PLANE TRIGONOMETRY

BY THE SAME AUTHOR.

INTRODUCTORY COURSE IN DIFFERENTIAL EQUATIONS, FOR STUDENTS IN CLASSICAL AND ENGINEERING COLLEGES. pp. xvi + 236. \$1.90.

AN ELEMENTARY COURSE IN THE INTEGRAL CALCULUS. pp. xiv + 288. \$2.00.

PLANE TRIGONOMETRY

FOR

COLLEGES AND SECONDARY SCHOOLS

BY

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LONGMANS, GREEN, AND CO.
91 AND 93 FIFTH AVENUE, NEW YORK
LONDON AND BOMBAY
1899

Norwood Press
J. S. Cushing & Co. — Berwick & Smith
Norwood Mass. U.S.A.

PREFACE.

Although there are already many excellent text-books on trigonometry, there appears still to be room for one which shall avoid the extremes of expansion and brevity. Some of the most thorough and scholarly of these contain a great variety of matters which it is impossible to consider in the time usually assigned to this study in school and college. On the other hand, the explanations given in many other works are so meagre that the student is perplexed and bewildered by the new ideas which are so abruptly forced upon him, and the difficulties of the teacher are greatly increased. The manner of presentation adopted in this volume necessitates more *reading matter*, and, consequently, a somewhat larger number of pages than is found in many of the recent text-books on trigonometry. This has seemed unavoidable, however, for the general consensus of opinion among those with whom the author has conferred, is that it is essential to explain in some detail the principles of the science, in order that it may be clearly and intelligently understood by an elementary student.

With regard to the scope of the book, it may be said that it deals with the subjects considered in the ordinary course in plane trigonometry in colleges and secondary schools. It discusses the topics usually required for teachers' certificates, for entrance to college, and for examinations in trigonometry in the first year of the college curriculum. It treats of all the topics that one who has taken a few months' course in trigonometry may be reasonably expected to know.

Careful consideration has been given both to the early difficulties and to the possible future needs of the beginner. The book differs somewhat from other text-books on this branch of mathematics both in the arrangement and in the manner of presentation. The oldest and simplest part of trigonometry, namely, the solution of triangles and the associated practical problems, is concluded before the more general and abstract portions of the study are introduced. The first chapters of the book contain little more about trigonometric ratios and angular analysis than is sufficient to enable the beginner to understand clearly the arithmetical part of the science, and its simple practical applications. This arrangement seems to have several advantages. The subject is rendered far less strange at the beginning, and, by means of practical, concrete examples, the student becomes familiar with the trigonometric functions before proceeding to the more general treatment. His progress is thus made easier and more rapid. Teachers who prefer a wider generality of treatment at the outset, however, can select the chapters in a different order from that followed in the text.

An endeavour has been made to introduce the several topics in such a way that the pupil may have, from the very start, an intelligent idea of each step in advance, as well as of the ultimate purpose of the study. In some cases, especially in Chapter II. (the first chapter on trigonometry), care has been taken to prepare the mind of the learner for the reception of new ideas, by the preliminary solution of easy familiar exercises. Throughout the work the author has endeavoured to make each step clear, and thus to prevent the appearance of that puzzled feeling which has such a depressing influence on those entering upon a new study. On the other hand, he has sought to develop independence of mind and the power of mental initiative on the part of the student. Suggestions as to practical methods of work are frequently introduced, and summaries are made in several places for the purpose of helping the pupil to get a better idea of the subject as a whole.

In the practical applications, marked attention has been given to the graphical method of solution, as well as to the method of computation. The former method serves as a check upon the latter, and affords practice in neat and careful drawing. What is perhaps more important, however, is that the students will thus become accustomed to a method which will be used by them in other studies, and which is often employed in practical work by engineers and others.

Logarithms are used almost at the beginning of the study as here presented. For this reason, and in order to avoid making a digression later on, an introductory chapter is devoted to a review on logarithms. Examples, simple ones as a rule, are given in the several articles. Questions and exercises suitable for practice and review on the separate chapters are placed at the end of the book instead of at the ends of the chapters. These collections will be found useful, both in the short reviews that may be required on the completion of each chapter and in the larger and more general reviews. Many of the examples have been taken from examination papers set in Great Britain and the United States.

Throughout the work there are many historical and other notes; and an historical sketch is given in the Appendix. It is believed that some knowledge of the historical development of trigonometry, and of the men of various times and races who have helped to advance the subject, will interest and stimulate those who are entering upon its study.

While writing this book, the author has received many valuable suggestions from Mr. J. A. Clark, B.S., of the Ithaca High School, and from several of his colleagues in the departments of mathematics and of engineering at Cornell University. He is indebted to Dr. G. A. Miller and Dr. J. V. Westfall, of the department of mathematics at Cornell University, for their kind assistance in the revision of the proof-sheets, and to Mr. E. A. Miller, B.S., for his friendly aid in working examples. The drawings have been made by Mr. A. T. Bruegel, M.M.E., formerly instructor in the kinematics of machinery at Cornell University, now of the Pratt Institute, Brooklyn, N.Y. The author uses this opportunity to express his thanks for the pains taken by Mr. Bruegel to make the figures a pleasing feature of the book.

D. A. MURRAY.

CORNELL UNIVERSITY,
August, 1899.

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PLANE TRIGONOMETRY.

CHAPTER I.

LOGARITHMS: REVIEW OF TREATMENT IN ARITHMETIC AND ALGEBRA.

1. There is a large amount of computation necessary in the solution of some of the practical problems in trigonometry. The labour of making extensive and complicated calculations can be greatly lessened by the employment of a table of logarithms, an instrument which was invented for this very purpose by John Napier (1550–1617), Baron of Merchiston in Scotland, and described by him in 1614. From Henry Briggs (1556–1631), who was professor at Gresham College, London, and later at Oxford, this invention received modifications which made it more convenient for ordinary practical purposes.*

Every good treatise on algebra contains a chapter on logarithms. This brief introductory review is given merely for the purpose of bringing to mind the special properties of logarithms which make them readily adaptable to the saving of arithmetical work. A little preliminary practice in the use of logarithms will be of advantage to any one who intends to study trigonometry. A review of logarithms as treated in some standard algebra is strongly recommended.

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* The logarithms in general use are known as Common logarithms or as Briggs's logarithms, in order to distinguish them from another system, which is also a modified form of Napier's system. The logarithms of this other modified system are frequently employed in mathematics, and are known as Natural logarithms, Hyperbolic logarithms, and also, but erroneously, as Napierian logarithms. See historical sketch in article *Logarithms* (Ency. Brit. 9th edition), by J. W. L. Glaisher.

2. Definition of a logarithm.

If $$a^x = N, \qquad (1)$$

then x *is the index of the power to which* a *must be raised in order to equal* N.

For some purposes, this idea is presented in these words: If $a^x = N$, then x *is the logarithm of* N *to the base* a.

The latter statement is taken as the *definition of a logarithm*, and is expressed by mathematical symbols in this manner, viz.:

$$x = \log_a N. \qquad (2)$$

Equations (1), (2), are equivalent; they are merely two different ways of stating a certain connection between the three quantities a, x, N. For example, the relations

$$2^3 = 8, \quad 5^4 = 625, \quad 10^{-3} = \tfrac{1}{1000} = .001,$$

may also be expressed by the equivalent logarithmic equations,

$$\log_2 8 = 3, \quad \log_5 625 = 4, \quad \log_{10} .001 = -3.$$

EXAMPLES.

1. Express the following equations in a logarithmic form:
$3^3 = 27$, $4^4 = 256$, $11^2 = 121$, $9^3 = 728$, $7^3 = 343$, $m^b = p$.

2. Express the following equations in the exponential form:
$\log_2 8 = 3$, $\log_5 625 = 4$, $\log_{10} 1000 = 3$, $\log_2 64 = 8$, $\log_n P = a$.

3. When the base is 2, what are the logarithms of 1, 2, 4, 8, 16, 32, 64, 128, 256?

4. When the base is 5, what are the logarithms of 1, 5, 25, 125, 625, 3125?

5. When the base is 10, what are the logarithms of 1, 10, 100, 1000, 10,000, 100,000, 1,000,000, .1, .01, .001, .0001, .00001, .000001?

6. When the base is 4, and the logarithms are 0, 1, 2, 3, 4, 5, what are the numbers?

7. When the base is 10, between what whole numbers do the logarithms of the following numbers lie: 8, 72, 235, 1140, 3470, .7, .04, .0035?

3. Properties of logarithms. Since a logarithm is the index of a power, it follows that the properties of logarithms must be derivable from the properties of indices; that is, from the laws of indices. The laws of indices are as follows (a, m, n, being any finite quantities):

(1) $a^m \times a^n = a^{m+n}$.

(2) $\frac{a^m}{a^n} = a^{m-n}$. $\left[\frac{a^m}{a^m} = a^{m-m} = a^0; \text{ also, } \frac{a^m}{a^m} = 1. \quad \therefore a^0 = 1\right]$.

(3) $(a^m)^n = a^{mn}$. (4) $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$.

Let $M = a^m$, whence, $\log_a M = m$; (1)

and let $N = a^n$, whence, $\log_a N = n$. (2)

It follows that

$MN = a^{m+n}$; whence, $\mathbf{\log_a MN = m + n = \log_a M + \log_a N}$. (3)

[If $P = a^p$, then $\log_a P = p$, $MNP = a^{m+n+p}$;

whence, $\log_a MNP = m + n + p = \log_a M + \log_a N + \log_a P$.]

Also, $$\frac{M}{N} = \frac{a^m}{a^n} = a^{m-n};$$

whence, $$\mathbf{\log_a \frac{M}{N} = m - n = \log_a M - \log_a N}. \quad (4)$$

Also, $M^r = (a^m)^r = a^{mr}$; whence, $\mathbf{\log_a M^r = rm = r \log_a M}$. (5)

Also, $$\sqrt[r]{M} = (a^m)^{\frac{1}{r}} = a^{\frac{m}{r}};$$

whence, $$\mathbf{\log_a \sqrt[r]{M} = \frac{1}{r} \cdot m = \frac{1}{r} \log_a M}. \quad (6)$$

The results (3)–(6) state the properties, or are *the laws of logarithms.* They may be expressed in words as follows:

(1) *The logarithm of the* **product** *of any number of factors is equal to the* **sum** *of the logarithms of the factors.*

(2) *The logarithm of the* **quotient** *of two numbers is equal to the logarithm of the numerator* **diminished** *by the logarithm of the denominator.*

(3) *The logarithm of the* **rth power** *of a number is equal to* **r times** *the logarithm of the number.*

(4) *The logarithm of the* **rth root** *of a number is equal to* $\frac{1}{r}$**th** *of the logarithm of the number.*

Hence, if the logarithms (*i.e.* the exponents of powers) of numbers be used instead of the numbers themselves, then the operations of *multiplication* and *division* are replaced by those of *addition* and *subtraction,* and the operations of *raising to powers* and *extracting roots,* by those of *multiplication* and *division.*

4. Common system of logarithms. Any positive number except 1 may be chosen as the base; and to the base chosen there corresponds a set or system of logarithms. In the common or decimal system the base is 10, and, as will presently appear, this system is a very convenient one for ordinary numerical calculations.* In what follows, the base 10 is not expressed, but it is always understood that 10 is the base. *The logarithm of a number in the common system is the answer to the question: "What power of 10 is the number?"*

Since

$$1=10^0,\quad 10=10^1,\quad 100=10^2,\quad 1000=10^3,\quad 10000=10^4, \cdots,$$

it follows that

$$\log 1= 0, \log 10= 1, \log 100= 2, \log 1000= 3, \log 10000= 4, \cdots.$$

This also shows that the logarithms of numbers

$$\left.\begin{array}{l}\text{between } 1 \text{ and } 10 \text{ lie between } 0 \text{ and } 1,\\ \text{between } 10 \text{ and } 100 \text{ lie between } 1 \text{ and } 2,\\ \text{between } 100 \text{ and } 1000 \text{ lie between } 2 \text{ and } 3, \text{ and so on.}\end{array}\right\} \quad (1)$$

For example,

$$9 = 10^{.95424}, \qquad 247 = 10^{2.39270}, \qquad 1453 = 10^{3.16227};$$

or $\log 9 = .95424, \log 247 = 2.39270, \log 1453 = 3.16227.$

Most logarithms are incommensurable numbers. (See Art. 9.) The decimal part of the logarithm is called the *mantissa,* the

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* The base of the natural system of logarithms is an incommensurable number, which is always denoted by the letter e and is approximately equal to 2.7182818284.

integral part of the logarithm is called the *index* or *characteristic.*

The two great advantages of the common system, as will now be shown, are:

(1) *The characteristic of a logarithm can be written on mere inspection;*

(2) *The position of the decimal point in a number affects the characteristic alone, the mantissa being always the same for the same sequence of figures.*

Since $\quad .1 = \frac{1}{10} = 10^{-1}, \quad .01 = \frac{1}{100} = 10^{-2},$

$$.001 = \frac{1}{1000} = 10^{-3}, \quad .0001 = \frac{1}{10000} = 10^{-4}, \cdots,$$

it follows that

$$\log .1 = -1, \log .01 = -2, \log .001 = -3, \log .0001 = -4, \text{ etc.} \quad (2)$$

From (1) and (2) comes the following **rule for finding the characteristic:**

When the number is greater than 1, the characteristic is positive and is one less *than the number of digits to the left of the decimal point; when the number is less than 1, the characteristic is negative, and is one* more *than the number of zeros between the decimal point and the first significant figure.*

When a change is made in the position of the decimal point in a number, the value of the number is changed by some integral power of 10. Its logarithm is then changed by a whole number only, and, consequently, *its mantissa is not affected.* For example,

$$25.38 = 2538 \times 10^{-2}, \qquad 2538000 = 2538 \times 10^{3};$$

and hence, $\log 25.38 = \log 2538 - 2, \ \log 2538000 = \log 2538 + 3.$

Accordingly, it is necessary to put only the *mantissas of sequences of integers* in the tables.

5. Negative characteristics. In common logarithms *the mantissa is always kept positive.* Thus, for example, $\log 25380 = 4.40449$; $\log .002538 = \log \frac{2538}{1000000} = \log 2538 - \log 1000000 = 3.40449 - 6$ $= -3 + .40449.$ (Never put -2.59551.)

This logarithm is usually written $\bar{3}.40449$, in order to show that the *minus* sign affects the characteristic alone. In order to avoid the use of negative characteristics, 10 is often added to the logarithm and -10 placed after it.

Thus $\bar{3}.40449$ is written $7.40449-10$.

The second form is more convenient for purposes of calculation.

Special care is necessary in dealing with logarithms because of the fact that the mantissa is always positive, while the characteristic may be either positive or negative. Some typical examples involving negative characteristics are given below.

Addition	*Subtraction*		*Multiplication*	
1. $\bar{3}.27412$	2. $\bar{3}.27412$	*i.e.* $7.27412-10$	3. $9.83471-10$	
4.51459	4.51459	4.51459	2	
1.78871		$2.75953-10$	$19.66942-20$	(1)
			i.e. $9.66942-10$	(2)

A result like (1) is always put in the form (2), in which the number placed after the logarithm is -10.

Ex. 3 may also be worked thus:

$$(-1+.83471)\times 2=-2+1.66942=\bar{1}.66942.$$

4. *Division.* $\bar{3}.27412 \div 4=(37.27412-40)\div 4=9.31853-10.$

As in Ex. 3 care is taken that, finally, the number after the logarithm be -10.

5. $\bar{2}.34175 \div 5=(48.34175-50)\div 5=9.66835-10.$

6. $\bar{4}.74752\times\frac{2}{3}=\frac{\bar{7}.49504}{3}=\frac{23.49504-30}{3}=7.83168-10.$

The method of finding the logarithms in the tables when the numbers are given, and the way to find the numbers when the logarithms are given, are usually explained in connection with the tables of logarithms.

6. Exercises in logarithmic computation. On looking at the laws of logarithms, (3)–(6), Art. 3, it is apparent that logarithms cannot assist in the operations of addition and subtraction. Logarithms are of no service in computing expressions of the forms

$M + N$, $M - N$. An expression is said to be *adapted to logarithmic computation* when it is expressed by means of factors only. Thus, $\dfrac{a^n \cdot b^{\frac{1}{r}}}{c^{10}x^s}$ is adapted to logarithmic computation, but $\dfrac{a+b-2a^2+19}{7a-5b}$ is not.

EXAMPLES.

1. Find $\dfrac{6837}{4341}$.

Let $R = \dfrac{6837}{4341}$. Then $\log R = \log 6837 - \log 4341$.

$$\begin{aligned} \log 6837 &= 3.83487 \\ \log 4341 &= \underline{3.63759} \\ \therefore\ \log R &= 0.19728 \qquad \therefore\ R = 1.575. \end{aligned}$$

2. Find $\sqrt{.005}$.

Let $R = \sqrt{.005}$. Then $\log R = \log (.005)^{\frac{1}{2}} = \frac{1}{2}\log .005 = \dfrac{\bar{3}.69897}{2}$

$$= \frac{17.69897 - 20}{2} = 8.84948 - 10 = \bar{2}.84948.$$

$\therefore\ R = .07071.$

3. Find $\sqrt[5]{742 \times .0769}$.

Let R be the value. Then $\log R = \log \sqrt[5]{742 \times .0769} = \log (742 \times .0769)^{\frac{1}{5}}$

$$= \tfrac{1}{5}\log (742 \times .0769) = \tfrac{1}{5}[\log 742 + \log .0769].$$

$$\begin{aligned} \log 742 &= 2.87040 \\ \log .0769 &= \underline{\bar{2}.88593} \\ \text{Dividing by 5,} \qquad 5\,&\underline{|\ 1.75633} \\ \therefore\ \log R &= \ \ .35126 \qquad \therefore\ R = 2.245. \end{aligned}$$

4. Find $\sqrt{\dfrac{456 \times 372}{350 \times 249}}$, *i.e.* $\left(\dfrac{456 \times 372}{350 \times 249}\right)^{\frac{1}{2}}$.

Let R be the value. Then $\log R = \frac{1}{2}(\log 456 + \log 372 - \log 350 - \log 249)$.

$$\begin{aligned} \log 456 &= 2.65896, & \log 350 &= 2.54407 \\ \log 372 &= \underline{2.57054}, & \log 249 &= \underline{2.39620} \\ &\ \ 5.22950 & &\ \ 4.94027 \end{aligned}$$

$$\begin{aligned} &\ \ \underline{4.94027} \\ \text{Dividing by 2,} \qquad 2\,&\underline{|\ .28923} \\ \therefore\ \log R = &\ \ .14462 \quad \text{(See Art. 9, Note 1.)} \\ \therefore\ R = &\ 1.395. \end{aligned}$$

5. Find the value of x in $34^x = 19$.

Since

$$\begin{aligned} 34^x &= 19, \\ \log 34^x &= \log 19, \\ x \log 34 &= \log 19, \\ x = \frac{\log 19}{\log 34} &= \frac{1.27875}{1.53148} = .83498, \text{ nearly.} \end{aligned}$$

6. Find the value of (*a*) $\frac{374}{267}$, (*b*) $\frac{29.76}{315.2}$, (*c*) $\frac{4.132}{59.83}$, (*d*) $\frac{.0417}{.4231}$.

7. Find the value of (*a*) $\frac{76.5 \times 83.21}{674.2}$, (*b*) $\frac{8.97 \times 6.36}{7.84}$, (*c*) $\frac{95.83 \times 76.49}{82.97}$

8. Find the value of (*a*) $\sqrt{63}$, (*b*) $\sqrt{630}$, (*c*) $\sqrt{6.3}$, (*d*) $\sqrt{.63}$, (*e*) $\sqrt{.063}$, (*f*) $\sqrt{.0063}$.

9. Find the value of $\sqrt{63.42 \times 74.95}$, $\sqrt{6.35 \times 10.87}$, $\sqrt{14.21 \times 17.29}$.

10. Find the value of $\sqrt{\frac{63.9 \times 72.11}{7.81 \times 6.95}}$, $\sqrt{\frac{31.21 \times 41.7}{11.39 \times 15.71}}$, $\sqrt{\frac{41.7 \times 85.6}{73.4 \times 97.8}}$.

11. Find the value of $2.5637^{\frac{3}{11}}$.

12. $\left(\frac{35}{113}\right)^{\frac{3}{8}}$

13. Find x from the equations:

(*a*) $3^x = 35$, (*b*) $5^x = 70$, (*c*) $10^x = 36$,

(*d*) $10^x = 127$, (*e*) $10^x = 765$, (*f*) $10^x = 1364$.

CHAPTER II.

TRIGONOMETRIC RATIOS OF ACUTE ANGLES.

7. The name Trigonometry is derived from two Greek words which taken together mean 'I measure a triangle.'* At the present time the measurement of triangles is merely one of several branches included in the subject of trigonometry. The more elementary part of trigonometry is concerned with the calculation of straight and circular lines, angles, and areas belonging to figures on planes and spheres. It consists of two sections, viz. Plane Trigonometry and Spherical Trigonometry. Elementary trigonometry has many useful applications, for instance, in the measurement of areas, heights, and distances. An acquaintance with its simpler results is very helpful, and sometimes indispensable, in even a brief study of such sciences as astronomy, physics, and the various branches of engineering. Some modern branches of trigonometry require a knowledge of advanced algebra. Their results are used in the more advanced departments of mathematics and in other sciences. This work considers only the simpler portions of trigonometry, and shows some of its applications.

The truths of elementary trigonometry are founded upon geometry, and are obtained and extended by the help of arithmetic and algebra. A knowledge of the principal facts of plane geometry, and the ability to perform the simpler processes of algebra, are necessary on beginning the study of plane trigonometry. Instruments for measuring lines and angles, and accuracy in computation are required in making its practical applications.

8. Ratio. Measure. On entering upon the study of trigonometry it is very necessary to have clear ideas concerning the terms *ratio* and *incommensurable numbers* as explained in arithmetic and algebra, for these terms play a highly important part

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* See historical sketch, p. 165.

in the subject. The study begins with an explanation of certain ratios which are used in it continually, and most of the numbers that appear in the solution of its problems are incommensurable.

If one quantity is half as great as another quantity in magnitude, it is said that the ratio of the first quantity to the second is as one to two, or one-half. This ratio is sometimes indicated thus, $1:2$; but more usually it is written in the fractional form, $\frac{1}{2}$. In this example the magnitude of the second quantity is twice that of the first, and the ratio of the second quantity to the first is $2:1$, or, adopting the more usual style, $\frac{2}{1}$, *i.e.* 2. *The ratio of two quantities* is simply the number which expresses the magnitude of the one when compared with the magnitude of the other. This ratio is obtained by finding *how many times* the one quantity contains the other, or by finding *what fraction* the one is of the other. It follows that a ratio is merely *a pure number*, and that it can be obtained only by comparing quantities of *the same kind*. Thus the ratio of the length 3 feet to the length 2 inches is $\frac{36}{2}$, *i.e.* 18; the ratio of the weight 2 pounds to the weight 3 pounds is $\frac{2}{3}$. But one cannot speak of the ratio of 3 weeks to 10 yards, for there is no sense in the questions: How many times does 3 weeks contain 10 yards? What fraction of 10 yards is 3 weeks?

When it is said that a line is ten inches long, this statement means that a line one inch long has been chosen for the unit of length, and that the first line contains ten of these units. Thus the *number* used in telling the length of a line is the ratio of the length of this line to the length of another line which has been chosen for the unit of length. The *measure* of any quantity, such as a length, a weight, a time, an angle, etc., is

$$\left\{\begin{array}{l}\text{the number of times the quantity contains}\\ \text{or, the fraction that the quantity is of}\end{array}\right\}$$

a certain quantity of the *same* kind which has been adopted as the unit of measurement. In other words, the measure of a quantity is the ratio of the quantity to the unit of measurement. For example, if half an inch is the unit of length, then the measure of a line 8 inches long is 16; if a foot is the unit of length, then the measure of the same line is $\frac{2}{3}$; if a second is the unit of time, then the measure of an hour is 3600; if an hour is the unit of time, then the measure of a second is $\frac{1}{3600}$.

If two quantities have a common unit of measurement, then their ratio is the ratio of their measures. For example, 1 pound being taken as the unit of weight, the ratio of a weight 3 pounds to a weight 7 pounds is $\frac{3}{7}$, which is also the ratio of the measures 3 and 7. In general, if a quantity P contains m units, and a quantity Q contains n units of the same kind as is used in the case of P, then the ratio

$$\frac{\text{quantity } P}{\text{quantity } Q} = \frac{m \text{ units}}{n \text{ units}} = \frac{m}{n}.$$

The last fraction $\frac{m}{n}$ is the ratio of the numbers m and n, which are the measures of the quantities P and Q respectively.

EXAMPLES.

1. What is the ratio of each of the following lengths to an inch, viz., 8 in., 2 ft., 3 ft. 6 in., 1.5 yd., 20 yd., a yd., b ft., c in.?

2. What is the ratio of each of the following lengths to a yard, viz., 6 yd., 3.75 yd., 8 ft., 2 ft. 6 in., 10 in., 5 in., a yd., b ft., c in.?

3. What is the measure of each of the following lengths, when a foot is the unit of length, viz., 1.5 mi., 17 yd., 3 yd. 2 ft., 8.5 ft., 2 ft. 6 in., 9 in., 2 in., a yd., b ft., c in.?

4. What is the measure of each of the following lengths, when 3 in. is the unit of length, viz., 2.5 yd., 1.5 ft., 8 in., a yd., b ft., c in.?

5. Express the ratio of 2.5 mi. to 10 yd.; and the ratio of $2\frac{1}{4}$ in. to $3\frac{1}{3}$ yd.

6. Compare the ratio of a foot to a yard with the ratio of a square foot to a square yard.

7. What is the unit of measurement in each of the following cases: when the measure of 2 ft. is 4, of 1 yd. is 72, of .5 in. is 4, of 2.5 ft. is .25?

N.B. *The following examples will be used again for purposes of illustration. The student is advised to draw figures neatly and accurately and to preserve the results carefully.*

8. In a right-angled triangle the base is 6 ft. and the hypotenuse 10 ft. What is the perpendicular? Calculate the following ratios, viz.:

$$\frac{\textit{perpendicular}}{\textit{hypotenuse}}, \quad \frac{\textit{base}}{\textit{hypotenuse}}, \quad \frac{\textit{perpendicular}}{\textit{base}},$$

$$\frac{\textit{base}}{\textit{perpendicular}}, \quad \frac{\textit{hypotenuse}}{\textit{base}}, \quad \frac{\textit{hypotenuse}}{\textit{perpendicular}}.$$

What are these ratios in a triangle whose base is 6 in., and hypotenuse 10 in.? What are they when the base is 6 yd., and the hypotenuse 10 yd.? When the base is 6 mi., and the hypotenuse 10 mi.? When the base is 12 ft., and the hypotenuse 20 ft.? When the base is 3 in., and the hypotenuse 5 in.? Compare, if possible, the *angles* in these triangles.

9. In a right-angled triangle whose base is 35 ft. and perpendicular 12 ft., what is the hypotenuse? For this triangle calculate the ratios specified in Ex. 8. Calculate these ratios for a triangle whose base is 70 yd., and perpendicular 24 yd. Compare, if possible, the angles in these triangles.

10. Calculate these ratios for the triangle whose hypotenuse is 29 ft., and perpendicular 21 ft.; for the triangle whose hypotenuse is 2.9 in., and perpendicular 2.1 in. Compare, if possible, the angles in these triangles.

9. Incommensurable quantities. Approximations. If the side of a square is one foot in length, then the length of a diagonal of the square is $\sqrt{2}$ feet. Thus the ratio of the diagonal to the side is $\sqrt{2}$, a number which cannot be expressed as the ratio of two whole numbers. Two quantities whose ratio can be expressed by means of two integers are said to be *commensurable* the one with the other; when their ratio cannot be so expressed, the one quantity is said to be *incommensurable* with the other. For example, the diagonal of a square is incommensurable with the side, and the length of a circle with its diameter.* The quantities in the examples, Art. 8, are commensurable. Numbers such as $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{10}$ are incommensurable with unity, and their values cannot be found exactly. Their values, however, can be found to two, to three, to four, in fact, to as many places of decimals as one please. The greater the number of places of decimals, the more nearly will the calculated values represent the true values of the numbers. In other words, the values of incommensurable numbers can be found *approximately*; and *the degree of approximation* (that is, the nearness to the exact values) will depend only on the carefulness and patience of the calculator. In practical problems there frequently is occasion for the exercise of judgment as to the degree of approximation that is necessary and sufficient. For example, in calculating a length in inches in ordinary engineer-

* See Appendix, Note C.

ing work there is no need to go beyond the third place of decimals, for engineers are satisfied when a measurement is correct to within $\frac{1}{64}$ of an inch. As a rule the results obtained in *practical* problems in mathematics are only approximate and not exact. There are two reasons for this: first, the *data* obtained by actual measurement can only be approximate, however excellent the instruments used in measuring may be, and however skilled and careful is the person who does the measuring; second, most of the *numbers used* in the subsequent computations *are incommensurable.*

The examples at the end of this article are intended to bring out more clearly the idea of an approximate result. The answers are to be calculated to three places of decimals. It is advisable to compare the values calculated to three places of decimals with the values calculated to two places of decimals, and to note the difference between them. The following facts are supposed to be known and will be taken for granted.

(*a*) In a right-angled triangle the square of the measure of the hypotenuse is equal to the sum of the squares of the measures of the other two sides.

(*b*) The ratio of the length of any circle to its diameter is a number which is the same for all circles.* The exact value of this ratio is incommensurable and is always denoted by the symbol π (read *pi*).† The approximate values commonly used for π are 3.1416, 3.14159, $\frac{355}{113}$ (*i.e.* 3.1415929 ...), $\frac{22}{7}$ (*i.e.* $3.\dot{1}4285\dot{7}$); of these values the last is the least accurate, but it is accurate enough for many practical purposes.

(*c*) The length of a circle of radius r is $2\pi r$ [by (*b*)]; and the enclosed area is πr^2.

NOTE 1. If a number be calculated to three or more places of decimals, then the closest approximation to, say, two places of decimals is obtained by leaving the number in the second place of decimals unchanged when the number in the third place is less than 5, and by increasing the number in the second place by unity when the number in the third place is greater than 5 or 5 followed by numbers; thus, *e.g.*, 3.72 for 3.724, 3.73 for 3.7261 and

* This ratio and facts (*c*) are considered in Note C, Appendix. The reading only requires a knowledge of elementary geometry.

† This symbol is the initial letter of *periphereia*, the Greek word for circumference. Its earliest appearances to denote this ratio are in Jones's *Synopsis Palmariorum Matheseos*, London, 1706, and in the *Introductio in analysin infinitorum*, published in 1748 by Leonhard Euler (1707–1783), a native of Switzerland, who was one of the greatest mathematicians of his time.

3.7257. When the number in the third place is 5 and this is followed by zeros only, the number in the second place is unchanged if it is even, and is increased by unity if it is odd; thus, *e.g.*, 3.78 for 3.775, 3.78 for 3.785. In a series of calculations the errors made by following this rule tend to balance one another.

NOTE 2. A quantity measured to two places of decimals is correct to the hundredth part of the unit employed, and a quantity measured to three places is correct to the thousandth part of the unit. For example, the length of a circle of 10 feet diameter is 31.4159 . . . feet. For this length 31.416 or 31.42 may be taken; the former result differs from the true result by less than one-thousandth of a foot, the latter by less than one-hundredth.

EXAMPLES.

1. A finds the square root of 3 correctly to two places of decimals, and B to three. How much closer than A does B come to the exact value of the square root of 3?

2. A circle is 50 ft. in diameter. In calculating its length A takes 3.1416 as the ratio of the length of a circle to the diameter, B takes 3.14159, and C takes 22 : 7. What are the differences (in inches) between their results?

3. The radius of a circle is 49.95 ft. How nearly will a person come to the length of the circle if he assumes the radius to be 50 ft.? [In this and the following example take $\pi = 22 : 7$.]

4. It is known that the diameter of a certain circle does not differ from 100 ft. by more than 2 in. What will be the outside limits of the error made in calculating the area when the diameter is taken as 100 ft.?

5. Find the difference between the calculations of the numbers of revolutions per mile made by a 50-in. bicycle, for $\pi = 22 : 7$ and $\pi = 3.1416$.

6. A lot is 75 ft. by 200 ft. Find the diagonal distance across the lot correctly to within a tenth of an inch.

7. Find the height of an equilateral triangle whose side is 20 yd.

8. The side of an isosceles triangle is 40 ft. and the base is 30 ft.; find the height.

9. What is the length of the diagonal of a square whose side is 20 ft.?

10. What is the length of the side of a square whose diagonal is 20 ft.?

N. B. *The following examples will be used again for purposes of illustration. The student is advised to draw figures and to preserve the results with those of* Exs. 8, 9, 10, Art. 8.

11. (*a*) In a right-angled triangle the hypotenuse is 12 ft. and the base is 6 ft.; calculate the ratios specified in Ex. 8, Art. 8.

(*b*) What are these ratios when the lengths in (*a*) are taken twice, three times, one-half as great? Compare, if possible, the *angles* in these triangles.

12. (*a*) In a right-angled triangle the base is 8 in. and the perpendicular 12 in.; calculate the ratios specified in Ex. 8, Art. 8.

(*b*) What are these ratios when the lengths in (*a*) are taken one-third, and four times as great? Compare, if possible, the *angles* in these triangles.

13. (*a*) In a right-angled triangle the hypotenuse is 35 yd. and the perpendicular is 15 yd.; calculate the ratios specified in Ex. 8, Art. 8.

(*b*) What are these ratios when the lengths in (*a*) are taken four times, six times, one-fifth as great? Compare, if possible, the *angles* in these triangles.

10. Linear measure. Drawing to scale. Direct measurement by means of drawings. Various systems of linear measurement are described in arithmetic. The system mostly used in English-speaking countries is that in which length is given in miles, yards, feet, or inches. The system which is in common use on the continent of Europe, and which is mainly employed in scientific measurements throughout the world, is the metric system. In this system lengths are given in centimetres, metres, etc., the centimetre being a hundredth part of a metre. A metre is equal to 39.37 . . . inches.*

Drawing to scale. It is often desirable to have a drawing on paper which shall serve to give an accurate idea of the relations of certain lines and positions. Maps and architects' plans are familiar examples of such drawings. In a map an inch may represent 1 mile, 10 miles, 100 miles, 500 miles, and so on, according to *the scale* on which the map is made; in a building plan an inch may represent 10 feet, 12 feet, and so on. The operation of drawing on paper lines that shall be a half, a quarter, a tenth, a thousandth, etc., part of the actual length of given lines, is called *drawing to scale.* In many cases the drawings of objects cannot be made full size; for instance, the map of a town, the floor plan of a church; these are drawn to a *reduced* scale. In other cases the drawings are made larger than the actual objects, for instance, the drawings of the minute things that live in a drop of water, the drawings of the various parts of a flower;

* The metric system has the great advantage of being a decimal system. At the present time committees of scientific societies in England and America are working to have the common system replaced by the metric.

these are drawn to an *enlarged* scale. *When a drawing is made to scale, the scale should always be indicated on it.* This may be done in various ways. Thus a mere statement may be made, *e.g.*,

1 inch to 10 feet;

or, the scale may be indicated by a *fraction* which gives the ratio of any line in the drawing to the actual line represented. The scale can also be *shown graphically* by means of a specially marked line. Both the latter methods are illustrated, for instance, on the map of the Kingdom of Saxony in *The Times Atlas:*

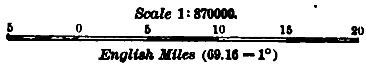

FIG. 1.

The scale should be expressed fractionally, that is, by expressing the ratio of a line in the drawing to the actual line represented. Thus in the first example above the scale is 1 : 120; in the second the scale is 1 : 870000.

When a drawing is made to scale, the distance between two objects can be measured directly, by merely measuring the distance between their corresponding points on the drawing. For instance, if 1 inch represents 120 feet, then 2.5 inches represents 300 feet. Another example: On the map of Saxony referred to above, the distance between Leipzig and Dresden is, approximately, $4\frac{1}{2}$ inches, and $4\frac{1}{2}$ inches $\times$ 870000 gives about 62 miles as the distance in an air-line between these cities. This method of finding distance can be used in solving many of the problems in trigonometry.* To find the length of the representative line in the drawing when the scale and the actual length are given, is an exercise in simple proportion; so, also, to find the actual length of a line when the scale and the length of the representative line are known.

* This is one of the methods which will be employed in this book in problems involving distance. Proficiency in drawing will be very helpful to the student.

EXAMPLES.

1. When an inch represents 10 ft., how long must the lines be that will represent 3 in., 6 in., 1 ft., 2 ft., 5 ft., 15 ft., 7.5 ft., 30 ft., 40 ft., 55 ft.? What is the scale?

2. When an inch represents 5 yd., how long must the lines be that will represent 2 yd., 4 yd., 7 yd., 11 yd., 3 yd. 2 ft., 4 yd. 1 ft. 8 in.? What is the scale?

3. When an inch represents 150 ft., what distances are represented by $\frac{1}{8}$ in., $\frac{1}{4}$ in., $\frac{1}{2}$ in., $1\frac{3}{4}$ in., $2\frac{1}{2}$ in., 4.8 in., 5.3 in.? What is the scale?

4. When an inch represents 10 mi., what distances are represented by 3 in., 7 in., $\frac{1}{4}$ in., $\frac{1}{2}$ in., $3\frac{1}{2}$ in.? What lengths on the drawings will represent 7 mi., 18 mi., 25 mi.? What is the scale?

5. What are the scales when 1 in. represents 100 ft., $\frac{1}{4}$ in. represents a mile, $\frac{1}{8}$ in. represents 20 ft., $\frac{1}{16}$ in. represents 15 yd., 1 in. represents 1 mi., 10 mi., 100 mi.?

6. Draw to a scale 1 : 240 (20 ft. to the inch) the circles in Exs. 2, 3, 4, 5, Art. 9.

7. On a map in Baedeker's Guide to Paris the distance between the nearest corners of the Eiffel Tower and Notre Dame Cathedral is $7\frac{3}{4}$ in. What is the distance between those points, the map being drawn to a scale 1 : 20000?

8. Make the comparison of angles asked for in Exs. 8, 9, 10, Art. 8; Exs. 11, 12, 13, Art. 9.

Suggested Exercises. Make drawings to scale of the floor plan of a dwelling house, of some other building, of some grounds. Find the distances between various points, such as diagonally opposite corners, by making measurements in the drawing and applying the scale. Compare the results obtained in this way with the results obtained by other methods. Other methods that may be used are: (1) making an off-hand estimate of the distance; (2) actually measuring the distance by "pacing off" or by using a rule or tape line; (3) making a computation. Let the student, from *his own* experience, form a judgment as to which of the four methods referred to is the easiest, and which the more exact. Find the air-line distances between places by measuring the distances between them on maps. Several maps may by used so as to have a variety of scales.

Note. The word *scale* also has another meaning in drawing and measurement. Engineers and draughtsmen use various kinds of rules called scales. The faces of these rules contain different numbers of divisions to an inch, one 10 divisions, one 20, one 30, and so on; and generally, one inch on each face is subdivided so that a small fraction of an inch may be set off or read. Some paper scales are on the protractor inserted in this book.

11. Degree measure. The protractor. It has been seen in geometry that: (1) When one line is perpendicular to another line, each of the angles made at their intersection is a right angle; (2) All right angles are equal to one another. In some geometrical propositions angles are compared, and one angle is shown to be greater or less than another. But geometry, with the exception of a few cases, does not show *by exactly how much* the one angle is greater or less than the other. In order to show this, measurement is necessary; and in order to measure, a unit angle of measurement must be chosen. The unit of angular magnitude which is generally used in practical work is the angle that is one-ninetieth part of a right angle. *This unit angle is called a degree.* All degrees are equal to one another, since all right angles are equal to one another. Each degree is subdivided into 60 equal parts called *minutes*, and each minute is subdivided into 60 equal parts called *seconds*. Hence comes the following *table of angular measure:*

60 seconds = 1 minute,
60 minutes = 1 degree,
90 degrees = 1 right angle.

The magnitude of an angle containing 37 degrees and 42 minutes and 35 seconds, say, is written thus: $37° 42' 35''$, read 37 degrees, 42 minutes, 35 seconds. This system of measurement is sometimes called *the rectangular system*, sometimes the *sexagesimal system*. In this chapter only acute angles, that is, angles which contain between 0° and 90°, are considered. Chapter V. considers angles of all magnitudes.

NOTE 1. An angle 1° is subtended by 1 in. at a distance 4 ft. 9.3 in., and by 1 ft. at a distance 57.3 ft. An angle $1'$ is subtended by 1 in. at a distance 286.5 ft., and by 1 ft. at a distance 3437.6 ft., about two-thirds of a mile. An angle $1''$ is subtended by 1 in. at a distance of nearly $3\frac{1}{4}$ mi., by 1 ft. at a distance a little greater than $39\frac{1}{2}$ mi., by a horizontal line 200 ft. long on the other side of the world, nearly 8000 mi. away. These facts can be verified later. See Ex. 3, Art. 83.

NOTE 2. Another system of angular measurement was advocated by Briggs and other mathematicians (see Art. 1), and was introduced in France at the time of the Revolution. In this system, which is a decimal one and called *the centesimal system*, a right angle is divided into 100 equal parts called *grades*, each grade into 100 equal parts called *minutes*, and each minute into 100 equal parts called *seconds*. It has not been generally adopted, on account of the immense amount of labour that would be necessary in order to change the mathematical tables computed for the other system.

NOTE 3. The sexagesimal system (from *sexagesimus*, sixtieth) was invented by the Babylonians, who constructed their tables of weights and measures on a scale of 60. Their tables of time (1 day = 24 hr., 1 hr. = 60 min., 1 min. = 60 sec.) and circular measure have come down to the present day. It has been suggested that their adoption of the scale of 60 is due to the fact that they reckoned the year at 360 days. "This led to the division of the circumference of a circle into 360 degrees, each degree representing the daily part of the supposed yearly revolution of the sun around the earth. Probably they knew that the radius could be applied to the circumference as a chord six times, and that each arc thus cut off contained 60 degrees. Thus the division into 60 parts may have suggested itself. . . . Babylonian science has made its impress upon modern civilization. Whenever a surveyor copies the readings from the graduated circle on his theodolite, whenever the modern man notes the time of day, he is, unconsciously perhaps, but unmistakably, doing homage to the ancient astronomers on the banks of the Euphrates." — Cajori, *History of Elementary Mathematics*, pp. 10, 11.

NOTE 4. Another system of angular measure is described in Chapter IX. See Art. 71.

The protractor. The protractor is an instrument used for measuring given angles and laying off required angles on paper. Protractors are of various kinds, of which the semicircular and the full-circled are the most common. The degrees are marked all round the edge. A paper protractor is inserted in this book for use in solving problems.* In order to draw a line that shall make a given angle with a given line at a given point, proceed as follows: Place the centre of the protractor at the given point and bring its diameter into coincidence with the given line, keeping the semicircle on the side on which the required line is to be drawn; prick off the required number of degrees with a sharp pencil or fine needle. The line joining the point thus fixed and the given point, is the line required. In order to measure a given angle with the protractor, place the centre at the vertex of the angle, and place the diameter in coincidence with one of the boundary lines of the angle; the number of degrees in the arc intercepted between the boundary lines of the angle is the measure of the angle.

* A horn protractor costs about 25 cents, and a small metal one about 50 cents. One who is neat and handy can make a paper protractor.

NOTE. Before proceeding further, the student should be able to draw with ease a right-angled triangle, having been given: (*a*) The hypotenuse and a side; (*b*) the two sides about the right angle; (*c*) the hypotenuse and one of the acute angles; (*d*) one of the sides about the right angle and the opposite angle; (*e*) one of the sides about the right angle and the adjacent angle. It is here taken for granted that these problems have been considered in a course in plane geometry or in a course of geometrical drawing.

EXAMPLES.

N. B. *The student is advised to do Exs. 1–6 carefully, and to preserve the results, for they will soon be required for purposes of illustration.*

1. Draw to scale the triangles considered in Exs. 8, 9, 10, Art. 8, and Exs. 11, 12, 13, Art. 9, and measure the angles.

2. Make drawings, on two different scales, of a right-angled triangle whose base is 20 ft. and adjacent acute angle is 55°. In each drawing measure the remaining parts and thence deduce the unknown parts of the original triangle. In each drawing calculate the ratios specified in Ex. 8, Art. 8.

3. Same as Ex. 2, for a right-angled triangle whose hypotenuse is 30 ft. and angle at base is 25°.

4. Same as Ex. 2, for a right-angled triangle whose base and perpendicular are 30 ft. and 45 ft. respectively.

5. Same as Ex. 2, for a right-angled triangle whose hypotenuse is 60 ft. and base is 45 ft.

6. Same as Ex. 2, for a right-angled triangle in which the base is 50 ft. and the angle opposite to the base is 40°.

7. What angles of a whole number of degrees can *easily* be constructed geometrically without the aid of the protractor? Make the constructions.

12. Trigonometric ratios defined for acute angles. The ratios referred to at the beginning of Art. 8 will now be explained so far as acute angles are concerned. (Before proceeding, the student

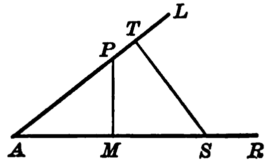

FIG. 2.

