cong \widetilde{H}^0(X; G) \oplus G$$
$$H^p(X; G) \cong \widetilde{H}^p(X; G), \quad p \geq 1.$$

**Theorem 14.12.** *(Alexander duality) Let $A$ be a proper closed nonempty subset of $S^n$. For any commutative ring $R$ with an identity element, for any $R$-module $G$, we have isomorphisms*

$$\widetilde{H}_p(S^n - A; G) \cong \begin{cases} \check{H}^{n-p-1}(A; G) & \text{if } p \neq n - 1 \\ \check{\widetilde{H}}^0(A; G) & \text{if } p = n - 1, \end{cases}$$

*or equivalently*

$$\check{\widetilde{H}}^{n-p-1}(A; G) \cong \widetilde{H}_p(S^n - A; G) \text{ for all } p \leq n.$$

*Proof.* The case $n = 0$ is easily handled, so assume $n > 0$. By Theorem 14.8 with $M = S^n$, $K = A$ and $L = \emptyset$, there are isomorphisms

$$\check{H}^{n-p-1}(A; G) \cong H_{p+1}(S^n, S^n - A; G) \text{ for all } p \leq n - 1.$$

We also have the long exact sequence of reduced homology of the pair $(S^n, S^n - A)$, which yields exact sequences

$$\widetilde{H}_{p+1}(S^n; G) \longrightarrow \widetilde{H}_{p+1}(S^n, S^n - A; G) \longrightarrow \widetilde{H}_p(S^n - A; G) \longrightarrow \widetilde{H}_p(S^n; G).$$

By Proposition 4.16 the reduced homology of $S^n$ is given by

$$\widetilde{H}_p(S^n; G) = \begin{cases} G & \text{if } p = n \\ (0) & \text{if } p \neq n, \end{cases}$$

It follows that we have isomorphisms

$$H_{p+1}(S^n, S^n - A; G) = \widetilde{H}_{p+1}(S^n, S^n - A; G) \cong \widetilde{H}_p(S^n - A; G)$$

for $p \neq n - 1$. If $p = n - 1$ we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(S^n) & \longrightarrow & \check{H}^0(A) & \longrightarrow & \check{\widetilde{H}}^0(A) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \vdots & & \\
H_n(S^n - A) & \xrightarrow{\ 0\ } & H_n(S^n) & \longrightarrow & H_n(S^n, S^n - A) & \longrightarrow & \widetilde{H}_{n-1}(S^n - A) & \xrightarrow{\ 0\ } &
\end{array}
$$

in which the left vertical solid arrow is an isomorphism by Poincaré duality, the right vertical solid arrow is an isomorphism by Theorem 14.8, the bottom row is exact by the long exact sequence of reduced homology, and the top one because

$$\check{H}^0(A) \cong \check{\widetilde{H}}^0(A) \oplus G$$

and $H^0(S^n) \cong H_n(S^n) \cong G$. We have zero maps on the bottom because the inclusion map $S^n - A \longrightarrow S^n$ factors through a contractible space $S^n - \{\text{pt}\}$. It is easy to see that the kernel of the map from $\check{H}^0(A)$ to $\widetilde{H}_{n-1}(S^n - A)$ is isomorphic to $H^0(S^n)$, so this map factors through $\check{\widetilde{H}}^0(A)$ as the dotted arrow, and using the commutative diagram and the fact that the rows are exact it is easy to show that the dotted arrow is an isomorphism.    □

**Remark:** This version involving Čech (or Alexander–Spanier) cohomology is a generalization of Alexander's original version that applies to a polyhedron in $S^n$, and only requires singular cohomology; see Munkres [38] (Chapter 8, Theorem 72.4).

An interesting corollary of Theorem 14.9 is the following generalization of the version of the Jordan curve theorem stated in Theorem 4.19. For comparison with Theorem 14.13 below think of $M$ as $S^n$ and of $A$ as $C$.

**Theorem 14.13.** *(Generalized Jordan curve theorem) Let $M$ be a connected, orientable, compact $n$-manifold, and assume that $H_1(M; R) = (0)$ for some ring $R$ (with unity). For any proper closed subset $A$ of $M$, the module $\check{H}^{n-1}(A; R)$ is a free $R$-module such that if $r$ is its rank, then $r + 1$ is equal to the number of connected components of $M - A$.*

*Proof.* The number of connected component of $M - A$ is equal to the rank $s$ of $H_0(M - A; R)$, and since $H_0(M - A; R) \cong \widetilde{H}_0(M - A; G) \oplus R$ we have $s = t + 1$ with $t = \text{rank}(\widetilde{H}_0(M - A; G))$.