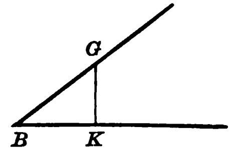

FIG. 3.

should glance over the work on Exs. 8–10, Art. 8; Exs. 11–13, Art. 9; Exs. 1–6, Art. 11.) Let A be any acute angle. In either one of the lines containing the angle take any point P and let fall a perpendicular PM to the other line. The three lines AP, AM, MP, can be taken by twos in three different ways, and hence *six ratios* can be formed with them, namely:

$$\frac{MP}{AP}, \quad \frac{AM}{AP}, \quad \frac{MP}{AM}, \quad \frac{AM}{MP}, \quad \frac{AP}{AM}, \quad \frac{AP}{MP}.$$

It is shown in Art. 13 that each of these ratios has the same value as in Fig. 2, no matter where the point P is taken on either one of the lines bounding an angle which is equal to A. *For the sake of convenience of reference, each one of these six ratios is given a particular name with respect to the angle A.* Thus:

$$\left.\begin{array}{l}
\dfrac{MP}{AP} \text{ is called the } \textbf{sine} \text{ of the angle } A; \\[2ex]
\dfrac{AM}{AP} \text{ is called the } \textbf{cosine} \text{ of the angle } A; \\[2ex]
\dfrac{MP}{AM} \text{ is called the } \textbf{tangent} \text{ of the angle } A; \\[2ex]
\dfrac{AM}{MP} \text{ is called the } \textbf{cotangent} \text{ of the angle } A; \\[2ex]
\dfrac{AP}{AM} \text{ is called the } \textbf{secant} \text{ of the angle } A; \\[2ex]
\dfrac{AP}{MP} \text{ is called the } \textbf{cosecant} \text{ of the angle } A.
\end{array}\right\} \quad . \quad . \quad . \quad (1)^*$$

These six ratios are known as *the trigonometric ratios* of the angle A. According to the definition of a ratio (Art. 8) they are merely *numbers*. For brevity they are written *sin A*, *cos A*, *tan A*,

* In Chapter V. the trigonometric ratios are defined for angles in general. The definitions give in this article will be found to follow immediately from those given in Art. 40.

cot A, sec A, cosec A (or *csc A*).* Thus *tan A* is read "*tangent A*," and means "the tangent of the angle *A*." The giving of names in (1) may be regarded as defining the trigonometric ratios. Definitions (1) may be expressed as follows:

$$\left.\begin{array}{lll} \dfrac{MP}{AP} = \sin A, & \dfrac{MP}{AM} = \tan A, & \dfrac{AP}{AM} = \sec A, \\[2ex] \dfrac{AM}{AP} = \cos A, & \dfrac{AM}{MP} = \cot A, & \dfrac{AP}{MP} = \operatorname{cosec} A, \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

These definitions can be given a slightly different form which is more general, and, accordingly, more useful in applications. In any right-angled triangle *AMP* (Fig. 2), *M* being the right angle, *with reference to the angle A* let *MP* be denoted as the opposite side, and *AM* as the adjacent side. Then these definitions take the form: —

---

* The term *sine* first appeared in the twelfth century in a Latin translation of an Arabian work on astronomy, and was first used in a published work by a German mathematician, *Regiomontanus* (1436–1476). The terms *secant* and *tangent* were introduced by a Dane, *Thomas Finck* (1561-1646), in a work published in 1583. The term *cosecant* seems to have been first used by *Rheticus*, a German mathematician and astronomer (1514-1576), in one of his works which was published in 1596. The names *cosine* and *cotangent* were first employed by *Edmund Gunter* (1581-1626), professor of astronomy at Gresham College, London, who made the first table of logarithms of sines and tangents, published in 1620, and introduced the Gunter's chain now used in land surveying. The abbreviations *sin*, *tan*, *sec*, were first used in 1626 by a Flemish mathematician, Albert Girard (1590-1634), and those of *cos*, *cot*, appear to have been earliest used by an Englishman, William Oughtred (1574–1660), in his *Trigonometry*, published in 1657. These contractions, however, were not generally adopted until after their reintroduction by Leonhard Euler (1707-1783), born in Switzerland of Dutch descent, in a work published in 1748. They were simultaneously introduced in England by Thomas Simpson (1710-1761), professor at Woolwich, in his *Trigonometry*, published in 1748. [See Ball, *A Short History of Mathematics*, pp. 215, 367.] When first used these names referred, not to *certain ratios connected with an angle*, but to *certain lines connected with circular arcs subtended by the angle*. This is explained in Art. 79, which the student can easily read at this time. See Art. 80, Notes 2, 3.

$$\left.\begin{aligned}\sin A &= \frac{\text{opposite side,}}{\text{hypotenuse,}}\\ \cos A &= \frac{\text{adjacent side,}}{\text{hypotenuse,}}\\ \tan A &= \frac{\text{opposite side,}}{\text{adjacent side,}}\\ \cot A &= \frac{\text{adjacent side,}}{\text{opposite side,}}\\ \sec A &= \frac{\text{hypotenuse,}}{\text{adjacent side,}}\\ \operatorname{cosec} A &= \frac{\text{hypotenuse,}}{\text{opposite side,}}\end{aligned}\right\} \quad \cdots\cdots \quad (3)$$

[The word *perpendicular* is sometimes used instead of *opposite side*, and *base* instead of *adjacent side.*]

It is necessary that these definitions be thoroughly memorized.

EXAMPLES.

N.B. *The student is requested to preserve the work and results of these Exs. for purposes of future reference.*

1. In *AMP* (Fig. 2) give the trigonometric ratios of angle *AMP*. Note what ratios of angles *A* and *P* are equal.

2. In Figs. 45 *a*, 45 *b*, Art. 46, give the trigonometric ratios of the various acute angles.

3. Find the trigonometric ratios of the acute angles in the triangles in Exs. 8–10, Art. 8; Exs. 11–13, Art. 9; Exs. 2–6, Art. 11.

4. In a triangle *PQR* right-angled at *Q*, the hypotenuse *PR* is 10 in. long, and the side *QR* is 7. Find the trigonometric ratios of the angles *P* and *Q*. Note what ratios of *P* and *Q* are equal.

5. For each of the angles in Ex. 4, and for each of any three of the angles in Ex. 3, calculate the following, and *make a note of the result.* [Let x denote the angle whose ratios are being considered.]

(1) $\sin x \operatorname{cosec} x$ (2) $\cos x \sec x$ (3) $\tan x \cot x$

(4) $\sin^2 x + \cos^2 x$ (5) $\sec^2 x - \tan^2 x$ (6) $\operatorname{cosec}^2 x - \cot^2 x$

(7) $\tan x - \dfrac{\sin x}{\cos x}$ (8) $\cot x - \dfrac{\cos x}{\sin x}$

6. Make the same calculations for angle *A* in Fig. 2, Art. 12.

13. Definite and invariable connection between (acute) angles and trigonometric ratios. It is important that the following principles be clearly understood:

(1) *To each value of an angle there corresponds but one value of each trigonometric ratio.*

(2) *Two unequal acute angles have different trigonometric ratios.*

(3) *To each value of a trigonometric ratio there corresponds but one value of an acute angle.*

(1) In Fig. 2, Art. 12, from any point S in AR draw ST perpendicular to AL. Let angle B (Fig. 3) be equal to A, and from any point G in one of the lines containing angle B draw GK perpendicular to the other line. Then, by definition (3), Art. 12,

$$\sin A \text{ (in } AMP) = \frac{MP}{AP}, \qquad \sin A \text{ (in } AST) = \frac{ST}{AS},$$

$$\sin B \,(= \sin A) = \frac{KG}{BG}.$$

But the triangles AMP, AST, BKG, are mutually equiangular. Hence the sides about the equal angles are proportional, and

$$\frac{MP}{AP} = \frac{ST}{AS} = \frac{KG}{BG}.$$

Therefore all angles equal to A have the same sine. In like manner, these angles can be shown to have the same tangent, secant, etc.*

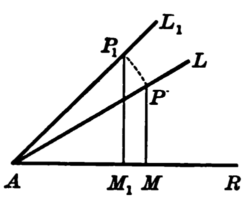

FIG. 4.

(2) Let RAL and RAL_1 be any two unequal acute angles, placed, for convenience, so as to have a common vertex A and a common boundary line AR. From any point P on AL draw PM perpendicular to AR. Take $AP_1 = AP$, and draw P_1M_1 perpendicular to AR. Then

* In Euclid's text on geometry, the properties of similar triangles are considered in Bk. VI. Pupils who study Euclid and have not reached Bk. VI. can be helped to understand these properties by means of a few exercises like those referred to in Ex. 3, Art. 12.

$$\sin RAL = \frac{MP}{AP}, \quad \sin RAL_1 = \frac{M_1P_1}{AP_1}.$$

But $$M_1P_1 > MP, \text{ and } AP_1 = AP;$$

hence $$\sin RAL_1 > \sin RAL.$$

In a similar manner the other ratios can be shown to be respectively unequal.

Ex. In this construction AP_1 is taken equal to AP. Why does this not affect the generality of the proof?

(3) This property follows as a corollary from (1) and (2).

The trigonometric ratios for angles from 0° to 90° are arranged in tables. In some tables the calculations are given to four places of decimals, in others to five, six, or seven places. There are also tables of the logarithms of the ratios (or of the logarithms increased by 10),* which vary in the number of places of decimals to which the calculations are carried out. The student is advised to examine a table of the trigonometric ratios at this time. A good exercise will consist in finding the logarithms of some of the sines, tangents, etc., adding 10 to each logarithm, and comparing the result with that given in the table of Logarithmic sines, tangents, etc. [What are denoted as Natural sines and cosines in the tables, are merely the actual sines and cosines, which have been discussed above; the so-called Logarithmic sines and cosines are the logarithms of the Natural sines and cosines with 10 added.] A book of logarithms and trigonometric ratios is the principal help and tool in solving most of the problems in practical trigonometry; and hence, proficiency in using the tables is absolutely necessary. The larger part of the numerical answers in this book have been obtained with the aid of a five-place table. Those who use six-place or seven-place tables will reach more accurate results.

EXAMPLES.

1. Compare each of the ratios of RAL_1 with the corresponding ratio of RAL.

2. Suppose that the line AR (Fig. 4) revolves about A in a *counter-clockwise direction*, starting from the position AM: show that, as the angle MAL

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* These are usually called Logarithmic sines, tangents, etc.

increases, its sine, tangent, and secant increase, and its cosine, cotangent, and cosecant decrease. *Test this conclusion by an inspection of a table of Natural ratios.*

3. Find by tables, sin 17° 40′, sin 43° 25′ 10″, sin 76° 43′, sin 83° 20′ 25″, cos 18° 10′, cos 37° 40′ 20″, cos 61° 37′, cos 72° 40′ 30″, tan 37° 40′ 20″, tan 79° 37′ 30″, cot 42° 30′, cot 72° 25′ 30″. Log sin 37° 20′, Log sin 70° 21′ 30″, Log cos 30° 20′ 20″, Log cos 71° 25′, Log tan 79° 30′ 20″, Log cot 48° 20′ 40″.

4. Find the angles corresponding to the following Natural and Logarithmic ratios:

sine = .15327,	sine = .62175,	sine = .82462,	sine = .84316,
cosine = .85970,	cosine = .61497,	cosine = .84065,	cosine = .60165.
tangent = .42482,	tangent = .60980,	tangent = 1.6820,	tangent = 2.4927,

Log sine = 9.79230,	Log sine = 9.94215,
Log cosine = 9.96611,	Log cosine = 9.74743,
Log tangent = 9.82120,	Log tangent = 10.37340.

14. Practical problems. The problems in this article are intended to help the learner to realize more clearly and strongly the meaning and the usefulness of the ratios which have been defined in Art. 12. The student is earnestly recommended to try to solve the first three problems below *without help* from the book. He will find this to be an advantage, whether he can solve the problems or not. If he *can* solve them, then he will have the pleasurable feeling that he is to some extent independent of the book; and he will thus be encouraged and strengthened for future work. Should he *fail* to solve them, he will have the advantage of a closer acquaintance with the difficulties in the problems, and so will observe more keenly how these difficulties are avoided or overcome. Throughout this course the student will find it to be of immense advantage if he will think and study over the subject-matter indicated in the headings of the articles and make some kind of an attack on the problems *before* appealing to the book for help. If he follows this plan, his progress, in the long run, will be easier and more rapid, and his mental power more greatly improved than if he is content merely to follow after, or be led by, the teacher or author.

EXAMPLES.

1. Construct the acute angle whose cosine is $\frac{2}{3}$. What are its other trigonometric ratios? Find the number of degrees in the angle.

The definition of the cosine of an angle shows that the required angle is equal to an angle in a certain right-angled triangle, namely, the triangle in which "the side adjacent to the angle is to the hypotenuse in the ratio 2 : 3." Thus the lengths of this side and hypotenuse can be taken as 2 and 3, 6 and 9, 200 and 300, and so on. Taking the lengths 2, 3, (these numbers being simpler and, accordingly, more convenient than the others), construct a right-angled triangle AST which has side $AS = 2$, and hypotenuse $AT = 3$. The angle A is the angle required, for $\cos A = \frac{2}{3}$.

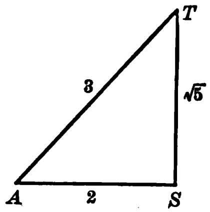

FIG. 5.

Now $$ST = \sqrt{3^2 - 2^2} = \sqrt{5} = 2.2361.$$

Hence, *the other ratios* are

$$\sin A = \frac{\sqrt{5}}{3} = .7454,\ \tan A = \frac{\sqrt{5}}{2} = 1.1180,\ \cot A = \frac{2}{\sqrt{5}} = .8944,$$

$$\sec A = \frac{3}{2} = 1.5000,\ \operatorname{cosec} A = \frac{3}{\sqrt{5}} = \dot{1}.3416.$$

The *measure of the angle* can be found in either one of two ways, viz.: (*a*) by measuring the angle with the protractor; (*b*) by finding in the table the angle whose cosine is $\frac{2}{3}$ or .6667. The latter method shows that $A = 48° 11' 22''$. [Compare the result obtained by method (*a*) with the value given by method (*b*).]

2. A right-angled triangle has an angle whose cosine is $\frac{2}{3}$, and the length of the hypotenuse is 50 ft. Find the angles and the lengths of the two sides.

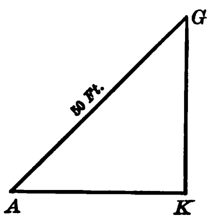

FIG. 6.

By method shown in Ex. 1, construct an angle A whose cosine is $\frac{2}{3}$. On one boundary line of the angle take a length AG to represent 50 ft. Draw GK perpendicular to the other boundary line.

$$\operatorname{Cos} A = \tfrac{2}{3} = .6666 \cdots,$$

$$\therefore\ A = 48° 11' 22'',$$

$$\therefore\ B = 90 - A = 41° 48' 38'',$$

$$\cos A = \frac{AK}{AG} = .6666 \cdots, \qquad \sin A = \frac{\sqrt{5}}{3}, \qquad \text{(Ex. 1)}$$

$$\therefore\ AK = 50 \times .6666 \cdots, \qquad \therefore\ \frac{KG}{AG} = \frac{\sqrt{5}}{3},$$

$$= 33.333 \cdots, \qquad \therefore\ KG = \frac{\sqrt{5}}{3} \times 50 = 37.27 \cdots.$$

The problem may also be solved graphically as follows. Measure angles A, G, with the protractor. Measure AK, KG directly in the figure.

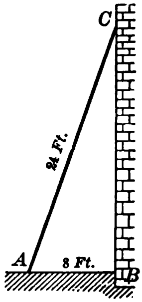

FIG. 7.

3. A ladder 24 ft. long is leaning against the side of a building, and the foot of the ladder is distant 8 ft. from the building in a horizontal direction. What angle does the ladder make with the wall? How far is the end of the ladder from the ground?

Graphical method. Let AC represent the ladder, and BC the wall. Draw AC, AB, to scale, to represent 24 ft. and 8 ft. respectively. Measure angle ACB with the protractor. Measure BC directly in the figure.

Method of computation.

$$BC = \sqrt{AC^2 - AB^2} = \sqrt{576 - 64} = \sqrt{512} = 22.63 \text{ ft.}$$

$$\sin ACB = \frac{AB}{AC} = \frac{8}{24} = .33333,$$

$$\therefore\ ACB = 19° \, 28' \, 16''.$$

4. Find tan 40° by construction and measurement. With the protractor lay off an angle SAT equal to 40°. From any point P in AT draw PR perpendicularly to AS. Then measure AR, RP, and substitute the values in the ratio, $\tan 40° = \frac{RP}{AR}$. Compare the result thus obtained with the value given for tan 40° in the tables.*

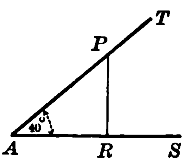

FIG. 8.

5. Construct the angle whose tangent is $\frac{5}{8}$. Find its other ratios. Measure the angle approximately, and compare the result with that given in the tables. Draw a number of right-angled, obtuse-angled, and acute-angled triangles, each of which has an angle equal to this angle.

6. Similarly for the angle whose sine is $\frac{2}{5}$; and for the angle whose cotangent is 3.

7. Similarly for the angle whose secant is $2\frac{1}{3}$; and for the angle whose cosecant is $3\frac{1}{3}$.

8. Find by measurement of lines the approximate values of the trigonometric ratios of 30°, 40°, 45°, 50°, 55°, 60°, 70°; compare the results with the values given in the tables.

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* The values of the ratios are calculated by an algebraic method, and can be found to any degree of accuracy that may be required.

If any of the following constructions asked for is impossible, explain why it is so.

9. Construct the acute angles in the following cases: (a) When the sines are $\frac{1}{2}$, 2, $\frac{2}{7}$, $\frac{3}{5}$; (b) when the cosines are $\frac{1}{2}$, $\frac{4}{5}$, 3, .3; (c) when the tangents are 3, 4, $\frac{2}{3}$, $\frac{1}{4}$; (d) when the cotangents are 4, 2, $\frac{2}{3}$, .7; (e) when the secants are 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, $4\frac{1}{2}$; (f) when the cosecants are 3, 2.5, .4, $\frac{3}{8}$.

10. Find the other trigonometric ratios of the angles in Ex. 9. Find the measures of these angles, (a) with the protractor, (b) by means of the tables.

11. What are the other trigonometric ratios of the angles: (1) whose sine is $\frac{a}{b}$; (2) whose cosine is $\frac{a}{b}$; (3) whose tangent is $\frac{a}{b}$; (4) whose cotangent is $\frac{a}{b}$; (5) whose secant is $\frac{a}{b}$; (6) whose cosecant is $\frac{a}{b}$?

12. A ladder 32 ft. long is leaning against a house, and reaches to a point 24 ft. from the ground. Find the angle between the ladder and the wall.

13. A man whose eye is 5 ft. 8 in. from the ground is on a level with, and 120 ft. distant from, the foot of a flag pole 45 ft. 8 in. high. What angle does the direction of his gaze, when he is looking at the top of the pole, make with a horizontal line from his eye to the pole?

14. Find the ratios of 45°, 60°, 30°, 0°, 90°, *before* reading the next article.

15. Trigonometric ratios of 45°, 60°, 30°, 0°, 90°. The ratios of certain angles which are often met will now be found.

A. **Ratios of 45°.** Let AMP be an isosceles right-angled triangle, and let each of the sides about the right angle be equal to a.

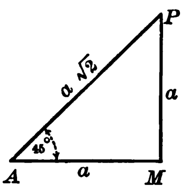

FIG. 9.

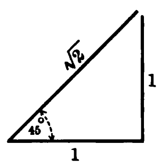

FIG. 10.

The angle

$$A = 45°, \text{ and } AP = a\sqrt{2}.$$

$$\therefore \sin 45° = \sin A = \frac{MP}{AP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

By using the same figure it can be shown that

$$\cos 45° = \frac{1}{\sqrt{2}}, \qquad \tan 45° = 1, \qquad \cot 45° = 1,$$

$$\sec 45° = \sqrt{2}, \qquad \text{cosec } 45° = \sqrt{2}.$$

The sides of triangle AMP are proportional to 1, 1, $\sqrt{2}$. Hence, in order to produce the ratios of 45° quickly, it is merely necessary to draw Fig. 10; from this figure the ratios of 45° can be read off at once.

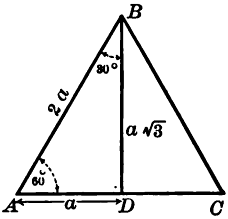

FIG. 11.

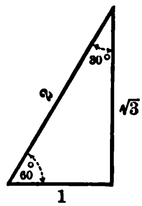

FIG. 12.

B. **Ratios of 30° and 60°.** Let ABC be an equilateral triangle. From any vertex B draw a perpendicular BD to the opposite side AC. Then angle $DAB = 60°$, angle $ABD = 30°$.

If $AB = 2a$, then $AD = a$, and $DB = \sqrt{4a^2 - a^2} = a\sqrt{3}$.

$$\therefore \sin 60° = \sin DAB = \frac{DB}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}.$$

By using the same figure it can be shown that

$$\cos 60° = \tfrac{1}{2}, \qquad \tan 60° = \sqrt{3}, \qquad \cot 60° = \frac{1}{\sqrt{3}},$$

$$\sec 60° = 2, \qquad \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}.$$

Also, $$\sin 30° = \sin ABD = \frac{AD}{AB} = \frac{a}{2a} = \tfrac{1}{2}.$$

Similarly,

$$\cos 30° = \frac{\sqrt{3}}{2}, \qquad \tan 30° = \frac{1}{\sqrt{3}}, \qquad \cot 30° = \sqrt{3},$$

$$\sec 30° = \frac{2}{\sqrt{3}}, \qquad \operatorname{cosec} 30° = 2.$$

In ADB the sides opposite to the angles 30°, 60°, 90°, are respectively proportional to 1, $\sqrt{3}$, 2. Hence, in order to produce the ratios of 30°, 60°, at a moment's notice, it is merely necessary to draw Fig. 12, from which these ratios can be immediately read off.

C. **Ratios of 0° and 90°.** *The algebraical note, Art. 76, may be read now.*

Let the hypotenuse in each of the right-angled triangles in Fig. 13 be equal to a.

$$\text{Sin } MAP = \frac{MP}{AP},$$

$$\cos MAP = \frac{AM}{AP},$$

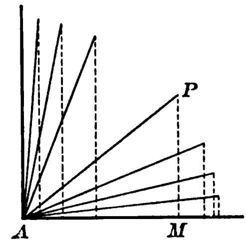

FIG. 13.

It is apparent from this figure that if the angle MAP approaches zero, then the perpendicular MP approaches zero, and the hypotenuse AP approaches to an equality with AM; so that, finally, if $MAP = 0$, then $MP = 0$, and $AP = AM$. Therefore, when $MAP = 0$, it follows that:

$$\sin 0^\circ = \frac{0}{a} = 0, \qquad \tan 0^\circ = \frac{0}{a} = 0, \qquad \sec 0^\circ = \frac{a}{a} = 1,$$

$$\cos 0^\circ = \frac{a}{a} = 1, \qquad \cot 0^\circ = \frac{a}{0} = \infty, \qquad \text{cosec } 0^\circ = \frac{a}{0} = \infty.$$

As MAP approaches 90°, AM approaches zero, and MP approaches to an equality with AP. Therefore, when $MAP = 90°$, it follows that:

$$\sin 90^\circ = \frac{a}{a} = 1, \qquad \tan 90^\circ = \frac{a}{0} = \infty, \qquad \sec 90^\circ = \frac{a}{0} = \infty,$$

$$\cos 90^\circ = \frac{0}{a} = 0, \qquad \cot 90^\circ = \frac{0}{a} = 0, \qquad \text{cosec } 90^\circ = \frac{a}{a} = 1.$$

EXAMPLES.

N. B. Read the first few lines of Art. 17 before attacking the problems.

Find the numerical value of

1. $\sin 60^\circ + 2\cos 45^\circ$.

2. $\sec^2 30^\circ + \tan^3 45^\circ$.

3. $\sin^3 60^\circ + \cot^3 30^\circ$.

4. $\cos 0^\circ \sin 45^\circ + \sin 90^\circ \sec^2 30^\circ$.

5. $4\cos^2 30^\circ \sin^2 60^\circ \cos^2 0^\circ$.

6. $3\tan^3 30^\circ \sec^3 60^\circ \sin^2 90^\circ \tan^2 45^\circ$.

7. $10\cos^4 45^\circ \sec^6 30^\circ$.

8. $2\sin^5 30^\circ \tan^3 60^\circ \cos^3 0^\circ$.

9. $x\cot^3 45^\circ \sec^2 60^\circ = 11\sin^2 90^\circ$; find x.

10. $x(\cos 30^\circ + 2\sin 90^\circ + 3\cos 45^\circ - \sin^2 60^\circ) = 2\sec 0^\circ - 5\sin 90^\circ$; find x.

16. Relations between the trigonometric ratios of an acute angle and those of its complement. When two angles added together make a right angle, the two angles are said to be *complementary*, and each angle is called *the complement* of the other.

For example, the acute angles in a right-angled triangle are complementary; the complement of A is $90° - A$; the complement of $27°$ is $63°$.

Ex. 1. What are the complements of $10°$, $12° 30'$, $47°$, $56° 27'$, $35°$?

Ex. 2. What angles are complementary to $23°$, $42°$, $51°$, $78°$, $86°$?

In Fig. 2, Art. 12, the angle APM is the complement of the angle A. Now,

$$\sin P = \frac{AM}{AP}, \quad \tan P = \frac{AM}{MP}, \quad \sec P = \frac{AP}{MP},$$

$$\cos P = \frac{MP}{AP}, \quad \cot P = \frac{MP}{AM}, \quad \operatorname{cosec} P = \frac{AP}{AM}.$$

Comparison of these ratios with the ratios of A in (2), Art. 12, shows that

$$\cos A = \sin P, \quad \tan A = \cot P, \quad \sec A = \operatorname{cosec} P,$$

$$\sin A = \cos P, \quad \cot A = \tan P, \quad \operatorname{cosec} A = \sec P.$$

These six relations can be expressed briefly:

Each trigonometric ratio of an angle is equal to the corresponding co-ratio of its complement.

Ex. Compare the ratios of $30°$ and $60°$; of $0°$ and $90°$.

17. Exponents in trigonometry.

When a trigonometric ratio has an exponent, a particular way of placing the exponent has been adopted. For example,

$$(\sin x)^2 \text{ is written } \sin^2 x.$$

There is no ambiguity in the second form, and the advantage is apparent. Thus, $\cos^{\frac{1}{3}} x$, $\tan^3 x$, $\sec^{\frac{3}{4}} x$, represent or mean $(\cos x)^{\frac{1}{3}}$, $(\tan x)^3$, $(\sec x)^{\frac{3}{4}}$. There is one exponent, however, which must not be written with the brackets removed. This exception is the exponent -1. Thus, for example, $(\cos x)^{-1}$, which means $\frac{1}{\cos x}$, must never be written $\cos^{-1} x$. The reason for this is that the symbol $\cos^{-1} x$ is used to represent something else. This symbol denotes *the angle whose cosine is* x, and is read thus, or is read "the anti-cosine of x," "the inverse cosine of x," "cosine minus one x." The number -1, which appears in $\cos^{-1} x$, is not an exponent at all, but is merely part of a symbol.

Suppose that (*a*) "the sine of the angle A is $\frac{3}{5}$."

The latter idea can also be expressed by saying

(*b*) "A is the angle whose sine is $\frac{3}{5}$";

or, more briefly, by saying,

(*c*) "A is the anti-sine of $\frac{3}{5}$."

The two ways, (*a*), (*c*), of expressing the same idea can be indicated still more briefly by equations, viz.,

$$\sin A = \tfrac{3}{5}, \quad A = \sin^{-1} \tfrac{3}{5}.$$

Thus, $(\sin x)^{-1}$ and $\sin^{-1} x$ mean very different things; for $(\sin x)^{-1}$ is $\dfrac{1}{\sin x}$, which is *a number*, and $\sin^{-1} x$ is *an angle*.

Note. The symbols $\sin^{-1} x$, $\cos^{-1} x$, $\cdots$, are considered in Art. 88.

Ex. Express $\sin A = \frac{2}{3}$, $\cos x = \frac{3}{4}$, $\tan C = 4$, $\sec A = 9$, $\operatorname{cosec} A = \frac{17}{10}$, in the inverse form.

18. Relations between the trigonometric ratios of an acute angle.

[**N.B.** Some relations between these ratios may have been noticed or discovered by the student in the course of his preceding work. If so, they should now be collected, so that they can be compared with the relations shown in this article.]

Some of the preceding exercises have shown that when *one* trigonometric ratio of an angle is known, the remaining *five* ratios can be easily determined. This at least suggests that the ratios are related to one another. In what follows, A denotes *any* acute angle.

A. Reciprocal relations between the ratios.

Inspection of the definitions (3), Art. 12, shows that:

$$\left.\begin{array}{llll}
(a)\ \sin A = \dfrac{1}{\operatorname{cosec} A}, & \operatorname{cosec} A = \dfrac{1}{\sin A}, & \text{or,} & \mathbf{\sin A \operatorname{cosec} A = 1;} \\[2ex]
(b)\ \cos A = \dfrac{1}{\sec A}, & \sec A = \dfrac{1}{\cos A}, & \text{or,} & \mathbf{\cos A \sec A = 1;} \\[2ex]
(c)\ \tan A = \dfrac{1}{\cot A}, & \cot A = \dfrac{1}{\tan A}, & \text{or,} & \mathbf{\tan A \cot A = 1.}
\end{array}\right\} \quad (1)$$

B. The tangent and cotangent in terms of the sine and cosine.

In the triangle AMP (Fig. 2, Art. 12),

$$\tan A = \frac{MP}{AM} = \frac{\frac{MP}{AP}}{\frac{AM}{AP}} = \frac{\sin A}{\cos A}; \quad \cot A = \frac{AM}{MP} = \frac{\frac{AM}{AP}}{\frac{MP}{AP}} = \frac{\cos A}{\sin A}. \quad (2), (3)$$

C. Relations between the squares of certain ratios.

In the triangle AMP (Fig. 2, Art. 12), indicating by $\overline{MP}^2$ the square of the length of MP,

$$\overline{MP}^2 + \overline{AM}^2 = \overline{AP}^2.$$

On dividing each member of this equation by $\overline{AP}^2$, $\overline{AM}^2$, $\overline{MP}^2$, in turn, there is obtained

$$\left(\frac{MP}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2 = \left(\frac{AP}{AP}\right)^2,$$

$$\left(\frac{MP}{AM}\right)^2 + \left(\frac{AM}{AM}\right)^2 = \left(\frac{AP}{AM}\right)^2,$$

$$\left(\frac{MP}{MP}\right)^2 + \left(\frac{AM}{MP}\right)^2 = \left(\frac{AP}{MP}\right)^2.$$

In reference to the angle A, these equations can be written:

$$\left.\begin{aligned} \sin^2 A + \cos^2 A &= 1, \\ \tan^2 A + 1 &= \sec^2 A, \\ 1 + \cot^2 A &= \operatorname{cosec}^2 A. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

NOTE 1. The relations shown above are true, not only for acute angles, but for all angles. This is shown in Art. 44.

NOTE 2. Relations (1) have a practical bearing on the construction and the use of tables. Thus, for example, since $\cos A = \frac{1}{\sec A}$, a table of natural cosines can be transformed into a table of natural secants by merely taking the reciprocals of the cosines. Again, in logarithmic computation, since $\sec A = \frac{1}{\cos A}$, $\log \sec A = -\log \cos A$.

NOTE 3. An equation involving trigonometric ratios is a *trigonometric equation*. Thus, for example, $\tan A = 1$. *One* angle which satisfies this equation is the *acute* angle $A = 45°$. Other solutions can be found after Arts. 84–87 have been taken up.

EXAMPLES.

A few simple exercises are given below, the solution of which brings in the relations shown in this article. These exercises are *algebraic* in character; collections of exercises of this kind are given also in other places in this book. In the following examples, the positive values of the radicals are to be taken. The meaning of the negative values is shown in Art. 44.

1. Given that $\sin A = \frac{1}{2}$, find the other trigonometric ratios of A by means of the relations shown in this article.

$$\operatorname{cosec} A = \frac{1}{\sin A} = 2\,;\ \cos A = \sqrt{1 - \sin^2 A} = \frac{\sqrt{3}}{2};\ \sec A = \frac{1}{\cos A} = \frac{2}{\sqrt{3}};$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\sqrt{3}};\quad \cot A = \frac{1}{\tan A} = \sqrt{3}.$$

These results may be verified by the method used in solving Exs. 1, 5–7, Art. 14.

2. Express all the ratios of angle A in terms of $\sin A$.

$$\sin A = \sin A\,;\ \cos A = \sqrt{1 - \sin^2 A}\,;\ \tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}};$$

$$\cot A = \frac{1}{\tan A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A};\ \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}};\ \operatorname{cosec} A = \frac{1}{\sin A}.$$

3. Prove that $\quad \dfrac{1}{1 - \sin A} + \dfrac{1}{1 + \sin A} = 2 \sec^2 A.$

$$\frac{1}{1 - \sin A} + \frac{1}{1 + \sin A} = \frac{2}{1 - \sin^2 A} = \frac{2}{\cos^2 A} = 2 \sec^2 A.$$

4. Prove that $\quad \sec^4 A - 1 = 2 \tan^2 A + \tan^4 A.$

$$\sec^4 A - 1 = (\sec^2 A)^2 - 1 = (1 + \tan^2 A)^2 - 1 = 2 \tan^2 A + \tan^4 A.$$

5. Solve the equation $4 \sin \theta - 3 \operatorname{cosec} \theta = 0$.

$$4 \sin \theta - \frac{3}{\sin \theta} = 0.$$

$$\therefore\ 4 \sin^2 \theta - 3 = 0,$$

$$\therefore\ \sin^2 \theta = \frac{3}{4},\quad \therefore\ \sin \theta = +\frac{\sqrt{3}}{2},\ \text{and } \sin \theta = -\frac{\sqrt{3}}{2}.$$

On taking the *plus* sign, *one* solution is the acute angle $\theta = 60°$; other solutions will be found later. For the *minus* sign there is also a set of solutions; these will be found later.

6. Solve $$2\sin^2\theta \operatorname{cosec}\theta - 5 + 2\operatorname{cosec}\theta = 0,$$

$$\frac{2\sin^2\theta}{\sin\theta} - 5 + \frac{2}{\sin\theta} = 0,$$

$$2\sin^2\theta - 5\sin\theta + 2 = 0,$$

$$(2\sin\theta - 1)(\sin\theta - 2) = 0.$$

$$\therefore\ \sin\theta = \tfrac{1}{2},\text{ and } \sin\theta = 2.$$

The acute angle whose sine is $\frac{1}{2}$ is 30°; hence $\theta = 30°$ is *one* solution. The sine cannot exceed unity; hence $\sin\theta = 2$ does not afford any solution.

7. Given $\cos A = \frac{3}{4}$, $\sin B = \frac{2}{3}$, $\tan C = 2$, $\cot D = \frac{4}{3}$, $\sec E = 3$, $\operatorname{cosec} F = 2.5$; find the other trigonometric ratios of A, B, C, D, E, F, by the algebraic method. Verify the results by the method used in Art. 14.

8. Find by the algebraic method the ratios required in Exs. 1, 5–7, 10, 11, Art. 14.

9. Express all the trigonometric ratios of an angle A in terms of: (*a*) $\cos A$; (*b*) $\tan A$; (*c*) $\cot A$; (*d*) $\sec A$; (*e*) $\operatorname{cosec} A$. Arrange the results and those of Ex. 2, neatly in tabular form.

Prove the following identities:

10. $(\sec^2 A - 1)\cot^2 A = 1$; $\cos A \tan A = \sin A$; $(1 - \sin^2 A)\sec^2 A = 1$.

11. $\sin^2\theta \sec^2\theta = \sec^2\theta - 1$; $\tan^2\theta - \cot^2\theta = \sec^2\theta - \operatorname{cosec}^2\theta$.

12. $$\frac{1}{\sec^2 A} + \frac{1}{\operatorname{cosec}^2 A} = 1; \qquad \frac{\sin A}{\operatorname{cosec} A} + \frac{\cos A}{\sec A} = 1;$$

$$(\tan\theta + \sec\theta)^2 = \frac{1 + \sin\theta}{1 - \sin\theta}.$$

13. $$\sec^2 A + \operatorname{cosec}^2 A = \tan^2 A + \cot^2 A + 2;$$

$$\frac{1 + \tan^2 A}{1 + \cot^2 A} = \frac{\sin^2 A}{\cos^2 A}; \qquad \frac{\operatorname{cosec} A}{\cot A + \tan A} = \cos A.$$

14. $$\frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A;$$

$$\frac{1}{\sec A - \tan A} = \sec A + \tan A; \quad \sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$$

Solve the following equations:

15. $2\sin\theta = 2 - \cos\theta$.

16. $\tan\theta + \cot\theta = 2$.

17. $\tan\theta + 3\cot\theta = 4$.

18. $6\sec^2\theta - 13\sec\theta + 5 = 0$.

19. $8\sin^2\theta - 10\sin\theta + 3 = 0$.

20. $\sin\theta + 2\cos\theta = 2.2$.

19. Summary. In this chapter important additions have been made to the knowledge concerning angles that one gained in geometry. A process of measuring angles has been introduced. The close connection between angles and the ratios of lines has been emphasized. It has been shown that each (acute) angle has, associated with it, a definite set of six numbers, called trigonometric ratios; and it has been seen that the sets of numbers are different for different angles. It has also been shown that the seven quantities (namely, the angle and the six numbers) are so related, that, *if one of the seven be given, then the remaining six can be determined.*

A few applications to the measurement of lines and angles have been made in some of the preceding articles. The next two chapters are taken up with a *formal* treatment of such applications. It should be stated, however, that any one who understands the contents of this chapter is *in possession of all the principles* which will be used in the next two chapters, and can proceed directly to the solution of the problems given there. The student is recommended to attack some of the exercises in Chapters III., IV., *before* reading the explanations given in the text. Attention may again be given to the first part of Art. 14.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter II. will be found at pages* 182, 183.

CHAPTER III.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

Before the solution of right-angled triangles is entered upon, a few remarks will be made on the solution of triangles in general. Some of the ideas expressed in Arts. 20–24 are applicable to practical problems throughout the book.

20. Solution of a triangle. Every triangle has three sides and three angles. These six quantities are called the *parts* or *elements* of a triangle. Sometimes one or several of the parts of a triangle are known; for instance, the three sides, two angles and a side, two sides, one side, three angles, and so on. In such cases the questions arise: Can the remaining parts be found or determined? and, if so, by what method shall this be done? The process of deducing the unknown parts of a triangle from the known, is called *solving the triangle*, or, *the solution of the triangle.* This Chapter and Chapter VII. are concerned with showing, in detail, methods of solving triangles. There are two methods which can be used to find (only approximately, in general) the unknown parts of a triangle when some of its parts are given. These methods are:

(*a*) **The graphical method;**

(*b*) **The method of computation.**

21. The graphical method. This method consists *in drawing* a triangle which has angles equal to the given angles, and sides proportional to, and thus representing the given sides, *and then measuring the remaining parts directly from the drawing.*

For example, a triangle has two sides whose lengths are 10 ft., 5 ft., and the included angle is 28° 30′; the third side and the other angles are required.

The graphical solution is as follows: Construct a triangle QPR having two sides, PQ, PR, representing 10 ft., 5 ft., respectively, on some convenient scale, and with their included angle, QPR, equal to 28° 30′, as shown in Fig. 14. Measure the angles PRQ, PQR with the protractor; measure the side RQ and, by reference to the scale, find the length represented by RQ. [The results thus obtained may be compared with those obtainable by the method of computation explained in Arts. 54, 57. The latter results are $R = 128° 26' 46''$, $Q = 23° 3' 14''$, $RQ = 6.092$.]

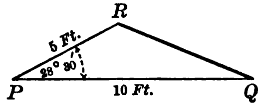

FIG. 14.

The conditions necessary and sufficient for constructing a triangle, and the methods of drawing triangles that satisfy given conditions, are shown in plane geometry and in geometrical drawing. It is obvious that *the graphical method can be employed only when the values of the parts given are consistent with one another, and when the parts given are sufficient in number to determine a definite triangle.* For instance, suppose that one is asked to find the remaining parts of a triangle one of whose sides is 10 inches long. In this case as many unequal triangles as one please, can be constructed, all of which will satisfy the given condition. Again, a given side and a given angle are insufficient *data* on which to proceed to find the remaining parts of a triangle, for there is an infinite number of unequal triangles which can have parts equal to the given parts. So also the method fails if three angles be given; for an infinite number of unequal triangles can be drawn whose angles are equal to the given angles. Again, let it be required to find the angles of a triangle whose sides are 10 feet, 40 feet, 60 feet. Such a triangle is impossible, since the length of one side (60 feet) is greater than the sum of the lengths of the other two. One more instance: let two given angles be 85° and 105° and the included side be 40 inches; this triangle is impossible, since the sum of the two given angles is greater than two right angles.

22. The method of computation. This method is applicable in precisely the same cases in which the preceding method can be employed; namely, in the cases in which the parts given are consistent with one another, and afford conditions sufficient to enable one to construct a definite triangle. This will be fully apparent later, when the various cases will be treated in detail. One of the principal purposes of this book is to show the different methods of computation applicable to various sets of given conditions. *One of the principal objects* of a student who is taking a first course in trigonometry *should be to acquire facility and, above all, accuracy in using these methods of computation.*

23. Comparison between the graphical method and the method of computation. The experience gained in some of the exercises in the preceding chapter has probably shown the student that he can attain much greater accuracy by using the method of computation than by using the graphical method. The accuracy of the results obtained by the latter method depends upon the carefulness and skill with which the figures are drawn and measured; in the other method, accuracy depends upon the care and patience employed in performing arithmetical work. While the results attainable by the graphical method, even in the case of skilled persons with excellent drawing instruments, are not as accurate as the results attainable by the other method, yet they are often accurate enough for practical purposes. When the computations required are long and complicated, the graphical method is much the more rapid of the two.

There are several reasons why it is advisable for the learner to use the graphical method, as well as the method of computation, in solving problems in this course. The first reason is that the former method *will serve as a check* upon the latter. With ordinary care the graphical method very quickly gives a fair approximation to the result. This result will sometimes show that there is an error in the result obtained by computation. A little error in arithmetic may yield a quantity which is ten times too great or too small; but this can be detected at once if the other method has also been used.* A second reason for using the graphical method is that this method incidentally *provides training in neat, careful, and accurate drawing;* this training will not only be a benefit in itself, but will be of very great advantage in other studies, and especially in the applied sciences. A third

* The results can also be tested by methods of computation, which will be shown in due course.

reason is that the pupil *will gain some knowledge and experience of a method that is used in other subjects*, for instance, in physics and in mechanics, and that is extensively employed by engineers in solving problems in which the computations required by the other method may be overwhelmingly cumbrous.

24. General directions for solving problems. A third method of approximating to the magnitude of lines and angles may be mentioned here, for it has often to be employed in practical life. In this method the student may suppose that he possesses neither measuring instruments, drawing materials, nor mathematical tables, and thereupon he *may give an off-hand estimate* concerning the magnitudes required. This method also serves as a check, by showing when great arithmetical blunders are committed. The pupil is advised to use all three methods in working each practical problem in this course, and to do so in the following order:

(1) *Make an off-hand estimate as to what the magnitude required may be, and write this estimate down;*

(2) *Solve the problem by the graphical method;*

(3) *Solve the problem by the slower but more accurate and reliable method of computation.* There may be some interest found in comparing the results obtained by these three methods. The exercise in judging linear and angular magnitudes afforded by the first method, the practice in neat and careful drawing necessary in the second, and the training in accurate computation given by the third, will each afford some benefit to the learner.

25. Solution of right-angled triangles. Let ABC be a right-angled triangle, C being the right angle. In what follows, a, b, c, denote the lengths of the sides opposite to the angles A, B, C, respectively. The sides and angles of ABC are connected by the following relations:

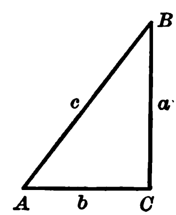

FIG. 15.

$$\left.\begin{array}{l}(1)\ A + B = 90°;\\ (2)\ c^2 = a^2 + b^2.\end{array}\right\} \ldots \text{(Geometry)}$$

$$\left.\begin{array}{ll}(3), (4) & \sin A = \dfrac{a}{c} = \cos B; \\ (5), (6) & \cos A = \dfrac{b}{c} = \sin B; \\ (7), (8) & \tan A = \dfrac{a}{b} = \cot B; \\ (9), (10) & \cot A = \dfrac{b}{a} = \tan B.\end{array}\right\} \quad . \quad . \quad \text{(Definitions (3), Art. 12)}$$

NOTE. To these may be added: $\sec A = \frac{c}{b} = \operatorname{cosec} B$, $\operatorname{cosec} A = \frac{c}{a} = \sec B$. These relations, however, are not needed, and are rarely used in solving triangles, for they are equivalent to (3)–(6), and few tables give secants and cosecants. See Art. 18, Note 2.

Equation (1) shows that no other element of the triangle can be derived from the two acute angles only. Each of the remaining equations, (2)–(10), involves *three* elements of the triangle, and at least two of these elements are sides. Hence, in order that a right-angled triangle be solvable, two elements must be known in addition to the right angle, and one of these must be a side. If any two of the elements involved in equations (2)–(10) are known, then a third element of the triangle can be found therefrom. Hence the following **general rule** can be used in solving right-angled triangles:

When in addition to the right angle, any two sides, or one of the acute angles and any one of the sides, of a right-angled triangle are known, and another element is required, write the equation involving the required element and two of the known elements, and solve the equation for the required element.

For example, suppose that a, c are known, and that A, B, b are required. In this case,

$$\sin A = \frac{a}{c}, \quad B = 90 - A, \quad b^2 = \sqrt{c^2 - a^2}.$$

The quantities A, B, b can be found from these equations. If an error has been made in finding A, then B will also be wrong. Hence *it is advisable to check the values found* by seeing whether they satisfy relations differing from those already employed.

For example, *check formulas* which may be used in this case are:

$$\frac{b}{a} = \tan B, \quad \frac{b}{c} = \cos A.$$

26. Checks upon the accuracy of the computation. As already pointed out, large errors can be detected by means of the off-hand estimate and by the use of the graphical method. The calculated results in any example can be *checked* or *tested* by employing relations which have not been used in computing the results, and examining whether the newly found values satisfy these relations. An instance has been given in the preceding article. The student is advised not to look up the answers until after he has tested his results in this way. Verification by means of check formulas is necessary in cases in which the answers are not given. The testing of the results also affords practice in the use of formulas and in computation. When a check formula is satisfied it is highly probable, but not absolutely certain, that the calculated results are correct.

27. Cases in the solution of right-angled triangles. All the possible sets of two elements that can be made from the three sides and the two acute angles of a right-angled triangle are the following:

(1) The two sides about the right angle.

(2) The hypotenuse and one of the sides about the right angle.

(3) The hypotenuse and an acute angle.

(4) One of the sides about the right angle, and an acute angle.

(5) The two acute angles. (This case has already been referred to.)

Some examples of these cases are solved. The *general method of procedure*, after making an off-hand estimate and finding an approximate solution by the graphical method, is as follows:

First: *Write all the relations (or formulas) which are to be used in solving the problem.*

Second: *Write the check formulas.*

Third: *In making the computations arrange the work as neatly as possible.*

This last is important, because, by attention to this rule, the work is presented clearly, and mistakes are less likely to occur. The computations may be made either with or without the help of logarithms. The calculations can generally be made more easily and quickly by using logarithms.

NOTE 1. Relations (3), (4), Art. 25, may be written: $a = c \sin A$, $a = c \cos B$. These relations may be thus expressed:

A side of a right-angled triangle is equal to the product of the hypotenuse and the cosine of the angle adjacent to the side.

A side of a right-angled triangle is equal to the product of the hypotenuse and the sine of the angle opposite to the side.

NOTE 2. Relations (7), (8), Art. 25, may be written: $a = b \tan A$, $a = b \cot B$. These relations may be thus expressed:

A side of a right-angled triangle is equal to the product of the other side and the tangent of the angle opposite to the first side.

A side of a right-angled triangle is equal to the product of the other side and the cotangent of the angle adjacent to the first side.

EXAMPLES.

1. In the triangle ABC, right-angled at C, $a = 42$ ft., $b = 56$ ft. Find the hypotenuse and the acute angles.

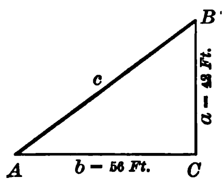

FIG. 16.

I. *Computation without logarithms.* [Four-place tables.]

$$\tan A = \frac{a}{b} = \frac{42}{56} = .7500. \qquad \therefore\ A = 36° 52'.2.$$

$$B = 90 - A. \qquad \therefore\ B = 53° \ 7'.8.$$

$$c = \sqrt{a^2 + b^2} = \sqrt{1764 + 3136}. \qquad \therefore\ c = 70 \text{ ft.}$$

Check: $a = c \cos B = 70 \times \cos 53° 7'.8 = 70 \times .6000 = 42$ ft.

II. *Computation with logarithms.*

Given: $a = 42$ ft.
$b = 56$ ft.

To find:* $A =$
$B =$
$c =$

Formulas: $\tan A = \dfrac{a}{b}$. (1)

$B = 90° - A$. (2)

$c = \dfrac{a}{\sin A}$. (3)

Checks: $\tan B = \dfrac{b}{a}$.

$a^2 = c^2 - b^2$
$= (c + b)(c - b)$.

---

* This is to be filled after the values of the unknown quantities have been found. It is advisable to indicate the given parts and the unknown parts clearly.

Logarithmic formulas: $\log \tan A = \log a - \log b$.

[See Note 6], $\log c = \log a - \log \sin A$.

$$\begin{aligned} \log a &= 1.62325 \\ \log b &= 1.74819 \\ \therefore\ \log \tan A &= 9.87506 - 10 \\ \therefore\ A &= 36^\circ\, 52'\, 12'' \\ \therefore\ B &= 53^\circ\ \ 7'\, 48'' \end{aligned} \qquad \begin{aligned} \log a &= 1.62325 \\ \log \sin A &= 9.77815 - 10 \\ \therefore\ \log c &= 1.84510 \\ \therefore\ c &= 70 \end{aligned}$$

The work can be more compactly arranged, as follows:

$$\begin{aligned} \log a &= 1.62325 \\ \log b &= 1.74819 \\ \therefore\ \log \tan A &= 9.87506 - 10 \\ \therefore\ A &= 36^\circ\, 52'\, 12'' \\ \therefore\ B &= 53^\circ\ \ 7'\, 48'' \\ \log \sin A &= 9.77815 - 10 \\ \therefore\ \log c &= 1.84510 \\ \therefore\ c &= 70 \end{aligned} \qquad \begin{aligned} &\textit{Checks:} \\ \log \tan B &= 10.12494 - 10 \\ \therefore\ B &= 53^\circ\, 7'\, 48'' \\ c + b &= 126 \\ c - b &= \ \ 14 \\ \log (c + b) &= 2.10037 \\ \log (c - b) &= 1.14613 \\ \therefore\ \log a^2 &= 3.24650 \\ \therefore\ \log a &= 1.62325 \end{aligned}$$

NOTE 1. The latter form is preferable when all the parts of a triangle are required.

NOTE 2. If there is difficulty in calculating *log a − log sin A* in the second form, write $\log \sin A$ on the edge of a piece of paper and place it immediately beneath $\log a$.

NOTE 3. The formula $\tan B = \frac{b}{a}$ can be used instead of (2). A check then is $A + B = 90^\circ$. Instead of (3), one of the following formulas can be used, viz.

$$c = \frac{b}{\sin B}, \quad c = \frac{a}{\cos B}, \quad c = \frac{b}{\cos A}.$$

There is often a choice of formulas that can be used in a solution.

NOTE 4. *In every example it is advisable to make a complete* ***skeleton scheme*** *of the solution, before using the tables and proceeding with the actual computation.* In the last exercise, for instance, such a skeleton scheme can be seen on erasing all the numerical quantities in the equations that follow the logarithmic formulas.

NOTE 5. Time will be saved if all the logarithms that can be found at one place in the tables, be written at one time. Thus, for example, in the preceding exercise find $\log \sin A$ immediately after A has been found.

NOTE 6. The logarithmic formulas can be written on a glance at the formulas such as (1), (2), (3). The writing of the logarithmic formulas may be dispensed with when the student has become familiar with calculation by logarithms. A glance at the original formulas will show how the logarithms are to be combined in the computation.

2. In a triangle ABC right angled at C, $c = 60$ ft., $b = 50$ ft.; find side a and the acute angles.

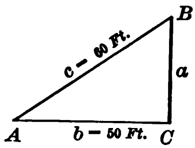

FIG. 17.

I. *Computation without logarithms.*

$$\cos A = \frac{b}{c} = \frac{50}{60} = .8333. \quad \therefore A = 33° \, 33'.75.$$

$$B = 90° - A. \quad \therefore B = 56° \, 26'.25.$$

$$a = c \sin A = 60 \times .5528 = 33.17 \text{ ft.}$$

Check: $a = b \tan A = 50 \times .6635 = 33.17.$

II. *Computation with logarithms.*

Given: $c = 60$ ft.
$b = 50$ ft.

To find: $A =$
$B =$
$a =$

Formulas: $\cos A = \dfrac{b}{c}.$
$B = 90° - A.$
$a = c \sin A.$

Checks: $a^2 = c^2 - b^2 = (c+b)(c-b).$
$a = b \tan A.$

Logarithmic formulas: $\log \cos A = \log b - \log c.$
(If necessary.) $\log a = \log c + \log \sin A.$

$$\begin{aligned}
\log b &= 1.69897 && (1)\\
\log c &= 1.77815 && (2)\\
\therefore \log \cos A &= 9.92082 - 10 && (3)\\
&= (1) - (2)\\
\therefore A &= 33° \, 33' \, 27''\\
\therefore B &= 56° \, 26' \, 33''\\
\log \sin A &= 9.74255 - 10 && (4)\\
\therefore \log a &= 1.52070 && (5)\\
&= (2) + (4)\\
\therefore a &= 33.16
\end{aligned}$$

$$\begin{aligned}
\log \tan A &= 9.82173 - 10 && (6)\\
\therefore \log a &= 1.52070 && (7)\\
&= (1) + (6)\\
c + b &= 110\\
c - b &= 10\\
\log (c + b) &= 2.04139\\
\log (c - b) &= 1\\
\therefore \log a^2 &= 3.04139\\
\therefore \log a &= 1.52070
\end{aligned}$$

NOTE. There is a slight difference between the results obtained by the two methods. This is due to the fact that the calculations have been made with a four-place table in one case, and with a five-place table in the other. A four-place table will give an angle correctly to within *one minute*; a five-place table will give it correctly to within *six seconds*, and sometimes, to within a second.

Ex. Make the computation I. with a five-place table.

3. In a triangle right angled at C, the hypotenuse is 250 ft., and angle A is $67° 30'$. Solve the triangle.

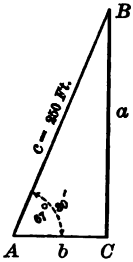

FIG. 18.

I. *Computation without logarithms.*

$B = 90° - A = 90° - 67° 30' = 22° 30'$.

$a = c \sin A = 250 \times \sin 67° 30' = 250 \times .9239 = 230.98$.

$b = c \cos A = 250 \times \cos 67° 30' = 250 \times .3827 = \ \ 95.68$.

Checks: $a^2 = c^2 - b^2$, or $a = b \tan A$.

II. *Computation with logarithms.*

Given: $c = 250$ ft. $\quad$ To find: $B =$
$A = 67° 30'$. $\quad$ $a =$
$b =$

Formulas: $B = 90° - A$. $\quad$ *Checks:* $a^2 = c^2 - b^2$
$a = c \sin A$. $\quad = (c + b)(c - b)$.
$b = c \cos A$.

Logarithmic formulas: $\log a = \log c + \log \sin A$.
$\log b = \log c + \log \cos A$.

$\therefore B = 22° 30'$	$c + b = 345.67$
$\log c = 2.39794$	$c - b = 154.33$
$\log \sin A = 9.96562 - 10$	$\log (c + b) = 2.53866$
$\log \cos A = 9.58284 - 10$	$\log (c - b) = 2.18845$
$\therefore \log a = 2.36356$	$\therefore \log a^2 = 4.72711$
$\therefore \log b = 1.98078$	$\therefore \log a = 2.36356$
$\therefore a = 230.97$	
$\therefore b = \ \ 95.67$	

4. In a triangle ABC right angled at C, $b = 300$ ft. and $A = 37° 20'$. Solve the triangle.

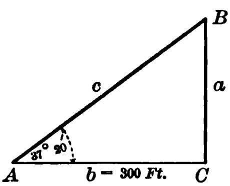

FIG. 19.

I. *Computation without logarithms.*

$B = 90 - A = 90° - 37° 30' = 52° 40'$.

$$c = \frac{b}{\cos A} = \frac{300}{.7951} = 377.3.$$

$a = b \tan A = 300 \times .7627 = 228.8$.

Checks: $a^2 = c^2 - b^2$, $a = c \sin A$.

II. *Computation with logarithms.*

Given: $A = 37° 20'$. $\quad$ To find: $B =$
$b = 300$ ft. $\quad$ $c =$
$a =$

Formulas: $B = 90° - A$.

$$c = \frac{b}{\cos A}.$$

$a = b \tan A$.

Checks: $a^2 = c^2 - b^2$
$= (c + b)(c - b)$.

$$\therefore B = 52° 40'$$
$$\log b = 2.47712$$
$$\log \cos A = 9.90043 - 10$$
$$\log \tan A = 9.88236 - 10$$
$$\therefore \log c = 2.57669$$
$$\therefore \log a = 2.35948$$
$$\therefore c = 377.3$$
$$\therefore a = 228.8$$

$$c + b = 677.3$$
$$c - b = 77.3$$
$$\log (c + b) = 2.83078$$
$$\log (c - b) = 1.88818$$
$$\therefore \log a^2 = 4.71896$$
$$\therefore \log a = 2.35948$$

N.B. *Check all results in the following examples.* The given elements belong to a triangle ABC which is right angled at C.

From the given elements solve the following triangles:

5. $c = 18.7,\quad a = 16.98.$

6. $a = 194.5,\quad b = 233.5.$

7. $c = 2934,\quad A = 31° 14' 12''.$

8. $a = 36.5,\quad B = 68° 52'.$

9. $a = 58.5,\quad b = 100.5.$

10. $c = 45.96,\quad a = 1.095.$

11. $c = 324,\quad A = 48° 17'.$

12. $b = 250,\quad A = 51° 19'.$

13. $c = 1716,\quad A = 37° 20' 30''.$

14. $a = 2314,\quad b = 1768.$

15. $b = 3741,\quad A = 27° 45' 20''.$

16. $c = 50.13,\quad a = 24.62.$

Solve Exs. 17–24 by two methods, viz.: (1) with logarithms;
(2) without logarithms.

17. $a = 40,\quad B = 62° 40'.$

18. $c = 9,\quad a = 5.$

19. $a = 4.5,\quad b = 7.5.$

20. $c = 15,\quad A = 39° 40'.$

21. $c = 12,\quad B = 71° 20'.$

22. $c = 12,\quad a = 8.$

23. $b = 15,\quad B = 42° 30'.$

24. $a = 8,\quad b = 12.$

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter III. will be found at page* 183.

CHAPTER IV.

APPLICATIONS INVOLVING THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

Some practical applications of trigonometry will now be given. It is not necessary that all the problems be solved, or all the articles be considered, before Chapter V. is taken up.

28. Projection of a straight line upon another straight line. If from a point P a perpendicular PO be drawn to the straight line ST, then O is called *the projection of the point P upon the line ST.* If perpendiculars be drawn from two points A, B, to

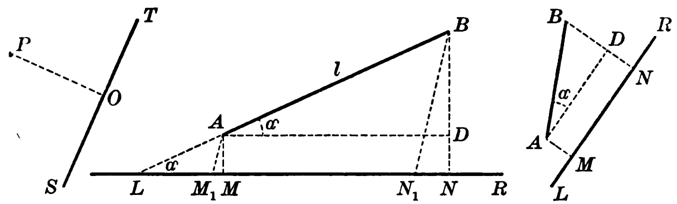

FIG. 20.

a line LR, and intersect LR in M, N, respectively, then MN is called *the projection of AB upon LR.*

Let l be the length of AB, and let α be the angle at which the two lines AB, LR are inclined to each other. Through A draw AD parallel to LR. Then

$$\textbf{Projection} = MN = AD = AB \cos DAB = \boldsymbol{l \cos \alpha}.$$

That is, *the projection of one straight line upon another straight line is equal to the product of the length of the first line and the cosine of the angle of inclination of the two lines.*

NOTE. The projection discussed here, is *orthogonal* (*i.e.* perpendicular) *projection.* If a pair of parallel lines AM_1, BN_1, *not perpendicular* to LR, be drawn through A, B, then M_1N_1 is an *oblique projection* of AB on LR.

EXAMPLES.

In working these examples use logarithms or not, as appears most convenient. Check the results.

1. A ladder 28 ft. long is leaning against the side of a house, and makes an angle 27° with the wall. Find its projections upon the wall and upon the ground.

2. What is the projection of a line 87 in. long upon a line inclined to it at an angle 47° 30′?

3. What are the projections: (*a*) of a line 10 in. long upon a line inclined 22° 30′ to it? (*b*) of a line 27 ft. 6 in. long upon a line inclined 37° to it? (*c*) of a line 43 ft. 7 in. long upon a line inclined 67° 20′ to it? (*d*) of a line 34 ft. 4 in. long upon a line inclined 55° 47′ to it?

29. Measurement of heights and distances. There are various instruments used for measuring angles. The *sextant* can be used for measuring the angle between the two lines drawn from the observer's eye to each of two distant objects. Horizontal and vertical angles can be measured with a *theodolite* or *engineer's transit.* When great accuracy is not required, vertical angles can be measured by means of a *quadrant.*

When an object is above the observer's eye, the angle between the line from the eye to the object, and the horizontal line through the eye and in the same vertical plane as the first line, is called **the angle of elevation** of the object, or simply *the elevation* of the object. When the object is below the observer's eye, this angle is called **the angle of depression** of the object, or simply *the depression* of the object.

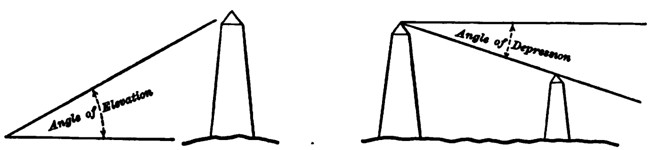

FIG. 21.

NOTE. In ordinary work engineers get angular measurements exact to within one minute, and in the best ordinary work to half a minute. In very particular work, like geodetic survey, they can get measurements exact to five seconds. For ordinary engineering work five-place tables are generally used; four-place tables are used in some kinds of work. See Art. 11, Note 1, Art. 27, Ex. 2, Note.

EXAMPLES.

A few simple examples are given here; others will be given later.

1. At a point 150 ft. from, and on a level with, the base of a tower, the angle of elevation of the top of the tower is observed to be 60°. Find the height of the tower.

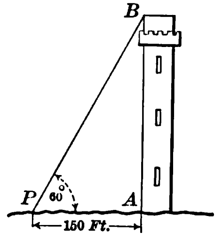

FIG. 22.

Let AB be the tower, and P the point of observation.

By the observations,

$$AP = 150 \text{ ft.}, \quad APB = 60°.$$

$AB = AP \tan 60° = 150 \times \sqrt{3} = 150 \times 1.7321 = 279.82$ ft.

2. In order to find the height of a hill, a line was measured equal to 100 ft., in the same level with the base of the hill, and in the same vertical plane with its top. At the ends of this line the angles of elevation of the top of the hill were 30° and 45°. Find the height of the hill.

Let P be the top of the hill, and AB the base line. The vertical line through P will meet AB produced in C. $AB = 100$ ft., $CAP = 30°$, $CBP = 45°$; the height CP is required. Let $BC = x$, and $CP = y$.

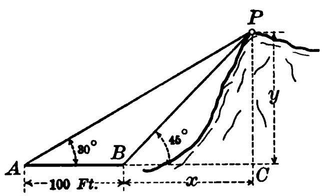

FIG. 23.

In triangle CAP,

$$\frac{CP}{AC} = \tan 30°;$$

in CBP,

$$\frac{CP}{BC} = \tan 45°.$$

Hence,
$$\frac{y}{x + 100} = \tan 30° = .57735, \tag{1}$$

and
$$\frac{y}{x} = \tan 45° = 1. \tag{2}$$

From (2), $x = y$. Substitution in (1) gives

$$y = (y + 100) \times .57735.$$

$$\therefore\ y(1 - .57735) = 57.735.$$

$$\therefore\ CP = y = \frac{57.735}{.42265} = 136.6 \text{ ft.}$$

3. A flagstaff 30 ft. high stands on the top of a cliff, and from a point on a level with the base of the cliff the angles of elevation of the top and bottom of the flagstaff are observed to be 40° 20′ and 38° 20′, respectively. Find the height of the cliff.

Let BP be the flagstaff on the top of the cliff BL, and let C be the place of observation. $BP = 30$ ft., $LCB = 38° 20'$, $LCP = 40° 20'$. Let $CL = x$, $LB = y$.

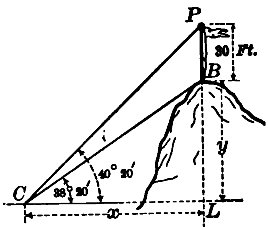

FIG. 24.

In LCB, $\quad \dfrac{LB}{LC} = \tan 38° 20'$;

i.e. $\quad \dfrac{y}{x} = .7907.$

In LCP, $\quad \dfrac{LP}{LC} = \tan 40° 20'$;

i.e. $\quad \dfrac{y + 30}{x} = .8491.$

Hence, on division, $\quad \dfrac{y}{y + 30} = \dfrac{7907}{8491}.$

On solving for y, $\quad LB = y = 406.18$ ft.

4. At a point 180 ft. from a tower, and on a level with its base, the elevation of the top of the tower is found to be 65° 40.5′. What is the height of the tower?

5. From the top of a tower 120 ft. high the angle of depression of an object on a level with the base of the tower is 27° 43′. What is the distance of the object from the top and bottom of the tower?

6. From the foot of a post the elevation of the top of a column is 45°, and from the top of the post, which is 27 ft. high, the elevation is 30°. Find the height and distance of the column.

7. From the top of a cliff 120 ft. high the angles of depression of two boats, which are due south of the observer, are 20° 20′ and 68° 40′. Find the distance between the boats.

8. From the top of a hill 450 ft. high, the angle of depression of the top of a tower, which is known to be 200 ft. high, is 63° 20′. What is the distance from the foot of the tower to the top of the hill?

9. From the top of a hill the angles of depression of two consecutive mile-stones, which are in a direction due east, are 21° 30′ and 47° 40′. How high is the hill?

10. For an observer standing on the bank of a river, the angular elevation of the top of a tree on the opposite bank is 60°; when he retires 100 ft. from the edge of the river the angle of elevation is 30°. Find the height of the tree and the breadth of the river.

11. Find the distance in space travelled in an hour, in consequence of the earth's rotation, by an object in latitude 44° 20′. [Take earth's diameter equal to 8000 mi.]

12. At a point straight in front of one corner of a house, its height subtends an angle 34° 45′, and its length subtends an angle 72° 30′; the height of the house is 48 ft. Find its length.

30. Problems requiring a knowledge of the points of the Mariner's Compass. The circle in the Mariner's Compass is divided into 32 equal parts, each part being thus equal to $360° \div 32$, *i.e.* $11\frac{1}{4}°$. The points of division are named as indicated on the figure.

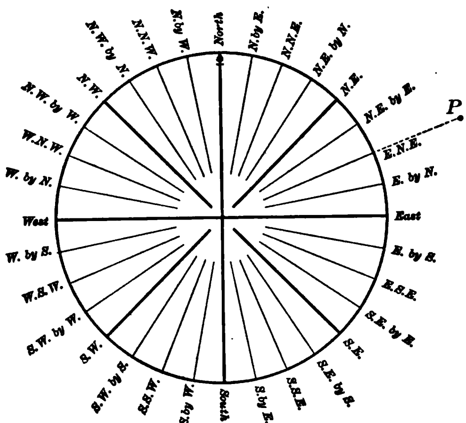

FIG. 25.

It will be observed that the points are named with reference to the points North, South, East, and West, which are called the *cardinal points.* Direction is indicated in a variety of ways. For instance, suppose C were the centre of the circle; then the point P in the figure is said to *bear* E.N.E. from C, or, from C the *bearing* of P is E.N.E. Similarly, C bears W.S.W. from P, or, the bearing of C from P is W.S.W. The point E.N.E. is 2 points North of East, and 6 points East of North. Accordingly, the phrases E. $22\frac{1}{2}°$ N., N. $67\frac{1}{2}°$ E., are sometimes used instead of E.N.E.

EXAMPLES.

1. Two ships leave the same dock at 8 A.M. in directions S.W. by S., and S.E. by E. at rates of 9 and $9\frac{1}{2}$ mi. an hour respectively. Find their distance apart, and the bearing of one from the other at 10 A.M. and at noon.

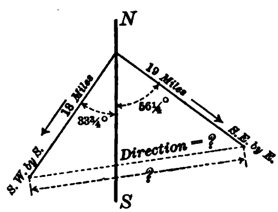

FIG. 26.

2. From a lighthouse L two ships A and B are observed in a direction N.E. and N. 20° W. respectively. At the same time A bears S.E. from B. If LA is 6 mi., what is LB?

31. Mensuration. Let ABC be any triangle, and let the lengths of the sides opposite the angle A, B, C be denoted by a, b, c, respectively. From any vertex C draw CD at right angles to the opposite

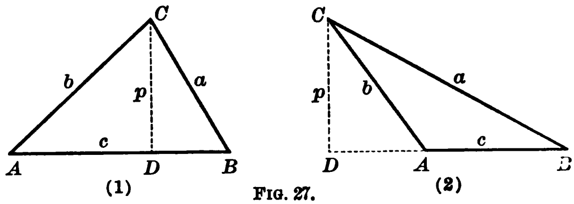

FIG. 27.

side AB. It has been shown in arithmetic and geometry, that the area of a triangle is equal to one-half the product of the lengths of any side and the perpendicular drawn to it from the opposite vertex. [In (1) A is acute, in (2) A is obtuse.]

$$\begin{aligned} \text{area } ABC \text{ (Fig. 1)} &= \tfrac{1}{2} AB \cdot DC; \\ &= \tfrac{1}{2} AB \cdot AC \sin A; \\ &= \tfrac{1}{2} bc \sin A. \end{aligned}$$

$$\begin{aligned} \text{area } ABC \text{ (Fig. 2)} &= \tfrac{1}{2} AB \cdot DC; \\ &= \tfrac{1}{2} AB \cdot AC \sin CAD; \\ &= \tfrac{1}{2} bc \sin (180 - A). \end{aligned}$$

It will be seen in Art. 45, that $\sin(180 - A) = \sin A$. Hence, *the area of a triangle is equal to one-half the product of any two sides and the sine of their contained angle.*

EXAMPLES.

1. Find the area of the triangle in which two sides are 31 ft. and 23 ft. and their contained angle 67° 30′.

$$\text{area} = \frac{31 \times 23}{2} \times \sin 67^\circ 30' = \frac{31 \times 23 \times .92388}{2} = 329.37 \text{ sq. ft.}$$

2. Find area of triangle having sides 125 ft., 80 ft., contained angle 28° 35′.

3. Find area of triangle having sides 125 ft., 80 ft., contained angle 151° 25′. [Draw figures carefully for Exs. 2, 3.]

4. Find area of parallelogram two of whose adjacent sides are 243, 315 yd., and their included angle 35° 40′.

5. Find area of parallelogram two of whose adjacent sides are 14, 15 ft., and included angle 75°.

6. Find area of triangle having sides 40 ft., 45 ft., with an included angle 28° 57′ 18″.

7. Write two other formulas for area ABC, similar to that derived above. Also, derive them.

32. Solution of isosceles triangles. In an isosceles triangle, the perpendicular let fall from the vertex to the base bisects the base and bisects the vertical angle. An isosceles triangle can often be solved on dividing it into two equal right-angled triangles.

EXAMPLES.

1. The base of an isosceles triangle is 24 in. long, and the vertical angle is 48°; find the other angles and sides, the perpendicular from the vertex and the area. Only the steps in the solution will be indicated. Let ABC be an isosceles triangle having base $AB = 24$ in., angle $C = 48°$. Draw CD at right angles to base; then CD bisects the angle ACB and base AB. Hence, in the right-angled triangle ADC, $AD = \frac{1}{2} AB = 12$, $ACD = \frac{1}{2} ACB = 24°$. Hence, angle A, sides AC, DC, and the area, can be found.

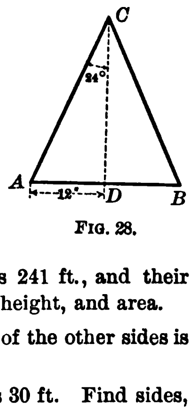

FIG. 28.

2. In an isosceles triangle each of the equal sides is 363 ft., and each of the equal angles is 75°. Find the base, perpendicular on base, and the area.

3. In an isosceles triangle each of the equal sides is 241 ft., and their included angle is 96°. Find the base, angles at the base, height, and area.

4. In an isosceles triangle the base is 65 ft., and each of the other sides is 90 ft. Find the angles, height, and area.

5. In an isosceles triangle the base is 40 ft., height is 30 ft. Find sides, angles, area.

6. In an isosceles triangle the height is 60 ft., one of equal sides is 80 ft. Find base, angles, area.

7. In an isosceles triangle the height is 40 ft., each of equal angles is 63°. Find sides and area.

8. In an isosceles triangle the height is 63 ft., vertical angle is 75°. Find sides and area.

33. Related regular polygons and circles. The knowledge of trigonometry thus far attained, is of service in solving many problems in which circles and regular polygons are concerned. Some of these problems are:

(*a*) Given the length of the side of a regular polygon of a given number of sides, to find its area; also, to find the radii of the inscribed and circumscribing circles of the polygon;

(*b*) To find the lengths of the sides of regular polygons of a given number of sides which are inscribed in, and circumscribed about, a circle of given radius.

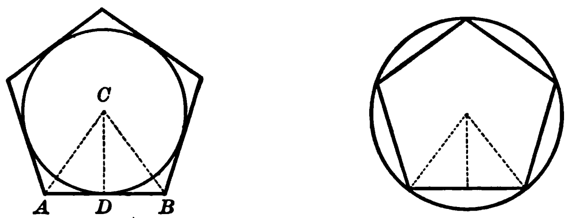

FIG. 29.

For example, let AB (Fig. 29) be a side, equal to $2a$, of a regular polygon of n sides, and let C be the centre of the inscribed circle. Draw CA, CB, and draw CD at right angles to AB. Then D is the middle point of AB.

By geometry, angle $ACD = \frac{1}{2} ACB = \frac{1}{2} \cdot \frac{360°}{n} = \frac{180°}{n}$.

Also, by geometry,

$$\left[\text{angle } DAC = \tfrac{1}{2}\left(\frac{2n-4}{n}\right)90° = \left(\frac{n-2}{n}\right)90°.\right]$$

Hence, in the triangle ADC, the side AD and the angles are known; therefore CD, the radius of the circle inscribed in the polygon, can be found. On making similar constructions, the solution of the other problems referred to above will be apparent. The perpendicular from the centre of the circle to a side of the inscribed polygon is called the apothem of the polygon.

EXAMPLES.

1. The side of a regular heptagon is 14 ft.: find the radii of the inscribed and circumscribing circles; also, find the difference between the areas of the heptagon and the inscribed circle, and the difference between the area of the heptagon and the area of the circumscribing circle.

2. The side of a regular pentagon is 24 ft. Find quantities as in Ex. 1.

3. The side of a regular octagon is 24 ft. Find quantities as in Ex. 1.

4. The radius of a circle is 24 ft. Find the lengths of the sides and apothems of the inscribed regular triangle, quadrilateral, pentagon, hexagon, heptagon, and octagon. Compare the area of the circle and the areas of these regular polygons; also compare the perimeters of the polygons and the circumference of the circle.

5. For the same circle as in Ex. 4, find the lengths of the sides of the circumscribing regular figures named in Ex. 4. Compare their areas and perimeters with the area and circumference of the circle.

6. If a be the side of a regular polygon of n sides, show that R, the radius of the circumscribing circle, is equal to $\frac{1}{2}a \operatorname{cosec} \frac{180°}{n}$; and that r, the radius of the circle inscribed, is equal to $\frac{1}{2}a \cot \frac{180°}{n}$.

7. If r be the radius of a circle, show that the side of the regular inscribed polygon of n sides is $2r\sin\frac{180°}{n}$; and that the side of the regular circumscribing polygon is $2r\tan\frac{180°}{n}$.

8. If a be the side of a regular polygon of n sides, R the radius of the circumscribing circle, and r the radius of the circle inscribed, show that area of polygon $= \frac{1}{4}na^2\cot\frac{180°}{n} = \frac{1}{2}nR^2\sin\frac{360°}{n} = nr^2\tan\frac{180°}{n}$.

34. Solution of oblique triangles. Since an oblique triangle can be divided into right-angled triangles by drawing a perpendicular from a vertex to the opposite side, it may be expected that knowledge concerning the solution of right-angled triangles will be of service in solving oblique triangles. This expectation will not be disappointed. An examination, which it is advisable for the student to make before proceeding farther, will show that all the sets of *data* from which a definite triangle can be drawn are those indicated in (1)–(4) below. *The ability to make the following geometrical constructions is presupposed:*

(1) To draw a triangle on being given two of its angles and a side opposite to one of them;

(2) To draw a triangle on being given two of its sides and an angle opposite to one of them;

(3) To draw a triangle on being given two of its sides and their included angle;

(4) To draw a triangle on being given its three sides.

In what follows, only the steps in the solutions will be indicated. The examples that are worked may be saved, so that the amount of labor required by the method of solution shown here can be compared with the amount required by another method which will be described later.

There are four cases in the solution of oblique triangles; these cases correspond to the four problems of construction stated above.

Case I. *Given two angles and a side opposite to one of them.*

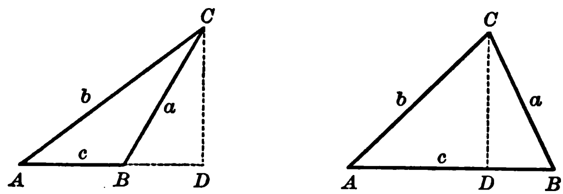

Fig. 30.

In ABC let A, B, a be known. Angle C and sides b, c are required. From C draw CD at right angles to AB or AB produced.

In triangle CBD, angle CBD and side CB are known. $\therefore$ BD and DC can be found.

Then, in triangle ACD, side DC and angle A are known. $\therefore$ AC and AD can be found.

Side $AB = AD - BD$ when B is obtuse, and $AB = AD + DB$ when B is acute.

$$\text{Angle } C = 180° - (A + B).$$

Another method of solution is given in Art. 55.

EXAMPLES.

1. Ex. 1, Art. 55.

2. Ex. 2, Art. 55.

3. Ex. 2, Art. 60.

4. Other Exs. in Arts. 55, 60.

Case II. *Given two sides and an angle opposite to one of them.*

N.B. The first part of the text in Art. 56 should be read at this time.

Let (Fig. 30) AC, BC, angle A be known. [In a certain case, as shown in Art. 56, two triangles can be drawn which satisfy the given conditions.] From C draw CD at right angles to AB or AB produced.

In ACD, AC and A are known. $\therefore$ AD, DC, angle ACD, can be found.

Then, in BCD, BC and CD are known. $\therefore$ BD, angle DBC, can be found.

In one figure, $AB = AD - BD$, angle $ABC = 180° - CBD$.

In other figure, $AB = AD + DB$. In both figures,

$$\text{angle } ACB = 180° - (CAB + ABC).$$

Another method of solution is given in Art. 56.

EXAMPLES.

1. Ex. 1, Art. 56.
2. Ex. 2, Art. 56.
3. Ex. 1, Art. 60.
4. Other Exs. in Arts. 56, 60.

CASE III. *Given two sides and their included angle.*

In ABC let b, c, A be known. Side a, B, C are required. From C draw CD at right angles to AB or AB produced.

In ACD, AC and angle A are known. $\therefore$ CD and AD can be found.

Then, in triangle CDB, CD is now known, and $BD = AD - AB$ or $AB - AD$. $\therefore$ Angle CBD can be found. Angle $ABC = 180° - CBD$ in figure on the left. Angle $ACB = 180° - (A + B)$.

Another method of solution is given in Art. 57.

EXAMPLES.

1. Ex. 1, Art. 57.
2. Ex. 2, Art. 57.
3. Ex. 1, Art. 61.
4. Other Exs. in Arts. 57, 61.
5. Ex. in Art. 21.

CASE IV. *Given the three sides.*

In ABC let a, b, c be known. The angles A, B, C are required. From any vertex C draw CD at right angles to AB or AB produced.

$$CD^2 = b^2 - AD^2; \tag{1}$$

also,
$$CD^2 = a^2 - DB^2 = a^2 - (c - AD)^2.$$

$$\therefore\ b^2 - AD^2 = a^2 - (c - AD)^2.$$

$$\therefore\ AD = \frac{b^2 + c^2 - a^2}{2c}. \tag{2}$$

Also, $DB = AD - c$ (one figure), or $c - AD$ (other figure).
Hence, in ACD, AC, AD are known. $\therefore$ A can be found.
Also, in CDB, CB, DB are known. $\therefore$ B can be found.

$$C = 180^\circ - (A + B).$$

Another method of solution is given in Art. 58.

EXAMPLES.

1. Ex. 1, Art. 58. **2.** Ex. 2, Art. 58.

3. Ex. 1, Art. 62. **4.** Other Exs. in Arts. 58, 62.

5. Solve some of the problems in Art. 63 by means of right-angled triangles.

34 a. The area of a triangle in terms of the sides. (See Fig. 30.) From (1), (2), Case IV., Art. 34,

$$CD^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2 = \frac{4c^2b^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

$$= \frac{[2cb + b^2 + c^2 - a^2][2cb - (b^2 + c^2 - a^2)]}{4c^2}$$

$$= \frac{[(b + c)^2 - a^2][a^2 - (b - c)^2]}{4c^2}$$

$$= \frac{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}{4c^2}.$$

Let $$a + b + c = 2s;$$

then $$2(s - a) = a + b + c - 2a = -a + b + c.$$

Similarly, $2(s - b) = a - b + c$; $2(s - c) = a + b - c$.

Then $$CD = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

$$\therefore \text{ Area } ABC = \tfrac{1}{2} AB \cdot CD = \sqrt{s(s - a)(s - b)(s - c)}.^*$$

Ex. Find the areas of the triangles in Exs. Case IV., Art. 34. Check the results by finding the areas by the method of Art. 31.

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* This is sometimes known as *Hero's Formula* for the area of a triangle. It was discovered by Hero (or Heron) of Alexandria, who lived about 125 B.C., and placed engineering and land surveying on a scientific basis.

34 *b*. Distance and dip of the visible horizon.

Let C be the centre of the earth, and let the radius be denoted by r.

Let P be a point above the earth's surface, and let its height PL be denoted by h.

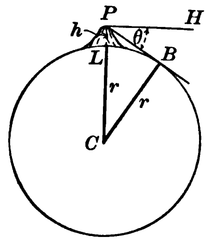

FIG. 30a.

Join P, C; draw PB from P to any point in the visible horizon; draw the horizontal line PH in the same plane with PC, PB. Then angle HPB is called the dip of the horizon. By geometry,

$$\text{angle } PBC = 90°.$$

$$PB^2 = PC^2 - CB^2 = (r+h)^2 - r^2 = 2\,rh + h^2.$$

Since h^2 is very small compared with $2\,rh$,

$$PB = \sqrt{2\,rh} \text{ approximately.}$$

Take $r = 3960$ mi., and let h be measured in feet. Then $\frac{h}{5280} =$ height of P in miles.

$$\therefore\ PB = \sqrt{2 \times 3960 \times \frac{h}{5280}} \text{ mi.} = \sqrt{\tfrac{3}{2}\,h} \text{ mi.}$$

Hence, the distance of the horizon in miles is approximately equal to the square root of one and one-half times the height in feet.

EXAMPLES.

1. A man whose eye is 6 ft. from the ground is standing on the sea-shore. How far distant is his horizon?

$$\text{Distance} = \sqrt{\tfrac{3}{2} \times 6} \text{ mi.} = 3 \text{ mi.}$$

2. Find the greatest distance at which the lamp of a lighthouse can be seen, the light being 80 ft. above the sea level.

3. Find the height of the lamp of a lighthouse above the sea level when it begins to be seen at a distance of 12 mi.

4. From the top of a cliff, 40 ft. above the sea level, the top of a steamer's funnel which is known to be 30 ft. above the water is just visible. What is the distance of the steamer?

5. Find the distance and dip of the horizon at the top of a mountain 3000 ft. high?

6. Find the distance and dip of the horizon at the top of a mountain $2\frac{1}{2}$ mi. high.

34 *c*. Examples in the measurement of land. In order to find the area of a piece of ground, a surveyor measures distances and angles sufficient to provide *data* for the computation. An account of his method of doing this, and of his arrangement of the *data* and the results in a simple, clear, and convenient form, belongs to special works on surveying. This article merely gives some examples which can be solved without any knowledge of professional details. The various rules for finding the area, of a triangle and a trapezoid, are supposed to be known. In solving these problems, the student should make the plotting or mapping an important feature of his work.

The Gunter's chain is generally used in measuring land. It is 4 rods or 66 feet in length, and is divided into 100 links.

An acre = 10 square chains = 4 roods = 160 square rods or poles. The points of the compass have been explained in Art. 30.

EXAMPLES.

1. A surveyor starting from a point A runs S. 70° E. 20 chains, thence N. 10° W. 20 chains, thence N. 70° W. 10 chains, thence S. 20° W. 17.32 chains to the place of beginning. What is the area of the field which he has gone around?

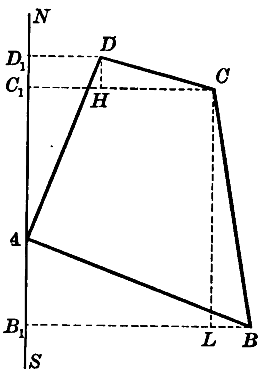

FIG. 30b.

Make a plot or map of the field, namely, $ABCD$. Here, AB represents 20 chains, and the bearing of B from A is S. 70° E. BC represents 20 chains, and the bearing C from B is N. 10° W., and so on. Through the most westerly point of the field draw a north-and-south line. This line is called *the meridian.* In the case of each line measured, find the distance that one end of the line is east or west from the other end. This easting or westing is called *the departure of the line.* Also find the distance that one end of the line is north or south of the other end. This northing or southing is called *the latitude of the line.* For example, in Fig. 30 *b*, the departures of AB, BC, CD, DA, are B_1B, BL, CH, DD_1, respectively; the latitudes of the boundary lines are AB_1, B_1C_1, C_1D_1, D_1A, respectively. It should be observed (Art. 36) that the algebraic sum of the departures of the boundary lines is zero, and so also is the algebraic sum of their latitudes. The following formulas are easily deduced:

Departure of a line = length of line × sine of the bearing;

Latitude of a line = length of line × cosine of the bearing.

By means of the departures, *the meridian distance of a point* (*i.e.* its distance from the north-and-south line) can be found. Thus the meridian distance of C is C_1C, and $C_1C = D_1D + HC$. Hence in Fig. 30 b, AB_1, B_1B, B_1C_1, C_1C, C_1D_1, D_1D can be computed. Now

$$\text{area } ABCD = \text{trapezoid } D_1DCC_1 + \text{trapezoid } C_1CBB_1 - \text{triangle } ADD_1 - \text{triangle } ABB_1.$$

The areas in the second member can be computed; it will be found that area $ABCD = 26$ acres.

NOTE. Sometimes the bearing and length of one of the lines enclosing the area is also required. These can be computed by means of the latitudes and departures of the given lines. The formulation of a simple rule for doing this is left as an exercise to the student.

2. In Ex. 1, deduce the length and bearing of DA from the lengths and bearings of AB, BC, CD.

3. A surveyor starts from A and runs 4 chains S. 45° E. to B, thence 5 chains E. to C, thence 6 chains N. 40° E. to D. Find the distance and bearing of A from D; also, the area of the field $ABCD$. Verify the results by going around the field in the reverse direction, and calculating the length and bearing of BA from the lengths and directions of AD, DC, CB.

4. A surveyor starts from one corner of a pentagonal field, and runs N. 25° E. 433 ft., thence N. 76° 55′ E. 191 ft., thence S. 6° 41′ W. 539 ft., thence S. 25° W. 40 ft., thence N. 65° W. 320 ft. Find the area of the field. Deduce the length and direction of one of the sides from the lengths and directions of the other four.

5. From a station within a hexagonal field the distances of each of its corners were measured, and also their bearings; required its plan and area, the distances in chains and the bearings of the corners being as follows: 7.08 N.E., 9.57 N. ½ E., 7.83 N.W. by W., 8.25 S.W. by S., 4.06 S.S.E. 7° E., 5.89 E. by S. 3½° E.

35. Summary. Chapter II. was concerned with defining and investigating certain ratios inseparably connected with (acute) angles, and attention was directed to the tables of these ratios and their logarithms. In Chap. III. it was shown how these definitions and tables can be used in finding parts of a right-angled triangle when certain parts are known. In Chap. IV. the knowledge gained in Chap. III. was employed in the solution of some of the many problems in which right-angled triangles appear. In Art. 34 it has been seen that this knowledge can serve for the solution of oblique triangles. It follows, then, that it can serve for the solution of problems in which oblique triangles appear, and, accordingly, for the solution of *all problems* *involving the measurement of straight lines only.* Consequently, the student is now able, without any additional knowledge of trigonometry, to solve the numerical problems in Chaps. VII., VIII. It is thus apparent that even a slight acquaintance with the ratios defined in Chap. II. has greatly increased the learner's ability to solve useful practical problems.

Oblique triangles can sometimes be solved in a more elegant manner than that pointed out in Art. 34. In order to show this, further consideration of angles and the trigonometric ratios is necessary. Consequently, in Chap. V. some important additions are made to the idea of a straight line and the idea of an angle; the *trigonometric ratios are defined in a more general way,* namely, for all angles, instead of for acute angles only, and *the principal relations of these ratios are deduced.* Chapter VI. treats of *the ratios of two angles in combination.* While it is necessary to consider these matters before proceeding to the solution of oblique triangles given in Chap. VII., it should be said that the knowledge that will be gained in Chaps. V., VI., VII., is necessary and important for other purposes besides the solution of triangles. In fact, the latter is one of the least important of the results obtained in these chapters.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chap. IV. will be found at page* 184.

CHAPTER V.

TRIGONOMETRIC RATIOS OF ANGLES IN GENERAL.

36. Directed lines. Let MN be a line unlimited in length in the directions of both M and N. Suppose that a point starts at P and moves along this line for some given distance. In order to mark where the point stops, it is necessary to know, not only this distance, but also the *direction* in which the point has moved from P. This direction may be indicated in various ways; by saying, for instance, that the point moves *toward the right* from P, or *toward the left* from P; that the point moves *toward* N, or *toward* M; that the point moves *in the direction of* N, or *in the direction of* M; and so on. Mathematicians, engineers, and others

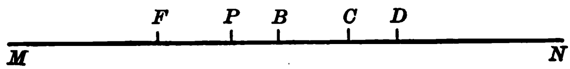

FIG. 31.

have agreed to use a particular method (and this practically comes to the adoption of a *particular rule*) for indicating the two opposite directions in which a point can move along a line, or in which distances along a line can be measured. This *convention*, or rule which has been adopted for the sake of convenience, is as follows:

Distances measured along a line, or along parallel lines, in one direction shall be called ***positive*** *distances, and shall be denoted by the sign* $+$*; distances measured in the opposite direction shall be called* ***negative*** *distances, and shall be denoted by the sign* $-$.

The convenience of this custom, fashion, or rule, will become apparent in the examples that follow.* In Fig. 31 let distances

* Advances in mathematics have often depended upon the introduction of a good custom which has at last been universally adopted and made a rule. Thus, for example, the custom of using exponents to show the power to which a quantity is raised, which was first introduced in the first half of the sixteenth century, and made gradual progress until its final establishment in the latter half of the seventeenth century, has been of great service in aiding the advances of algebra.

measured in the direction of N be taken *positively;* then distances measured in the direction of M will be taken *negatively.* On directed lines the direction in which a line is measured, or in which a point moves on a line, is indicated by *the order of the letters* naming the line. Thus, for example, if a point moves from B to C, the distance passed over is read BC. In this reading, the starting point is indicated by the first letter B, and the stopping point, by the last letter C. After the same fashion, CB means the distance from C to B. If, for instance, there are 3 units of length between B and C, then $BC = +3$, $CB = -3$.

EXAMPLES.

1. Suppose a point (Fig. 31) moves from P to B, thence to C, thence to D, thence to F. Let the number of units of length between P and B, B and C, C and D, F and D, be 2, 3, 2, 10, respectively. The point starts at P and stops at F; hence the distance from the starting point to the stopping point is PF. In this case the point's trip from P to F is made in several steps as indicated above. That is, on properly indicating the lines passed over,

$$\begin{aligned} PF &= PB + BC + CD + DF \\ &= 2 + 3 + 2 - 10 \quad [\because FD = +10, \text{ then } DF = -10.] \\ &= -3. \end{aligned}$$

This shows that the final position of the moving point is three units to the left of P. This example also shows one great *convenience of the rule of signs in measurement*, namely, that by attending to this rule and to the proper naming of the lines passed over by a moving point, *one immediately obtains the result of the successive movements.*

NOTE. In the following examples, in lines that lie east and west, let measurements toward the east be taken positively; in lines that lie north and south, let measurements toward the north be taken positively.

2. A man travelling on an east and west line goes east 20 mi., then east 16 mi., then west 18 mi., then east 30 mi. What is his final distance from the starting point? [Draw a figure, and indicate the successive trips by letters.]

3. A man travelling on an east and west line goes west 20 mi., then east 10 mi., then east 25 mi., then east 30 mi., then west 45 mi. Do as in Ex. 2.

4. A man travelling on a north and south line goes north 100 mi., then south 60 mi., then south 110 mi., then north 200 mi., then north 15 mi., then south 247 mi. Do as in Ex. 2.