By the long exact sequence of reduced homology of the pair $(M, M - A)$ we have the exact sequence

$$H_1(M; R) \longrightarrow H_1(M, M - A; R) \longrightarrow \widetilde{H}_0(M - A; R) \longrightarrow \widetilde{H}_0(M; R).$$

Since $H_1(M; R) = (0)$ and since $M$ is connected $\widetilde{H}_0(M; R) = (0)$ so we get the isomorphism

$$\widetilde{H}_0(M - A; R) \cong H_1(M, M - A; R).$$

By Lefschetz duality (Theorem 14.9) we have

$$H_1(M, M - A; R) \cong \check{H}^{n-1}(A; R),$$

and thus

$$\check{H}^{n-1}(A; R) \cong \widetilde{H}_0(M - A; R),$$

which shows that $\check{H}^{n-1}(A; R)$ is a free $R$-module whith rank $r = t = s - 1$, where $s$ is the number of connected component of $M - A$. $\qquad \square$

Recall that given two topological spaces $X$ and $Y$ we say that there is an *embedding* of $X$ into $Y$ if there is a homeomorphism $f \colon X \to Y$ of $X$ onto its image $f(X)$. As a corollary of Theorem 14.13 we get the following result.

**Proposition 14.14.** *Let $M$ be a connected, orientable, and compact $n$-manifold $M$. If $H_1(M; \mathbb{Z}) = (0)$, then no nonorientable compact $(n-1)$-manifold $N$ can be embedded in $M$.*

*Proof.* If the $(n-1)$-manifold $N$ is nonorientable, then by Proposition 7.8 $H^{n-1}(N; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and since $N$ is a manifold $H^{n-1}(N; \mathbb{Z}) \cong \check{H}^{n-1}(N; \mathbb{Z})$, so $\check{H}^{n-1}(N; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, which contradicts Theorem 14.13 (since $\mathbb{Z}/2\mathbb{Z}$ is not free). $\qquad \square$

Proposition 14.14 implies that $\mathbb{RP}^{2n}$ cannot be embedded into $S^{2n+1}$. In particular $\mathbb{RP}^2$ cannot be embedded into $S^3$.

More applications of duality are presented in Bredon [4] (Chapter 8, Section 10). In particular, it is shown that for all $n \geq 2$ (not just even) the real projective space $\mathbb{RP}^n$ cannot be embedded in $S^{n+1}$.

We conclude this chapter by stating a generalization of Alexander–Lefschetz duality for cohomology with compact support.

## 14.5 Alexander–Lefschetz Duality for Cohomology with Compact Support

The Alexander–Spanier cohomology modules with compact support $H_{\text{A-S},c}(X, A; G)$ were defined in Section 14.7. Alexander–Lefschetz duality (Theorem 14.8) can be generalized to arbitrary closed pairs $(K, L)$ (not necessarily compact), using the modules $H_{\text{A-S},c}(X, A; G)$ instead of the modules $H_{\text{A-S}}(X, A; G)$, in a way which is reminiscent of the general Poincaré Duality Theorem (Theorem 7.13).

**Theorem 14.15.** *(Alexander–Lefschetz duality) Let $M$ be a $R$-orientable manifold where $R$ is any commutative ring with an identity element. For any $R$-module $G$, for any pair $(K, L)$ of closed subsets of $M$ such that $L \subseteq K$, there is an isomorphism*

$$H^p_{A\text{-}S,\, c}(K, L; G) \cong H_{n-p}(M - L, M - K; G) \text{ for all } p \geq 0.$$

Theorem 14.15 is proved in Spanier [47] (Chapter 6, Section 9, Theorem 10) and in Dold [9] (Chapter VIII, Section 7, Proposition 7.14). It should be noted that Spanier's proof provides an isomorphism in the other direction (from homology to cohomology) and does not involve the cap product. However, Dold's version uses a version of the cap product obtained by a limit argument.

# Chapter 15

# Spectral Sequences

# Bibliography

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra.* Addison Wesley, third edition, 1969.

[2] Raoul Bott and Tu Loring W. *Differential Forms in Algebraic Topology.* GTM No. 82. Springer Verlag, first edition, 1986.

[3] Nicolas Bourbaki. *Elements of Mathematics. Commutative Algebra, Chapters 1-7.* Springer–Verlag, 1989.

[4] Glen E Bredon. *Topology and Geometry.* GTM No. 139. Springer Verlag, first edition, 1993.

[5] Glen E Bredon. *Sheaf Theory.* GTM No. 170. Springer Verlag, second edition, 1997.

[6] Jean-Luc Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization.* Birkhäuser, first edition, 1993.

[7] Henri Cartan and Samuel Eilenberg. *Homological Algebra.* Princeton Math. Series, No. 19. Princeton University Press, 1956.

[8] Jean Dieudonné. *A History of Algebraic and Differential Topology, 1900-1960.* Birkhäuser, first edition, 1989.

[9] Albrecht Dold. *Lectures on Algebraic Topology.* Springer Classics in Mathematics. Springer Verlag, first edition, 1995.

[10] C.H. Dowker. Homology groups of relations. *Annals of Mathematics*, 56:84–95, 1952.

[11] David S. Dummit and Richard M. Foote. *Abstract Algebra.* Wiley, second edition, 1999.

[12] Samuel Eilenberg and Norman Steenrod. *Foundations of Algebraic Topology.* Princeton Math. Series, No. 15. Princeton University Press, 1952.

[13] David Eisenbud. *Commutative Algebra With A View Toward Algebraic Geometry.* GTM No. 150. Springer–Verlag, first edition, 1995.