37. Trigonometric definition of an angle. Angles unlimited in magnitude. Positive and negative angles. In books on plane geometry a plane angle is defined in various ways, namely, as the inclination of two lines to one another, which meet together, but are not in the same direction; or, as the figure formed by two straight lines drawn from the same point; or, as the amount of divergence of two lines which meet in a point, or would meet if produced; or, as the opening between two straight lines which meet; or, as the difference in direction of two lines which meet; and so on. In these definitions an angle is always regarded as less than two right angles. A definition according to which angles are less restricted, is adopted in trigonometry.

Trigonometric definition of an angle. The angle between two lines which intersect is the **amount of turning** which a line revolving about their point of intersection makes, when it begins its revolution at the position of one of the two lines and stops in the position of the other line. Thus, for example, the angle between OX and OQ is the amount of turning which is made by a line OP revolving about O when OP starts revolving from the position OX and stops its revolution at the position OQ. The line OX at which the revolution begins, is called the **initial line**; the line OQ at which the revolution ends, is called the **terminal line**; when the turning line OP has reached the terminal position OQ, OP is said to have *described the angle* XOQ.

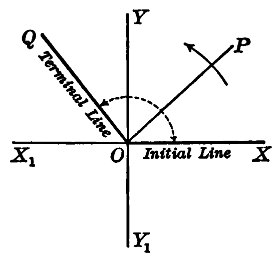

FIG. 32.

Let YOY_1 be at right angles to X_1OX. When OP has revolved until it lies in the position OY, it has described a right angle, or 90°; when it has revolved until it lies in the position OX_1, it has described two right angles, or 180° (this is usually termed "a straight angle" or "a flat angle"); when OP keeps on turning until it is in the position OY_1, it has described three right angles, or 270°; when OP has again reached the position OX, that is, when it has made one complete revolution, it has described four right angles, or 360°.

Angles unlimited in magnitude. Now OP may start revolving from OX, make one complete revolution, continue to revolve, and *then* cease revolving when it has again reached the position OQ. This is indicated in Fig. 33. Or, OP may make two complete revolutions before it comes to rest in the position OQ; or, it may make three revolutions, or four, or as many as one please, before ceasing its revolution at the position OQ. An angle of 360° is described each time that OP makes a complete revolution, and OP can make as many revolutions as one please. According to the trigonometric definition of an angle, therefore, **angles are unlimited in magnitude.**

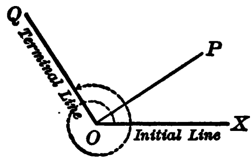

FIG. 33.

Moreover, when this definition of an angle is adopted, **the same figure can represent an infinite number of different angles.** Any two of these angles differ from each other by a whole number of complete revolutions. For instance, Fig 34 may represent 60°, 360°+60° or 420°, $2 \cdot 360° + 60°$ or 780°, $3 \cdot 360° + 60°$ or 1140°, ⋯, $n \cdot 360° + 60°$, in which n denotes any whole number. *Any two of these angles differ by a multiple of* 360°. Angles which have the same initial and terminal lines may be called **coterminal angles.**

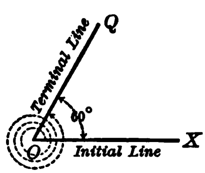

FIG. 34.

Positive and negative angles. The revolving line OP (Fig. 32) may revolve about O in the same direction as that in which the hands of a watch revolve, or it may revolve in the opposite direction. The following *convention* (see Art. 36) has been adopted for the sake of distinguishing these two opposite directions:

When the turning line revolves in a **counter-clockwise direction,** *the angles described are said to be* **positive,** *and are given the* ***plus*** *sign; when the turning line revolves in a* **clockwise direction,** *the angles described are said to be* **negative,** *and are given the* ***minus*** *sign.*

Thus, for example, Fig. 34 represents the angles $+ 60°$, $- 300°$; further, this figure represents the angles $60° \pm n \cdot 360°$, in which n denotes any whole number. The angle $-300°$ is included in these angles, for, on putting -1 for n, there is obtained $60° - 360°$, *i.e.* $- 300°$. (Negative angles are also unlimited in magnitude.)

As in the case of lines, *the sign of an angle can be denoted by the order of the letters used in naming the angle.* Thus XOQ denotes the angle formed by revolving OX toward OQ, and QOX denotes the angle formed by revolving OQ toward OX. Accordingly, $QOX = -XOQ$.

Quadrants. In Fig. 32, XOY, YOX_1, X_1OY_1, Y_1OX, are called *the first, second, third, and fourth quadrants,* respectively. When the turning line ceases its revolution at some position between OX and OY, the angle described is said to be *an angle in the first quadrant;* when the final position of the turning line is between OY and OX_1, the angle described is said to be *in the second quadrant;* and so on for the third and fourth quadrants.

For example, the angles 30°, – 345°, 395°, 725° are all in the first quadrant; the angles – 60°, 340°, 710° are all in the fourth quadrant; the angle – 225° is in the second quadrant, and the angle 225° is in the third quadrant.

NOTE. While all acute angles are in the first quadrant, all angles which are in the first quadrant are not acute.

EXAMPLES.

NOTE. When it is necessary, the number of revolutions and their direction may be indicated on the figure in the manner shown in Fig. 34.

Lay off the following angles with the protractor: In the case of each angle name the least positive angle that has the same terminal line. Name the quadrants in which the angles are situated. In the case of each angle name the four smallest positive angles that have the same terminal line.

1. 137°, 785°, 321°, 930°, 840°, 1060°, 1720°, 543°, 3657°.

2. – 240°, – 337°, – 967°, – 830°, – 750°, – 1050°, – 7283°.

3. – 47° + 230° + 37°, 420° – 470° + 210° – 150°, 230° – 47° + 37°, 230° + 37° – 47°.

38. Supplement and complement of an angle. *The supplement of an angle* is that angle which must be *added to it* in order to make two right angles, or 180°; *the complement of an angle* is that angle which must be *added to it* in order to make one right angle, or 90°. Thus, if A be *any* angle, then

$$\text{supplement of angle } A = 180° - A,$$

$$\text{complement of angle } A = 90° - A.$$

EXAMPLES.

1. What are the complements and supplements of 40°, 227°, – 40°?

$$\text{complement of } 40° = 90° - 40° = 50°;$$
$$\text{supplement of } 40° = 180° - 40° = 140°.$$
$$\text{complement of } 227° = 90° - 227° = -137°;$$
$$\text{supplement of } 227° = 180° - 227° = -47°.$$
$$\text{complement of } -40° = 90° - (-40°) = 130°;$$
$$\text{supplement of } -40° = 180° - (-40°) = 220°.$$

2. By means of a figure verify the results obtained in Ex. 1.

3. What are the complements of – 230°, 150°, – 40°, 340°, 75°, 83°, 12°, – 295°, – 324°, 200°, 240°, – 110°, – 167°?

4. What are the supplements of the angles in Ex. 3?

5. Verify the results in (3), (4), by drawing figures.

39. The convention of signs on a plane. Articles 36, 37 contain statements of the conventions adopted regarding *the algebraic signs to be given to distances* measured on parallel straight lines, *and to angles* described by the revolution of a turning line. A figure, such as Figs. 32, 35, will be frequently used in the articles that follow. In this figure, OX is the initial line, the turning line revolves about O, and YOY_1 is at right angles to X_1OX. *The following convention has been adopted regarding the lines which will be used:*

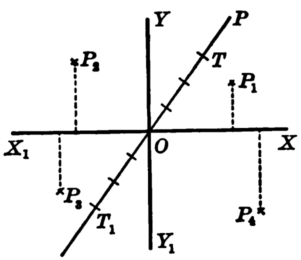

FIG. 35.

Horizontal lines measured in the direction of X are taken positively;
Horizontal lines measured in the direction of X_1 are taken negatively;
Vertical lines measured upward are taken positively;
Vertical lines measured downward are taken negatively.

The distance of points, such as P_1, P_2, P_3, P_4, from X_1X, is always measured *from* X_1X *toward* the points.

Any turning line (or oblique line) as OP is measured positively from O toward the *end* of the turning line which lies in the direction of X from O when the turning line coincides with the initial line. Thus a distance $+3$ on OP will terminate at T, distant 3 units from O, and a distance -3 on OP will terminate at T_1, distant 3 units from O, but in the direction opposite to the former. This is sometimes briefly expressed in the words: *the turning line carries its positive direction with it in its revolution.*

40. General definition of the trigonometric ratios. The remarks in this article apply to each of the four figures below. In each figure, O is the point about which the angle is described, OX is the initial line, and OP is the terminal line. The first figure represents any angle in the first quadrant; the second figure represents any angle in the second quadrant; the third figure, any angle in the third quadrant; and the fourth figure, any angle in the fourth quadrant. In each figure the angle will be called A.

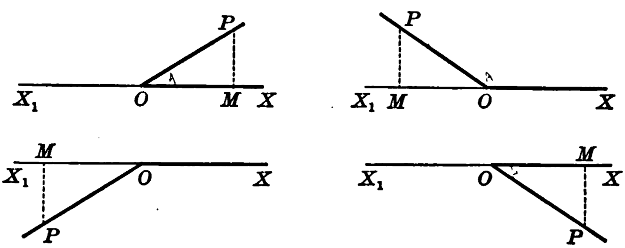

FIG. 36.

Let P be any point in OP, the terminal line of any angle A. From P draw PM at right angles to the initial line OX, or to the initial line produced in the negative direction. In each figure, **OM** is the distance measured along X_1OX *from* the point O *to* the foot of the perpendicular MP, and **MP** is the distance *from* X_1OX *to* the point P. Following are the definitions of the trigonometric ratios; these definitions apply to the angles represented in Fig. 36, and, accordingly, to all angles whatsoever. [*Particular attention should be paid to the order of the letters used in naming the lines, for this order indicates the direction in which the line is measured.* See Art. 36.]

The ratio $\frac{MP}{OP}$ is called the *sine* of the angle A.

The ratio $\frac{OM}{OP}$ is called the *cosine* of the angle A.

The ratio $\frac{MP}{OM}$ is called the *tangent* of the angle A.

The ratio $\frac{OM}{MP}$ is called the *cotangent* of the angle A.

The ratio $\frac{OP}{OM}$ is called the *secant* of the angle A.

The ratio $\frac{OP}{MP}$ is called the *cosecant* of the angle A.

These definitions may be briefly stated:

$$\left.\begin{array}{lll} \sin A = \frac{MP}{OP}. & \tan A = \frac{MP}{OM}. & \sec A = \frac{OP}{OM}. \\ \cos A = \frac{OM}{OP}. & \cot A = \frac{OM}{MP}. & \operatorname{cosec} A = \frac{OP}{MP}. \end{array}\right\} \quad (1)$$

Inspection will show that the definitions of the trigonometric ratios for acute angles given in Art. 12, are in accordance with these general definitions.

N.B. The *projection* definitions of the trigonometric ratios are given in Note B, Appendix.

41. The algebraic signs of the trigonometric ratios for angles in the different quadrants. Figures 36 show that if the angle A is in the first, second, third, fourth quadrants, then the algebraic sign of MP is $+$, $+$, $-$, $-$, respectively, and the algebraic sign of OM is $+$, $-$, $-$, $+$, respectively. As stated in Art. 39, OP is always taken positively. Hence, on paying regard to the algebraic signs of OM, MP, OP, in the several quadrants, it will be seen that the ratios of the angles in these quadrants are positive or negative, as indicated in the following table:

QUADRANT.	I.	II.	III.	IV.
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−
Cotangent	+	−	+	−
Secant	+	−	−	+
Cosecant	+	+	−	−

EXAMPLES.

The student is advised to preserve his work on these examples. If he regards his results attentively, he will probably discover some useful facts, and *be able to deduce some useful theorems*, concerning angles in general. Any preceding results, such as those in Art. 15, may be used as an aid in solving these exercises.

1. Find the ratios of 945°.

$$945^\circ = 2 \times 360^\circ + 225^\circ.$$

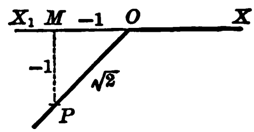

FIG. 37.

$\therefore$ OP, the terminal line of angle 945°, has the position shown in Fig. 37. For this position of the terminal line, OM and MP are both negative. As shown in Art. 15, the lines OM, MP, OP, in this figure are respectively proportional to 1, 1, $\sqrt{2}$. These are indicated on the figure *with their proper algebraic signs*. It is immediately apparent that

$$\sin 945^\circ = -\frac{1}{\sqrt{2}}, \quad \cos 945^\circ = -\frac{1}{\sqrt{2}}, \quad \tan 945^\circ = +1, \quad \cot 945^\circ = +1,$$

$$\sec 945^\circ = -\sqrt{2}, \quad \operatorname{cosec} 945^\circ = -\sqrt{2}.$$

2. Construct, and find the ratios of, 420°, 780°, 1140°.

3. Construct, and find the ratios of, 120°, 480°, 240°, 600°, − 60°, 300°, 660°, − 720°.

4. Construct, and find the ratios of, 150°, 410°, 210°, −150°, 330°, −390°.

5. Construct, and find the ratios of, 45°, 765°, 135°, − 225°, 225°, 585°, − 405°, 1035°.

6. Construct, and find the ratios of, − 754°, 487°, − 245°.

7. Compare the ratios of 90° − 30°, 90° − 60°, 90° − 45°, 90° − 135°, 90° − 240°, 90° − 300°, with the ratios of 30°, 60°, 45°, 135°, 240°, 300°, respectively.

8. Compare the ratios of $90° + 30°$, $90° + 60°$, $90° + 45°$, $90° + 135°$, $90° + 240°$, $90° + 300°$, with the ratios of $30°$, $60°$, $45°$, $135°$, $240°$, $300°$, respectively.

9. Compare the ratios of $180° - 30°$, $180° - 60°$, $180° - 45°$, $180° - 135°$, $180° - 240°$, $180° - 300°$, with the ratios of $30°$, $60°$, $45°$, $135°$, $240°$, $300°$, respectively. So, also, the ratios of $-30°$, $-60°$, $-45°$, etc.

10. Are any general relations *indicated* by the results of Exs. 7, 8, 9? If so, state these relations. Try to prove them.

42. To represent the angles geometrically when the ratios are given. In constructing the angles in this article it is necessary to bear in mind that, according to the definitions given in Art. 40:

When MP is *positive*, it can be drawn in the *first and second quadrants;*

When MP is *negative*, it can be drawn in the *third and fourth quadrants;*

When OM is *positive*, it is to be drawn in the direction OX;

When OM is *negative*, it is to be drawn in the direction OX_1;

and OP is to be taken positively.

EXAMPLES.

1. Represent by a figure the angles which have sines equal to $\frac{3}{4}$. Calculate their other ratios. Let A denote an angle whose sine is $\frac{3}{4}$; *i.e.* let $\sin A = \frac{3}{4}$. But $\sin A = \frac{MP}{OP}$ (Art. 40). Hence, $MP : OP = 3 : 4$; and if $OP = 4$, then $MP = 3$. Now, MP can be drawn positively in both the first and second quadrants. Hence the problem amounts to finding a point in the first quadrant and a point in the second quadrant, each at a distance 4 from O and a distance 3 from X_1OX. The result is indicated in Fig. 38. The student can make the construction for himself. The angles having sines equal to $\frac{3}{4}$, accordingly, include *all the angles* which have OP for a terminal line, *and all the angles* which have OP_1 for a terminal line. By Art. 37 each of these two sets of angles consists of an infinite number of angles, any two of which differ from one another by a whole number, positive or negative, of complete revolutions. A general algebraic expression which includes all these angles, is deduced in Art. 85.

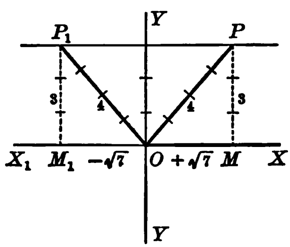

Fig. 38.

Figure 38 shows that for angles having OP for a terminal line, cosine is $\frac{\sqrt{7}}{4}$, tangent is $\frac{3}{\sqrt{7}}$, etc.; and that for angles having OP_1 for a terminal line, cosine is $\frac{-\sqrt{7}}{4}$, tangent is $-\frac{3}{\sqrt{7}}$, etc. Since the given sine is positive, it is apparent that the angles required, must be in the first and second quadrants. (See Art. 41.)

2. Represent by a figure all the angles which have tangents equal to $-\frac{3}{4}$. Let A denote an angle whose tangent is $-\frac{3}{4}$; *i.e.* let $\tan A = -\frac{3}{4}$. [This may be written, $\frac{+3}{-4}$ or $\frac{-3}{+4}$.] But $\tan A = \frac{MP}{OM}$ (Art. 40). Hence, if $MP = 3$, then $OM = -4$, and if $MP = -3$, then $OM = 4$. When MP is positive and OM is negative, OP can lie only in the second quadrant. When MP is negative and OM is positive, OP can lie only in the fourth quadrant. Figure 39 represents the angles. The student can make the construction for himself. Thus, the angles whose tangents are equal to $-\frac{3}{4}$, consist of the set of angles, infinite in number, which have OP for a terminal line, and the set of angles, infinite in number, which have OP_1 for a terminal line.

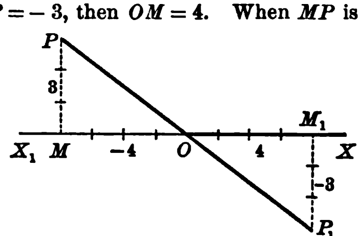

FIG. 39.

3. Calculate the other ratios of the angles in Ex. 2.

4. Represent geometrically all the angles whose cosine is $\frac{3}{4}$. Calculate their other ratios.

5. So, also, when the cosine is $-\frac{2}{3}$.

6. So, also, when the tangent is $\frac{4}{3}$. [Note. $\frac{4}{3} = \frac{+4}{+3} = \frac{-4}{-3}$.]

7. So, also, when the sine is $-\frac{3}{5}$.

8. So, also, when the secant is $\frac{5}{3}$.

9. So, also, when the secant is $-\frac{4}{3}$.

10. So, also, when the cosecant is -2; $\frac{7}{4}$.

N.B. The student is now strongly recommended to delay the reading of the next article until after he has reviewed the properties stated in Art. 13, and, if possible, determined what are *the correct corresponding statements for angles in general.*

43. Connection between angles and the trigonometric ratios. For the same terminal position of the revolving line OP each of the ratios, $\frac{MP}{OP}$, etc., in (1) Art. 40, is always the same, no matter where the point P is taken on the revolving line. Thus, for example, let any other point P_1 (Fig. 40 a) be taken on the ter-

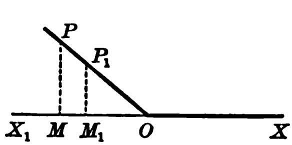

FIG. 40a.

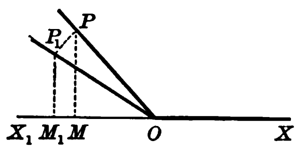

FIG. 40b.

minal line, and let P_1M_1 be drawn perpendicular to X_1OX. Then $\frac{MP}{OP}=\frac{M_1P_1}{OP_1}$. That is, the sine of any angle whose terminal line is OP, has a fixed definite value. The same can be shown for the other ratios. Hence, as already shown in Art. 13 for angles between 0° and 90°,

(1) *To each angle there corresponds but one value of each trigonometric ratio.*

Now let OP revolve a little from OP into the position OP_1 (Fig. 40 b). For convenience keep OP_1 equal to OP. Draw P_1M_1 at right angles to X_1OX. Then,

$$\text{the sine of the angle whose terminal line is } OP = \frac{MP}{OP},$$

$$\text{and the sine of the angle whose terminal line is } OP_1 = \frac{M_1P_1}{OP_1}.$$

Since M_1P_1 is not equal to MP, it follows that these two sines are unequal. Hence, the sine of an angle changes when the angle changes. The same can be shown for the other ratios. Hence,

(2) *The ratios of an angle change when the angle changes.*

The variation in the ratios as the angle increases, is discussed in Art. 77.

Ex. 1. In the above, OP_1 is taken equal to OP. Why does this not affect the generality of the deduction?

Ex. 2. Trace the changes in the trigonometric ratios as the turning line revolves from 0° to 360°. Compare your results with those of Art. 15 C, and those given in the table at the end of Art. 77.

It has been shown in Art. 37 that an infinite number of angles have the same terminal line. It follows that in each of the figures in Art. 40, $\frac{MP}{OP}$ is the sine of an infinite number of angles. The same is true in the case of the other ratios. Moreover, the geometrical solutions in Art. 42 show that there are *two sets of angles corresponding to each given ratio*, and that *each set is infinite in number*, and has a particular terminal line. Hence,

(3) *To each value of a trigonometric ratio there corresponds an infinite number of angles.*

NOTE. The student will see, by turning to Arts. 84–87, that all angles which have the same sine, can be given in a simple formula, and that the same fact is true in the case of each of the other ratios. The deduction of these formulas, while easily possible at this place, is postponed in order to permit the early completion of the solution of triangles.

44. Relations between the trigonometric ratios of an angle. The relations between the trigonometric ratios of *an acute angle* were set forth in Art. 18. It will now be shown that these relations also hold for the ratios *of any angle.*

A. Inspection of the definitions (1), Art. 40, shows *the reciprocal relations,* namely:

$$\mathbf{\sin A \operatorname{cosec} A = 1; \quad \cos A \sec A = 1; \quad \tan A \cot A = 1.} \qquad (1)$$

B. In each of the figures in Art. 40,

$$\tan A = \frac{MP}{OM} = \frac{\frac{MP}{OP}}{\frac{OM}{OP}} = \frac{\sin A}{\cos A}; \quad \cot A = \frac{OM}{MP} = \frac{\frac{OM}{OP}}{\frac{MP}{OP}} = \frac{\cos A}{\sin A}. \qquad (2)$$

C. In each of the figures in Art. 40,

$$\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2.$$

On dividing both members of this equation by $\overline{OP}^2$, $\overline{OM}^2$, $\overline{MP}^2$, in turn, and following the same process as that adopted in Art. 18, it results that

$$\mathbf{\sin^2 A + \cos^2 A = 1; \quad \sec^2 A = 1 + \tan^2 A; \quad \operatorname{cosec}^2 A = 1 + \cot^2 A.} \qquad (3)$$

From the first of relations (3) it follows that

$$\cos A = \pm\sqrt{1 - \sin^2 A}.$$

This shows that, corresponding to *a given sine,* there are *two cosines* which are numerically equal, and opposite in algebraic sign. Ex. 1, Art. 42, illustrates this. This is also manifest in the table of signs in Art. 41. As indicated in this table, the sine is positive in the first and second quadrants, and then the cosine is positive and negative, respectively; the sine is negative in the third and fourth quadrants, and then the cosine is negative and positive, respectively. The signs of the remaining ratios corresponding to a given sine will be apparent on a short geometrical inspection, or by a glance at this table of signs. *When any single ratio is given, there is an ambiguity as to the signs of some of the other ratios.* Thus, to take another instance, it follows from the second of (3) that

$$\tan A = \pm\sqrt{\sec^2 A - 1}.$$

The secant of A is positive in the first and fourth quadrants, and then the tangent is positive and negative respectively; the secant of A is negative in the second and third quadrants, and then the tangent is negative and positive respectively. The double sign which appears in these relations was referred to in the examples, Art. 18. The student is advised to review and work the examples, for angles in general, in Art. 18.

EXAMPLES.

1. Given that $\sin A = \frac{3}{4}$; find the other ratios of A by means of the relations shown in this article.

[In Ex. 1, Art. 42, this problem is solved *geometrically;* here it will be solved *algebraically.*]

$$\cos A = \pm\sqrt{1 - \sin^2 A} = \frac{\pm\sqrt{7}}{4}; \quad \sec A = \frac{1}{\cos A} = \frac{4}{\pm\sqrt{7}};$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{4}{3}; \quad \tan A = \frac{\sin A}{\cos A} = \frac{3}{\pm\sqrt{7}}; \quad \cot A = \frac{1}{\tan A} = \frac{\pm\sqrt{7}}{3}.$$

Since the given sine is positive, the corresponding angles are in the first and second quadrants. Hence the double values of the calculated ratios are paired as follows:

$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\operatorname{cosec} A$
$\frac{3}{4}$	$\frac{+\sqrt{7}}{4}$	$\frac{3}{\sqrt{7}}$	$\frac{\sqrt{7}}{3}$	$\frac{4}{\sqrt{7}}$	$\frac{4}{3}$
$\frac{3}{4}$	$\frac{-\sqrt{7}}{4}$	$-\frac{3}{\sqrt{7}}$	$\frac{-\sqrt{7}}{3}$	$\frac{4}{\sqrt{7}}$	$\frac{4}{3}$

Find the other ratios *algebraically*, and verify the results *geometrically*, when:

2. $\cos A = -\frac{2}{3}$. **3.** $\tan A = \frac{5}{7}$. **4.** $\sec A = 4$. **5.** $\operatorname{cosec} A = -5$.

6. $\sin A = -\frac{1}{7}$. **7.** $\cos A = \frac{4}{7}$. **8.** $\tan A = -3$. **9.** $\cot A = \frac{2}{5}$.

Find the other ratios algebraically, and verify the results geometrically, when angle A satisfies the following *pairs of conditions:*

10. $\sin A = \frac{1}{2}$ and $\tan A = -\frac{1}{\sqrt{3}}$. **11.** $\tan A = \sqrt{3}$ and $\sec A = -2$.

12. $\cos A = -\frac{2}{3}$ and $\sin A = +\frac{\sqrt{5}}{3}$. **13.** $\sin A = -\frac{3}{5}$ and $\tan A = \frac{3}{4}$.

Give a geometrical solution of the following trigonometric equations:

14. $\sin A = \frac{2}{3}$. **15.** $\cos A = -\frac{3}{7}$. **16.** $\tan A = 4$. **17.** $\sec A = 5$.

Name the four least angles, and also the four least positive angles, that satisfy the equations:

18. $\sin\theta = \frac{1}{\sqrt{2}}$. **19.** $\tan\theta = \sqrt{3}$. **20.** $\cos\theta = -\frac{1}{\sqrt{2}}$. **21.** $\cot\theta = -\sqrt{3}$.

45. Ratios of $90° - A$, $180° - A$, $90° + A$, $-A$, compared with the ratios of A, A being any angle. The student may have suspected, from Exs. 7–10, Art. 41, that there is a close connection between the ratios of an angle A on the one hand, and the ratios of the angle $-A$ and of angles differing from A and $-A$ by multiples of 90°, on the other. He may have discovered already what the connection is. This connection, which is set forth in this article, is interesting in the study of angles, and has an important bearing on the construction of trigonometric tables, and on the solution of triangles.

In each figure in this article OP is the terminal line of the angle A, and OP_1 is the terminal line of the related angle which is under consideration; for the purpose of easy comparison, OP_1 is always taken equal to OP; MP, M_1P_1, are the perpendiculars drawn from the initial line to P, P_1, respectively. The deductions made in the simplest case, namely, when A is an angle in the first quadrant, are true for all angles. The student is advised to consider only the simplest case, when first he considers the subject of this article, and then to draw the figures and make the deductions for himself, in the cases in which angle A is in the second, third, and fourth quadrants, respectively.

NOTE. A compound angle, $90° - A$, for instance, can be described by revolving the turning line *forward* through $90°$, and then *backward* through an angle equal to A; or, these steps may be taken in a reverse order, namely, by revolving the turning line *backward* through an angle equal to A, and then *forward* through $90°$. Similarly, for the compound angles $90° + A$, $180° \pm A$, etc.

For the sake of clearness of construction, it is better not to take the terminal line of A nearly midway between X_1OX and Y_1OY.

A. **Ratios of $90° - A$.** Describe the angles A, $90° - A$. Let OP, OP_1, be the terminal lines of A, $90° - A$, respectively. In Figs.

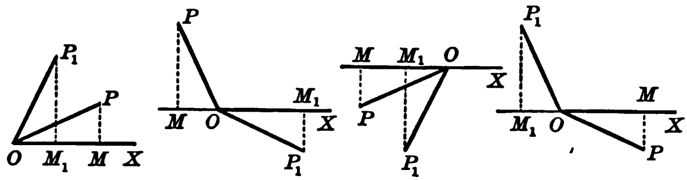

FIG. 41a. FIG. 41b. FIG. 41c. FIG. 41d.

41 *a*, 41 *b*, 41 *c*, 41 *d*, A is an angle in the first, second, third, and fourth quadrants, respectively.

Take OP_1 equal to OP, and draw MP, M_1P_1, at right angles to the initial line. In the triangles M_1OP_1, MOP, (in each figure) the angles at M_1, M, are right angles, angle $M_1OP_1 =$ angle OPM, and $OP_1 = OP$. Hence these two triangles are equal, and

$$OM_1 = MP,\ M_1P_1 = OM.$$

The figures also show that, for A in each quadrant, OM_1, MP have the same algebraic sign, and M_1P_1, OM have the same sign. Hence, for all angles A,

$$\sin(90° - A) = \frac{M_1P_1}{OP_1} = \frac{OM}{OP} = \cos A;$$

$$\cos(90° - A) = \frac{OM_1}{OP_1} = \frac{MP}{OP} = \sin A;$$

$$\tan(90^\circ - A) = \frac{M_1P_1}{OM_1} = \frac{OM}{MP} = \cot A;$$

$$\cot(90^\circ - A) = \frac{OM_1}{M_1P_1} = \frac{MP}{OM} = \tan A;$$

$$\sec(90^\circ - A) = \frac{OP_1}{OM_1} = \frac{OP}{MP} = \operatorname{cosec} A;$$

$$\operatorname{cosec}(90^\circ - A) = \frac{O_1P_1}{M_1P_1} = \frac{OP}{OM} = \sec A.$$

Hence, the ratio of any angle is the same as the co-ratio of its complement. Compare with Art. 16. The relations for tangent, secant, cotangent, cosecant, can also be deduced from those of sine and cosine by means of Art. 44 (1), (2). Thus, for example,

$$\tan(90^\circ - A) = \frac{\sin(90^\circ - A)}{\cos(90^\circ - A)} = \frac{\cos A}{\sin A} = \cot A.$$

B. **Ratios of $180^\circ - A$.** Describe the angles A, $180^\circ - A$. Let OP, OP_1 be the terminal lines of A and $180^\circ - A$ respectively. In Figs. 42 *a*, 42 *b*, 42 *c*, 42 *d*, A is an angle in the first, second, third, fourth quadrants, respectively. Take OP_1 equal to OP, and draw MP, M_1P_1, at right angles to the initial line. In the two triangles OM_1P_1, OMP, in each figure, the angles at M_1, M, are

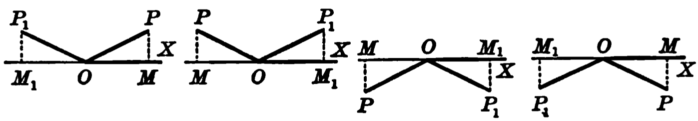

FIG. 42a. FIG. 42b. FIG. 42c. FIG. 42d.

right angles, angle $M_1OP_1 =$ angle MOP, $OP_1 = OP$. Hence, $OM_1 = OM$, and $M_1P_1 = MP$. In each figure, OM_1, OM have opposite algebraic signs, and M_1P_1, MP, have the same sign. Hence, for all angles A,

$$\sin(180^\circ - A) = \frac{M_1P_1}{OP_1} = \frac{MP}{OP} = \sin A;$$

$$\cos(180^\circ - A) = \frac{OM_1}{OP_1} = \frac{-OM}{OP} = -\cos A.$$

So also, $\tan(180° - A) = -\tan A$; $\cot(180° - A) = -\cot A$; $\sec(180° - A) = -\sec A$; $\operatorname{cosec}(180° - A) = \operatorname{cosec} A$.

The last four relations can be deduced by means of the figures, or by means of relations (1), (2), Art. 44. Hence, *any ratio of an angle is equal in magnitude to the same ratio of its supplement; the sines of supplementary angles have the same algebraic sign, and so have the cosecants; the other ratios of supplementary angles have opposite signs.*

C. **Ratios of $90° + A$.** Describe the angles A, $90° + A$. Let OP, OP_1, be the terminal lines of A, $90° + A$, respectively. In Figs. 43 *a*, *b*, *c*, *d*, A is an angle in the first, second, third, fourth quadrants, respectively. Take $OP_1 = OP$, and draw MP, M_1P_1, at right angles to the initial line. In the two triangles, OM_1P_1, OMP, (in each figure) the angles at M_1, M, are right angles,

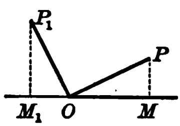

FIG. 43a.

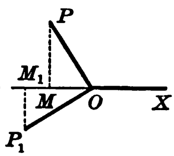

FIG. 43b.

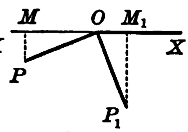

FIG. 43c.

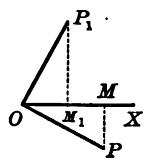

FIG. 43d.

angle M_1OP_1 = angle OPM, $OP_1 = OP$. Hence, $M_1P_1 = OM$, and $OM_1 = MP$. In each figure M_1P_1, OM, have the same algebraic signs, and OM_1, MP, have opposite signs. Hence, for all angles A,

$$\sin(90° + A) = \frac{M_1P_1}{OP} = \frac{OM}{OP} = \cos A;$$

$$\cos(90° + A) = \frac{OM_1}{OP_1} = -\frac{MP}{OP} = -\sin A.$$

So also, $\tan(90° + A) = -\cot A$; $\cot(90° + A) = -\tan A$;

$\sec(90° + A) = -\operatorname{cosec} A$; $\operatorname{cosec}(90° + A) = \sec A$.

These four relations can be deduced from the figures, or by means of (1), (2), Art. 44.

D. **Ratios of $-A$.** Describe the angles A, $-A$. Let OP, OP_1, be the terminal lines of A, $-A$, respectively. In Figs. 44 *a*,

b, c, d, A is in the first, second, third, fourth quadrants, respectively. Take $OP_1 = OP$, and draw PM, P_1M_1, at right angles to the initial line. In the two triangles, OM_1P_1, OMP, (in each figure) the angles at M_1, M, are right, angle $M_1OP_1 =$ angle MOP,

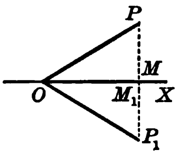

FIG. 44a.

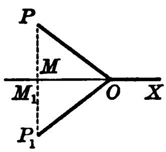

FIG. 44b.

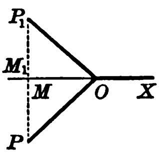

FIG. 44c.

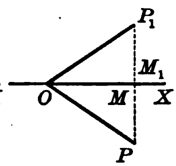

FIG. 44d.

$OP_1 = OP$. Hence, $M_1P_1 = MP$, $OM = OM_1$. In each figure, OM_1, OM, have the same sign, and M_1P_1, MP, have opposite signs. Hence, for all angles A,

$$\sin(-A) = \frac{M_1P_1}{OP_1} = -\frac{MP}{OP} = -\sin A;$$

$$\cos(-A) = \frac{OM_1}{OP_1} = \frac{OM}{OP} = \cos A.$$

So also, $\tan(-A) = -\tan A$; $\cot(-A) = -\cot A$;

$$\sec(-A) = \sec A; \quad \operatorname{cosec}(-A) = -\operatorname{cosec} A.$$

The last four relations can be deduced from the figures, or by means of (1), (2), Art. 44.

Ex. 1. Show that $\sin(180° + A) = -\sin A$,

$\cos(180° + A) = -\cos A$, $\tan(180° + A) = \tan A$, etc.,

when A denotes an angle in any one of the four quadrants.

Ex. 2. Deduce the relations between each of the following angles and angle A, viz. $270° - A$, $270° + A$, $360° - A$, $360° + A$, $n \cdot 360° \pm A$, n being any whole number.

By means of the relations shown in this article, the ratios of any angle can be expressed in terms of the ratios of an angle between 0° *and* 45°. Thus, for example,

$$\sin 700^\circ = \sin(360^\circ + 340^\circ) = \sin 340^\circ = \sin(-20^\circ) = -\sin 20^\circ;$$

$$\begin{aligned}\tan 975^\circ = \tan(2\cdot 360^\circ + 255^\circ) = \tan 255^\circ &= \tan(180^\circ + 75^\circ)\\ &= \tan 75^\circ = \cot 15^\circ;\end{aligned}$$

$$\begin{aligned}\operatorname{cosec}(-1160^\circ) = -\operatorname{cosec} 1160^\circ &= -\operatorname{cosec}(3\cdot 360^\circ + 80^\circ)\\ &= -\operatorname{cosec} 80^\circ = -\sec 10^\circ.\end{aligned}$$

$$\therefore \sin 700^\circ = -.34202;\ \tan 975^\circ = 3.7321;$$

$$\operatorname{cosec}(1160^\circ) = -\sec 10^\circ = \frac{-1}{\cos 10^\circ} = \frac{-1}{.98481} = -1.015.$$

This property, and the property that the ratio of an angle is the co-ratio of its complement, account for the arrangement and extent of the trigonometric tables.

EXAMPLES.

1. Express the ratios of the angles in Exs. 1–7, Art. 41, in terms of ratios of angles between 0° and 45°. Also find the ratios.

2. Do likewise for the angles in Exs. 1, 3, Art. 38. Also find the ratios.

3. Do likewise for the angles in Exs. 1, 2, 3, Art. 37. Also find the ratios.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chap. V. will be found at page* 186.

CHAPTER VI.

TRIGONOMETRIC RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES.

N.B. *Another way of making the derivations shown in Arts.* 46–48 *is given in Note B of the Appendix. The method of projection, as it is called, used in Note B, is preferred by many.*

46. Derivation of the sine and cosine of the sum of two angles when each of the angles is less than a right angle. In this article and the following one, careful regard must be paid to the *directions* in which lines and angles are measured, and to the *order of the letters* used in measuring them. See Arts. 36, 37, 40.

To deduce sin $(A + B)$ and cos $(A + B)$. Let A and B be two angles each of which is less than a right angle. Let the turning line revolve from the initial line OX, and about O describe the

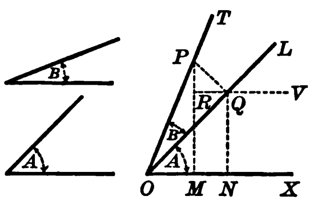

FIG. 45a.

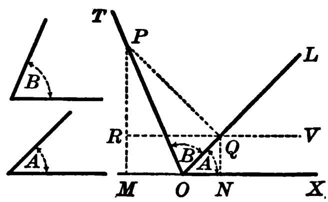

FIG. 45b.

angle XOL equal to A, and then revolve forward from the position OL and describe the angle LOT equal to B. Thus, angle $XOT = A + B$. [In Fig. 45 *a*, $A + B$ is less than 90°; in Fig. 45 *b*, $A + B$ is greater than 90°.] Take any point P on OT, the terminal line of the angle $(A + B)$, and draw PQ at right angles to OL, the terminal line of the angle A. From P, Q, draw PM, QN, at right angles to the initial line; and through Q draw VQR parallel to OX and intersecting MP in R.

$$\sin(A+B) = \sin XOP = \frac{MP}{OP} = \frac{NQ+RP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP}$$

$$= \frac{OQ \sin A}{OP} + \frac{QP \sin VQP}{OP}. \quad \text{(Definitions, Art. 40.)}$$

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos QOP = \cos B; \quad \frac{QP}{OP} = \sin QOP = \sin B;$$

$$\sin VQP = \sin(180^\circ - PQR) = \sin PQR$$

$$= \cos RQO = \cos XOQ = \cos A.$$

$$\therefore \mathbf{\sin(A+B) = \sin A \cos B + \cos A \sin B}. \quad (1)$$

$$\cos(A+B) = \cos XOP = \frac{OM}{OP} = \frac{ON - RQ}{OP}$$

$$= \frac{ON + QR}{OP} = \frac{ON}{OP} + \frac{QR}{OP}$$

$$= \frac{OQ \cos A}{OP} + \frac{QP \cos VQP}{OP}.$$

(Definitions, Art. 40.)

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos B; \quad \frac{QP}{OP} = \sin B;$$

$$\cos VQP = \cos(180^\circ - PQR) = -\cos PQR = -\sin RQO = -\sin A.$$

$$\therefore \mathbf{\cos(A+B) = \cos A \cos B - \sin A \sin B}. \quad (2)$$

EXAMPLES.

1. Sin 75° = sin (30° + 45°) = sin 30° cos 45° + cos 30° sin 45°

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

2. Find cos 75° by putting 30° + 45° for 75° and using formula (2).

3. Deduce the sine and cosine of 15° from the results in Exs. 1, 2.

4. Find $\sin 90°$, $\cos 90°$, by putting $90° = 30° + 60°$. Also by putting $90° = 45° + 45°$. Also by putting $90° = 75° + 15°$.

5. Find $\sin 120°$, $\cos 120°$, by putting $120° = 60° + 60°$; $120° = 90° + 30°$; $120° = 75° + 45°$.

6. Find $\sin 150°$, $\cos 150°$, by putting $150° = 75° + 75°$; $150° = 90° + 60°$.

7. Find $\sin 135°$, $\cos 135°$, by putting $135° = 75° + 60°$; $135° = 90° + 45°$.

8. Given $\sin x = \frac{1}{4}$, $\sin y = \frac{2}{5}$, x and y both in the first quadrant; find $\sin(x+y)$, $\cos(x+y)$.

47. Derivation of the sine and cosine of the difference of two angles when each of the angles is less than a right angle. The construction and derivation are *very similar* to that made in the preceding article.

To deduce $\sin(A-B)$ and $\cos(A-B)$. Let A and B be two angles each of which is less than a right angle, and let A be the greater. Let the turning line revolve from the initial line OX,

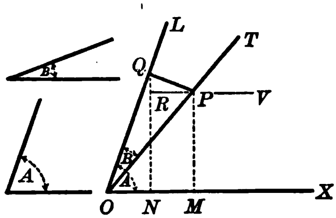

FIG. 46.

and about O describe the angle XOL equal to A, and then revolve backward from the position OL and describe the angle LOT equal to $-B$. Then angle $XOT = A - B$. Take any point P on OT, the terminal line of the angle $(A-B)$, and draw PQ at right angles to OL, the terminal line of the angle A. From P, Q, draw PM, QN, at right angles to the initial line; and through P draw RPV parallel to OX and intersecting NQ in R.

$$\sin(A-B) = \sin XOP = \frac{MP}{OP} = \frac{NQ - RQ}{OP} = \frac{NQ}{OP} - \frac{RQ}{OP}$$

$$= \frac{OQ \sin A}{OP} - \frac{PQ \sin VPQ}{OP}. \quad \text{(Definitions, Art. 40.)}$$

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos QOP = \cos(-B) = \cos B;$$

$$-\frac{PQ}{OP} = \frac{QP}{OP} = \sin QOP = \sin(-B) = -\sin B;$$

$$\sin VPQ = \sin(180° - QPR) = \sin QPR = \cos RQP = \cos A.$$

$$\therefore\ \mathbf{\sin(A - B) = \sin A \cos B - \cos A \sin B.} \qquad (3)$$

$$\cos(A - B) = \cos XOP = \frac{OM}{OP} = \frac{ON + RP}{OP} = \frac{ON}{OP} - \frac{PR}{OP}$$

$$= \frac{OQ \cos A}{OP} - \frac{PQ \cos VPQ}{OP}. \quad \text{(Definitions, Art. 40.)}$$

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos B, \text{ and } -\frac{PQ}{OP} = -\sin B, \text{ as shown above;}$$

$$\cos VPQ = \cos(180° - QPR) = -\cos QPR = -\sin RQP = -\sin A.$$

$$\therefore\ \mathbf{\cos(A - B) = \cos A \cos B + \sin A \sin B.} \qquad (4)$$

If B is greater than A, then the formula,

$$\sin(B - A) = \sin B \cos A - \cos B \sin A,$$

can be deduced as above. Since

$$\sin(A - B) = -\sin(B - A),$$

then $$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

It is shown in Art. 48 *that the formulas* (1), (2), (3), (4), *are true for all values of A and B.* These formulas are called *the addition* and *subtraction formulas* or *theorems* in trigonometry. They are of such great importance, and so many thorems can be deduced by means of them, that they are called the *fundamental* formulas of trigonometry.* They should be memorized.

NOTE. Arts. 48, 49, may be omitted, if deemed advisable, until after the solution of triangles is completed. Art. 48 can also be shown *geometrically*.

---

* Adrian Romanus (1561–1625), professor of mathematics and medicine at the University of Louvain, was the first to prove the formula for $\sin(A + B)$. The formulas for $\cos(A \pm B)$ and $\sin(A - B)$ were given by Pitiscus (1561–1613), a German mathematician and astronomer, in his Trigonometry published in 1595.

EXAMPLES.

1. Derive sin 15°, cos 15°, on putting 60° − 45° for 15°.
2. Derive sin 15°, cos 15°, on putting 45° − 30° for 15°.
3. Find $\sin(x - y)$, $\cos(x - y)$ in the cases in Ex. 8, Art. 46.

48. Proof of addition and subtraction formulas for all values of A and B. These formulas have been proved in Arts. 46, 47, for values of A and B which are less than a right angle. In Art. 45 c it has been shown that for *any* angle, say X,

$$\cos X = \sin(90° + X), \quad \sin X = -\cos(90° + X).$$

Hence, $\cos A = \sin(90° + A)$, $\sin A = -\cos(90° + A)$,

$$\cos(A+B) = \sin(90° + A + B), \quad \sin(A+B) = -\cos(90° + A + B).$$

The substitution of these values for $\cos A$, $\sin A$, $\cos(A + B)$, $\sin(A + B)$, in (1), Art. 46, gives

$$-\cos(90° + A + B) = -\cos(90° + A)\cos B + \sin(90° + A)\sin B;$$

$$\therefore \cos(\overline{90° + A} + B) = \cos(90° + A)\cos B - \sin(90° + A)\sin B. \quad (1)$$

The substitution of the same values in (2), Art. 46, gives

$$\sin(\overline{90° + A} + B) = \sin(90° + A)\cos B + \cos(90° + A)\sin B. \quad (2)$$

Hence, formulas (1), (2), Art. 46, are true when one of the angles is increased by a right angle. In a similar way, these formulas can be shown to remain true when one of the angles in (1), (2), of this article is increased by a right angle. It is thus evident that the formulas are true, no matter how many right angles are added to either one or both of the angles. It can easily be shown that $\sin A = \cos(A - 90°)$, $\cos A = -\sin(A - 90°)$. Then, in the same way as that just employed, it can be shown that the formulas (1), (2), Art. 46, hold when either one or both of the angles is diminished by integral multiples of 90°. Hence, formulas (1), (2), Art. 46, are true for angles in any quadrant, that is, for all angles. In a similar way, formulas (3), (4), Art. 47, can be shown to be universally true.

49. Each fundamental formula contains the others. From any one of the four fundamental formulas, the remaining three can be derived. Thus for example:

In (1) Art. 46, change A into $90° - A$; then

$$\sin(90° - A + B) = \sin(90° - A)\cos B + \cos(90° - A)\sin B.$$

From this,

$$\sin(90° - \overline{A - B}), \text{ i.e. } \cos(A - B) = \cos A \cos B + \sin A \sin B.$$

In (1) Art. 46, change B into $(-B)$; then

$$\begin{aligned}\sin(A - B) &= \sin A \cos(-B) + \cos A \sin(-B)\\ &= \sin A \cos B - \cos A \sin B.\end{aligned}$$

In (1) Art. 46, change A into $(90° + A)$; then

$$\sin(90° + A + B) = \sin(90° + A)\cos B + \cos(90° + A)\sin B,$$

whence, $\cos(A + B) = \cos A \cos B - \sin A \sin B.$

Ex. 1. From formula (2), Art. 46, derive the other three fundamental formulas.

Ex. 2. So also, from formula (3), Art. 47.

Ex. 3. So also, from formula (4), Art. 47.

50. Ratio of an angle in terms of the ratios of its half angle. In this article and Arts. 51, 52, a few deductions will be made from the addition and subtraction formulas, which have been shown to be true for all angles. These deductions are necessary for the explanations concerning triangles, as well as useful for other purposes. More ample opportunity will be afforded later for working exercises involving the use of these formulas. The fundamental formulas may be brought together:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B. \tag{1}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B. \tag{2}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B. \tag{3}$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \tag{4}$$

Let $B = A$; then, from (1),

$$\sin(A + A) = \sin A \cos A + \cos A \sin A;$$

that is,
$$\mathbf{\sin 2A = 2 \sin A \cos A.} \tag{5}$$

Similarly, from (3),
$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A.} \tag{6}$$

Since $\cos^2 A + \sin^2 A = 1$, it follows that

$$\mathbf{\cos 2A = 1 - 2\sin^2 A.} \tag{7}$$

and
$$\mathbf{\cos 2A = 2\cos^2 A - 1.} \tag{8}$$

In formulas (1)–(8), A, B, denote any angles whatsoever. These formulas occur so often, and are so useful, that it is well to translate them into words. Thus,

$$\left.\begin{aligned}
\text{sine } \textit{sum} \text{ of any two angles} &= \text{sin } \textit{first} \cdot \text{cosine } \textit{second} \\
&\quad + \text{cosine } \textit{first} \cdot \text{sine } \textit{second} \\
\text{sine } \textit{difference} \text{ of any two angles} &= \text{sine } \textit{first} \cdot \text{cosine } \textit{second} \\
&\quad - \text{cosine } \textit{first} \cdot \text{sine } \textit{second} \\
\text{cosine } \textit{sum} \text{ of any two angles} &= \text{cosine } \textit{first} \cdot \text{cosine } \textit{second} \\
&\quad - \text{sine } \textit{first} \cdot \text{sine } \textit{second} \\
\text{cosine } \textit{difference} \text{ of any two angles} &= \text{cosine } \textit{first} \cdot \text{cosine } \textit{second} \\
&\quad + \text{sine } \textit{first} \cdot \text{sine } \textit{second}
\end{aligned}\right\} \text{A}$$

Since A is one-half of $2A$, formulas (5)–(8) can be translated as follows:

$$\text{sine any } \textit{angle} = 2 \text{ sine } \textit{half-angle} \cdot \text{cosine } \textit{half-angle},$$

$$\begin{aligned}
\text{cosine any } \textit{angle} &= (\text{cosine } \textit{half-angle})^2 - (\text{sine } \textit{half-angle})^2, \\
&= 1 - 2(\text{sine } \textit{half-angle})^2, \\
&= 2(\text{cosine } \textit{half-angle})^2 - 1.
\end{aligned}$$

EXAMPLES.

1. Find $\cos 22\frac{1}{2}°$ from $\cos 45°$.

$$2\cos^2 22\tfrac{1}{2}° = 1 + \cos 45° \text{ by (8)};$$

$$\therefore \cos^2 22\tfrac{1}{2}° = \tfrac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{2}}{2\sqrt{2}} = \frac{1 + 1.4142}{2 \times 1.4142} = .8536;$$

$$\therefore \cos 22\tfrac{1}{2}° = .9239.$$

2. Express $\cos 4x$ in terms of $\sin x$ and $\cos x$.

$\cos 4x = 2\cos^2 2x - 1 = 2(2\cos^2 x - 1)^2 - 1 = 8\cos^4 x - 8\cos^2 x + 1.$

3. Deduce $\sin 30°$, $\cos 30°$, from $\cos 60°$.

4. Deduce $\sin 75°$, $\cos 75°$, from $\cos 150°$. [Logarithms may be helpful.]

5. Deduce $\sin 22\frac{1}{2}°$, from $\cos 45°$.

6. Deduce $\sin 15°$, $\cos 15°$, from $\cos 30°$.

7. Express $\cos 6x$, $\sin 6x$, in terms of ratios of $3x$.

8. Express $\cos 3x$, $\sin 3x$, in terms of ratios of $\frac{3}{2}x$.

9. Express $\sin\frac{3}{4}x$, $\cos\frac{3}{4}x$, in terms of ratios of $\frac{3}{8}x$.

10. Express $\cos 6x$, $\sin 6x$, in terms of ratios of $12x$.

11. Express $\cos 3x$, $\sin 3x$, in terms of ratios of $6x$.

12. Express $\sin\frac{3}{4}x$, $\cos\frac{3}{4}x$, in terms of ratios of $\frac{3}{2}x$.

13. Show that $\sin(n+1)A + \sin(n-1)A = 2\sin nA\cos A$, and

$$\cos(n+1)A + \cos(n-1)A = 2\cos nA\cos A;$$ and

hence, express $\sin 2A$, $\cos 2A$, in terms of $\sin A$, $\cos A$.

51. Tangents of the sum, and difference of two angles, and of twice an angle. Let A, B, be any two angles. It is required to find $\tan(A+B)$ and $\tan(A-B)$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

On dividing each term of the numerator and the denominator of the second member by $\cos A\cos B$, there is obtained

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}. \qquad (1)$$

In the same way it can be shown that

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}. \qquad (2)$$

Formula (2) can also be deduced from (1) by changing B into $-B$.

If $B = A$, then (1) becomes

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \qquad (3)$$

Formulas (1), (2), (3), can be translated into words, as follows:

$$\text{tangent } sum \text{ any two angles} = \frac{\text{sum of tangents}}{1 - \text{product of tangents}};$$

$$\text{tangent } difference \text{ any two angles} = \frac{\text{difference of tangents}}{1 + \text{product of tangents}};$$

$$\text{tangent any angle} = \frac{2 \text{ tangent half-angle}}{1 - (\text{tangent half-angle})^2}.$$

EXAMPLES.

1. $\tan P = 2$, $\tan Q = \frac{1}{3}$. Find $\tan (P + Q)$, $\tan (P - Q)$.

$$\tan (P + Q) = \frac{2 + \frac{1}{3}}{1 - 2 \cdot \frac{1}{3}} = 7; \quad \tan (P - Q) = \frac{2 - \frac{1}{3}}{1 + 2 \cdot \frac{1}{3}} = 1.$$

2. Find $\tan 75°$ by means of $\tan 45°$, $\tan 30°$.
3. Find $\tan 15°$ by means of $\tan 60°$, $\tan 45°$.
4. Find $\tan 22\frac{1}{2}°$ from $\tan 45°$.
5. Find $\tan 37\frac{1}{2}°$ from $\tan 75°$.
6. Derive $\cot (A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$, $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}$.

52. Sums and differences of sines and cosines. The set of formulas (1)–(4), Art. 50, can be transformed into two other sets which are very useful. From (1), (2), (3), (4), Art. 50, on addition and subtraction, there is obtained:

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B. \tag{1}$$

$$\sin (A + B) - \sin (A - B) = 2 \cos A \sin B. \tag{2}$$

$$\cos (A + B) + \cos (A - B) = 2 \cos A \cos B. \tag{3}$$

$$\cos (A + B) - \cos (A - B) = -2 \sin A \sin B. \tag{4}$$

If $$A + B = P,$$

and $$A - B = Q,$$

then $$2A = P + Q, \text{ and } A = \tfrac{1}{2}(P + Q),$$

$$2B = P - Q, \text{ and } B = \tfrac{1}{2}(P - Q).$$

Substitution of these values of A, B, in (1)–(4) gives

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}. \tag{5}$$

$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}. \tag{6}$$

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}. \tag{7}$$

$$\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}. \tag{8}$$

Formulas (1)–(4) with the members transposed, are useful for transforming *products* of sines and cosines *into sums and differences;* formulas (5) to (8) are useful for transforming *sums and differences* of sines and cosines into *products.* These formulas may be translated into words: — *Of any two angles,*

$$\left.\begin{array}{lr}
2\ \textit{sin}\ \text{one} \cdot \textit{cos}\ \text{the other} = \textit{sin}\ \text{sum} + \textit{sin}\ \text{difference}, \quad . \quad . \quad . & (1') \\
2\ \textit{cos}\ \text{one} \cdot \textit{sin}\ \text{the other} = \textit{sin}\ \text{sum} - \textit{sin}\ \text{difference}, \quad . \quad . \quad . & (2') \\
2\ \textit{cos}\ \text{one} \cdot \textit{cos}\ \text{the other} = \textit{cos}\ \text{sum} + \textit{cos}\ \text{difference}, \quad . \quad . \quad . & (3') \\
2\ \textit{sin}\ \text{one} \cdot \textit{sin}\ \text{the other} = \textit{cos}\ \text{difference} - \textit{cos}\ \text{sum}, \quad . \quad . \quad . & (4')
\end{array}\right\} \text{B}$$

$$\left.\begin{array}{rr}
\text{the } \textit{sum of two sines} = 2\ \textit{sin}\ \text{half sum} \cdot \textit{cos}\ \text{half difference}, & (5') \\
\text{the } \textit{difference of two sines} = 2\ \textit{cos}\ \text{half sum} \cdot \textit{sin}\ \text{half difference}, & (6') \\
\text{the } \textit{sum of two cosines} = 2\ \textit{cos}\ \text{half sum} \cdot \textit{cos}\ \text{half difference}, & (7') \\
\text{the } \textit{difference of two cosines} = -2\ \textit{sin}\ \text{half sum} \cdot \textit{sin}\ \text{half difference}, & (8')
\end{array}\right\} \text{C}$$

The difference between the first members of A, Art. 50, and C should be noted.

N.B. Arts. 92–95 are similar in character to, and are merely a continuation of, Arts. 50–52. If deemed advisable, Arts. 91–95 can be taken up now. The student is advised to glance at them after solving the following exercises:

EXAMPLES.

1. Show that $\dfrac{\cos x - \cos y}{\cos x + \cos y} = -\tan \tfrac{1}{2}(x+y) \tan \tfrac{1}{2}(x-y)$.

$$\frac{\cos x - \cos y}{\cos x + \cos y} = \frac{-2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)}{2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)} = -\tan \tfrac{1}{2}(x+y) \tan \tfrac{1}{2}(x-y).$$

2. Show that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$.

$$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{1+(2\cos^2 A - 1)} = \frac{\sin A}{\cos A} = \tan A.$$

3. Show that $2\sin(A+45^\circ)\sin(A-45^\circ) = \sin^2 A - \cos^2 A$.

$$2\sin(A+45^\circ)\sin(A-45^\circ) = \cos(\overline{A+45^\circ} - \overline{A-45^\circ}) - \cos(\overline{A+45^\circ} + \overline{A-45^\circ}),$$ Art. 52, B (4′)

$$= \cos 90^\circ - \cos 2A = \sin^2 A - \cos^2 A.$$

4. Show that $\sin 5A \sin A = \sin^2 3A - \sin^2 2A$.

$$\sin 5A \sin A = \tfrac{1}{2}[\cos(5A - A) - \cos(5A + A)],$$ Art. 52, B (4′)

$$= \tfrac{1}{2}(\cos 4A - \cos 6A),$$

$$= \tfrac{1}{2}[1 - 2\sin^2 2A - (1 - 2\sin^2 3A)] = \sin^2 3A - \sin^2 2A.$$

5. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$.

$$\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \frac{2\sin\frac{1}{2}(3A+A)\cos\frac{1}{2}(3A-A)}{2\cos\frac{1}{2}(3A+A)\cos\frac{1}{2}(3A-A)},$$ Art. 52, C (5′), (7′)

$$= \frac{\sin 2A}{\cos 2A} = \tan 2A.$$

Prove the following statements:

6. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$.

7. $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

8. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

9. $\cot A - \cot 2A = \operatorname{cosec} 2A$.

10. $\sin(A+B)\sin(A-B) = \cos^2 B - \cos^2 A$.

11. $1 + \tan 2A \tan A = \sec 2A$.

12. $\tan(45^\circ + A) = \dfrac{1+\tan A}{1-\tan A}$.

13. $\tan(45^\circ - A) = \dfrac{1-\tan A}{1+\tan A}$.

14. $\left(\sin\dfrac{A}{2} + \cos\dfrac{A}{2}\right)^2 = 1 + \sin A$.

15. $\left(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right)^2 = 1 - \sin A$.

16. $\dfrac{1-\cos 2A}{\sin 2A} = \tan A$.

17. $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$.

18. $\dfrac{1+\cos A}{\sin A} = \cot\dfrac{A}{2}$.

19. $\dfrac{2\tan A}{1+\tan^2 A} = \sin 2A$.

20. $\dfrac{\text{cosec}^2}{\text{cosec}^2 A - 2} = \sec 2A.$

21. $\dfrac{2 - \sec^2 A}{\sec^2 A} = \cos 2A.$

22. $\dfrac{\sin(A+B)}{\cos A \cos B} = \tan A + \tan B.$

23. $\dfrac{\sin(A-B)}{\cos A \cos B} = \tan A - \tan B.$

24. $\dfrac{\cos(A-B)}{\cos A \sin B} = \cot B + \tan A.$

25. $\dfrac{\cos(A+B)}{\sin A \cos B} = \cot A - \tan B.$

26. $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}.$

27. $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{1 - \tan x \tan y}{1 + \tan x \tan y}.$

28. Given $\sin x = \frac{1}{2}$, $\sin y = \frac{2}{3}$. Find $\sin(x+y)$, $\sin(x-y)$, $\cos(x+y)$, $\cos(x-y)$, $\sin 2x$, $\sin 2y$, $\cos 2x$, $\cos 2y$, $\tan 2x$, $\tan 2y$, $\tan(x+y)$, $\tan(x-y)$, when (*a*) both x, y, are in the first quadrant; (*b*) x is in the first, y in the second; (*c*) x in the second, y in the first; (*d*) both in the second quadrant. Check results by means of the tables.

29. Given $\sin x = \frac{1}{5}$, $\sin y = \frac{3}{7}$. Do as in Ex. 28.

30. Given $\sin x = \frac{3}{5}$, $\sin y = -\frac{2}{3}$. Find the ratios named in Ex. 28, when x is in the first quadrant and y in the third, x in the first and y in the fourth, x in the second and y in the third, x in the second and y in the fourth.

N.B. *Examples suitable for exercise and review on the subject-matter of this chapter will be found in Arts.* 91–95, *and at page* 187.

CHAPTER VII.

SOLUTION OF TRIANGLES IN GENERAL.

53. Cases for solution. In Art. 34 oblique triangles were solved by means of right-angled triangles. In this chapter some relations of the sides and angles of any triangle (whether right-angled or oblique) will be derived; methods of solution will be shown, which are applicable to the solution of both right-angled and oblique triangles, and which are independent of the special aid that can be afforded by right-angled triangles. In Art. 54 the chief relations between the sides and angles of a triangle will be deduced. These relations constitute the foundation for the remainder of the chapter. In Arts. 55–58 solutions of triangles are obtained without the use of logarithms; in Arts. 60–62 logarithms are employed in finding the solutions.

In order that a triangle may be constructed, three elements, one of which must be a side, are required. Hence, there are four cases for construction and solution, namely, when the given parts are as follows:

I. **One side** and **two angles.**

II. **Two sides** and **the angle opposite to one of them.**

III. **Two sides** and **their included angle.**

IV. **Three sides.**

Before proceeding, the student should test his ability to construct a triangle readily in each of these cases.

In the discussions that follow, the triangle is denoted by ABC, the angles by A, B, C, and the lengths of their opposite sides by a, b, c, respectively.*

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* The formulas are greatly simplified by the adoption of this notation, which was first introduced by Leonhard Euler (1707–1783).

54. Fundamental relations between the sides and angles of a triangle. The law of sines. The law of cosines.

I. *The law of sines.* From C in the triangle ABC draw CD at right angles to opposite side AB, and meeting AB or AB produced in D. (In Fig. 47 a B is acute, in Fig. 47 b B is obtuse, and in

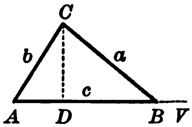


FIG. 47a.

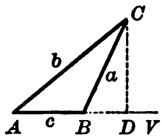


FIG. 47b.

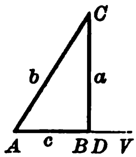


FIG. 47c.

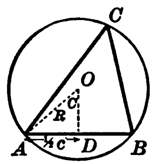


FIG. 48.

Fig. 47 c B is a right angle.) Produce AB to V. In what follows, AB is taken as the positive direction.

In CDA, $\qquad DC = b \sin A.$

In CDB (Figs. 47 a, b), $\qquad DC = a \sin VBC$ (Definition, Art. 40.)

$$= a \sin B.$$

$$[\because \sin VBC = \sin(180° - VBC), \text{ Art. 45, } = \sin CBA]$$

In Fig. 47 c, $\qquad DC = BC = a = a \sin B.$

$$(\because B = 90°, \text{ and } \sin 90° = 1)$$

Therefore, in all three triangles,

$$a \sin B = b \sin A.$$

Hence,
$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly, on drawing a line from B at right angles to AC, it can be shown that

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Hence, in any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad (1)$$

In words: *The sides of any triangle are proportional to the sines of the opposite angles.*

Each of the fractions in (1) gives *the length of the diameter of the circle described about ABC.* Let O (Fig. 48) be the centre and R the radius of the circle described about ABC. Draw OD at right angles to any one of the sides, say AB. Draw AO. Then $AD = \frac{1}{2}c$, $AOD = C$, by geometry. In triangle AOD,

$$AD = AO \sin AOD; \; i.e. \; \tfrac{1}{2}c = R \sin C.$$

$$\therefore 2R = \frac{c}{\sin C}. \qquad (2)$$

Ex. 1. Explain why the circumscribing circle of a triangle depends only upon *one side and its opposite angle.*

Ex. 2. Derive the law of sines by drawing a perpendicular from A to BC.

Ex. 3. Derive $2R = \frac{a}{\sin A}$, $2R = \frac{b}{\sin B}$, by means of figures.

II. *The law of cosines.* An expression for the length of the side of a triangle in terms of the cosine of the opposite angle and the lengths of the other two sides, will now be deduced. The angle A is acute in Fig. 49 a, obtuse in Fig. 49 b, right in Fig. 49 c. From C draw CD at right angles to AB. The direction AB is taken as positive.

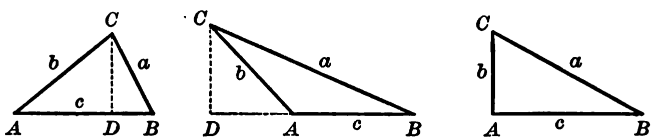


FIG. 49a. FIG. 49b. FIG. 49c.

In Figs. 49 a, 49 b, $\overline{BC}^2 = \overline{DC}^2 + \overline{DB}^2$.

In Fig. 49 a, $DB = AB - AD$;

in Fig. 49 b, $DB = DA + AB = -AD + AB$.

Hence, in both figures, $\overline{BC}^2 = \overline{DC}^2 + (AB - AD)^2$

$$= \overline{DC}^2 + \overline{AD}^2 + \overline{AB}^2 - 2\,AB \cdot AD.$$

In Fig. 49 *a*, $AD = AC \cos BAC$;

in Fig. 49 *b*, $AD = AC \cos BAC$ (Art. 40).

Also, $\overline{DC}^2 + \overline{AD}^2 = \overline{AC}^2$.

Hence, in both figures, $\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AC \cdot AB \cos A$;

that is, $$\boldsymbol{a^2 = b^2 + c^2 - 2\,bc \cos A.} \tag{3}$$

This formula also holds for Fig. 49 *c*; for there,

$$\cos A = \cos 90° = 0.$$

Similar formulas for b, c, can be derived in like manner, or can be obtained from (3) by symmetry:

$$b^2 = c^2 + a^2 - 2\,ca \cos B,\quad c^2 = a^2 + b^2 - 2\,ab \cos C. \tag{3'}$$

These formulas can be expressed in words: *In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these two sides multiplied by the cosine of their included angle.*

NOTE. In Fig. 49 *a*, A is acute and $\cos A$ is positive; in Fig. 49 *b*, A is obtuse and $\cos A$ is negative. Hence formula (3) shows that in Fig. 49 *a*, a^2 is less than $b^2 + c^2$, and that in Fig. 49 *b*, a^2 is greater than $b^2 + c^2$. In Fig. 49 *c*, $a^2 = b^2 + c^2$.

Relation (3) may be expressed as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \tag{4}$$

and similarly for $\cos B$, $\cos C$.

Ex. Derive the formulas for b^2 and for c^2.

Each of the relations (1), (3), (3′), involves *four* of the six elements of a triangle. If any *three* of the elements in any one of these relations are known, then the *fourth* element can be found by solving the equation. Inspection shows that relations (1) are serviceable in the solution of Cases I., II., Art. 53, and that relations (3), (3′), are serviceable in the solution of Cases III., IV., Art. 53. The student is advised to try to work some of the examples in Arts. 55–58 before reading the text of the articles. (See Arts. 20–24, 34.)

54 *a*. Substitution of sines for sides, and of sides for sines.

Since $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, the sines of the opposite angles can be substituted for the sides of triangles, and *vice versa*, when they are involved *homogeneously* in the numerator and denominator of a fraction, or in both members of an equation.

Thus, on putting $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = x$, it follows that $a = x \sin A$, $b = x \sin B$, $c = x \sin C$.

Then, for example, $\frac{a^2}{b+c} = \frac{ax \sin A}{x \sin B + x \sin C} = \frac{a \sin A}{\sin B + \sin C}$.

EXAMPLES.

1. Show that in any triangle $\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}$.

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \sin \frac{1}{2} C \cos \frac{1}{2} C} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C},$$

for $\qquad \sin \frac{1}{2}(A+B) = \cos \frac{1}{2} C$, since $\frac{1}{2}(A+B) + \frac{1}{2} C = 90°$.

2. Derive two other relations similar to that in Ex. 1.

3. Show that

$$\frac{3a^2 + 2b^2}{abc} = \frac{3 \sin^2 A + 2 \sin^2 B}{a \sin B \sin C} = \frac{3 \sin^2 A + 2 \sin^2 B}{b \sin A \sin C} = \frac{3 \sin^2 A + 2 \sin^2 B}{c \sin A \sin B}.$$

55. Case I. Given one side and two angles. In triangle ABC, suppose that A, B, a are known; it is required to find C, b, c. In this case (see Fig. 47 *a*, Art. 54),

$$C = 180° - (A + B);$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}, \text{ whence } b = \frac{a}{\sin A} \cdot \sin B;$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}, \text{ whence } c = \frac{a}{\sin A} \cdot \sin C.$$

Checks: $a^2 = b^2 + c^2 - 2bc \cos A$, $\frac{b}{\sin B} = \frac{c}{\sin C}$, the result in Ex. 1, Art. 54 *a*. Other checks will be discovered later.

EXAMPLES.

1. ... triangle PQR ... $= 12$ in., ... Solution: $R =$

$Q = 40°$. $PR =$

$P = 75°$. $RQ =$

$R = 180° - (40° + 75°) = 65°$.

$$\frac{RQ}{\sin P} = \frac{PQ}{\sin R}.$$

$$RQ = \frac{PQ}{\sin R} \cdot \sin P.$$

$$= \frac{12}{\sin 65°} \cdot \sin 75°$$

$$= \frac{12}{.9063} \times .9659.$$

$$= 13.24 \times .9659.$$

$$= 12.8 \text{ in.}$$

... $= 60$ in. Solve the triangle.

... $b = 5\frac{1}{2}$ ft. Find a.

... $a = 102$. Solve the triangle.

... $a = b$ 42. Solve the triangle.

... and an angle opposite to one of ... b, A be known, and C, B, c be ... be constructed with the given ...

FIG. 50.

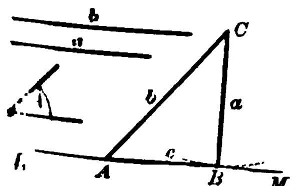


FIG. 51.

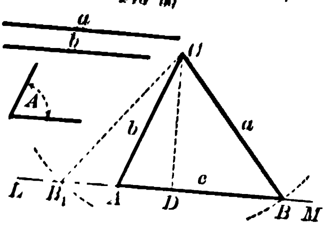


FIG. 52.

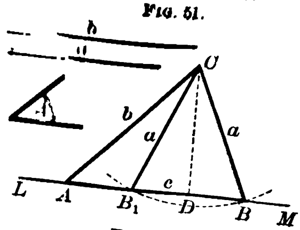


FIG. 53.

elements. At any point A of a straight line LM, unlimited in length, make angle MAC equal to angle A, and cut off AC equal to b. About C as a centre, and with a radius equal to a, describe a circle. This circle will either:

(1) Not reach to LM, as in Fig. 50.

(2) Just reach to LM, thus having LM for a tangent, as in Fig. 51.

(3) Intersect LM in two points, as in Figs. 52, 53.

Each of these possible cases must be considered. In each figure, from C draw CD at right angles to AM; then $CD = b \sin A$.

In case (1), Fig. 50, $CB < CD$, and there is no triangle which can have the given elements. Hence, *the triangle is impossible when $a < b \sin A$.*

In case (2), Fig. 51, $CB = CD$. Hence, *the triangle which has elements equal to the given elements is right-angled when $a = b \sin A$.*

In case (3), Figs. 52, 53, $CB > CD$; that is, $a > b \sin A$. If $a > b$, then the points B, B_1, in which the circle intersects LM, are on opposite sides of A, as in Fig. 52, and *there is one triangle which has three elements equal to the given elements*, namely, ABC. If $a < b$, then the points of intersection B, B_1, are on the same side of A, as in Fig. 53, and *there are two triangles which have elements equal to the given elements*, namely, ABC, AB_1C. For, in ABC, angle $BAC = A$, $AC = b$, $BC = a$; in AB_1C, angle $B_1AC = A$, $AC = b$, $B_1C = a$. *Both triangles must be solved.* In this case, Fig. 53, the given angle is opposite to the smaller of the two given sides. Hence, *there may be two solutions when the given angle is opposite to the smaller of the two given sides.* The words "*may be*" are used, for in cases (1), (2), the given angle is opposite to the smaller of the two given sides. Case II. is sometimes called **the ambiguous case** in the solution of triangles.

The ambiguity in Case II. is also apparent in the trigonometric solution. The angle B is found by means of the relation,

$$\frac{\sin B}{b} = \frac{\sin A}{a}; \text{ or, } \sin B = \frac{b}{a} \sin A. \qquad (1)$$

The angle B is thus determined from its sine. Now there is always an *ambiguity* when an angle of a triangle is determined from its sine alone, for $\sin x = \sin(180° - x)$. Figure 53 shows the two angles which have the same sine, namely, ABC, AB_1C. In Fig. 52, the given condition, namely, that $b < a$, shows that $B < A$; accordingly, only the acute angle corresponding to $\sin B$ can be taken. If, in equation (1), $b \sin A > a$, then $\sin B > 1$, and, accordingly, B is impossible and there is no solution. If, in equation (1), $b \sin A = a$, then $\sin B = 1$, and $B = 90°$. The consideration of the trigonometric equation (1) leads, therefore, to the same results as the preceding geometrical investigation.

Checks: $A + B + C = 180°$, and, as in Case I. Other checks will be found later.

EXAMPLES.

1. Solve the triangle STV, given: $ST = 15$, $VT = 12$, $S = 52°$.

$$\frac{\sin V}{ST} = \frac{\sin S}{VT};$$

$$\frac{\sin V}{15} = \frac{\sin 52°}{12} = \frac{.78801}{12} = .065668.$$

$$\therefore \sin V = 15 \times .065668 = .98502.$$

$$\therefore V = 80° \, 4' \, 20'', \text{ or } 180° - 80° \, 4' \, 20'', \textit{ i.e. } 99° \, 55' \, 40''.$$

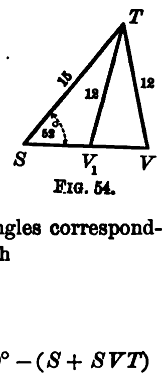


FIG. 54.

Both values of V must be taken, since the given angle is opposite to the smaller of the given sides. The two triangles corresponding to the two values of V are STV, STV_1, Fig. 54, in which

$$SVT = 80° \, 4' \, 20'', \quad SV_1T = 99° \, 55' \, 40''.$$

In STV_1

$$\text{angle } STV_1 = 180° - (S + SV_1T) = 28° \, 4' \, 20''.$$

$$\frac{SV_1}{\sin STV_1} = \frac{V_1T}{\sin S};$$

$$\frac{SV_1}{.47059} = \frac{12}{.78801} = 15.228.$$

$$\therefore SV_1 = 7.17.$$

In STV

$$\text{angle } STV = 180° - (S + SVT) = 47° \, 55' \, 40''.$$

$$\frac{SV}{\sin STV} = \frac{VT}{\sin S};$$

$$\frac{SV}{.74230} = 15.228.$$

$$\therefore SV = 11.3.$$

The solutions are:

$$\left.\begin{aligned} V_1 &= 99° \, 55' \, 40'' \\ T &= 28° \, \ \ 4' \, 20'' \\ SV_1 &= 7.17 \end{aligned}\right\}; \qquad \left.\begin{aligned} V &= 80° \, \ \ 4' \, 20'' \\ T &= 47° \, 55' \, 40'' \\ SV &= 11.3 \end{aligned}\right\}.$$

In the ambiguous case, care must be taken that the calculated sides and angles are combined properly.

2. Solve ABC, given: $a = 29$ ft., $b = 34$ ft., $A = 30° 20'$.
3. Solve ABC when $a = 30$ ft., $b = 24$ ft., $B = 65°$.
4. Solve ABC when $a = 30$ in., $b = 24$ in., $A = 65°$.
5. Solve ABC when $a = 15$ ft., $b = 8$ ft., $B = 23° 25'$.

57. Case III. Given two sides and their included angle. In the triangle ABC, a, b, C are known, and it is required to find A, B, c. In this case, c can be determined from the relation $c^2 = a^2 + b^2 - 2ab\cos C$, Art. 54; angle A can be determined from the relation

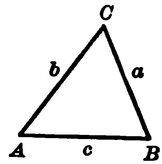


FIG. 55.

$$\frac{\sin A}{a} = \frac{\sin C}{c};$$

angle B can be determined from the relation

$$A + B + C = 180°, \text{ or from } \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Checks: $a^2 = b^2 + c^2 - 2bc\cos A$, $b^2 = a^2 + c^2 - 2ac\cos A$, the result in Ex. 1, Art. 54 a; other checks will be found later.

EXAMPLES.

1. In triangle PQR, $p = 8$ ft., $r = 10$ ft., $Q = 47°$. Find q, P, R.

$$q^2 = p^2 + r^2 - 2pr\cos Q$$
$$= 64 + 100 - 2 \times 8 \times 10 \times .6820 = 54.88.$$
$$\therefore q = 7.408.$$
$$\sin P = \frac{p\sin Q}{q} = \frac{8 \times .7314}{7.408} = .7898. \quad \therefore P = 52° 10'.$$
$$\sin R = \frac{r\sin Q}{q} = \frac{10 \times .7314}{7.408} = .9873. \quad \therefore R = 80° 50'.$$

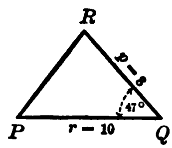


FIG. 56.

2. Solve ABC, given: $a = 34$ ft., $b = 24$ ft., $C = 59° 17'$.
3. Solve ABC, given: $a = 33$ ft., $c = 30$ ft., $B = 35° 25'$.
4. Solve RST, given: $r = 30$ ft., $s = 54$ ft., $T = 46°$.
5. Solve PQR, given: $p = 10$ in., $q = 16$ in., $R = 97° 54'$.

58. Case IV. Three sides given. If the sides a, b, c are known in the triangle ABC, then the angles A, B, C can be found by means of the relations (3), Art. 54.

Checks: Relations (1), Art. 54: $A + B + C = 180°$. Other checks will be shown later.

EXAMPLES.

1. In ABC, $a = 4$, $b = 7$, $c = 10$; find A, B, C.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{49 + 100 - 16}{2 \times 7 \times 10} = \frac{133}{140} = .9500. \qquad \therefore A = 18° 12'.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{100 + 16 - 49}{2 \times 10 \times 4} = \frac{67}{80} = .8375. \qquad \therefore B = 33° 7' 30''.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 49 - 100}{2 \times 4 \times 7} = \frac{-35}{56} = -.6250. \quad \therefore C = 128° 40' 52''.$$

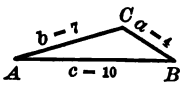


FIG. 57.

Angle C is in the *second* quadrant since its cosine is negative.

Check: $18° 12' + 33° 7' 30'' + 128° 40' 52'' = 180° 0' 22''$. The discrepancy is due to the fact that four-place tables were used in the computation. Had five-place tables been used, the discrepancy would have been less.

2. In PQR, $p = 9$, $q = 24$, $r = 27$. Find P, Q, R.
3. In RST, $r = 21$, $s = 24$, $t = 27$. Find R, S, T.
4. In ABC, $a = 12$, $b = 20$, $c = 28$. Find A, B, C.
5. In ABC, $a = 80$, $b = 26$, $c = 74$. Find A, B, C.
6. Solve Ex. 1, using five-place tables.

59. The aid of logarithms in the solution of triangles. It was pointed out in Art. 6 that an expression is adapted for logarithmic computation when, and only when, it is decomposed into factors. In Cases I., II., Arts. 55, 56, the expressions used in solving the triangle can be computed with the help of logarithms. On the other hand, the side opposite to the given angle in Case III., Art. 57, and the angles in Case IV., Art. 58, are found by evaluating expressions which are not adapted to the use of logarithms. Other relations between the sides and angles of a triangle will be found in Arts. 61, 62. By these relations the computations in Cases III., IV., can be made both without and with logarithms. *These relations are useful not merely for purposes of computation; they are important in themselves, and valuable because many important properties of triangles can be deduced from them.*

The explanations given in Arts. 55–57 are presupposed in Arts. 60–62. The general directions to be observed in working the problems are as follows:

1. *Write down all the formulas which will be used in the computation.*

2. *Express these formulas in the logarithmic form.*

[As soon as the student perceives that this step does not afford any additional assistance, it may be omitted. See Art. 27, Ex. 1, Note 6.]

3. *Make a skeleton scheme, and arrange the arithmetical work neatly and clearly.*

The skeleton schemes in the worked examples that follow, are apparent when the numbers are omitted.*

Checks: The various formulas can serve as checks on the results of one another. The relations derived in Exs. 1, 2, Art. 54 *a*, are also useful as checks.

60. The use of logarithms in Cases I., II. An example worked out, will give sufficient explanation.

EXAMPLES.

1. In ABC, given: $a = 447$, $b = 576$, $A = 47° 35'$. To find: $B =$, $C =$, $c =$ (Write the results here.)

Since $a < b$, there may be two solutions. Construction shows there are two solutions.

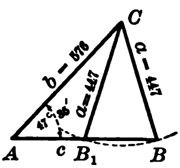


FIG. 58.

Formulas: $\sin ABC = \frac{b}{a} \sin A = \sin AB_1C.$

$ACB = 180° - (A + ABC)$. $ACB_1 = 180° - (A + AB_1C)$.

$AB = \frac{a}{\sin A} \sin ACB.$ $AB_1 = \frac{a}{\sin A} \sin ACB_1.$

$$\therefore \log \sin ABC = \log b + \log \sin A - \log a = \log \sin AB_1C;$$

$$\log AB = \log a + \log \sin ACB - \log \sin A;$$

$$\log AB_1 = \log a + \log \sin ACB_1 - \log \sin A.$$

---

* Cologarithms are not used in the solutions in the text. In extensive computations the use of cologarithms is favoured by many computers; but it seems best for *beginners* in trigonometry first to become accustomed to the obvious and direct method of working with logarithms.

$$\begin{aligned}\log a &= 2.65031\\ \log b &= 2.76042\\ \log \sin A &= \underline{9.86821 - 10}\\ \therefore\ \log \sin B &= 9.97832 - 10\end{aligned}$$

$$\begin{aligned}\therefore\ ABC &= 72^\circ\,2'\,45''\\ \therefore\ ACB &= 60^\circ\,22'\,15''\\ \log\sin ACB &= 9.93914 - 10\\ \therefore\ \log AB &= 2.72124\\ \therefore\ AB &= 526.3\end{aligned}$$

$$\begin{aligned}\text{and } AB_1C &= 107^\circ\,57'\,15''\\ \therefore\ ACB_1 &= 24^\circ\,27'\,45''\\ \log\sin ACB_1 &= 9.61710 - 10\\ \therefore\ \log AB_1 &= 2.39920\\ \therefore\ AB_1 &= 250.7\end{aligned}$$

In obtaining $\log AB$, for instance, $\log\sin ACB$ may be written on the margin of a slip of paper, placed under $\log a$, the addition made, $\log\sin A$ placed beneath, and the subtraction made.

Solve the triangle ABC, when the following elements are given:

2. $A = 63^\circ\,48'$, $B = 49^\circ\,25'$, $a = 825$ ft.
3. $B = 128^\circ\,3'\,49''$, $C = 33^\circ\,34'\,47''$, $a = 240$ ft.
4. $A = 78^\circ\,30'$, $b = 137$ ft., $a = 65$ ft.
5. $a = 275.48$, $b = 350.55$, $B = 60^\circ\,0'\,32''$.
6. $c = 690$, $a = 464$, $A = 37^\circ\,20'$.
7. $a = 690$, $b = 1390$, $A = 21^\circ\,14'\,25''$.

61. Relation between the sum and difference of any two sides of a triangle. The Law of Tangents. Use of logarithms in Case III. In any triangle ABC, for any two sides, say a, b,

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \qquad \text{[By equation (1), Art. 54.]}$$

$$\therefore\ \frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}. \qquad \text{[By composition and division.]}$$

$$= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}. \qquad \text{[Art. 52, Formulas (5), (6).]}$$

$$\therefore\ \frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}. \qquad \text{[Art. 44, A, B.]} \qquad (1)$$

That is, *the difference of any two sides of a triangle is to their sum as the tangent of half the difference of their opposite angles is* *to the tangent of half their sum.* This is sometimes called the **law of tangents.**

Now $A+B=180° - C$, and, consequently, $\frac{1}{2}(A+B)=90° - \frac{C}{2}$. Hence, $\tan\frac{1}{2}(A+B)=\cot\frac{C}{2}$, and, accordingly, relation (1) may be written

$$\tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\tfrac{1}{2}C. \qquad (2)$$

Formulas for b, c, and c, a, similar to the formulas for a, b in (1), (2), can be derived in the same way as (1), (2), have been derived. These formulas can also be written down immediately, on noticing the symmetry in formulas (1), (2).

Ex. *Write* the formulas for sides b, c and c, a. Derive these formulas.

Case III. In a triangle ABC, a, b, C, are known, and c, B, A, are required. Here, $\frac{1}{2}(A+B)=90° - \frac{1}{2}C$; also, $\frac{1}{2}(A-B)$ can be found by (2). Hence, A and B can be found; for

$$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B), \text{ and } B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B).$$

The side c can then be found by (1), Art. 54. (In using (1), (2), write the greater side and the greater angle first, in order that the difference may be positive.) *Formulas* (1), (2), *can also be used as a check in the cases discussed in the preceding articles.* Other checks will be shown in the next article.

EXAMPLES.

1. In triangle ABC, Given: $b = 472$, $c = 324$, $A = 78° 40'$. Find: $B =$, $C =$, $a =$

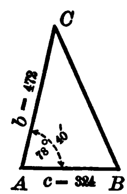


FIG. 59.

Formulas:

$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A.$$

$$\tfrac{1}{2}(B+C)=90° - \tfrac{1}{2}A.$$

$$B=\tfrac{1}{2}(B+C)+\tfrac{1}{2}(B-C).$$

$$C=\tfrac{1}{2}(B+C)-\tfrac{1}{2}(B-C).$$

$$a=\frac{b\sin A}{\sin B};\ \text{or} =\frac{c\sin A}{\sin C}.$$

Checks: $A+B+C=180°$, formulas in preceding articles, and formulas shown in the next article.

$\log\tan\frac{1}{2}(B-C)=\log(b-c)+\log\cot\frac{1}{2}A-\log(b+c),$

$$\log a=\log b+\log\sin A-\log\sin B;\text{ or }=\log c+\log\sin A-\log\sin C.$$

$b=472$	$\log(b-c)=\ \ 2.17026$	$\log b=2.67394$
$c=324$	$\log(b+c)=\ \ 2.90091$	$\log\sin A=9.99145-10$
$A=78°\ 40'$	$\log\cot\frac{1}{2}A=\underline{10.08647-10}$	$\log\sin B=\underline{9.95223-10}$
$b-c=148$	$\therefore\ \log\tan\frac{1}{2}(B-C)=\ \ 9.35582-10$	$\therefore\ \log a=2.71316$
$b+c=796$	$\therefore\ \frac{1}{2}(B-C)=12°\ 47'\ 1''$	$\therefore\ a=516.6$
$\frac{1}{2}A=39°\ 20'$	$\frac{1}{2}(B+C)=50°\ 40'$	
	$\therefore\ B=63°\ 27'\ 1''$	
	$\therefore\ C=37°\ 52'\ 59''$	

Check: $A+B+C=78°\ 40'+63°\ 27'\ 1''+37°\ 52'\ 59''=180°.$

NOTE. Formulas (1), (2), are adapted to logarithmic computation; but the computations can be made without the aid of logarithms.

2. Solve ABC, given $b=352$, $a=266$, $C=73°$.

3. Solve PQR, given $p=91.7$, $q=31.2$, $R=33°\ 7'\ 9''$.

4. Solve ABC, given $a=960$, $b=720$, $C=25°\ 40'$.

5. Solve ABC, given $b=9.081$, $c=3.6545$, $A=68°\ 14'\ 24''$.

6. Solve Exs. 1, 5, Art. 57, using the formulas of this article, without logarithms.

62. Trigonometric ratios of the half angles of a triangle. Use of logarithms in Case IV. In any triangle ABC,

$$\cos A=\frac{b^2+c^2-a^2}{2bc}.\qquad\text{[Art. 54 (4).]}$$

Now $$1-\cos A=2\sin^2\tfrac{1}{2}A,$$

and $$1+\cos A=2\cos^2\tfrac{1}{2}A.\qquad\text{[Art. 50 (7), (8).]}$$

$$\text{Also, }1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-(b^2+c^2-a^2)}{2bc}$$

$$=\frac{a^2-(b^2+c^2-2bc)}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{[a-(b-c)][a+(b-c)]}{2bc}.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}A=\frac{(a-b+c)(a+b-c)}{2bc}\qquad(1)$$

Also, $1+\cos A = 1 + \dfrac{b^2+c^2-a^2}{2\,bc} = \dfrac{2\,bc + (b^2+c^2-a^2)}{2\,bc}$

$$= \frac{(b+c)^2-a^2}{2\,bc}.$$

$$\therefore\ 2\cos^2 \tfrac{1}{2} A = \frac{(b+c+a)(b+c-a)}{2\,bc}. \qquad (2)$$

Let $\qquad a+b+c=2\,s;$

then $\qquad 2(s-c)=(a+b+c)-2\,c=a+b-c.$

Similarly $\qquad 2\,(s-b)=a-b+c,$

$\qquad 2\,(s-a)=-a+b+c.$

The substitution of these values in (1) and (2) gives

$$2\sin^2 \tfrac{1}{2} A = \frac{2\,(s-b)\cdot 2\,(s-c)}{2\,bc};\quad 2\cos^2 \tfrac{1}{2} A = \frac{2\,s\cdot 2\,(s-a)}{2\,bc}.$$

$$\therefore\ \sin^2 \tfrac{1}{2} A = \frac{(s-b)(s-c)}{bc}; \qquad (3)$$

$$\cos^2 \tfrac{1}{2} A = \frac{s\,(s-a)}{bc}. \qquad (4)$$

Since $\tan^2 \frac{1}{2} A = \sin^2 \frac{1}{2} A \div \cos^2 \frac{1}{2} A$, it follows that

$$\tan^2 \tfrac{1}{2} A = \frac{(s-b)(s-c)}{s\,(s-a)}. \qquad (5)$$

$$\therefore\ \sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}};\quad \cos \tfrac{1}{2} A = \sqrt{\frac{s\,(s-a)}{bc}};$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s\,(s-a)}}.$$

NOTE. By geometry, $b+c>a$. Hence, $-a+b+c>0$, and, accordingly, $s-a$ is positive. Similarly, $s-b$, $s-c$, are also positive. Therefore, the quantities under the radical signs are positive. The positive sign must be given to the radical, for A is less than 180°, and consequently $\frac{1}{2} A$ lies between 0° and 90°.

Similar formulas hold for $\frac{1}{2}B$ and $\frac{1}{2}C$. They can be deduced in the same manner as those for $\frac{1}{2}A$; or, they can be written immediately, from the symmetry apparent in the formulas (3)–(5). The student is advised to derive the similar formulas for $\frac{1}{2}B$, $\frac{1}{2}C$, viz.:

$$\sin^2 \tfrac{1}{2} B = \frac{(s-a)(s-c)}{ac}; \qquad \cos^2 \tfrac{1}{2} B = \frac{s(s-b)}{ac}; \tag{3'}$$

$$\sin^2 \tfrac{1}{2} C = \frac{(s-a)(s-b)}{ab}. \qquad \cos^2 \tfrac{1}{2} C = \frac{s(s-c)}{ab}. \tag{4'}$$

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2} B &= \frac{(s-a)(s-c)}{s(s-b)}; \\ \tan^2 \tfrac{1}{2} C &= \frac{(s-a)(s-b)}{s(s-c)}. \end{aligned}\right. \tag{5'}$$

Formula (5) can be given a more symmetrical form; for, on multiplying and dividing its second member by $(s-a)$,

$$\tan^2 \tfrac{1}{2} A = \frac{(s-a)(s-b)(s-c)}{s(s-a)^2};$$

whence
$$\tan \tfrac{1}{2} A = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{6}$$

If
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \tag{7}$$

then
$$\tan \tfrac{1}{2} A = \frac{r}{s-a}. \tag{8}$$

Similarly,
$$\tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}. \tag{8'}$$

When all the sides are known, the angles can be found by means of formulas (3)–(5′) or, by (7)–(8′). When *all* the angles are required, the tangent formulas are better, since fewer logarithms are required than in (3), (4), (3′), (4′). It will be shown in Art. 69 that *r is the radius of the circle inscribed in the triangle.*

EXAMPLES.