[14] David Eisenbud and Joseph Harris. *Schemes: The Language of Modern Algebraic Geometry.* Wadsworth & Brooks/Cole, first edition, 1992.

[15] Otto Forster. *Lectures on Riemann Surfaces*. GTM No. 81. Springer Verlag, first edition, 1981.

[16] Jean H. Gallier. *Geometric Methods and Applications, For Computer Science and Engineering*. TAM, Vol. 38. Springer, second edition, 2011.

[17] Jean H. Gallier and Dianna Xu. *A Guide to the Classification Theorem for Compact Surfaces*. Geometry and Computing, Vol. 9. Springer, first edition, 2013.

[18] Roger Godement. *Topologie Algébrique et Théorie des Faisceaux*. Hermann, first edition, 1958. Second Printing, 1998.

[19] Marvin J. Greenberg and John R. Harper. *Algebraic Topology: A First Course*. Addison Wesley, first edition, 1981.

[20] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. Wiley Interscience, first edition, 1978.

[21] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Mathematical Journal*, 9:119–221, 1957.

[22] Alexander Grothendieck and Jean Dieudonné. Eléments de Géométrie Algébrique, III: Etude Cohomologique des Faisceaux Cohérents (Première Partie). *Inst. Hautes Etudes Sci. Publ. Math.*, 11:1–167, 1961.

[23] Robert C. Gunning. *Introduction to Holomorphic Functions of Several Variables, Volume III: Homological Theory*. Wadsworth & Brooks/Cole, first edition, 1990.

[24] Robin Hartshorne. *Algebraic Geometry*. GTM No. 52. Springer Verlag, first edition, 1977. Fourth Printing.

[25] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, first edition, 2002.

[26] Friedrich Hirzebruch. *Topological Methods in Algebraic Geometry*. Springer Classics in Mathematics. Springer Verlag, second edition, 1978.

[27] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*. Grundlehren der Math. Wiss., Vol. 292. Springer–Verlag, first edition, 1994.

[28] Serge Lang. *Algebra*. Addison Wesley, third edition, 1993.

[29] Saunders Mac Lane. *Homology*. Grundlehren der Math. Wiss., Vol. 114. Springer–Verlag, third edition, 1975.

[30] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext. Springer Verlag, first edition, 1992.

[31] Ib Madsen and Jorgen Tornehave. *From Calculus to Cohomology. De Rham Cohomology and Characteristic Classes.* Cambridge University Press, first edition, 1998.

[32] William S. Massey. *A Basic Course in Algebraic Topology.* GTM No. 127. Springer Verlag, first edition, 1991.

[33] H. Matsumura. *Commutative Ring Theory.* Cambridge University Press, first edition, 1989.

[34] Peter J. May. *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics. The University of Chicago Press, first edition, 1999.

[35] John W. Milnor and James D. Stasheff. *Characteristic Classes.* Annals of Math. Series, No. 76. Princeton University Press, first edition, 1974.

[36] Shigeyuki Morita. *Geometry of Differential Forms.* Translations of Mathematical Monographs No 201. AMS, first edition, 2001.

[37] David Mumford. *The Red Book of Varieties and Schemes.* LNM No. 1358. Springer Verlag, second edition, 1999.

[38] James R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, first edition, 1984.

[39] Raghavan Narasimham. *Compact Riemann Surfaces.* Lecture in Mathematics ETH Zürich. Birkhäuser, first edition, 1992.

[40] Joseph J. Rotman. *An Introduction to Homological Algebra.* Academic Press, first edition, 1979.

[41] Joseph J. Rotman. *Introduction to Algebraic Topology.* GTM No. 119. Springer Verlag, first edition, 1988.

[42] Pierre Samuel. *Algebraic Theory of Numbers.* Dover, first edition, 2008.

[43] Hajime Sato. *Algebraic Topology: An Intuitive Approach.* Mathematical Monographs No 183. AMS, first edition, 1999.

[44] Jean-Pierre Serre. Faisceaux algébriques cohéhents. *Annals of Mathematics*, 61:197–278, 1955.

[45] Igor R. Shafarevich. *Basic Algebraic Geometry 2.* Springer Verlag, second edition, 1994.

[46] Stephen S. Shatz and Jean H. Gallier. Algebra. Technical report, University of Pennsylvania, Math. Department, Philadelphia, PA 19104, 2016. Book in Preparation.

[47] Edwin H. Spanier. *Algebraic Topology.* Springer, first edition, 1989.

[48] B.R. Tennison. *Sheaf Theory*. LMS No. 20. Cambridge University Press, first edition, 1975.

[49] Loring W. Tu. *An Introduction to Manifolds*. Universitext. Springer Verlag, first edition, 2008.

[50] Frank Warner. *Foundations of Differentiable Manifolds and Lie Groups*. GTM No. 94. Springer Verlag, first edition, 1983.

[51] Charles A. Weibel. *Introduction to Homological Algebra*. Studies in Advanced Mathematics, Vol. 38. Cambridge University Press, first edition, 1994.