1. In triangle ABC, $a = 25.17$, $b = 34.06$, $c = 22.17$. Find A, B, C.

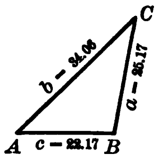


FIG. 60.

Formulas: $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}; \quad \tan \tfrac{1}{2} B = \frac{r}{s-b}; \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

$$\therefore \log r = \tfrac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) - \log s].$$

$$\log \tan \tfrac{1}{2} A = \log r - \log(s-a); \quad \log \tan \tfrac{1}{2} B = \log r - \log(s-b);$$

$$\log \tan \tfrac{1}{2} C = \log r - \log(s-c).$$

Check: $A + B + C = 180°.$

$a = 25.17$	$\log s = 1.60959$	$\log \tan \frac{1}{2} A = 9.64465 - 10$
$b = 34.06$	$\log(s-a) = 1.19117$	$\frac{1}{2} A = 23° 48' 28''$
$c = 22.17$	$\log(s-b) = 0.82217$	$\log \tan \frac{1}{2} B = 10.01365 - 10$
$2s = 81.40$	$\log(s-c) = 1.26788$	$\frac{1}{2} B = 45° 54'$
$s = 40.70$	$\therefore \log r^2 = 1.67163$	$\log \tan \frac{1}{2} C = 9.56794 - 10$
$s - a = 15.53$	$\therefore \log r = 0.83582$	$\frac{1}{2} C = 20° 17' 35''$
$s - b = 6.64$		
$s - c = 18.53$		

$$\therefore A = 47° 36' 56'', \quad B = 91° 48', \quad C = 40° 35' 10''$$

Check: $A + B + C = 180° 0' 6''.$

2. Solve ABC, given $a = 260$, $b = 280$, $c = 300$.
3. Solve ABC when $a = 26.19$, $b = 28.31$, $c = 46.92$.
4. Solve PQR, given $p = 650$, $q = 736$, $r = 914$.
5. Solve RST, given $r = 1152$, $s = 2016$, $t = 2592$.
6. Solve Exs. 1, 4, Art. 58, using formulas (3)–(8′), without logarithms.

63. Problems in heights and distances. Some problems in heights and distances have been solved in Art. 29 by the aid of right-angled triangles. Additional problems of the same kind will now be given, in the solution of which oblique-angled triangles may be used. It is advisable to draw the figures neatly and accurately. The graphical method should also be employed.

EXAMPLES.

1. Another solution of Ex. 2, Art. 29.

In the triangle ABP (Fig. 23), $AB = 100$ ft., $BAP = 30°$, $PBA = 180° - 45° = 135°$. Hence the triangle can be solved, and BP can be found. When BP shall have been found, then in the triangle CBP, BP is known and $BP = 45°$; hence CP can be found. The computation is left to the student.

2. Another solution of Ex. 3, Art. 29. In the triangle CBP (Fig. 24), $BP = 30$ ft., $BCP = 40° 20' - 38° 20' = 2°$, $PBC = 90° + LCB = 128° 20'$. Hence CBP can be solved and the length of CB can be found. When CB shall have been found, then, in the triangle LCB, angle $C = 38° 20'$, CB is known, and hence LB can be found. The computation is left to the student.

3. Find the distance between two objects that are invisible from each other on account of a wood, their distances from a station at which they are visible being 441 and 504 yd., and the angle at the station subtended by the distance of the objects being 55° 40′.

4. The distance of a station from two objects situated at opposite sides of a hill are 1128 and 936 yd., and the angle subtended at the station by their distance, is 64° 28′. What is their distance?

5. Find the distance between a tree and a house on opposite sides of a river, a base of 330 yd. being measured from the tree to another station, and the angles at the tree and the station formed by the base line and lines in the direction of the house being 73° 15′ and 68° 2′, respectively. Also find the distance between the station and the house.

6. Find the height of a tower on the opposite side of a river, when a horizontal line in the same level with the base and in the same vertical plane with the top is measured and found to be 170 ft., and the angles of elevation of the top of the tower at the extremities of the line are 32° and 58°, the height of the observer's eye being 5 ft.

7. Find the height of a tower on top of a hill, when a horizontal base line on a level with the foot of the hill and in the same vertical plane with the top of the tower is measured and found to be 460 ft.; and at the end of the line nearer the hill the angles of elevation of the top and foot of the tower are 36° 24′, 24° 36′, and at the other end the angle of elevation of the top of the tower is 16° 40′.

8. A church is at the top of a straight street having an inclination of 14° 10′ to the horizon; a straight line 100 ft. in length is measured along the street in the direction of the church; at the extremities of this line the angles of elevation of the top of the steeple are 40° 30′, 58° 20′. Find the height of the steeple.

9. The distance between the houses C, D, on the right bank of a river and invisible from each other, is required. A straight line AB, 300 yd. long, is measured on the left bank of the river, and angular measurements are taken as follows: $ABC = 53° 30'$, $CBD = 45° 15'$, $CAD = 37°$, $DAB = 58° 20'$. What is the length CD?

10. A tower CD, C being the base, stands in a horizontal plane; a horizontal line AB on the same level with the base is measured and found to be 468 ft.; the horizontal angles BAC, ABC, are equal to 125° 40′, 12° 35′, respectively, and the vertical angles CAD, CBD, are equal to 30° 20′, 11° 50′, respectively. Find the height of the tower and its distances from A and B.

11. A base line AB 850 ft. long is measured along the straight bank of a river; C is an object on the opposite bank; the angles BAC, ABC, are observed to be $63° 40'$, $37° 15'$, respectively. Find the breadth of the river.

12. A tower subtends an angle α at a point on the same level as the foot of the tower and, at a second point, h feet above the first, the depression of the foot of the tower is β. Show that the height of the tower is $h \tan \alpha \cot \beta$.

13. The elevation of a steeple at a place due south of it is 45°, and at another place due west of it the elevation is 15°. If the distance between the two places be a, prove that the height of the steeple is $a(\sqrt{3}-1) \div 2\sqrt[4]{3}$.

14. The elevation of the summit of a hill from a station A is α; after walking c feet toward the summit up a slope inclined at an angle β to the horizon the elevation is γ. Show that the height of the hill above A is $c \sin \alpha \sin (\gamma - \beta) \operatorname{cosec} (\gamma - \alpha)$ ft.

64. Summary. The preceding discussions on the solution of triangles have shown that a triangle may be solved in the following ways:

I. By the graphical method. [Arts. 10, 14, 21–24.]

II. If the triangle is right-angled, it can be solved, either with or without logarithms, by the methods shown in Arts. 25–27.

III. If the triangle is oblique, it can be divided into right-angled triangles, each of which can be solved by either of the methods II. [Art. 34.]

IV. The triangle, whether right-angled or oblique, can be solved without using logarithms, by means of formulas (1), (3), Art. 54; (1) or (2), Art. 61; (3)–(8), Art. 62.

V. The triangle, whether right-angled or oblique, can be solved with the use of logarithms, by means of formulas (1), Art. 54; (1) or (2), Art. 61; (3)–(8), Art. 62.

Checks: Any formula not employed in the computation can be employed as a check; that is, as a test for the correctness of the result.

Two things are necessary on the part of one who wishes to do well in the solution of triangles:

(1) The formulas referred to above should be clearly understood and readily derived.

(2) The arithmetical work required should be done accurately.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of this chapter will be found at pages* 189–193.

CHAPTER VIII.

SIDE AND AREA OF A TRIANGLE. CIRCLES CONNECTED WITH A TRIANGLE.

65. Length of a side of a triangle in terms of the adjacent sides and the adjacent angles. In this proof, regard is paid to the conventions about signs, described in Arts. 36, 37. Let ABC be any triangle. From A draw AD perpendicular to BC, or BC produced. The positive direction of BC is in the direction of V. [At the first reading, only Fig. 61 a may be regarded.]

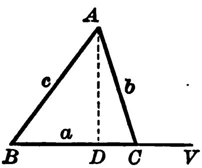


FIG. 61a.

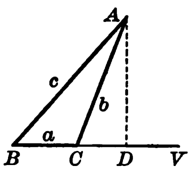


FIG. 61b.

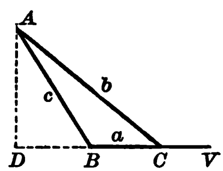


FIG. 61c.

$$BC = BD + DC,$$
$$= -CD + BD,$$
$$= -CA \cos VCA + BA \cos VBA. \quad \text{[Art. 40.]}$$

But $\qquad VCA = 180° - ACB.$

$$\therefore \cos VCA = \cos(180° - ACB) = -\cos ACB. \quad \text{[Art. 45.]}$$
$$\therefore BC = CA \cos ACB + BA \cos CBA;$$

i.e. $$\boldsymbol{a = b \cos C + c \cos B.} \tag{1}$$

Therefore, in any triangle each side is equal to the sum of the products of each of the other sides by the cosine of the angle which it makes with the first side.

When C is a right angle, (1) reduces to $a = c \cos B$.

EXAMPLE. Write the corresponding formulas for b and c. Derive these formulas.

66. Area of a triangle. Suppose that the area of a triangle ABC is required. Let the length of the perpendicular DC from

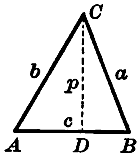


FIG. 62a.

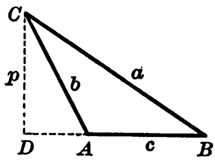


FIG. 62b.

C to AB, or AB produced, be denoted by p, and let the area be denoted by S. The following cases may occur:

I. *One side and the perpendicular on it from the opposite angle known,* say (c, p).

$$S = \tfrac{1}{2} cp. \quad \text{[By geometry.]} \qquad (1)$$

II. *Two sides and their included angle known,* say, b, c, A. (See Figs. 62 *a*, 62 *b*.)

$$S = \tfrac{1}{2} cp = \tfrac{1}{2} c \cdot AC \sin BAC. \qquad \text{[Art. 40.]}$$

$$\therefore\ S = \tfrac{1}{2} bc \sin A. \quad \text{[Compare Art. 31.]} \quad (2)$$

III. *Three sides known.*

$$S = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} bc \cdot 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A, \qquad \text{[Art. 50.]}$$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}. \quad \text{[Art. 62.]}$$

$$\therefore\ S = \sqrt{s(s-a)(s-b)(s-c)}. \qquad (3)$$

That is, *the area of a triangle is equal to the square root of the product of half the sum of the sides by the three factors formed by subtracting each side in turn from this half sum.* See Art. 34 *a* for another derivation of this formula.

IV. *One side and the angles known,* say, a, A, B, C.

$$S = \tfrac{1}{2} ab \sin C. \qquad \text{Now } b = \frac{a \sin B}{\sin A}.$$

$$\therefore\ S = \frac{1}{2}\,\frac{a^2 \sin C \sin B}{\sin A}. \qquad (4)$$

EXAMPLE. **Write and also derive the similar formulas in *b* and *c*.**

EXAMPLES.

1. Find the areas of the triangles in Exs. 1–5, Art. 61.

2. Find the areas of the triangles in Exs. 1–5, Art. 62.

3. Find the areas of the triangles in Exs. 2, 3, Art. 60.

67. Area of a quadrilateral in terms of its diagonals and their angle of intersection.

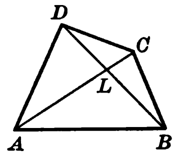


FIG. 63.

Area $ABCD$ = area ADC + area ABC.

$$\begin{aligned}\text{Area } ADC &= \text{area } ALD + \text{area } CLD\\ &= \tfrac{1}{2}\, AL \cdot LD \sin ALD\\ &\quad + \tfrac{1}{2}\, CL \cdot LD \sin CLD \quad [\text{Art. 66 (2).}]\\ &= \tfrac{1}{2}(AL + LC) DL \sin ALD, \text{ (since } \sin CLD = \sin ALD)\\ &= \tfrac{1}{2}\, AC \cdot DL \sin ALD.\end{aligned}$$

Similarly,

$$\text{area } ABC = \tfrac{1}{2}\, AC \cdot BL \sin BLC = \tfrac{1}{2}\, AC \cdot BL \sin ALD.$$

$$\therefore \text{ area } ABCD = \tfrac{1}{2}\, AC(DL + LB) \sin DLA = \tfrac{1}{2}\, AC \cdot BD \sin DLA.$$

$\therefore$ area of a quadrilateral is equal to one-half the product of the two diagonals and their angle of intersection.

EXAMPLES.

1. Find the area of a quadrilateral whose diagonals are 108, 240 ft. long, and inclined to each other at an angle $67° 40'$. Find the sides and angles of a parallelogram having these diagonals.

2. So also when the diagonals are 360, 570 ft. long, and their inclination is $39° 47'$.

3. The diagonals of a parallelogram are 347 and 264 ft., and its area is 40,437 sq. ft. Find its sides and angles.

4. Solve an isosceles trapezoid, knowing the parallel sides $a = 682.7$ metres, $c = 1242.6$ metres, and the non-parallel equal sides $b = d = 986.4$ metres. Find the angles, the area, the lengths and angle of inclination of the diagonals.

68. The circumscribing circle. Let the radius of the circle described about a triangle ABC be denoted by R. It has been shown (see equation (2), Art. 54) that

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}. \tag{1}$$

That is, *the radius of the circumscribing circle of any triangle is equal to half the quotient of any side by the sine of the opposite angle.*

From (2), Art. 66, $\sin A = \frac{2S}{bc}$. Substitution of this in the first of equations (1), gives

$$R = \frac{abc}{4S}. \tag{2}$$

69. The inscribed circle. Let the radius of the circle inscribed in a triangle ABC be denoted by r. Join the centre O and the points of contact L, M, N. By geometry, the angles at L, M, N are right angles. Draw OA, OB, OC.

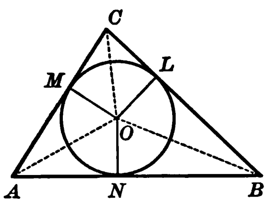


FIG. 64.

$$\text{Area } BOC + \text{area } COA + \text{area } AOB = \text{area } ABC.$$

$$\therefore\ \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = \sqrt{s(s-a)(s-b)(s-c)}, \text{ or } S.$$

$$\therefore\ \tfrac{1}{2}(a+b+c)r = S,$$

i.e.
$$sr = S.$$

$$\therefore\ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{S}{s}. \tag{3}$$

That is, *the length of the radius of the inscribed circle of a triangle is equal to the number of units in its area divided by half the sum of the lengths of its sides.* See reference in Art. 62.

NOTE. Formula (8), Art. 62, can be readily derived from *Fig.* 64. By geometry, $AN = MA$, $BL = NB$, $CM = LC$.

Now
$$\tan \tfrac{1}{2} A = \tan BAO = \frac{NO}{AN}.$$

But $NO = r$, and

$$AN = (AN + BL + CM) - (BL + LC) = s - a.$$

$$\therefore\ \tan \tfrac{1}{2} A = \frac{r}{s-a}.$$

70. The escribed circles. An *escribed circle* of a triangle is a circle that touches one of the sides of the triangle and the other two sides produced.

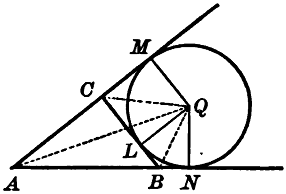


FIG. 65.

Let r_a denote the radius of the escribed circle touching the side BC opposite to the angle A. Join the centre Q and the points of contact L, M, N. By geometry, the angles at L, M, N are right angles. Draw QA, QB, QC.

$$\text{Area } ABQ + \text{area } CAQ - \text{area } BCQ = \text{area } ABC.$$

$$\therefore\ \tfrac{1}{2} r_a c + \tfrac{1}{2} r_a b - \tfrac{1}{2} r_a a = S,$$

$$\therefore\ \tfrac{1}{2}(c + b - a) r_a = S;$$

i.e.

$$(s - a) r_a = S.$$

$$\therefore\ \boldsymbol{r_a = \frac{S}{s-a}}.$$

Similarly,

$$r_b = \frac{S}{s-b};\quad r_c = \frac{S}{s-c}.$$

Other interesting relations between the sides, angles, and related circles, of a triangle, are indicated in the exercises in the latter part of the book.

EXAMPLES.

1. Find the radii of the circumscribed, inscribed, and escribed circles of some of the triangles in Arts. 55–58.

2. Find the radii of these related circles of some of the triangles in Exs. 1–3, Art. 66.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of this Chapter will be found at pages* 193, 194.

CHAPTER IX.

RADIAN MEASURE.

71. The radian defined. The system of measuring angles with a degree as the unit angle, was described in Art. 11. Since the time of the Babylonians this system has been the common practical method employed. Another method of measuring angles was introduced early in the last century. This method is used to some extent in practical work, and is universally used in the higher branches of mathematics. It is employed, on account of

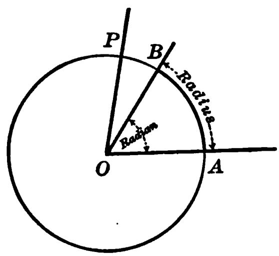


FIG. 66.

its great convenience, in the larger and more important part of what is now called trigonometry, namely, the part which is not concerned with the measurement of lines and angles, but which pursues investigation of the properties of the quantities that, so far in this book, have been called the trigonometric ratios. A very little knowledge of the trigonometric ratios is sufficient for the solution of triangles. The more detailed and extended study of angles and their six related numbers, constitutes part of what is sometimes called *Higher Trigonometry*, but, more generally, *Analytical Trigonometry*. This subject is a large one, and has close connections with many other branches of modern mathematics.

The system of angular measurement now to be described, is sometimes referred to as the *theoretical system* of measurement. **In this system the unit angle is the angle which at the centre of a circle subtends an arc equal in length to the radius.** This unit angle is called **a radian.** Thus, if a circle with any radius be described about O as a centre, and an arc AB be taken equal in length to the radius, then the angle AOB is a radian.

72. The value of a radian. In order that a quantity may be used as a unit of measurement, it must have a fixed value; that is, using the customary mathematical phrase, it must be a *constant* quantity. The proof that a radian has a fixed value, or is a constant quantity, depends upon two geometrical facts, viz.:

(*a*) In the same circle two angles at the centre are in the same ratio as their intercepted arcs.

(*b*) The ratio of a circumference of a circle to its diameter is the same for all circles. [See Art. 9 (*b*).]

For the proof of (*a*), reference may be made to any plane geometry; for instance, to Euclid VI., 33.* The proof of (*b*) is not contained in all geometries; for instance, Euclid does not give it.† Accordingly, an outline of such a proof and the calculation of π are given in Note C of the Appendix. This note should now be studied by those whose course in plane geometry has not included

* The truth of theorem (*a*) can easily, by an inductive method, be made evident to students who have not proved the theorem in plane geometry. Thus, on taking angles which are twice, three times, four times, one-half, one-third, etc., of a given angle, it can be seen that their respective arcs bear the same relations to one another.

† Euclid lived about 323–283 B.C. Archimedes (287?–212 B.C.), the greatest mathematician of antiquity, measured the length of the circle and the area contained by it, and also measured the surface of the sphere. He showed that the ratio of the circle to its diameter lies between $\frac{223}{71}$ and $\frac{22}{7}$. In 1794 a French mathematician, Adrien Marie Legendre (1752–1833), published his *Elements of Geometry*, in which the works of Euclid and Archimedes on elementary geometry are blended together. The elementary text-books now in use on the continent of Europe and in the United States, are written mainly on Legendrean lines; the geometrical text-books generally studied throughout the British Empire, are editions of Euclid's *Elements*.

the measurement of the circle. Theorems (*a*) and (*b*) are assumed in what follows.

$$\text{In Fig. 66, } \frac{\textit{the radian } AOB}{4 \textit{ right angles}} = \frac{\textit{arc } AB}{\textit{circumference of circle}} \quad [\text{By } (a).]$$

$$= \frac{r}{2\pi r} = \frac{1}{2\pi} \quad [\text{By } (b).]$$

$$\therefore \textbf{ the radian} = \frac{1}{2\pi} \times 4 \text{ right angles} = \frac{2}{\pi} \times \textbf{right angle.} \quad (1)$$

Since all right angles are equal, and since each radian is a fixed fraction, namely, $\frac{2}{\pi}$, of a right angle, it follows that *all radians are equal.* It will be remembered that the unit in the common practical system is one-ninetieth of a right angle.

From (1),

$$\textbf{A radian} = \frac{180^\circ}{\pi} \quad (2)$$

$$= \frac{180^\circ}{3.14159 \cdots} = 57^\circ\ 17'\ 44''.81 \text{ approximately}^*$$

$$= 206265'' \text{ approximately.}$$

Ex. With a protractor lay off an angle approximately equal to a radian. Compare it with angle 60°. An angle 60°, at centre of a circle, is subtended by *a chord* equal in length to the radius; a radian is subtended by *an arc* equal in length to the radius.

73. The radian measure of an angle. Measure of a circular arc. *The radian measure of an angle is the ratio of the angle to a radian.* [See Art. 8.] For instance, if an angle A is twice a radian, then its radian measure is 2; if an angle B is two-thirds of a radian, then its radian measure is $\frac{2}{3}$. This is expressed thus:

$$A = 2 \text{ radians} = 2^r; \quad B = \tfrac{2}{3} \text{ radians} = \tfrac{2}{3}^r. \quad (3)$$

Here, r is used as the symbol for radians just as ° is used as the symbol for degrees in 23°. In general discussions the radian

* The value of the radian has been calculated by J. W. L. Glaisher to 41 places of decimals of a second. [*Proc. Lond. Math. Soc.*, Vol. IV. (1871–73), pp. 308–312.]

measure of an angle is often expressed by Greek letters; thus, the angles α, β, θ, ϕ, etc., contain α, β, θ, ϕ, etc., radians. In these cases the symbol r is usually omitted, but it is always *understood* that the radian is the unit of measurement.

If the circular arc subtended by an angle is equal in length to *twice* the radius, then the radian measure of the angle is obviously *two;* if the arc is *one-half* the length of the radius, then the angle contains *half* a radian. *The radian measure of an angle may be given* **a second definition,** which depends on Theorem (a), Art. 72. Let AOP, Fig. 66, be any angle, and AOB be a radian. Describe a circle with any radius OA, equal to r, about the vertex O as a centre. Let arc AB be equal to the radius, and draw OB. Then angle AOB is a radian, by the definition in Art. 71.

Now, $$\frac{\text{angle } AOP}{\text{radian } AOB} = \frac{\text{arc } AP}{\text{arc } AB} \quad \text{[Th. } (a)\text{, Art. 72.]} \tag{4}$$

i.e. $$\frac{\text{angle } AOP}{\text{the radian}} = \frac{\text{arc } AP}{\text{the radius}}. \tag{5}$$

That is, the number of radians in an angle, or **the radian measure of an angle,** is the answer to the question: **how many times does any circular arc subtended by it, contain the radius?** Thus, for example, the radian measures of the angles which subtend circular arcs equal in length to 2, 3, 1.5, .825 radii are 2, 3, 1.5, .825, respectively.

The circular arc subtended by $360° = 2\pi r$;

hence, $$\text{radian measure of } 360° = \frac{2\pi r}{r} = 2\pi,$$

and $$\text{radian measure of } 180° = \pi. \tag{6}$$

This shows that an angle 2π radians is described each time that the revolving line makes a complete revolution. Relation (6), namely,

$$\mathbf{180° = \pi} \text{ radians,} \tag{7}$$

connects the two systems of angular measurement. By means of (7), an angle expressed in the one system can be expressed in the other. The word *radians* is usually omitted from (7), but is always understood. Relation (7) may also be deduced directly from (1). Just as angles are considered as unlimited in magnitude, so arcs are considered as unlimited in length.

NOTE 1. The term *circular measure* is often used for *radian measure*, and c is used as the symbol for radians. Thus (3) is written $A = 2^c$, $B = \frac{2}{3}^c$.

Ex. 1. Express 30° in radian measure.

Since $$180° = \pi,\ 1° = \frac{\pi}{180};$$

$$30° = \tfrac{30}{180}\pi = \frac{\pi}{6}. \quad \text{Also, } 30° = \frac{\pi}{6} = \frac{3.14159^r \cdots}{6} = .52359^r \cdots.$$

The term *radians* and the symbol for radian is usually omitted from the second members of these equations, but is always *understood* to be there.

Ex. 2. Express 45°, 60°, 135°, 210°, 300°, 330°, 270°, 225°, $-75°$, 63°, 27°, $-33°$, $-150°$, in radian measure, (*a*) as fractions of π, (*b*) numerically, on putting $\pi = \frac{22}{7}$.

Ex. 3. Express the angle $\frac{9}{10}\pi$ in degrees.

Here, "$\frac{9}{10}\pi$" means "$\frac{9}{10} \times \frac{22}{7}$ *radians*."

Since $$\pi = 180°,\ \tfrac{9}{10}\pi = \tfrac{9}{10} \times 180° = 162°.$$

Ex. 4. Express the angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{5}, \frac{3}{4}\pi, \frac{7}{8}\pi, 10\pi, 4\pi, 3\pi, 6\pi, \frac{5}{2}\pi, \frac{5}{3}\pi$, in degrees, and their complements and supplements, in radians.

Ex. 5. Express the angles $-\frac{3}{4}\pi, -5\pi, -\frac{4}{3}\pi, -\frac{11}{3}\pi, -25\pi$, in degrees.

Ex. 6. Express 2^r (2 radians) in degrees.

Since $$\pi = 180°,$$

$$\therefore\ 1^r = \frac{180°}{\pi};$$

$$\therefore\ 2^r = 2 \times \frac{180°}{\pi} = 114°\,35'\,29.6'' \text{ approximately.}$$

Ex. 7. Express $\frac{1^r}{2}, 4^r, 3^r, \frac{1^r}{3}, \frac{1^r}{5}, 5^r, 10^r, \frac{1^r}{10}$, in degrees.

Measure of an arc. Since, by (5),

$$\frac{\text{subtended circular arc}}{\text{radius}} = \text{number of radians in the angle,}$$

then **length of arc = radius × number of radians in the angle.**

If a denote the length of any arc AP, r the radius, θ the radian measure of angle AOP, then

$$a = r\theta. \tag{8}$$

In words: **The length of any circular arc is equal to the product of the radius and the radian measure of its subtended central angle.** For example, the arc of $360° = 2\pi$ radii, arc of $180° = \pi$ radii, etc. These arcs are usually referred to as the arcs 2π, π, etc.; but it is always *understood* that *the radius is the unit of measurement.* The symbol π, which always denotes the incommensurable number 3.14159 ⋯, can thus be used in *three* connections in trigonometry:

(1) With other numbers, as a number simply.

(2) With reference to *angles;* in which case it denotes an angle containing π radians, *i.e.* 3.14159 ⋯ radians.

(3) With reference to *arcs;* in which case it denotes an arc containing 3.14159 radii. This is an arc subtended by a central angle of π radians.

The expression $180° = \pi$ does not mean $180° = 3.14159 \cdots$; it means $180° = 3.14159$ *radians.*

The expression "arc π" does not mean arc 3.1416; it means "arc of 3.1416 *radii.*" In any particular instance, the context will show to what π refers, whether to *angle* or *arc.*

It is evident from the second definition of radian measure that, like the trigonometric ratios, **the radian measure of an angle is also a ratio of one line to another,** namely, the ratio of the subtended circular *arc* to its *radius.*

NOTE 2. *If the radius be taken as unit length, then, by* (8) *or* (5), *the number of units of length in the arc is the same as the number of radians in the angle.*

EXAMPLES.

8. What is the radian measure of the angle which at the centre of a circle of radius $1\frac{1}{2}$ yd. subtends an arc of 8 in.? Also express the angle in degrees.

Let θ denote the radian measure of the angle. Then

$$\theta = \frac{\text{arc}}{\text{rad}} = \frac{8 \text{ in.}}{1.5 \text{ yd.}} = \frac{8}{54} = \frac{4}{27}.$$

Since $$\pi = 180°,$$

$$\therefore\ 1^r = \frac{180}{\pi};$$

$$\therefore\ (\tfrac{4}{27})^r = \tfrac{4}{27} \times \tfrac{7}{22} \times 180° = 8°\,29' \text{ approximately.}$$

9. Give the trigonometric ratios of

$$\frac{\pi}{6},\ \frac{\pi}{4},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \pi,\ \tfrac{3}{4}\pi,\ -\tfrac{5}{3}\pi,\ -\tfrac{7}{4}\pi,\ -\tfrac{5}{6}\pi.$$

10. Find the numerical values of (*a*) $\sin^2\dfrac{\pi}{6} + \cos^2\tfrac{7}{4}\pi + \tan^2\dfrac{2\pi}{3}$,

(*b*) $3\sin\tfrac{5}{6}\pi\cos\tfrac{17}{3}\pi\tan\tfrac{23}{4}\pi$, (*c*) $2\sin\tfrac{23}{3}\pi\cos\tfrac{23}{3}\pi\tan\tfrac{23}{3}\pi$.

11. Find the number of radians (*a*) as fractions of π, (*b*) numerically (on putting $\pi = \frac{22}{7}$), in each interior and exterior angle of the following regular polygons: pentagon, hexagon, heptagon, octagon, decagon, dodecagon, quindecagon.

12. Find the number of radians and the number of degrees in the following angles subtended at the centres of circles: (1) arc 10 in., radius 3.5 in.; (2) arc $\frac{1}{7}$ ft., radius 2 ft.; (3) arc 1 mi., radius 7920 mi.; (4) arc 250 mi., radius 8000 mi.; (5) arc 10 yd., radius 10 mi.; (6) arc $\frac{1}{10}$ mi., radius 10 ft.

13. What are the radii when an arc 10 in. in length subtends central angles containing 1, 2, 4, 6, 8, 12, 15, 20, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{8}$, radians respectively?

14. What are the radii when an arc 10 in. in length subtends central angles containing 1°, 2°, 3°, 16°, 28°, 120°, 30′, 20′, 10′, 10″, 20″, 45″, respectively?

15. In a circle whose radius is 10 in., what are the lengths of the arcs subtended by central angles containing 1, 4, 7, 8, 12, .5, .375, .125, radians respectively?

16. In the circle in Ex. 15, what are the lengths of the arcs subtended by central angles containing 2°, 25°, 48°, 135°, 250°, 30′, 45′, 30″, 50″, respectively?

17. What are the areas of the circular sectors in Exs. 13, 15? [See Note C, 5.]

N.B. *Questions and exercises suitable for practice and review on the subject-matter of this Chapter will be found at pages* 194, 195.

CHAPTER X.

ANGLES AND TRIGONOMETRIC FUNCTIONS.

74. Chapters II., V., contain little more about the trigonometric ratios than is needed in the solution of triangles. In this and the following chapters a further study of these ratios is made. Although the results of this study are not applicable to such ordinary practical uses as the measurement of triangles, heights, and distances, yet they are very interesting in themselves, and help to give a better and fuller understanding of the connection between angles and trigonometric ratios. These results are also useful in further mathematical work, and in the study of various branches of mechanical and physical science. In reading Chapters X., XI., acquaintance will be made, or renewed, with some important general ideas of mathematics.

75. Function. Trigonometric functions. If a number is so related to one or more other numbers, that its values depend upon their values, then it is *a function* of these other numbers. Thus the circumference of a circle is a function of its radius; the area of a rectangle is a function of its base and height; the area of a triangle is a function of its three sides.

NOTE. The values of such expressions as $2x - 5$, $x^2 - 4x + 7$, $\log_{10} x$, 2^x, depend upon the values given to x. These expressions are, accordingly, functions of x. A function of x is usually denoted by one of the symbols $f(x)$, $F(x)$, $\phi(x)$, etc., which are read "the f-function of x," "the F-function of x," "the *Phi*-function of x," etc.

The trigonometric ratios of an angle depend upon the value (*i.e.* magnitude) of the angle. On this account the trigonometric ratios are very often called the *trigonometric functions*. They are also frequently called the *circular functions*.

The trigonometric (or circular) functions include not only the *six* functions previously discussed, namely, *sine, cosine, tangent, cotangent, secant, cosecant,* but also three others, viz.:

versed sine of $A = 1 - \cos A$, written **vers** A,

coversed sine of $A = 1 - \sin A$, written **covers** A,

suversed sine of $A = 1 + \cos A$, written **suvers** A.

The versed sine is used not unfrequently; the latter two are rarely used.

EXAMPLES.

Find the remaining eight trigonometric functions when:

1. $\sin A = .3$.
2. $\cos A = .4$.
3. $\tan A = -3$.
4. $\cot A = .7$.
5. $\sec A = -3$.
6. $\operatorname{cosec} A = .8$.
7. $\operatorname{vers} A = 1.5$.
8. $\operatorname{vers} A = .5$.
9. Show that $\dfrac{\sqrt{2 \operatorname{vers} A - \operatorname{vers}^2 A}}{1 - \operatorname{vers} A} = \tan A$.
10. Show that $\cos\theta \operatorname{vers}\theta\,(1 + \sec\theta) = \sin^2\theta$.

76. Algebraical note.

It will be useful to have an idea of the meaning of the word *limit* as used in mathematics. In the geometrical series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots,$$

the sum of 2 terms is $1\frac{1}{2}$, of 3 terms is $1\frac{3}{4}$, of 4 terms is $1\frac{7}{8}$, of 5 terms is $1\frac{15}{16}$. The sum of the series varies with the number of terms taken; and the greater the number of terms taken, the more nearly does their sum approach 2. It is stated in arithmetic and algebra that the sum of an infinitely great number of terms of this series is $1 + (1 - \frac{1}{2})$, *i.e.* 2. This simply means that, by making the number of terms as great as one please, the sum can be made to approach as nearly as one please to 2; or, in other words, the greater the number of terms taken, the more nearly does their sum approach the value 2. This idea is expressed in mathematics in slightly different language: "The *limit* of the sum of this series is 2." In geometry (see Note C) it is shown that if a regular polygon be inscribed in a circle, the length of the perimeter of the polygon approaches nearer and nearer to the length of the circle as the number of the sides of the polygon is increased; also the area of the polygon approaches nearer and nearer to the area of the circle. The length of the circle is said to be the *limit* of the length of the perimeter of the inscribed polygon, and the area of the circle is said to be the *limit* of the area of the polygon, as the number of its sides is indefinitely increased.

Definition. If a varying quantity approaches nearer and nearer to a fixed quantity (or given *constant*), so that the difference between the two quanti-

... ase, then the fixed quantity ... uity.

... are required in some of the articles that ...

... inite, and its absolute value either ... nominator decreases, then the ... either remains ... creases; e.g.

... 100 a, ...

... does $\frac{a}{x}$ become; ... approaches zero.

... as its limit, then ... limit. In ... word *infinity* ... number which is ... just stated,

... If x approaches zero ... The same idea is also ex-

... is zero, is infinity.

... that as x increases, ... decreases (a remaining ... approaches an infinitely great value, $\frac{a}{x}$ approaches

$$x = \infty, \text{ then } \frac{a}{x} = 0; \text{ or, } \underset{x \to \infty}{\text{Limit}} \frac{a}{x} = 0.$$

... approaches zero, so long as a does not

$$\underset{x \to 0}{\text{Limit}} \frac{x}{a} = 0.$$

77. Changes in the trigonometric functions as the angle increases from 0° to 360°. For convenience the revolving line will be kept constant in length in the following explanations. The student should try to deduce the changes in the functions for himself, especially after reading about the changes in the sine.

Change in sin A as A increases from 0° to 360°.

If OP be any position of the revolving line, then

$$\sin XOP = \frac{MP}{OP}.$$

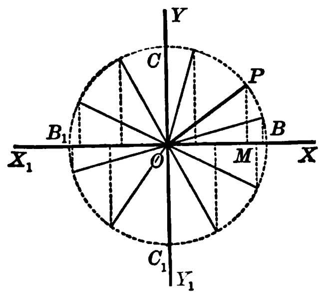


FIG. 67.

Now OP is kept the same in length, say length a, as XOP increases from 0° to 360°. Hence, in order to trace changes in the sine as the angle changes, it is necessary to consider only the changes in MP. Let the angle be denoted by A.

When $A = 0$, OP coincides with OB, and $MP = 0$. $\therefore \sin 0° = \frac{0}{a} = 0$.

As OP revolves from OX to OY, MP increases in length and is positive.

When $A = 90°$, OP coincides with OC, and $MP = a$. $\therefore \sin 90° = \frac{a}{a} = 1$.

Hence, as the angle A increases from 0° to 90°, its sine increases from 0 to 1.

As OP revolves from OY to OX_1, MP decreases in length and is positive.

When $A = 180°$, OP coincides with OB_1, and $MP = 0$. $\therefore \sin 180° = \frac{0}{a} = 0$.

Hence, as the angle A increases from 90° to 180°, its sine decreases from 1 to 0.

As OP revolves from OX_1 to OY_1, MP increases in length and is negative; *i.e.* MP really decreases.

When $A = 270°$, OP coincides with OC_1, $MP = a$ and is negative.

$$\therefore \sin 270° = \frac{-a}{a} = -1.$$

Hence, as the angle A increases from 180° to 270°, its sine decreases from 0 to -1.

As OP revolves from OY_1 to OX, MP decreases in length and is negative; *i.e.* MP really increases.

When $A = 360°$, OP coincides with OB, and $MP = 0$. $\therefore \sin 360° = \frac{0}{a} = 0.$

Hence, as the angle A increases from 270° to 360°, its sine increases from -1 to 0.

If OP continues to revolve, then the sine again undergoes the same changes in the same order, and does so during each successive revolution.

Change in cos A as A increases from 0° to 360°.

In Fig. 67, $\cos XOP = \dfrac{OM}{OP}$. Hence, in order to trace the changes in the cosine as the angle changes, it is necessary to consider only the changes in OM, since OP is kept at a constant length a.

When $A = 0°$, OP coincides with OB, and $OM = a$. $\therefore \cos 0° = \frac{a}{a} = 1.$

As OP revolves from OX to OY, OM decreases in length and is positive.

When $A = 90°$, OP coincides with OC, and $OM = 0$. $\therefore \cos 90° = \frac{0}{a} = 0.$

Hence, as the angle A increases from 0° to 90°, its cosine decreases from 1 to 0.

As OP revolves from OY to OX_1, OM increases in length and is negative, *i.e.* OM really decreases.

When $A = 180°$, OP coincides with OB_1, $OM = a$, and is negative.

$$\therefore \cos 180° = \frac{-a}{a} = -1.$$

Hence, as the angle increases from 90° to 180°, its cosine decreases from 0 to -1.

On proceeding in the same manner, the student will discover that:

As A increases from 180° to 270°, $\cos A$ increases from -1 to 0;

As A increases from 270° to 360°, $\cos A$ increases from 0 to 1.

If OP continues to revolve, then the cosine again undergoes the same changes in the same order, and does so during each successive revolution.

Change in tan A as A increases from 0° to 360°.

In Fig. 67, $\tan XOP = \dfrac{MP}{OM}$. Hence, in order to trace the changes in the tangent as the angle changes, it is necessary to consider the changes in MP and OM.

When $A = 0°$, OP coincides with OB, $MP=0$, $OM=a$. $\therefore \tan A = \frac{0}{a} = 0$.

As OP revolves from OX to OY, MP increases and OM decreases, and both are positive; hence, tan A increases.

When $A = 90°$, OP coincides with OC, $MP = a$, $OM = 0$.

$$\therefore \tan 90° = \frac{a}{0} = \infty.$$

As OP revolves from OY to OX_1, MP decreases and is positive, OM increases in length and is negative; hence, tan A decreases in magnitude and is negative; *i.e.* tan A really increases. [When OP passes at OY from the first quadrant into the second, the value of the tangent changes from $+\infty$ to $-\infty$, for OM changes its sign from + to −.]

When $A = 180°$, OP coincides with OB_1, $MP = 0$, $OM = -a$.

$$\therefore \tan 180° = \frac{0}{-a} = 0.$$

Hence, as A increases from 90° to 180° tan A increases from $-\infty$ to 0.

On proceeding in the same manner the student will discover that:

As A increases from 180° to 270°, tan A increases from 0 to $+\infty$;

As A increases from 270° to 360°, tan A increases from $-\infty$ to 0.

If OP continues to revolve, then the tangent again undergoes the same changes in the same order, and does so during each successive revolution.

In the same way as above, the student can trace the changes in *sec A, cosec A, cot A,* as A increases from 0° to 360°. The changes in these functions can also be deduced from the results obtained for sin A, cos A, tan A, and the relations

$$\sec A = \frac{1}{\cos A}, \ \operatorname{cosec} A = \frac{1}{\sin A}, \ \cot A = \frac{1}{\tan A}.$$

The results are collected in the following table: *

* This method of indicating the changes in the trigonometric functions is that given in Loney's *Plane Trigonometry*, p. 57.

Y

In the second quadrant the			In the first quadrant the		
sine	decreases from	1 to 0	sine	increases from	0 to 1
cosine	decreases from	0 to -1	cosine	decreases from	1 to 0
tangent	increases from	$-\infty$ to 0	tangent	increases from	0 to ∞
cotangent	decreases from	0 to $-\infty$	cotangent	decreases from	∞ to 0
secant	increases from	$-\infty$ to -1	secant	increases from	1 to ∞
cosecant	increases from	1 to ∞	cosecant	decreases from	∞ to 1
X_1			O		X
In the third quadrant the			In the fourth quadrant the		
sine	decreases from	0 to -1	sine	increases from	-1 to 0
cosine	increases from	-1 to 0	cosine	increases from	0 to 1
tangent	increases from	0 to ∞	tangent	increases from	$-\infty$ to 0
cotangent	decreases from	∞ to 0	cotangent	decreases from	0 to $-\infty$
secant	decreases from	-1 to $-\infty$	secant	decreases from	∞ to 1
cosecant	increases from	$-\infty$ to -1	cosecant	decreases from	-1 to $-\infty$

Y_1

NOTE. It should be observed that *the algebraic sign of each function changes when the function passes through either of the values zero and infinity.*

Ex. **Trace the changes in the versed sine as the angle changes from 0° to 360°.**

78. Periodicity of the trigonometric functions. It has been seen in Arts. 40–44 that all angles coterminal with XOP have the same ratios. That is, the same ratios as XOP has, are obtained each time that the revolving line returns to the position OP, no matter how many complete revolutions in the positive or negative direction it may make in the meantime. In the last article it was pointed out that *the sine*, for instance, *always goes through all its changes* (the *cycle* of changes, namely, 0 to 1, 1 to 0, 0 to -1, -1 to 0) *in the same order when the turning line revolves from the position OX through the angle* 360° *or* 2π. According to the opening remarks of this article, *all angles which differ by any integral multiple* (positive or negative) *of* 2π *radians have the same sine.* These facts are expressed mathematically by saying:

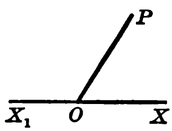


FIG. 68.

The sine is a periodic function, and **the period of the sine is 2π.**

Similar considerations show that *the cosine, secant, cosecant, are periodic functions, and that each of them has a period* 2π.

The *tangent* and *cotangent*, however (Art. 77), go through all their changes, while the angle increases by 180° or π radians. Hence, *the period of the tangent and cotangent is* π.

NOTE 1. These properties may be expressed as follows, m denoting any positive or negative whole number, and x being any angle:

$$\sin x = \sin(2\,m\pi + x), \quad \cos x = \cos(2\,m\pi + x),$$

and similar for $\sec x$, $\operatorname{cosec} x$;

$$\tan x = \tan(m\pi + x), \quad \cot x = \cot(m\pi + x).$$

NOTE 2. (*Algebraic.*) When a function $f(x)$ has the property that $f(x) = f(x + k)$, in which x can have any value and k is a constant, the function $f(x)$ is said to be a *periodic function.* If k is the *least quantity* for which this equation is true, then k is called *the period of the function.*

If $f(x) = f(x + k)$, then $f(x) = f(x + nk)$, n being any positive or negative whole number. For $f(\overline{x + k}) = f(\overline{x + k} + k) = f(x + 2\,k)$, and so on. Also, since $f(x) = f(x + k)$ for all values of x, this equation holds when $x - k$ is put for x; that is, $f(x - k) = f(x)$. Similarly, $f(x - 2\,k) = f(x - k) = f(x)$, and so on.

NOTE 3. It has been shown above that each of the trigonometric functions has but *one* period, namely, π, in the case of the tangent and cotangent, and 2π in the case of each of the other functions. Hence, the trigonometric functions are *singly periodic functions.* Functions which have more than one period appear in some branches of higher mathematics. For instance, certain functions called *elliptic functions* have two periods, and, accordingly, are said to be *doubly periodic.* *See Questions on Chap. X., Ex. 16.*

79. The old or line definitions of the trigonometric functions. The trigonometric functions were formerly considered as belonging to arcs rather than to angles, and were certain *lines* related to these arcs. Let APB be a circle described with any radius R about O as a centre. Let OA, OB, be at right angles to each other, and let AP be any arc having A for its initial point. Draw OP; from P draw PM at right angles to OA; through A draw a tangent AT to meet OP produced in T; through B draw a tangent BT_1 to meet OP produced in T_1; from P draw PM_1 at right angles

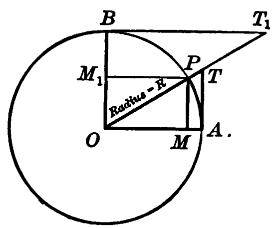


FIG. 69.

to OB. The lines MP, AT, OT, AM, were called respectively, the *sine, tangent, secant, coversed sine, of the arc AP;* and M_1P, BT_1, OT_1 (the sine, tangent, secant, of the complementary arc PB) were called respectively, the *cosine, cotangent, cosecant, of the arc* AP. These definitions are expressed in words as follows:

The *sine of an arc* is a straight line drawn from one extremity of the arc perpendicular to the radius passing through the other extremity. The *tangent of an arc* is a straight line touching the arc at one extremity, and limited by the radius produced through the other extremity. The *secant of an arc* is the straight line joining the centre of the circle, and the further extremity of the tangent drawn at the origin of the arc.

The *sine, tangent, and secant of the complement of an arc* are called the *cosine, cotangent,* and *cosecant* of that arc.

Since the arc measures the angle at the centre (the number of degrees in this arc is the same as the number of degrees in the subtended angle), these lines were also called the *sine, cosine,* ···, of the central angle AOP measured by the arc AP. These lines were known as *the trigonometric lines.*

NOTE. By "the length of a line" is meant the *number* of units of length which it contains. The lengths of these lines *depend on the length of the radius of the circle,* as well as on the magnitude of the central angle subtended by the arc. Hence it was *necessary to specify the radius* when the functions were discussed. This inconvenience has led to the adoption of the ratio definitions.

In Fig. 69 let R denote the length of the radius. Then, on using the *ratio* definitions,

$$\sin AOP = \frac{MP}{R}, \quad \tan AOP = \frac{AT}{R}, \quad \sec AOP = \frac{OT}{R}.$$

Hence, the *ratio* definitions of the trigonometric functions can be derived from the *line* definitions by dividing the lengths of the lines by the length of the radius. *If the length of the* **radius is unity,** *then the lengths of the* **lines** *used in the line definitions are* **equal to the ratios** *in the ratio definitions.* This suggests a geometrical or graphical method of representing the trigonometric functions, which is shown in the next article.

80. Geometrical representation of the trigonometric functions. Let a circle of radius equal to unity be drawn. This circle is called a **unit-circle.** Let the construction described in Art. 79 be made for each of the angles AOP_1, AOP_2, AOP_3, $\cdots$. *In any circle* the lines M_1P_1, M_2P_2, M_3P_3, $\cdots$, are proportional to, and hence *represent* the sines of these angles,

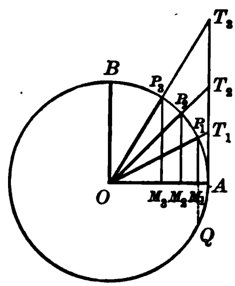


FIG. 70.

the lines AT_1, AT_2, AT_3, $\cdots$,
represent the tangents of these angles,

the lines OT_1, OT_2, OT_3, $\cdots$,
represent the secants of these angles,

and so on for the other ratios. *In the unit-circle,* however, *the measures of these lines,* the radius being the unit of length, *are the very same numbers as the respective ratios* mentioned. In the unit-circle also, *the linear measure of the arc is the same as the radian measure of the angle which it subtends.* [See Art. 73, Note 2.]

SUGGESTED EXERCISES. (1) By means of *the lines* on the unit-circle, trace the changes in the trigonometric functions as the angle changes from 0° to 90°. Compare the results with those of Art. 77.

(2) For particular values of the angle AOP_1, measure the lengths of the related lines on the unit-circle, and compare the results with the values given in the tables of natural sines and tangents.

NOTE 1. The origin of the terms *circular* functions, *tangent, secant,* is apparent from Art. 79.

NOTE 2. The name *sine* comes from the Latin word *sinus,* which was the translation of the Arabic word for this trigonometric function. The Arabic word for the sine resembled a word meaning an indentation or gulf.

NOTE 3. In trigonometry the Greeks used the *whole* chord P_1Q instead of the *half*-chord or sine. For example, Ptolemy, the celebrated astronomer who flourished about 125–151 A.D., gave a table of chords in Book I. of the *Almagest,* his work on astronomy. The Hindoos, on the other hand, always used the half-chord or sine. The Arabian astronomer, Al Battani (or Albatagnius) (877–929), in his work *The Science of the Stars,* like the Hindoos determined angles "by the semi-chord of twice the angle," *i.e.* by the *sine* of the angle, taking the radius as unity. The translation of this work into Latin in the twelfth century introduced the word *sine* into trigonometry. The *Hindoo sine* was finally adopted in Europe in preference to the *Greek chord* in the fifteenth century. [See Art. 12, foot-note.]

81. Graphical representation of functions.

Graphical representation. The different values which a varying quantity takes, are often represented by means of a curve. Many illustrations can be given of the graphical representation of various things whose values can be denoted by means of *numbers*. For example, the curve in Fig. 71 shows the record of the barometer at Ithaca from May 22 to May 29, 1899.

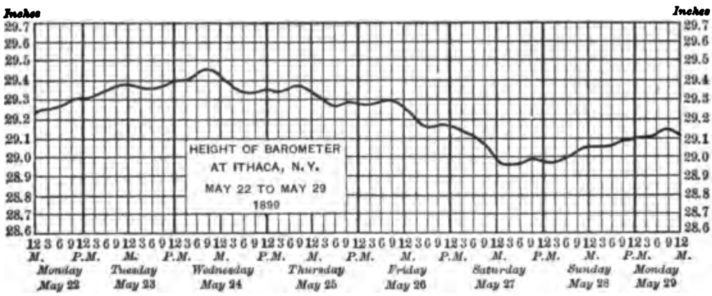
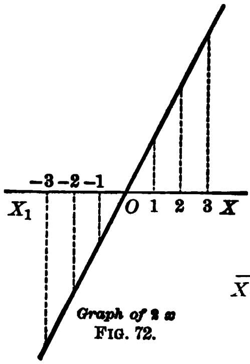


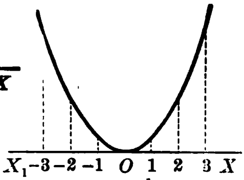
FIG. 71.

In this figure an hour is represented by a certain length, and the lengths representing hours are measured along a horizontal line. Each inch of height of the barometer is also represented by a certain length. At the points corresponding to the successive times perpendiculars are drawn, the lengths of which represent the heights of the barometer at the respective times. (In the figure the position of the horizontal line marked 29, represents the upper ends of heights of 29 inches.) The smooth curve drawn through the extremities of the perpendiculars is the *barometric curve* or *curve of barometric heights* for the period May 22 to May 29, 1899. This curve will give to most persons a clearer and more vivid idea of the range and variation of the height of the barometer during this period than a column of numbers of inches of heights is likely to give. If the scales used in representing the hours and the heights of the barometer were changed, then the curve would be somewhat altered, but its *general appearance would remain the same.*

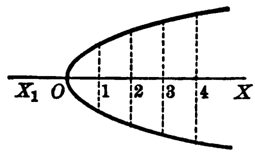
The graph of a function. The graph of a function of x, say $f(x)$, is obtained in the following way: Take a horizontal line X_1OX; choose a point O, from which, distances representing the different values of x are measured along the line; measure positive values of x toward the right from O, and negative values toward the left. At particular points of X_1OX, at convenient distances apart, draw perpendiculars to represent the values of $f(x)$ at the respective points. Draw the perpendiculars upward from X_1OX when the values of $f(x)$ are positive, and downward when these values are negative. The smooth curve drawn through the extremities of these perpendiculars is the graph of $f(x)$. The nearer the perpendiculars are to one another, the better is the graph. For example, the function $2x$ is represented (for certain values of x) by Fig. 72, the function $\frac{1}{4}x^2$, by Fig. 73, the function $\sqrt{x}$, by Fig. 74. The pupil is advised to construct these graphs by following the method just described above.



Graph of 2x
FIG. 72.



Graph of $\frac{1}{4}x^2$
FIG. 73.



Graph of $\sqrt{x}$
FIG. 74.

Exs. Draw the graphs for $3x$, $\frac{1}{2}x$, $4x+5$, $4x-5$, $\frac{1}{4}x^2$, $\frac{1}{8}x^3$. $\sqrt{25-x^2}$.

NOTE. The notion of representing a function by a curve is the fundamental notion in algebraic geometry, or, as it is usually termed, analytic geometry. This geometry was invented, in the form in which it is now known, by the philosopher and mathematician, René Descartes (1596–1650), and first published by him in 1637. This article may be regarded as a short lesson in the subject.

82. Graphs of the trigonometric functions.

Graph of sin θ. In order to draw the graph of *sin* θ take distances, measured from O along the line X_1OX, to represent the number of radians in the angle θ. At points (not too far apart) on X_1OX draw perpendiculars to represent the sines of the angles corresponding to these points. The smooth curve drawn through the extremities of these perpendiculars will be the graph of the sine. Thus, for example, let *a radian* be represented by a *unit length*, and let *the ratio unity* be also represented by a *unit length*. Then (see Fig. 75) angle π (*i.e.* 180°) is represented by $OM_1 = 3\frac{1}{7}$. The perpendiculars at O and M_1 are zero, since $\sin 0 = 0$ and $\sin\pi = 0$. Erect perpendiculars equal to $\cdots$, $\sin 30°$, $\sin 45°$, $\sin 60°$, $\sin 90°$, $\cdots$, for instance, at the points corresponding to $\cdots$, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \cdots$, (*i.e.* $\cdots$, 30°, 45°, 60°, 90°, $\cdots$,) respectively. Do the same at points between L and M_1, and draw the smooth curve OG_1M_1 through the extremities of the perpendiculars. The successive perpendiculars from π to 2π are the same in length as those from 0 to π, but negative. From 2π to 4π the values of the sine are repeated in the same order as from 0 to 2π. Hence, the graph of the sine can be obtained by merely successively reproducing the double undulation $OG_1M_1G_2M_2$, as indicated in Fig. 75. This is called the *curve of sines, sine curve*, or *sinusoid.*

NOTE 1. The unit circle (Art. 80) will be of service in drawing the graphs of the sine and the other trigonometric functions. For, if the scales for radians and ratios be those adopted above, then the *horizontal distances* from O will be equal to the *lengths of the arcs* (Art. 73, Note 2), and the *lengths of the perpendiculars* will be the *lengths of the lines in the line definitions* (Art. 79).

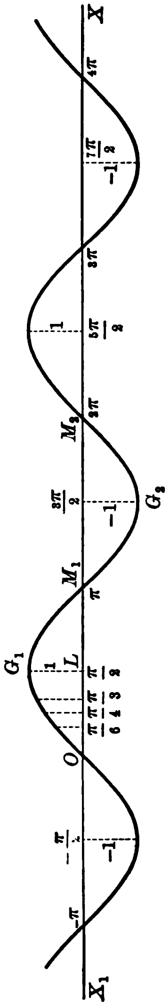
NOTE 2. If π radians (*i.e.* 180°) be represented by a length different from that adopted in Fig. 75, then the graph of the sine will differ somewhat from Fig. 75, but its main features will be the same as in that figure. Figures 76 and 77 show portions of the graph of $\sin\theta$ when π is represented on two other scales, while the $\sin\frac{\pi}{2}$ (*i.e.* 1) is represented by a unit length. Hence the curve of sines, or the sinusoid, may be defined as the curve in which horizontal distances measured on a certain line are proportional to an angle, and the perpendiculars to this line are proportional to its sine.

Ex. Draw the graphs for $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$.

Graph of $\cos\theta$. On using the same scales for radians and ratios as those adopted in Fig. 75, the graph of $\cos\theta$ takes the form shown in Fig. 78. It is the same as the graph of $\sin\theta$ in Fig. 75 would be if O and the other points in X_1OX were all moved a distance $\frac{1}{2}\pi$ toward the right. This might have been expected, since the sine of an angle is equal to the cosine of its complement. The values of the sine and the cosine alike range from $+1$ to -1.

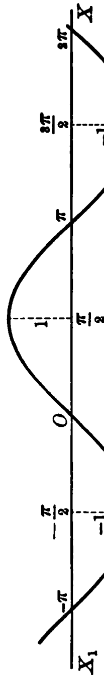
Graph of $\tan\theta$. On using the same scales for radians and ratios as have been adopted in Fig. 75, the graph of $\tan\theta$ takes the form shown in Fig. 79.

Graph of $\sec\theta$. On using the same scales for radians and ratios as have been adopted in Fig. 75, the graph of $\sec\theta$ takes the form shown in Fig. 80.



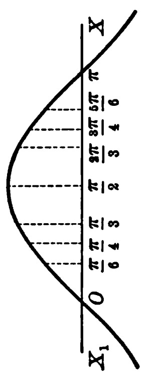
Graph of sin θ

FIG. 75.



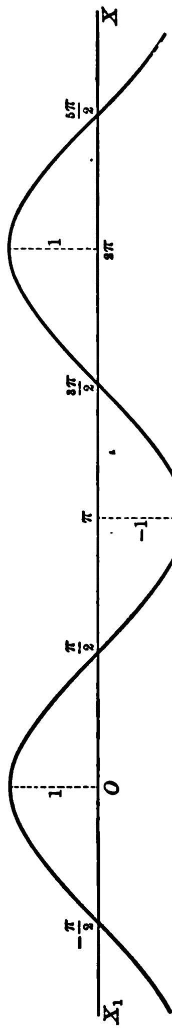
Graph of sin θ

FIG. 77.



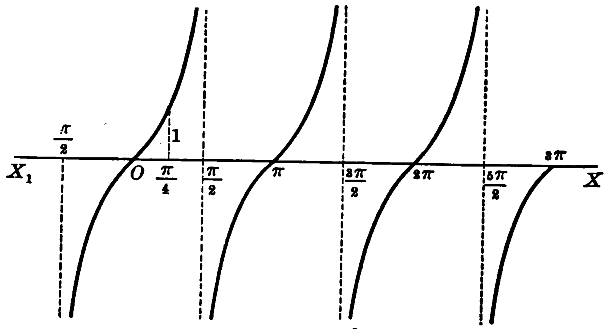
Graph of sin θ

FIG. 76.



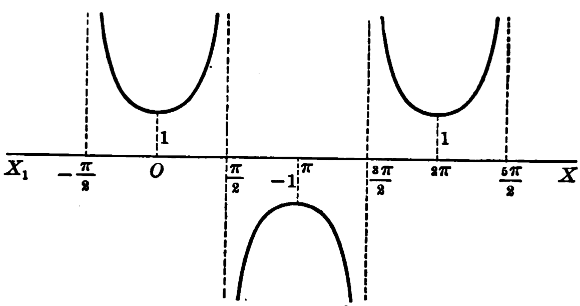
Graph of cos θ

FIG. 78.



Graph of $\tan\theta$

FIG. 79.



Graph of $\sec\theta$

FIG. 80.

EXAMPLES.

1. Draw the graph for $\cot\theta$, using the scales adopted in Fig. 75.

2. Draw the graph for $\operatorname{cosec}\theta$, using the scales adopted in Fig. 75.

3. Construct various graphs for $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$, by varying the scales used in representing radians and ratios.

4. Draw graphs for:

(*a*) $\sin x + \cos x$, (*b*) $\sin x - \cos x$, (*c*) $\sin 2x$, (*d*) $\cos 2x$.

83. Relations between the radian measure, the sine, and the tangent of an acute angle.

A. If θ be the radian measure of an acute angle, then $\sin\theta < \theta < \tan\theta$.

Let angle $AOP = \theta$, make the angle AOR equal to θ, and with any radius r describe the arc QBR about O as a centre. Draw the chord QR intersecting OB in M, and draw the tangents at Q and R. By geometry, arc $QB =$ arc BR, QR is at right angles to OB, $MQ = MR$, the tangents at Q and R intersect at a point T on OA, $QT = RT$.

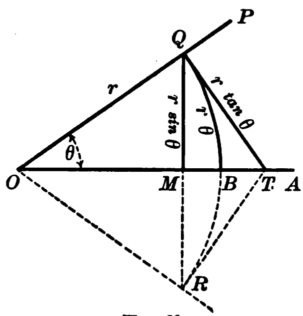


FIG. 81.

NOTE. If $r = 1$, that is, if QB is an arc of a unit circle, then the linear measures of MQ, BQ, TQ are equal to $\sin\theta$, θ, $\tan\theta$, respectively.

It can be easily shown *by mechanical means* that

$$MQ < BQ < TQ. \tag{1}$$

For suppose that pegs are placed at Q, R, T, and that a string is drawn taut from Q to R; suppose that another string is drawn from Q to R, but constrained to lie on the circular arc QR, like a string stretched along the tire of a wheel. Also let a third string be drawn taut from R to Q, but passed over the peg at T. Then it is obvious that the first of the three strings is the shortest, and the third is the longest. The second string cannot be drawn away from the arc QBR without being stretched, and if peg T were removed, the string QTR would lie loosely on the arc QBR. Since $QMR < QBR < QTR$, then $MQ < BQ < TQ$; that is, $r\sin\theta < r\theta < r\tan\theta$. Hence, $\sin\theta < \theta < \tan\theta$.

The truth of **A** may be perceived from the following mathematical consideration. Draw the chord QB. Evidently,

$$\text{area triangle } OQB < \text{area sector } OQB < \text{area triangle } OQT.$$

That is, $\frac{1}{2}\,OB \cdot MQ < \frac{1}{2}\,OQ \cdot \text{arc}\,BQ < \frac{1}{2}\,OQ \cdot QT;$

or, $\frac{1}{2}\,r \cdot r \sin\theta < \frac{1}{2}\,r \cdot r\theta < \frac{1}{2}\,r \cdot r \tan\theta.$

Hence, $$\sin\theta < \theta < \tan\theta. \qquad (1)$$

B. When angle θ approaches zero, each of the ratios $\frac{\sin\theta}{\theta}$, $\frac{\tan\theta}{\theta}$, approaches unity as a limit. On dividing each of the members of the inequality (1) by $\sin\theta$, there is obtained

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

But when θ approaches zero, $\cos\theta$ approaches unity as a limit (Art. 77). Hence, when θ approaches zero, $\frac{\theta}{\sin\theta}$ must also approach unity as a limit; that is, the limit of $\frac{\sin\theta}{\theta}$ is 1.

On dividing the members of (1) by $\tan\theta$, there is obtained

$$\cos\theta < \frac{\theta}{\tan\theta} < 1.$$

As before, when θ approaches zero, $\cos\theta$ approaches unity as a limit, and hence $\frac{\theta}{\tan\theta}$ approaches unity as a limit; *i.e.* the limit of $\frac{\tan\theta}{\theta}$ is 1. These results may be briefly expressed:

$$\underset{\theta=0}{\text{Limit}}\left(\frac{\sin\theta}{\theta}\right) = 1; \quad \underset{\theta=0}{\text{Limit}}\left(\frac{\tan\theta}{\theta}\right) = 1. \qquad (2)$$

These are two of the most important theorems in elementary trigonometry; they are frequently employed both in practical work and in pure mathematics.

A very important corollary to (2) is the following:

If θ be the radian measure of a very small angle, then θ can be used for $\sin\theta$ and $\tan\theta$ in calculations.

For instance, $\sin 10''$ to 12 places of decimals is .000048481368. This is also the radian measure of $10''$ to 12 places of decimals. The radian measures, sines, and tangents, of angles from $0°$ to $6°$, agree in the first three places of decimals. For

$$\text{radian measure } 6° = (.10472) = .105;\ \sin 6° = (.10453) = .105;$$
$$\tan 6° = (.10510) = .105.$$

EXAMPLES.

1. Find the angle subtended by a man 6 ft. high at a distance of half a mile.

Here, $\theta = \tan\theta = \dfrac{6}{2640} = \dfrac{1}{440}.$

Now $\dfrac{1^r}{440} = \dfrac{1}{440} \times \dfrac{180^\circ}{\pi} = \dfrac{7 \times 180^\circ}{22 \times 440} = 7' \, 48''.6.$

FIG. 82.

2. What must be the height of a tower, in order that it subtend an angle 1° at a distance of 4000 ft.?

$\dfrac{x}{4000} = \tan 1^\circ = \text{radian measure } 1^\circ = \dfrac{\pi}{180} = \dfrac{22}{7 \times 180}.$

$\therefore \; x = \dfrac{22 \times 4000}{7 \times 180} = 69.84 \text{ ft.}$

FIG. 83.

3. Verify the statements made in Art. 11, Note 1. (Take $\pi = 3.14159265$.)

4. The moon's mean angular diameter as observed at the earth is 31′ 5″, and its actual diameter is about 2160 miles. Find the mean distance of the moon. How many full moons would make a chaplet across the sky?

5. Taking the earth's equatorial radius as 3963 mi., find the angular semi-diameter of the earth as it would appear if observed from the moon. Compare the relative apparent sizes of the moon as seen from the earth, and the earth as seen from the moon.

6. The semi-diameter of the earth as seen from the sun is very nearly 8″.8. (See Art. 11, Note 1.) What is the sun's distance from the earth, the radius of the earth being assumed as 4000 miles?

7. At least how many times farther away than the sun is the nearest fixed star α Centauri, at which the mean distance between the earth and sun (about 92,897,000 miles) subtends an angle something less than 1″? How long, at least, will it take light to come from this star to the earth?

8. Find approximately the distance at which a coin an inch in diameter must be placed so as just to hide the moon, the latter's angular diameter being taken 31′ 5″.

9. The inclination of a railway to a horizontal plane is 50′. Find how many feet it rises in a mile.

10. Find the angle subtended by a circular target 4 ft. in diameter at a distance of 1000 yd.

11. Find the height of an object whose angle of elevation at a distance of 900 yd. is 1°.

12. Find the angle subtended by a pole 20 ft. high at a distance of a mile.

13. Exs. 5, 6, Art. 34 *b*.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter X. will be found at page* 195.

CHAPTER XI.

GENERAL VALUES. INVERSE TRIGONOMETRIC FUNCTIONS. TRIGONOMETRIC EQUATIONS.

84. General values. Articles 40–43 should be reviewed carefully before this chapter is taken up. It has been seen in these articles that all co-terminal angles have the same trigonometric ratios. It was also pointed out in Art. 43 that two sets of co-terminal angles, each set being infinite in number, correspond to any given ratio. For example, in Art. 42, Ex. 1, Fig. 38, any angle whose terminal line is either OP or OP_1 has a sine, $\frac{3}{4}$; in Ex. 2, Fig. 39, any angle whose terminal line is either OP or OP_1 has a tangent, $-\frac{3}{4}$. One of the objects of this chapter is to derive *expressions* or *formulas* that will include all angles which have the same sine, cosine, tangent, cotangent, secant, cosecant, respectively. These general expressions are sometimes called *general values*. The student is advised to deduce, after reading Art. 85, the general values for cosine, tangent, etc., without the help of the book.

85. General expression for all angles which have the same sine. Let s be the given value of the sine. It is required to find an expression that will represent and include every angle whose sine is s. All the angles whose sines are equal to s can be repre-

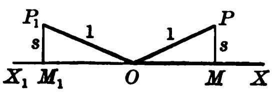


FIG. 84.

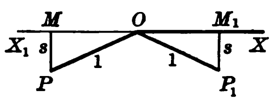


FIG. 85.

sented geometrically, as shown in Art. 42, and indicated in Figs. 84, 85. In Fig. 84, s is positive; in Fig. 85, s is negative.

Let XOP be the least positive angle whose sine is s. Let

$XOP = A$; then $XOP_1 = 180° - A$. Every angle whose terminal line is either OP or OP_1 has its sine equal to s. Now all angles having OP for a terminal line are obtained by adding all numbers of complete revolutions (positive and negative) to XOP. Hence, these angles are represented by

$$m \cdot 360° + A, \text{ i.e. } 2m \cdot 180° + A, \tag{1}$$

in which m is any positive or negative whole number.

Similarly, all angles having OP_1 for a terminal line are represented by

$$m \cdot 360° + (180° - A), \text{ i.e. } (2m + 1)\,180° - A. \tag{2}$$

An expression that will include both sets of angles, (1) and (2), will now be obtained. In the expression (1), the *coefficient* of 180° is *even*, and the *sign* of A is *positive;* in (2), the *coefficient* of 180° is *odd*, and the *sign* of A is *negative.* Hence, n being any positive or negative whole number, the expression

$$\boldsymbol{n \cdot 180° + (-1)^n A}, \tag{3}$$

includes the angles in (1) and (2). This is, accordingly, the general expression for all the angles which have the same sine as A. If radian measure is used, and $XOP = \alpha$, then (3) takes the form

$$\boldsymbol{n\pi + (-1)^n \alpha}. \tag{4}$$

The result may be thus expressed:

$$\sin A = \sin\{n \cdot 180° + (-1)^n A\}, \quad \sin \alpha = \sin\{n\pi + (-1)^n \alpha\}. \tag{5}$$

Since $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$, *the general expression for all angles which have the same cosecant is the same as the general expression for all angles which have the same sine.*

EXAMPLES.

1. Find an expression to include all angles which have the same sine as 135°.

By (3), (4), the expression is $n \cdot 180° + (-1)^n 135°$, or $n\pi + (-1)^n \dfrac{3\pi}{4}$.

2. Find *the general value* of the angle whose sine is $+\frac{1}{\sqrt{2}}$. Give the four least positive angles which have sines equal to $+\frac{1}{\sqrt{2}}$.

The least angle whose sine is $+\frac{1}{\sqrt{2}}$ is 45°, *i.e.* $\frac{\pi}{4}$. Hence the general value is

$$n \cdot 180° + (-1)^n 45°, \text{ i.e. } n\pi + (-1)^n \frac{\pi}{4}.$$

To find the *four* least positive angles, put $n = 0, 1, 2, 3$, in this expression. This gives 45°, 135°, 405°, 495°, *i.e.* $\frac{\pi}{4}, \frac{3}{4}\pi, \frac{9}{4}\pi, \frac{11}{4}\pi$. These four angles can also be obtained by means of a figure.

3. Given that $\sin\theta = \frac{\sqrt{3}}{2}$; find the general value of θ, and find the *four* least positive values of θ.

4. As in Ex. 3 when $\sin\theta = -\frac{1}{2}$.

5. As in Ex. 3 when $\sin\theta = .95372$.

6. As in Ex. 3 when $\sin\theta = .39741$.

7. As in Ex. 3 when $\sin\theta = -.57833$.

86. General expression for all angles which have the same cosine. Let c be the given value of the cosine. It is required to find an expression to include every angle whose cosine is c. All the angles that have c for a cosine can be represented geometrically, as shown in Figs. 86, 87. In Fig. 86, c is positive; in Fig. 87, c is negative.

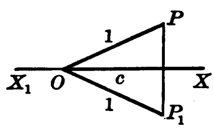


FIG. 86.

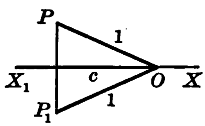


FIG. 87.

Let XOP be the least positive angle whose cosine is c, and let $XOP = A$ (in degree measure) $= \alpha$ (in radian measure). Angle $XOP_1 = -A = -\alpha$, also has its cosine equal to c. All angles whose terminal line is OP, have cosines equal to c. All these angles are included in

$$n \cdot 360° + A, \text{ i.e. } 2n\pi + \alpha, \tag{1}$$

in which n denotes any positive or negative whole number. Also, all angles whose terminal line is OP_1, have cosines equal to c. All these angles are included in

$$n \cdot 360° - A, \text{ i.e. } 2n\pi - \alpha, \tag{2}$$

n being as before. Both the expressions, (1), (2), are evidently included in

$$n \cdot 360^\circ \pm A, \text{ or } 2n\pi \pm \alpha, \tag{3}$$

in which n is any positive or negative whole number. Hence (3) is the general expression for all angles which have the same cosine as A or α. The result may be thus expressed:

$$\cos A = \cos(n \cdot 360^\circ \pm A); \ \cos\alpha = \cos(2n\pi \pm \alpha). \tag{4}$$

Since $\sec\theta = \frac{1}{\cos\theta}$, *the general expression for all angles which have the same secant is the same as the general expression for all angles which have the same cosine.*

EXAMPLES.

1. What is the general value of the angles which have the cosine, $-\frac{1}{2}$? Give the *three* least positive angles.

The least positive angle whose cosine is, $-\frac{1}{2}$, is 120°. Hence, the general value is, by (3), $n \cdot 360^\circ \pm 120^\circ$, *i.e.* $2n\pi \pm \frac{2}{3}\pi$. On putting $n = 0$ and 1, the three least positive angles are found to be 120°, 360° − 120°, or 240°, 360 + 120°, or 480°. These three angles may also be found by means of a figure.

2. Given that $\cos\theta = \frac{\pm\sqrt{3}}{2}$: find the general value of θ, and find the four least positive values of θ.

3. As in Ex. 2 when $\cos\theta = .99106$. 4. As in Ex. 2 when $\cos\theta = .46690$.

5. As in Ex. 2 when $\cos\theta = -.72637$. 6. As in Ex. 2 when $\cos\theta = -.40141$.

87. General expression for all angles which have the same tangent. Let t be the given value of the tangent. It is required to find an expression to include all angles which have the same tangent t.

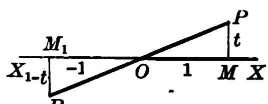


FIG. 88.

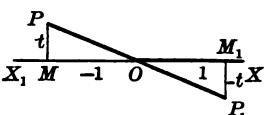


FIG. 89.

All the angles which have the same tangent t can be represented geometrically as in Figs. 88, 89. In Fig. 88, *the tangent t* is positive, in Fig. 89, it is negative.

Let $XOP = A$ (in degrees) $= \alpha$ (in radians). Then

$$XOP_1 = 180° + A = \pi + \alpha.$$

Each angle which has either OP or OP_1 for its terminal line, has its tangent equal to t. All the angles which have OP for a terminal line are included in the expression $m \cdot 360° + A$, that is, in

$$2m \cdot 180° + A, \text{ or } 2m\pi + \alpha, \tag{1}$$

in which m denotes any positive or negative whole number.

All the angles which have OP_1 for a terminal line are included in the expression $m \cdot 360° + (180° + A)$, that is, in

$$(2m+1)180° + A, \text{ or } (2m+1)\pi + \alpha. \tag{2}$$

Both these sets of angles, (1) and (2), are included in the expression

$$\boldsymbol{n \cdot 180° + A, \text{ or } n\pi + \alpha,} \tag{3}$$

in which n denotes any positive or negative whole number. Hence (3) is the general expression for all angles which have the same tangent as A or α. The result may be thus expressed:

$$\tan A = \tan(n \cdot 180° + A); \ \tan \alpha = \tan(n\pi + \alpha). \tag{4}$$

Since $\cot \theta = \dfrac{1}{\tan \theta}$, *the general expression for all angles which have the same cotangent is the same as the general expression for all angles which have the same tangent.*

EXAMPLES.

1. Find the general value of θ when $\tan \theta = 1$. The least positive angle whose tangent is 1, is $\frac{\pi}{4}$. Hence $\theta = n\pi + \frac{\pi}{4}$, in which n is any positive or negative whole number.

Find the general value of θ, and the four least positive values of θ when:

2. $\tan \theta = \sqrt{3}$. **3.** $\tan \theta = .36727$. **4.** $\tan \theta = 2.2998$.

5. $\tan \theta = .71769$. **6.** $\tan \theta = -.90040$. **7.** $\tan \theta = -2.6511$.

8. Find the general expression for all angles which have the same sine *and* cosine.

88. Inverse trigonometric functions. It has been seen that, on the one hand, the value of the sine depends on the value of the angle, and, on the other hand, the value of the angle depends on the value of the sine. If the angle is given, the sine can be determined; if the sine is given, the angle can be expressed. Hence, on the one hand, *the sine is a function of the angle*, and, on the other hand, *the angle is a function of the sine*. The latter function is said to be **the inverse** function of the former. The same holds in the case of each of the other trigonometric functions. Inverse functions are usually denoted by the symbol described below.

The two statements: *the sine of the angle θ is m,* (1)

θ is the angle whose sine is m, (2)

are briefly expressed:
$$\sin\theta = m, \tag{3}$$
$$\theta = \sin^{-1} m. \tag{4}$$

The symbols $\sin^{-1} m$, $\cos^{-1} m$, $\tan^{-1} m$, $\cdots$, are called *inverse trigonometric functions*, or *anti-trigonometric functions*, or *inverse circular functions*. The symbol "*$\sin^{-1} m$*" is read, "angle whose sine is m," "anti-sine of m," "inverse sine of m," "sine minus one m." It should be carefully remembered that *here*, -1 *is not an algebraical exponent*, but is merely part of a mathematical symbol; $\sin^{-1} m$ does not denote $(\sin m)^{-1}$, that is, $\frac{1}{\sin m}$; $\sin^{-1} m$ denotes each and every angle whose sine is m. *The trigonometric functions are pure numbers; the inverse circular functions are angles*, and are denoted by *the number of degrees or radians in these angles*. For instance, if $\theta = \frac{\pi}{4}$ in (3), then $m = +\frac{1}{\sqrt{2}}$;

$$\text{if } m = +\frac{1}{\sqrt{2}} \text{ in (4), then } \theta = \sin^{-1}\left(\frac{1}{+\sqrt{2}}\right) = n\pi + (-1)^n \frac{\pi}{4}$$
$$= n \cdot 180^\circ + (-1)^n 45^\circ,$$

in which n is any whole number. This example illustrates what has already been noted in Arts. 42, 43, 78, namely:

For a given value of the angle θ, $\sin\theta$ or m has *a single definite value.*

For a given value of the sine m, $\sin^{-1} m$ or θ has *an infinite number of values.*

The same is the case with each of the other inverse trigonometric functions. Thus the **trigonometric functions are single-valued,** and the **inverse circular functions are multiple-valued.**

For example, if $\cos\theta = \frac{\sqrt{3}}{2}$, then $\theta = \cos^{-1}\frac{\sqrt{3}}{2} = 2n\pi \pm \frac{\pi}{6}$(Ex. 2, Art. 86); if $\theta = \tan^{-1} 1$, then $\theta = n\pi + \frac{\pi}{4}$ (Ex. 1, Art. 87), in which n denotes any whole number. The *smallest numerical value* of an inverse trigonometric function is called the **principal value** of the inverse function. For instance, the principal value of $\sin^{-1}\frac{1}{2}$ is 30°, of $\tan^{-1}(-1)$ is $-45°$, of $\cos^{-1}(-\frac{1}{2})$ is 120°, of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is $-60°$.

NOTE 1. In some books the symbols *arc sin x, arc cos x, arc tan x*, ⋯, are used for inverse trigonometric functions. These symbols are read, "arc sine x," ⋯. The derivation of these names is apparent from Art. 79.

NOTE 2. *Algebraic.* If y is a function of x, say $f(x)$, then x also depends on y, and hence, is some function of y. This function of y is called the *inverse function of* $f(x)$ *or* y, and is usually denoted by $f^{-1}(y)$. For instance, if $y = f(x) = x^2$, then $x = f^{-1}(y) = \pm\sqrt{y}$.

It will be observed in this simple example that, while the function of x has a single value, the inverse function has two values. In other words, y is a *single-valued* function of x, and x is a *two-valued* function of y. As shown above, if $y = \sin x$, then $x = \sin^{-1} y$; y is a single-valued function of x, but x is a multiple-valued function of y.

It appears from Notes 1, 2, that the English notation for inverse trigonometric functions avoids the old geometrical conceptions of trigonometric functions, and is also more general in character. The inverse trigonometric functions are frequently met in calculus and applied mathematics.

89. Sum and difference of two anti-tangents. Exercises on inverse functions.

Find $\tan^{-1} m + \tan^{-1} n$, and $\tan^{-1} m - \tan^{-1} n$.

Let $x = \tan^{-1} m$, and $y = \tan^{-1} n$.

Then $\tan x = m$, $\tan y = n$.

Now $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ (Art. 51) $= \frac{m+n}{1-mn}$.

$$\therefore x + y = \tan^{-1}\frac{m+n}{1-mn}; \text{ i.e. } \tan^{-1} m + \tan^{-1} n = \tan^{-1}\frac{m+n}{1-mn}. \quad (1)$$

In a similar manner it can be shown that

$$\tan^{-1} m - \tan^{-1} n = \tan^{-1}\frac{m-n}{1+mn}. \quad (2)$$

EXAMPLES.

1. Find $\tan^{-1} 2 \pm \tan^{-1} \frac{1}{3}$. (Compare Ex. 1, Art. 51.)

$$\tan^{-1} 2 + \tan^{-1} \tfrac{1}{3} = \tan^{-1} \frac{2 + \frac{1}{3}}{1 - 2 \cdot \frac{1}{3}} = \tan^{-1} 7 = n \cdot 180° + 81° \, 52' \, 11''.5.$$

$$\tan^{-1} 2 - \tan^{-1} \tfrac{1}{3} = \tan^{-1} \frac{2 - \frac{1}{3}}{1 + 2 \cdot \frac{1}{3}} = \tan^{-1} 1 = n\pi + \frac{\pi}{4}.$$

By the tables, taking acute angles only, $\tan^{-1} 2 = 63° \, 26' \, 4''.3$, $\tan^{-1} \frac{1}{3} = 18° \, 26' \, 6''$, the sum is $81° \, 52' \, 10''.3$, and the difference is $44° \, 59' \, 58''.3$. The slight discrepancy between the results obtained by the two methods is due to the fact that the angles found by the tables are only approximately correct.

In the following examples test or verify the result in the manner shown in Ex. 1.

2. Find $\tan^{-1} 7 \pm \tan^{-1} 3$.

3. Find $\tan^{-1} 2 + \tan^{-1} .5$.

4. Find $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$.

5. Find $\tan^{-1} 3 + \tan^{-1} 2 + \tan^{-1} .6$.

(SUGGESTION. Find $\tan^{-1} 3 + \tan^{-1} 2$, then combine the result with $\tan^{-1} .6$.)

6. Find $2 \tan^{-1} 1.5$, $2 \tan^{-1} 3$, $2 \tan^{-1} 2$, $3 \tan^{-1} .2$.

7. Show that $2 \tan^{-1} m = \tan^{-1} \dfrac{2m}{1 - m^2}$. Show that $2\theta = \tan^{-1} \left(\dfrac{2 \tan \theta}{1 - \tan^2 \theta} \right)$.

8. Show that $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \dfrac{\pi}{4}$ when the angles are between 0° and 90°.

9. Find $\sin (\sin^{-1} \frac{1}{2} + \sin^{-1} \frac{1}{3})$ when the angles are between 0° and 90°.

10. When the angles are between 0° and 90°, show that:

(*a*) $\sin (\sin^{-1} m \pm \sin^{-1} n) = m\sqrt{1 - n^2} \pm n\sqrt{1 - m^2}$.

(SUGGESTION. Let $x = \sin^{-1} m$, $y = \sin^{-1} n$.)

(*b*) $\cos (\sin^{-1} m \pm \sin^{-1} n) = \sqrt{1 - n^2} \sqrt{1 - m^2} \mp mn$.

(*c*) $\sin (\sin^{-1} m \pm \cos^{-1} n) = mn \pm \sqrt{1 - m^2} \sqrt{1 - n^2}$.

(*d*) $\cos (\sin^{-1} m \pm \cos^{-1} n) = n\sqrt{1 - m^2} \mp m\sqrt{1 - n^2}$.

11. Find $\sin (\sin^{-1} \frac{2}{3} + \sin^{-1} \frac{4}{7})$, $\cos (\sin^{-1} \frac{1}{7} - \cos^{-1} \frac{1}{8})$, $\sin (\cos^{-1} \frac{2}{5} - \cos^{-1} \frac{3}{10})$, $\sin (\tan^{-1} 4 - \cos^{-1} \frac{2}{3})$, $\tan (\sec^{-1} 3 - \sin^{-1} \frac{1}{2})$, (*a*) when the angles are between 0° and 90°, (*b*) when this restriction is not imposed.

12. Two lines, AB, AC, intersect a horizontal line at B, C, making angles whose tangents are $\frac{5}{3}$, $\frac{2}{7}$. Find the angle BAC.

13. Two lines, LM, LN, make angles whose tangents are $\frac{1}{3}$, 2, with a horizontal line. Find the angle MLN.

90. Trigonometric equations. Trigonometric equations have appeared in many of the preceding articles. When an angle, θ say, is the unknown quantity in a trigonometric equation, *the complete solution is the general value of θ which satisfies the equation.* For example, if α be an angle whose sine is s, then the solution of the equation,

$$\sin\theta = s, \text{ that is, of } \theta = \sin^{-1}s,$$

is $$\theta = n\pi + (-1)^n\alpha, \; n \text{ being any integer.}$$

EXAMPLES.

(See the definition of principal value in Art. 88.)

1. Solve the equation $\cos\theta = \dfrac{\sqrt{3}}{2}$.

The principal value of θ is $\dfrac{\pi}{6}$. Hence the complete solution is $\theta = 2n\pi \pm \dfrac{\pi}{6}$, n being any integer. (See Ex. 2, Art. 86.)

2. Solve the equation $\theta = \tan^{-1}1$.

The principal value is $\dfrac{\pi}{4}$. $\therefore\ \theta = n\pi + \dfrac{\pi}{4}$. (See Ex. 1, Art. 87.)

3. Solve the equation $\sin x\cos x = -\frac{1}{4}\sqrt{3}$.

$\therefore\ \sin x\sqrt{1-\sin^2 x} = -\frac{1}{4}\sqrt{3}$. $\qquad \therefore\ \sin^2 x(1-\sin^2 x) = \frac{3}{16}$.

$\therefore\ \sin^4 x - \sin^2 x + \frac{3}{16} = 0$. $\qquad \therefore\ (\sin^2 x - \frac{3}{4})(\sin^2 x - \frac{1}{4}) = 0$.

$\therefore\ \sin^2 x = \frac{3}{4}$; $\qquad \sin^2 x = \frac{1}{4}$.

Whence $\qquad$ (a) $\sin x = \pm\sqrt{\frac{3}{4}}$; $\qquad$ (b) $\sin x = \pm\frac{1}{2}$.

The given equation shows that $\sin x$ and $\cos x$ have opposite algebraic signs. Hence, x can only be in the second and fourth quadrants.

$\therefore$ In (a), $x = 120°$, $300°$, etc., its general value is $n\cdot 180 - 60°$, where n is any positive integer.

In (b), $x = 150°$, $330°$, etc.; its general value is $n\cdot 180° - 30°$, n being any positive integer.

4. Solve the equation $\sin 5\theta + \sin\theta = \sin 3\theta$.

$\therefore\ 2\sin 3\theta\cos 2\theta = \sin 3\theta$. $\quad \therefore\ \sin 3\theta(2\cos 2\theta - 1) = 0$.

$\therefore$ (a) $\sin 3\theta = 0$, $\quad$ (b) $2\cos 2\theta - 1 = 0$.

From (a), $3\theta = 0°$, $180°$, etc.; the general value of 3θ is $n\pi$ (n being any integer).

$\therefore\ \theta = 0°$, $60°$, etc.; the general value of 3θ is $\dfrac{n\pi}{3}$.

From (b), $\cos 2\theta = \frac{1}{2}$. $\quad \therefore\ 2\theta = \pm 60°$, etc.; its general value is $2n\pi \pm \dfrac{\pi}{3}$.

$\therefore\ \theta = \pm 30°$, etc.; its general value is $n\pi \pm \dfrac{\pi}{6}$.

Find solutions of these equations:

5. $3(\sec^2\theta + \cot^2\theta) = 13.$

6. $\cot\theta - \tan\theta = 2.$

7. $\sec x + \tan x = 2.$

8. $\sec^2 x + \tan x = 7.$

9. $\sec^2 x - \tan x = 3.$

10. $\cos\theta - \cos 7\theta = \sin 4\theta.$

11. $2\sin x + 5\cos x = 2.$

12. $\sin 2\theta + \sin 4\theta = \sqrt{2}\cdot\cos\theta.$

13. $4\sin\theta\cos 2\theta = 1.$

14. $\tan^4 A - 4\tan^2 A + 3 = 0.$

15. $3(\tan^2\theta + \cot^2\theta) = 10.$

16. $\cos^{-1}x - \sin^{-1}x = \cos^{-1}x\sqrt{3}.$

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter XI. will be fouud at pages* 197–199.

CHAPTER XII.

MISCELLANEOUS THEOREMS AND EXERCISES.

91. Chapters II.–VIII. were devoted to the oldest and the simplest application of trigonometry; namely, the measurement of triangles. Angles and the trigonometric functions connected with angles were more fully discussed in Chapters IX.–XI. This chapter does not introduce any new principles. Most of its articles may be regarded as exercises on the relations shown in Chapters II.–VIII., and more especially on the properties announced in Arts. 44, 50–52. The articles just mentioned should be reviewed. Some of the results in the exercises in this chapter are useful and important; but *the student should direct attention mainly to the methods whereby the results are obtained,* so that he can proceed quickly and confidently to the solutions of similar exercises. These solutions require a ready and an accurate knowledge of (that is, an intelligent familiarity with) the formulas deduced in the earlier chapters. It is on this account, perhaps, that such exercises are regarded with favor by examiners.

92. Functions of twice an angle. Functions of half an angle.

Relations (5)–(8), Art. 50, (3), Art. 51, give the sine, cosine, and tangent of twice an angle in terms of the functions of the angle. On rearranging (7), (8), Art. 50, there is obtained,

$$\sin A = \sqrt{\frac{1-\cos 2A}{2}}, \quad \cos A = \sqrt{\frac{1+\cos 2A}{2}}; \tag{1}$$

whence,
$$\tan A = \frac{\sin A}{\cos A} = \sqrt{\frac{1-\cos 2A}{1+\cos 2A}}. \tag{2}$$

On putting $\frac{1}{2}x$ for A, these relations take the forms

$$(a)\ \sin \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}, \qquad (b)\ \cos \tfrac{1}{2}x = \sqrt{\frac{1+\cos x}{2}},$$
$$(c)\ \tan \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{1+\cos x}}. \tag{3}$$

In (1), (2), (3), angles A and x denote any angles.

EXERCISES.

1. Express the results (1), (2), (3), *in words.*

2. Find $\sin 45°$, given that $\cos 90° = 0$.

From (3) *a*, $$\sin 45° = \sqrt{\frac{1-\cos 90°}{2}} = \frac{1}{\sqrt{2}}.$$

3. Find $\sin 22° 30'$, $\cos 22° 30'$, $\tan 22° 30'$ by means of (1), (2). Compare the values with those given in the tables.

93. Functions of three times an angle. Functions of an angle in terms of functions of one-third the angle.

To express $\tan 3A$ in terms of $\tan A$. Let A denote any angle.

$$\tan 3A = \tan(2A + A)$$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \dfrac{2\tan^2 A}{1-\tan^2 A}}. \quad \text{(Art. 51.)}$$

$$\therefore \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}. \qquad (1)$$

On putting x for $3A$, (1) becomes

$$\tan x = \frac{3\tan\frac{1}{3}x - \tan^3\frac{1}{3}x}{1 - 3\tan^2\frac{1}{3}x}.$$

To express $\sin 3A$ in terms of $\sin A$.

$$\sin 3A = \sin(2A + A) = \sin 2A\cos A + \cos 2A \sin A \qquad \text{[Art. 50, (1)]}$$
$$= 2\sin A\cos^2 A + (1 - 2\sin^2 A)\sin A$$
$$= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A.$$

$$\therefore \sin 3A = 3\sin A - 4\sin^3 A. \qquad (2)$$

In a similar way, $\cos 3A$ can be expressed in terms of $\cos A$.

$$\cos 3A = 4\cos^3 A - 3\cos A. \qquad (3)$$

EXERCISES.

1. Derive formula (3).
2. On substituting x for $3A$, write (2), (3).
3. Express formulas (1), (2), (3), and the results of Ex. 2, *in words.*
4. Assuming the value of $\sin 30°$, calculate $\sin 90°$.

5. From $\cos 30°$, derive $\cos 90°$; from $\tan 30°$, derive $\tan 90°$.

6. Derive $\sin 180°$, $\cos 180°$, $\tan 180°$, from $\sin 60°$, $\cos 60°$, $\tan 60°$, respectively.

7. Derive $\sin 75°$, $\cos 75°$, $\tan 75°$, from $\sin 25°$, $\cos 25°$, $\tan 25°$, respectively, as given in the tables.

8. Derive $\sin 37° 30'$, $\cos 37° 30'$, $\tan 37° 30'$, from the ratios of 75°.

94. Functions of the sum of three angles.

$$\left.\begin{aligned}\tan(A+B+C) &= \tan(\overline{A+B}+C) = \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C},\\ &= \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\cdot\tan C}\\ &= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.\end{aligned}\right\} \quad (1)$$

COR. 1. If $A=B=C$, (1) reduces to (1), Art. 93.

COR. 2. If $A+B+C=180°$, then $\tan(A+B+C)=0$, and, accordingly, the numerator of (1) is equal to zero. Hence, if A, B, C, are the three angles of a triangle,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C. \quad (2)$$

COR. 3. If $A+B+C=90°$, then $\tan(A+B+C)=\infty$, and, accordingly, the denominator of (1) is equal to zero. Hence,

$$\tan A\tan B+\tan B\tan C+\tan C\tan A=1, \text{ when } A+B+C=90°. \quad (3)$$

EXERCISES.

1. Show that $\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$.

If $A+B+C=180°$, the first member is zero. Division of the second member by $\cos A\cos B\cos C$ will give relation (2) above.

2. Show that $\cos(A+B+C)=\cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C$. What does this become when

$$A+B+C=180°?$$

If $A+B+C=90°$, the first member is zero. Division of the second member by $\cos A\cos B\cos C$ will give relation (3) above.

3. If $A+B+C=180°$, prove that

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Since $A+B+C=180°$, $\dfrac{A+B}{2}=90°-\dfrac{C}{2}.$

$$\begin{aligned}\cos A+\cos B+\cos C &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\cos C && \text{[Art. 52 (7)]}\\ &= 2\sin\frac{C}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2} && \text{[Art. 50 (7)]}\\ &= 1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)\\ &= 1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)\\ &= 1+2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2} && \text{[Art. 52 (8)]}\\ &= 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\end{aligned}$$

4. If $A+B+C=180°$, prove that

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

5. If $A+B+C=180°$, prove that

$$\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

6. Also, that $\sin(A+B)\sin(B+C)=\sin A\sin C$.

7. Also, that $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$.

8. If $A+B+C=90°$, show that

$$\sin 2A+\sin 2B-\sin 2C=4\sin A\sin B\cos C.$$

9. Find $\tan 4A$, $\tan 5A$, $\tan 6A$, $\tan 7A$, in terms of $\tan A$.

95. Identities. In the following exercises it is required that the first member be changed into the second member. When it is difficult to do this, help is sometimes afforded by taking some steps in changing the second member into the first. The direct steps to be taken from the first member to the second may be indicated by this means. No general directions can be given concerning the making of these transformations. The two following suggestions, however, are frequently useful:

(*a*) Since $\sin^2 A+\cos^2 A=1$, unity can be substituted for the first expression, and the first expression can be substituted for unity.

(*b*) The change of $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, into their values in terms of the sine and cosine, is sometimes helpful.

The examples in Art. 52 belong to this class.

EXERCISES.

1. Show that $\dfrac{1-\cos 2A}{1+\cos 2A}=\tan^2 A.$

$$\frac{1-\cos 2A}{1+\cos 2A}=\frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A-1)}=\frac{\sin^2 A}{\cos^2 A}=\tan^2 A.$$

2. Show that $\tan^2 A=\dfrac{1-\cos 2A}{1+\cos 2A}.$

$$\tan^2 A=\frac{\sin^2 A}{\cos^2 A}=\frac{\frac{1}{2}(1-\cos 2A)}{\frac{1}{2}(1+\cos 2A)}=\frac{1-\cos 2A}{1+\cos 2A}.$$

NOTE. The fact that $\tan^2 A=\dfrac{\sin^2 A}{\cos^2 A}$, suggests that the numerator in Ex. 1 be expressed in terms of the sine, and the denominator in terms of the cosine. In Ex. 2, the plan of transformation is more obvious.

Prove the following identities:

3. $\dfrac{\sec^2 B}{2-\sec^2 B}=\sec 2B.$
4. $1-2\sin^2(45^\circ-A)=\sin 2A.$
5. $\cos^2 A+\sin^2 A\cos 2B=\cos^2 B+\sin^2 B\cos 2A.$
6. $1+\cot 2\theta\cot\theta=\operatorname{cosec} 2\theta\cot\theta.$
7. $4\sin A\sin(60^\circ+A)\sin(60^\circ-A)=\sin 3A.$
8. $\cos 5\theta=\cos(3\theta+2\theta)=16\cos^5\theta-20\cos^3\theta+5\cos\theta.$
9. $\sin 5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta.$
10. $\tan(45^\circ+A)-\tan(45^\circ-A)=2\tan 2A.$
11. $\cos^4 B-\sin^4 B=\cos 2B.$
12. $\dfrac{\sin 3A-\cos 3A}{\sin A+\cos A}=2\sin 2A-1.$
13. $\dfrac{\sin x+\sin 2x}{1+\cos x+\cos 2x}=\tan x.$
14. $4(\cos^6 x+\sin^6 x)=1+3\cos^2 2x.$
15. $\sin 4A=4\sin A\cos^3 A-4\cos A\sin^3 A.$
16. $\cos 4A=1-8\cos^2 A+8\cos^4 A.$

96. For an acute angle of θ radians, $\cos\theta>1-\dfrac{\theta^2}{2}$, $\sin\theta>\theta-\dfrac{\theta^3}{4}$.

By Art. 50, (7), $\cos\theta=1-2\sin^2\dfrac{\theta}{2}$; by Art. 83, $\sin\dfrac{\theta}{2}<\dfrac{\theta}{2}.$

Hence, $\cos\theta>1-2\left(\dfrac{\theta}{2}\right)^2$, *i.e.* $\cos\theta>1-\dfrac{\theta^2}{2}.$

By Art. 50, (5),

$$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}=2\tan\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right).$$

But by Art. 83, $\tan\frac{\theta}{2} > \frac{\theta}{2}$, and $\sin\frac{\theta}{2} < \frac{\theta}{2}$.

Hence $\quad \sin\theta > \frac{2\theta}{2}\left\{1 - \left(\frac{\theta}{2}\right)^2\right\}$; *i.e.* $\sin\theta > \theta - \frac{\theta^3}{4}$.

97. One method of computing the trigonometric functions. A method of computing the trigonometric functions of angles which are in an arithmetic progression having the common difference D'', will now be shown.

$$\sin(n+1)D'' + \sin(n-1)D'' = 2\sin nD''\cos D''. \quad \text{[Art. 52, (5).]}$$

$$\therefore \quad \sin(n+1)D'' = 2\sin nD''\cos D'' - \sin(n-1)D''. \qquad (1)$$

Also $\quad \cos D'' = \sqrt{1 - \sin^2 D''}$.

Hence, if the sines of the angles D'', $2\,D''$, $3\,D''$, up to nD'' be known, then $\sin(n+1)D''$ can be computed by formula (1). The other functions can be derived from the sine.

The functions for angles from 0° to 45° will serve for the angles from 45° to 90°, since the ratio of an angle is the co-ratio of its complement. When the functions have been computed for angles up to 30°, the computations for angles greater than 30° can be made more easily. For, if A is an angle less than 30°,

$$\sin(30° + A) + \sin(30° - A) = 2\sin 30°\cos A = \cos A.$$

$$\therefore \qquad \sin(30° + A) = \cos A - \sin(30° - A). \qquad (2)$$

Similarly, $\quad \cos(30° + A) = \cos(30° - A) - \sin A. \qquad (3)$

In formula (1) suppose that $D'' = 10''$, and let its radian measure be denoted by θ.

Then $\qquad \sin 10'' < \theta, \; > \theta - \frac{\theta^3}{4}. \qquad$ [Arts. 83, 96.]

Since $\qquad 180° = \pi,$

$$10'' = \frac{10\,\pi}{180 \times 60 \times 60} = \frac{3.1415926535}{64800} = .000048481368 \cdots \text{ radians.}$$

$$\therefore \; \sin 10'' < .00004848 \cdots, \; > [.00004848 \cdots - \tfrac{1}{4}(.00004848 \cdots)^3].$$

Hence, to 12 places of decimals, $\sin 10'' = .000048481368$.

From this, $\sin 20''$ can be found by (1); then $\sin 30''$, then $\sin 40''$, and so on.

The functions of several angles can be found independently of the method just shown. Formulas involving these angles, and

Euler's and Legendre's *verification formulas,* may be used to test the accuracy of the tables. The latter formulas are (see Exs. 7–10, Ch. XII.),

$$\sin(36°+A)-\sin(36°-A)-\sin(72°+A)+\sin(72°-A)=\sin A, \quad (4)$$

$$\cos(36°+A)+\cos(36°-A)-\cos(72°+A)-\cos(72°-A)=\cos A. \quad (5)$$

EXERCISES.

1. Test the tables of natural sines and cosines by means of formulas (4), (5), taking A equal to 4°, 10°, 15°, and other values. **2.** Assuming the functions of 1° as known, calculate the sines of 2°, 3°, 4°, 5°, 6°, by formula (1). **3.** By means of formulas (2). (3), calculate the sines and cosines of 33°, 37°, 41°, 47°, 53°, 67°, and other angles.

98. Trigonometry defined. Branches of trigonometry. Before concluding this text-book it may be well to indicate to the student the relation of the part of trigonometry treated in the preceding pages to the subject as a whole, and also to try to give him a little idea of another branch of trigonometry; namely, analytical trigonometry.

In Chapters II.–IX., plane angles, the solution of plane triangles, and applications connected therewith were discussed. This is what is usually known as *plane trigonometry.* The study of solid angles, the solution of spherical triangles, and the associated practical applications, constitute *spherical trigonometry.* These branches of mathematics are founded on geometrical considerations, and may be looked upon as applications of algebra to geometry. Pure mathematics is sometimes regarded as consisting of two great branches; namely, *geometry* and *analysis.* Analysis includes algebra, infinitesimal calculus, and other subjects which employ the symbols, rules, and methods of algebra, and do not rest upon conceptions of space. (Geometrical ideas may be used in analysis, however, for the sake of exposition and illustration, and, on the other hand, algebra may be employed in expounding the principles of geometry.) Since the eighteenth century, trigonometry has also been treated as a *branch of analysis.**

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* The meaning of the word "analysis" thus used in mathematics, should not be confounded with the ordinary meaning of the word, or with the meaning attached to the term "analysis" in logic.

Analytical (or algebraical) *trigonometry* treats of the general relations of angles and their trigonometric functions without any reference to measurement. It discusses, among other things, the development of exponential and logarithmic series, the connections between trigonometric and exponential functions, the expansions of an angle and its trigonometric functions into infinite series, the calculation of π, the summation of series, and the factorization of certain algebraic expressions. The properties stated in formulas, (1)–(3) Art. 44, (1)–(8) Art. 50, (1)–(8) Art. 52, (1)–(3) Art. 93, are analytical properties, and *can be derived without the aid of geometry.* Analytical trigonometry includes hyperbolic trigonometry; that is, the treatment of what are called the hyperbolic functions.

While the trigonometric functions may be defined and discussed on a geometrical basis, as done in this book (and this is the easiest way for beginners), it may be stated that they can also be defined and their properties deduced on a purely algebraic basis. It is beyond the scope of this work to show this, but the student may obtain a little light on the subject by reading Notes A and D. It may be stated further, that, under certain restrictions, some of the most important theorems and properties found in analytical trigonometry can be derived easily in an elementary course in the infinitesimal calculus. It has been pointed out that the trigonometric functions can be defined in a purely geometrical manner, and in a purely algebraic manner; they can also be given definitions depending on the infinitesimal calculus, and their properties deduced therefrom. Finally, it may be said that trigonometry is merely a brief chapter in the modern Theory of Functions, and may be defined as the science of *singly periodic functions* (see Art. 78). For a treatment of trigonometry, either as a part of algebra, or, as "an elementary illustration of the application of the Theory of Functions," see Lock, *Higher Trigonometry;* Loney (Part II.), *Analytical Trigonometry;* W. E. Johnson, *Treatise on Trigonometry,* Chaps. XII.–XXII.; Casey, *A Treatise on Plane Trigonometry;* Levett and Davison, *Elements of Plane Trigonometry* (Parts II., III., Real Algebraical Quantity, Complex Quantity); Hayward, *Vector Algebra and Trigonometry;* Hobson, *A Treatise on Plane Trigonometry;* Chrystal, *Algebra,* Part I., Chap. XII.; Part II., Preface, and Chaps. XXIX., XXX.

APPENDIX.

NOTE A.

HISTORICAL SKETCH.

The most ancient mathematical writing known at the present time is an Egyptian papyrus preserved in the British Museum. It is the work of Ahmes, an Egyptian priest who lived at least seventeen hundred years B.C., and is believed to have been founded on older works dating as far back as 3400 B.C. The treatise is concerned with practical mathematics, and merely gives rules for making geometrical constructions and determining areas. The area of an isosceles triangle is obtained by taking the product of half the base and one of the sides. The area of a circle is found by deducting from the diameter one-ninth of its length, and squaring the remainder — a proceeding which is equivalent to taking $\pi = 3.1604 \cdots$.

The ancient Greeks brought geometry to a high state of perfection, but showed little aptitude for algebra and trigonometry. They were not inclined to be satisfied with approximate results, and regarded the practical application of mathematics as degrading to the science. Trigonometry was invented to supply practical needs, and its development, in the earlier stages, was due to men of the Egyptian, the Hindoo, and the Semitic races.

Astronomy was one of the studies most cultivated by the ancients, but astronomy could not advance, or even become a science, without the aid of trigonometry. Hipparchus of Nicæa in Bithynia, the greatest astronomer of antiquity, who flourished about 160–120 B.C., is regarded as the founder of trigonometry, which he developed solely as a necessary part of astronomy. Moreover, trigonometry continued to exist, for the most part, merely as a handmaid of astronomy for over eighteen hundred years. On this account, the theorems of spherical trigonometry were developed earlier than those of plane trigonometry. Of the writings of Hipparchus, all but one have been lost; but it is known that he constructed a table of chords, which serves the same purpose as a table of natural sines. Hero of Alexandria, who flourished some time between 155 and 100 B.C., and is supposed to have been a native Egyptian, found the area of a triangle in terms of its sides, and placed engineering and land-surveying on a scientific basis. Ptolemy, a native of Egypt, the records of whose observations cover the period 127–151 A.D., wrote the *Syntaxis Mathematica* (called the *Almagest* by the Arabs), a work founded on the investigations of Hipparchus. This was regarded as a kind of astronomical Bible for thirteen hundred years, until the Ptolemaic theory, namely, that the sun, planets, and stars revolve around the earth, was shown to be erroneous by Copernicus and Galileo. The *Almagest* is divided into thirteen books. Book I. treats of plane and spherical trigonometry, contains a very accurate table of chords, probably derived from Hipparchus, and shows the method of forming the table. It develops spherical before plane trigonometry, and does not give the solution of plane triangles. "Whereas the Ptolemaic system (of astronomy) was . . . overthrown, the theorems of Hipparchus and Ptolemy, on the other hand, will be, as Delambre * says, forever the basis of trigonometry." †

Whatever advance was made in trigonometry during the thousand years after Ptolemy, was due to the Hindoos and Arabs. The Hindoos had tables of the half-chords, or sines, and found that the arc equal in length to the radius contained 3438′. Aryabhatta (476–530 A.D.?) wrote a work containing sections on astronomy, spherical and plane trigonometry. This contained tables of natural sines of the angles in the first quadrant at intervals of $3\frac{3}{4}°$, the sine being defined as the semi-chord of twice the angle. He gave 3.1416 as the value of π. Other writers were Brahmagupta, born 598, and Bhaskara, about 1150, who gave some trigonometric formulas. The Hindoos knew how to solve plane and spherical right triangles.

During the period of the Dark Ages in Europe, the sciences of the Greeks and Hindoos were preserved, and, to some slight extent, improved by the Arabs. The latter studied trigonometry only for the sake of astronomy. The term *sine* is due to the celebrated Arabian astronomer *Al Battani* (*Albatagnius*), a native of Syria, who died about 930 A.D. Another Arabian astronomer, *Abû 'l Wafâ* (940–998), a native of Persia, was the first to introduce the tangent of the arc into the science; he calculated a table of tangents. Among the Western Arabs, to whom the development of the subject is indebted, were *Ibn Yûnos* of Cairo (died 1008), and *Gabir ben Aflah*, who was born at Seville and who died at Cordova in the latter part of the eleventh century. The latter wrote an astronomy in nine books, the first of which is devoted to trigonometry; he also contributed to the advancement of spherical trigonometry.

The next stage in the history of trigonometry is marked by the introduction of the Arabian works into Europe, and the development of the arithmetical part of the subject, especially the calculation of tables. This was largely

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* Jean Baptiste Delambre (1749–1822), a French mathematician who derived important formulas in spherical trigonometry.

† Ency. Brit., Art. *Ptolemy*.

the work of German astronomers, and chiefly of *Regiomontanus* and *Rheticus*. *Georg Purbach* (1423–1461), professor of mathematics and astronomy at the University of Vienna, wrote a table of natural sines computed for intervals of ten minutes, which was published in 1541. *Regiomontanus* (John Müller) (1436–1476), a native of Franconia, who was one of the greatest mathematicians that Germany has ever produced, in conjunction with Purbach made a translation of the *Almagest*, which was published in 1496. In this he substituted sines for chords, and gave a table of natural sines. He reinvented the tangent, and made a table of natural tangents for all degrees of the quadrant; this was published in 1490. In 1464 he wrote his *De Triangulis*, which was the earliest modern systematic exposition of plane and spherical trigonometry. This was printed in 1533, and a second edition appeared in 1561. The only functions introduced were sines and cosines. *Copernicus* (1473–1543), born in Prussia, wrote a short text-book on the subject about 1500, which was published in 1542. *Rheticus* (Georg Joachim) (1514–1576), a native of the Tyrol, professor of mathematics at Heidelberg, constructed tables (published in 1596) which are the basis of those still in use. He introduced secants and cosecants, and found the values of $\sin 2\theta$, $\sin 3\theta$ in terms of $\sin\theta$, $\cos\theta$. Hitherto the trigonometric functions had been considered as lines related to circular arcs. Rheticus was the first who constructed the right triangle and used the ratio definitions which depend directly on the angle. These definitions were not adopted, however, and, although introduced two hundred years later by Euler in 1748, they did not come into common use until after the middle of the present century. *Pitiscus* (1561–1613), professor of mathematics at Heidelberg, made important corrections in and additions to the tables of Rheticus. His trigonometry, published in 1599, contained formulas for $\cos(A \pm B)$, $\sin(A - B)$. *Adrian Romanus* (1561-1625), a Belgian mathematician, professor at the University of Louvain, first found the formula for $\sin(A + B)$. *François Vieta* (1540–1603), the greatest French mathematician of the sixteenth century, extended the tables of Rheticus. He made one of the earliest attempts to find the value of π by means of infinite series, and was the first who made any considerable application of algebra to trigonometry. In his work, *Ad Angulares Sectiones*, he gave formulas for $\sin n\theta$, $\cos n\theta$, in terms of $\sin\theta$, $\cos\theta$. *John Napier* (1550–1617) discovered the important formulas in spherical trigonometry which are commonly called Napier's Analogies. His invention of logarithms greatly lessened the arithmetical work necessary in astronomy and trigonometry, and thus ushered in a new era in the history of these sciences. *Edmund Gunter* (1581-1626), professor of astronomy at Gresham College, London, gave the first tables of logarithms of sines and tangents. He first used the terms cosine, cotangent, cosecant. *Albert Girard* (1590–1634), a Flemish mathematician, published a trigonometry in which the contractions sin, tan, sec were used. *William Oughtred* (1575–1660), an English mathematician, wrote a trigonometry, published in 1657, containing abbreviations for sine, cosine, but they did not come into general use until Euler reintroduced them nearly a century later.*

Thus far, trigonometry had been confined to the bounds set by the ancients, namely, to expressing the relations between the sides and angles of plane and spherical triangles, to the solution of triangles, and to the calculation of tables. Trigonometry had been founded on geometrical conceptions, and was regarded mainly as an appendage of geometry and astronomy. In the seventeenth and eighteenth centuries, however, a new branch of the subject, namely, analytical trigonometry, was created, chiefly by the genius of *De Moivre* and *Euler*. In the new development of the science, the symbols, rules, and methods of algebra were employed, and geometrical conceptions were disregarded. [See Art. 98 and Note D.] The older trigonometry still retains its position as a necessary department of applied mathematics. The modern analytical (or algebraical) side of the subject, however, has been so highly developed since the middle of the eighteenth century, and its results are so much employed in other branches of mathematical and physical science, that it may be regarded as the larger and more important part of trigonometry.

The new development began with the discovery and investigation of exponential, logarithmic, and trigonometric series. The chief investigators of infinite series were: John Wallis (1616–1703), professor of geometry at Oxford; James Gregory (1638–1675), professor of mathematics at Edinburgh; Nicolaus Mercator, died 1687, a native of Holstein, who settled in England; Isaac Newton (1642–1727); Gottfried William Leibnitz (1646–1716). Several of these series greatly simplified the calculation of π; some of them were obtained by means of the infinitesimal calculus invented by Newton and Leibnitz. Before 1669, Newton obtained the series for the arc in powers of the sine, and the series for the sine and cosine in powers of the arc. In 1670, Gregory discovered the series for the arc in powers of the tangent, and the series for the tangent and secant in powers of the arc; Leibnitz discovered the first of these independently in 1673.

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* "To England falls the honour of having produced the earliest European writers on trigonometry." (Cajori, *History of Mathematics*, p. 135.) *Thomas Bradwardine* (1290?–1349), archbishop of Canterbury, *Richard of Wallingford* (1292?–1336), abbot of St. Albans, *John Mauduith* (about 1310), fellow of Merton College, Oxford, who were mathematicians and astronomers, left writings containing trigonometry and tables drawn from Arabic sources. The earliest English books in which spherical trigonometry is used, are those of *Thomas Digges* (died 1595), one of the foremost English mathematicians of the sixteenth century. The earliest book in which plane trigonometry is introduced, is a work published by *Thomas Blundeville* in 1594.

John Bernoulli (1667–1748), a native of Switzerland, originated the idea of trigonometric functions, and treated trigonometry as a branch of analysis. He was the first to obtain real results by using the symbol $\sqrt{-1}$. *Abraham de Moivre* (1667–1754), a French Huguenot who settled in London, did much to advance analytical trigonometry, by his use of (so-called) imaginary quantities, and the discovery of the great fundamental theorems, which are called by his name. (See Note D.) *Johann Heinrich Lambert* (1728–1777), a native of Alsace, developed de Moivre's theorems, introduced the functions called hyperbolic sine and cosine, and showed their connection with the hyperbola. He also found that π is incommensurable. Modern trigonometry is indebted most of all to Leonhard Euler (1707–1783), a native of Switzerland. In his *Introductio in Analysin Infinitorum*, published in 1748, he systematized and generalized what was then known about algebra and trigonometry. He discussed the expressions of functions in series, and treated trigonometry as a branch of analysis. The latter was effected by regarding trigonometric functions, not as straight lines belonging to arcs, and thus depending on the radius of a circle, but as ratios, and thus as functions of the angle only. He reintroduced the abbreviations now used. [This was done simultaneously in England by Thomas Simpson (1710–1761), professor of mathematics at Woolwich, in his trigonometry, also published in 1748.] Euler first showed the connection between exponential and trigonometric functions (see Note D), and discovered many of their analytical properties.* Since the time of Euler, analytical trigonometry has benefited by the immense advances made in the theory of functions of complex quantities; that is, quantities of the form $x + \sqrt{-1}\,y$. It is now coming to be regarded, more properly and more logically, as an elementary chapter in the modern theory of functions. See Chrystal, *Algebra*, Part II., p. vii.†

NOTE B.

1. Projection definition of the trigonometric ratios. [Supplementary to Art. 40.] In Fig. 20, Art. 28, MN is the projection of AB on LR, and NM is the projection of BA on LR. In naming the projection, the points obtained by projection are taken in the *same* order as the corresponding points in the original line. It is apparent that, for any line, *the projections upon a series of parallel lines are equal.* For instance, $AD = MN$, Fig. 20. This may also be seen by drawing a series of lines parallel to LR and projecting AB upon them.

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* The first English book in which trigonometry received an analytical treatment was that of Robert Woodhouse (1773–1827), professor at Oxford, which was published in 1809.

† The principal sources from which this historical sketch has been drawn, are Hobson, Article *Trigonometry* (*Encyclopædia Britannica*, 9th edition), Ball, *A Short History of Mathematics*, Cajori, *A History of Mathematics*.

Suppose that in Fig. 36, Art. 40, YOY_1 be drawn at right angles to X_1OX. Then

OM is the projection of the turning line OP upon OX,

MP is equal to the projection of the turning line OP upon OY.

In two cases in Fig. 36, the projection of OP on OX is in the direction opposite to OX, that is, it is *negative;* in two cases, the projection of OP on OY is opposite to the direction of Y, that is, it is *negative.*

The definitions, Art. 40, may now be stated as follows:

$$\sin A = \frac{\text{proj. } OP \text{ on } OY}{OP}. \quad \tan A = \frac{\text{proj. } OP \text{ on } OY}{\text{proj. } OP \text{ on } OX}. \quad \sec A = \frac{OP}{\text{proj. } OP \text{ on } OX}.$$

$$\cos A = \frac{\text{proj. } OP \text{ on } OX}{OP}. \quad \cot A = \frac{\text{proj. } OP \text{ on } OX}{\text{proj. } OP \text{ on } OY}. \quad \operatorname{cosec} A = \frac{OP}{\text{proj. } OP \text{ on } OY}.$$

These differ from (1), Art. 40, merely in the fact that *names* are given to OM and MP. The properties shown in Chap. V. follow from these definitions.

2. Theorem on projection. The projection of one side of a polygon upon any straight line is equal to the algebraic sum of the projections of the other sides.

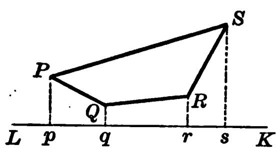

FIG. 90.

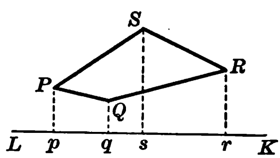

FIG. 91.

Let $PQRS$ be any polygon. Draw parallel lines Pp, Qq, Rr, Ss, from its vertices to any straight line LK. Then it is apparent that

$$ps = pq + qr + rs;$$

i.e. $$\text{proj. } PS = \text{proj. } PQ + \text{proj. } QR + \text{proj. } RS.$$

In Fig. 91, rs, the projection of RS, is negative. This proposition is true whether the projection be oblique or orthogonal. The theorem may also be stated thus:

The projection of a broken line upon a straight line is equal to the projection of the line drawn from the initial point to the terminal point of the broken line. Thus, the projection of the broken line $PQRS$ upon any straight line is equal to the projection of PS upon the same line.

3. The sine and cosine of the sum of two angles. [Supplementary to Art. 46.]

Let the construction be made as indicated in Art. 46. Then

$$\sin(A+B) = \frac{\text{proj. } OP \text{ on } OY}{OP} = \frac{\text{proj. } OQ \text{ on } OY}{OP} + \frac{\text{proj. } QP \text{ on } OY}{OP}.$$

[Theorem in (2).]

$$= \frac{\text{proj. } OQ \text{ on } OY}{OQ} \cdot \frac{OQ}{OP} + \frac{\text{proj. } QP \text{ on } OY}{QP} \cdot \frac{QP}{OP}$$

$$= \sin A \cos B + \sin VQP \sin B$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\cos(A+B) = \frac{\text{proj. } OP \text{ on } OX}{OP} = \frac{\text{proj. } OQ \text{ on } OX}{OP} + \frac{\text{proj. } QP \text{ on } OX}{OP}$$

$$= \frac{\text{proj. } OQ \text{ on } OX}{OQ} \cdot \frac{OQ}{OP} + \frac{\text{proj. } QP \text{ on } OX}{QP} \cdot \frac{QP}{OP}$$

$$= \cos A \cos B + \cos VQP \cdot \sin B$$

$$= \cos A \cos B - \sin A \sin B.$$

In the projection proof of the addition formulas for the sine and cosine, A and B can have any magnitudes, positive or negative. The formulas for $\sin(A-B)$, $\cos(A-B)$, can also be derived by substituting $-B$ for $+B$ in the addition formulas.

NOTE C.

[Supplementary to Arts. 9, 72.]

ON THE LENGTH AND AREA OF A CIRCLE.

1. The main purpose of this note is to outline a method of approximating to the value of π; that is, to the ratio of the length of a circle to its diameter. This method depends only on elementary geometry.* There are simpler and more expeditious methods of finding π, but they require a greater knowledge of mathematics than beginners in trigonometry generally possess.

By the methods of elementary geometry, as shown in the texts of Euclid and others, regular polygons of 3, 4, 5, 6, 15 sides can be inscribed in, and circumscribed about a given circle. Moreover, inscribed and circumscribing regular polygons of 2, 4, 8, 16, ⋯, times each of those numbers of sides can also be constructed by successively bisecting the arcs subtended by the sides, and joining the consecutive points of division. This process can evidently be

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* A section on the mensuration of the circle is given in many geometries. Reference may be made to the geometries of Beman and Smith (Ginn & Co.), Gore (Longmans, Green, & Co.), Phillips and Fisher (Harpers), and others.

carried on until the inscribed and circumscribing polygons have an infinitely great number of sides; that is, regular polygons of 3.2^n, 4.2^n, 5.2^n, 15.2^n sides, n being any positive integer, can be inscribed in, or circumscribed about, a given circle.

2. Outline of a proof of the theorem that the lengths of circles are proportional to their diameters.

(*a*) The length of a circle is greater than the perimeter of an inscribed polygon, and is less than the perimeter of a circumscribing polygon of any finite number of sides.

(*b*) As the number of sides of a regular polygon inscribed in, or circumscribed about, a circle is increased, the length of the perimeter of the polygon approaches nearer and nearer to the length of the circle. In other words, by increasing the number of sides, the difference between the length of the perimeter of the polygon and the length of the circle may be made as small as one please, and this difference approaches zero when the number of sides approaches infinity.

(*c*) Let any two circles be taken, and let the radii be R, r. Let AB be a side of a regular polygon of n sides inscribed in the circle having centre O

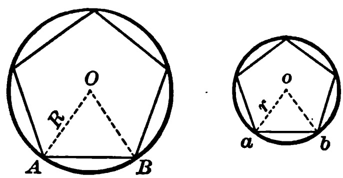


FIG. 92.

and radius R, and let ab be the side of a regular polygon of n sides inscribed in the circle having centre o and radius r. Let P denote the perimeter of the first polygon, p that of the second; let C denote the length of the first circle, and c that of the second. Then

$$P = C - D, \quad p = c - d,$$

where D, d may each be made smaller than any assignable quantity by making the number of sides, n, infinitely great.

The polygons are similar, since they are regular and have the same number of sides. Hence, by geometry,

$$\frac{P}{p} = \frac{OA}{oa} = \frac{R}{r};$$

that is,

$$\frac{C-D}{c-d} = \frac{R}{r}.$$

From this, $$rC - rD = Rc - Rd;$$

whence, $$rC - Rc = rD - Rd.$$

Now, let n become infinitely great. Then the second member becomes smaller than any assignable quantity, since r, R, each remains finite, and d, D, each approaches zero. Hence, when n is infinitely great,

$$rC - Rc = 0. \tag{1}$$

From (1),
$$\frac{C}{c} = \frac{R}{r}, \text{ and } \frac{C}{R} = \frac{c}{r}. \tag{2}$$

The first of equations (2) may be expressed in words: lengths of circles are to one another as their radii. According to the second equation, the length of the first circle is to its radius as the length of the second circle is to *its* radius. But these are any two circles. Hence, the ratio of the length of a circle to its radius, and, consequently, to its diameter, is constant. The ratio of the length of a circle to its diameter, which ratio is denoted by π, will now be approximately determined. [See Arts. 9 (*b*), (*c*), 72.]

3. The formulas used in this determination of π are deduced in problems ***A, B,*** that follow:

A. *Given the radius of a regular inscribed polygon, to compute the side of a similar circumscribing polygon.*

Let AB be the side of the inscribed polygon, and $OC = R$, the radius of the circle; let LM be a side of the similar circumscribing polygon. Let LM be obtained by producing OA, OB, to intersect the tangent drawn at C, the middle point of the arc AB.

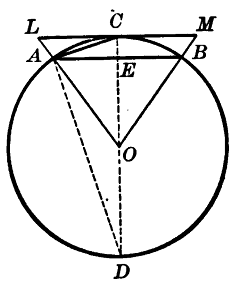


FIG. 93.

The triangles LCO, AEO, are similar. Hence,

$$\frac{LC}{AE} = \frac{OC}{OE}.$$

$$\therefore LC = \frac{OC \times AE}{OE} = \frac{R \times AE}{OE}.$$

$$\therefore LM = \frac{R \times AB}{OE}.$$

In the right-angled triangle OAE,

$$OE = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{R^2 - \frac{\overline{AB}^2}{4}} = \frac{1}{2}\sqrt{4R^2 - \overline{AB}^2}.$$

$$\therefore LM = \frac{2R \times AB}{\sqrt{4R^2 - \overline{AB}^2}}. \tag{1}$$

B. ***Given the radius and the side of a regular inscribed polygon, to compute the side of the regular inscribed polygon of double the number of sides.***

In Fig. 93, let AB be the side of a regular inscribed polygon of n sides. Draw AC; then AC is the side of a regular inscribed polygon of double the number of sides, namely, $2n$ sides. It is required to express AC in terms of the radius R and the side AB. Produce CO to D and draw DA. The triangles ACD, ACE, are similar, since the angles DAC, AEC, are equal, both being right angles, and the angle ACE is common to both triangles. Hence,

$$CD : AC = AC : CE.$$

$$\therefore \overline{AC}^2 = CD \cdot CE = CD(CO - EO) = 2R(R - EO) = R(2R - 2EO).$$

But $$EO = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{R^2 - \frac{\overline{AB}^2}{4}} = \frac{1}{2}\sqrt{4R^2 - \overline{AB}^2}.$$

$$\therefore \overline{AC}^2 = R(2R - \sqrt{4R^2 - \overline{AB}^2}).$$

$$\therefore AC = \sqrt{R(2R - \sqrt{4R^2 - \overline{AB}^2})} \qquad (2)$$

4. To determine approximately the ratio of the circumference of a circle to its diameter. If the radius is 1, the length of the circle is 2π, and the length of the semicircle is π. Hence the length of the semi-perimeter of each inscribed and circumscribing regular polygon, is an approximate value of π, and approaches nearer and nearer to π, the greater the number of sides in the polygon. The side of the *inscribed square* of the circle of radius 1 is $\sqrt{2}$; its semi-perimeter is 2.8284271. Successive applications of (2), Art. 3, give the sides of the inscribed polygons of 8, 16, 32, ... sides, and successive applications of (1) give the sides of the similar circumscribing polygons. The successive semi-perimeters are obtained by taking one-half the product of the length of a side and the number of sides in the polygons. The results of the computation are given in the following table. The table also gives the results when the initial polygon taken, is the *inscribed hexagon*. *The figures in bold type show the approximations.*

Lengths of semi-perimeters of regular inscribed and circumscribing polygons of circle of radius = 1.

Number of Sides.	Inscribed.	Circumscribing.	Number of Sides.	Inscribed.	Circumscribing.
4	2.8284271	4.0000000	6	**3**	**3.**4641016
8	**3.**0614675	**3.**3137085	12	**3.1**058285	**3.**2153903
16	**3.1**214452	**3.1**825979	24	**3.1**326286	**3.1**596599
32	**3.1**365485	**3.1**517249	48	**3.1**393502	**3.14**60862
64	**3.14**03312	**3.14**41184	96	**3.141**0319	**3.14**27146
128	**3.141**2773	**3.14**22236	192	**3.141**4524	**3.141**8730
256	**3.141**5138	**3.141**7504	384	**3.1415**576	**3.141**6627
512	**3.1415**729	**3.141**6321	768	**3.1415**838	**3.141**6101
1024	**3.1415**877	**3.141**6025	1536	**3.14159**04	**3.14159**70
2048	**3.14159**14	**3.14159**51			
4096	**3.14159**23	**3.14159**33			
8192	**3.141592**6	**3.141592**8			

5. Area of a circle. Area of a circular sector. The area of a circumscribing polygon of a circle is equal to one-half the product of the lengths of the perimeter and the radius. When the number of sides of the polygon increases indefinitely, the perimeter of the polygon approaches the length of the circle as its limit, and the area of the polygon approaches the area contained by the circle as its limit. Hence,

$$\text{area of circle} = \tfrac{1}{2} \text{ length of circle} \times \text{length of radius};$$

i.e.
$$\textbf{area of circle} = \tfrac{1}{2} \times 2\,\pi R \times R = \boldsymbol{\pi R^2}.$$

Since the area of a sector of a circle has the same ratio to the area of the circle that the arc of the sector has to the length of the circle,

$$\text{area of circular sector} = \tfrac{1}{2} \text{ length of arc} \times \text{length of radius}.$$

Hence, if θ is the radian measure of the angle of the sector,

$$\textbf{area sector} = \tfrac{1}{2} R\theta \times R = \frac{1}{2}\boldsymbol{R^2\theta}.$$

[For example, see Art. 73, Ex. 17.]

6. Historical Note. The problem to find a square whose area is equal to that of a given circle, which is commonly known as "squaring the circle," or "the quadrature of the circle," has long been of interest to mathematicians and others. Since the area of a circle is one-half the radius by the length of the circle, and the ratio of the length of the circle to the diameter is a constant, it follows that "squaring the circle" comes to determining this ratio.

The ancient peoples used 3 as the value of π; see 1 Kings vii. 23, 2 Chron. iv. 2. Ahmes used 3.1604; Archimedes showed, by the method described above, and by successively inscribing and circumscribing regular polygons of 6, 12, 24, 48, 96 sides, that π is between $3\frac{10}{71}$ and $3\frac{1}{7}$. Ptolemy used $3\frac{17}{120}$, and Aryabhatta, 3.1416. Adrian of Metz, in 1527, by using polygons up to 1536 sides, showed that the ratio is between $\frac{377}{120}$ and $\frac{333}{106}$. By taking the mean of the numerators for a new numerator, and the mean of the denominators for a new denominator, he obtained the value $\frac{355}{113}$, which is correct to six places of decimals. In 1579, Vieta by using polygons of 32,316 (*i.e.* 6×2^{16}) sides, got the value of π correctly to ten places. His method is not the same as that of Archimedes. (Professor Newcomb has remarked that the value of π to *ten* places of decimals would give the circumference of the earth correctly to within a fraction of an inch, if the diameter were accurately known.) In 1593, Adrian Romanus of Louvain computed π to 15 places of decimals and Ludolph van Ceulen (d. 1610), a German residing in Holland, calculated it to 35 places. Hence π is often referred to in Germany as "the Ludolphian number."

The discoveries of trigonometric series (see Note A) made the work of computers easier and more mechanical. In 1699, Abraham Sharp (1651–1742), found π to 71 places, and in 1706, John Machin, died 1751, professor of astronomy at Gresham College, London, extended the value to 100 places. Fautet de Lagny (1660–1734) carried it, in 1719, to 127 places; Baron Georg Vega (1756–1802), in 1794, to 136 places; Z. Dase of Vienna, in 1844, to 200 places; William Rutherford (1798?–1871), Royal Military Academy, Woolwich, in 1853, to 440 places; Richter, in 1854, to 500 places; and W. Shanks, in 1873, to 707 places of decimals. The laborious calculations of the "π-computers" have neither theoretical nor practical value.

About 1761 Lambert showed that π is *incommensurable*, and in 1882, F. Lindemann, in Freiburg, showed that it is *transcendental*, that is, it cannot be a root of any algebraic equation with integral coefficients. See article, "Squaring the circle," Ency. Brit., 9th edition.

N.B. The ratio π is often calculated approximately by means of Gregory's series (discovered in 1670) and certain identities, namely,

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots \text{ to } \infty, \quad \frac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239},$$

$$\frac{\pi}{4} = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{5} + \tan^{-1}\tfrac{1}{8}, \qquad \frac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{70} + \tan^{-1}\tfrac{1}{99},$$

in which the principal values of the inverse tangents are taken.

The student is advised to verify these identities, and to find an approximate value of π by means of Gregory's series; also to verify the remark made above concerning the circumference of the earth. Also to show that the method of Ahmes (see Note A) for finding the area of a circle is equivalent to taking $\pi = 3.1604 \cdots$.

NOTE D.

DE MOIVRE'S THEOREM, AND OTHER RESULTS IN ANALYTICAL TRIGONOMETRY.

[In what follows i denotes $\sqrt{-1}$.]

1. De Moivre's Theorem. For all values of n, positive and negative, integral and fractional,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(*a*) *When n is a positive integer.*

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2). \qquad (1)$$

On multiplying each member of (1) by $\cos\theta_3 + i\sin\theta_3$, there is obtained,

$$\begin{aligned}
&(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\\
&\qquad=\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3)\\
&\qquad=\{\cos(\theta_1+\theta_2)\cos\theta_3 - \sin(\theta_1+\theta_2)\sin\theta_3\}\\
&\qquad\qquad+ i\{\sin(\theta_1+\theta_2)\cos\theta_3 + \cos(\theta_1+\theta_2)\sin\theta_3\}\\
&\qquad=\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3).
\end{aligned}$$

In a similar way, the product of four or more such factors can be found. Thus, for n factors,

$$\begin{aligned}
&(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)\\
&\qquad=\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n) \qquad (2)
\end{aligned}$$

If $\theta_1=\theta_2=\cdots=\theta_n=\theta$, then (2) becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \qquad (3)$$

(*b*) ***When n is a negative integer, say,*** $-m$.

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{-m} &= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}\\
&= \frac{1}{\cos m\theta + i\sin m\theta}\cdot\frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}\\
&= \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta. \qquad (4)
\end{aligned}$$

(*c*) In (3) let $n\theta = \phi$; then $\theta = \frac{\phi}{n}$, and (3) becomes

$$\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right)^n = \cos\phi + i\sin\phi.$$

On transposing and taking the nth root of each member of this equation, there is obtained

$$(\cos\phi + i\sin\phi)^{\frac{1}{n}} = \cos\frac{\phi}{n} + i\sin\frac{\phi}{n}. \qquad (5)$$

(The second member of (5) is *one* of the n roots of the first member.)

(*d*) ***When n is a fraction,*** $\frac{p}{q}$.

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{\frac{p}{q}} &= [(\cos\theta + i\sin\theta)^p]^{\frac{1}{q}} = (\cos p\theta + i\sin p\theta)^{\frac{1}{q}}\\
&= \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta. \qquad (6)
\end{aligned}$$

(The second member of (6) is *one* of the q roots of the first member.)

For all the roots of the first members of (5), (6), see one of the works referred to in Art. 98. For a geometrical representation of the factors considered above and the results (1)–(6), and for a proof of these results, see Hobson, *Plane Trigonometry*, Chap. XIII.; Chrystal, *Algebra*, Part I., Chap. XII.

2. Following are some of the theorems proved in analytical trigonometry, which will be met by those who read only a little farther in mathematics:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \text{ to } \infty. \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ to } \infty. \tag{2}$$

Expansions (1) and (2) were first shown by Newton in 1669.

If e^x denote the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ to ∞, (3)

then
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{4}$$

Formulas (4) were first given by Euler. The expansions (1), (2), (3), are also derived in works on the differential calculus. They are convergent for all finite values of x. Either (1), (2), or (3), (4), may be taken as *definitions* of the sine and cosine.

Hyperbolic functions. The hyperbolic sine and cosine of x, denoted by *sinh* x, *cosh* x, may be defined in either one of the following ways, namely,

$$\left.\begin{aligned} \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \text{ to } \infty \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ to } \infty \end{aligned}\right\} \quad \ldots\ldots \tag{5}$$

and
$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}. \tag{6}$$

A geometrical definition may also be given to the hyperbolic functions. In this definition, they are related to the hyperbola in a manner analogous to a way in which the trigonometric (circular) functions are related to a circle. It may be said that the formulas or definitions (1)–(6) may be applied to all numbers x, real, pure imaginary, or complex; *i.e.* quantities of the form $a + b\sqrt{-1}$. (When x is real in (1)–(4), it denotes the radian measure of the angle.) See Chrystal, Part II., Chap. XXIX.

EXERCISES.

1. Substitute 1 for x in the series (3), and thus deduce 2.71828 as an approximate value of e.

2. (*a*) Write the series for e^{ix} and e^{-ix} by substituting ix and $-ix$ for x in (3).

(*b*) Then find the value of $\cos x$ in (4), and compare the result with (1).

(*c*) Then find the value of $\sin x$ in (4), and compare the result with (2).

3. Using formulas (4), show that $\cos^2 x + \sin^2 x = 1$.

4. Substitute ix for x in (1) and (2), and compare the results with (5). Substitute ix for x in (4), and compare the results with (6). Each substitution shows that $\cos(ix) = \cosh x$, and $\sin(ix) = i \sinh x$.

5. Show by means of the formulas (1)–(4) that the cosine of an angle of magnitude zero is unity, and that the sine of such an angle is zero.

6. By means of (1), (2), find approximate values of $\sin 10°$, $\cos 10°$, $\sin 15°$, $\cos 15°$, $\sin 20°$, $\cos 20°$, $\sin 30°$, $\cos 30°$. [First, express the angles in radian measure.]

QUESTIONS AND EXERCISES FOR PRACTICE AND REVIEW.

It is not intended that all these exercises be worked by any one person, or by any one class. It is advisable to consider only a few of them on the completion of each chapter, and to use them chiefly in the general reviews. In each set there are a number of direct questions on the principles and theorems explained in the corresponding chapter; these questions will enable the pupil to examine himself concerning his knowledge of the text. Teachers will often do well to take examples from other sources.

CHAPTER I.

1. (*a*) Define and illustrate *logarithms*, *characteristic*, *mantissa*. (*b*) What is meant by the base of a system of logarithms? (*c*) What is meant by a system of logarithms? (*d*) Show that in all systems $\log 1 = 0$, and that the logarithms of all proper fractions are negative.

2. What are the advantages gained by the use of logarithms calculated to the base 10? Show that the characteristic of any logarithm to the base 10 may be found by inspection.

3. What are the logarithms to base 3, of 81, $\frac{1}{81}$, $\sqrt[4]{729}$?

4. Find, by using logarithms, the values of the following quantities: (*a*) $\sqrt{375}$, $\sqrt{37.5}$, $\sqrt{3.75}$, $\sqrt{.375}$, $\sqrt{.0375}$; (*b*) $\sqrt[3]{12.5}$, $\sqrt[3]{1.25}$, $\sqrt[3]{.125}$, $\sqrt[3]{.0125}$, $\sqrt[3]{.00125}$; (*c*) $\sqrt[5]{784}$, $\sqrt[7]{93}$, $\sqrt[9]{834}$; (*d*) $(19)^{\frac{4}{5}}$, $(212)^{\frac{2}{3}}$, $(31.7)^{\frac{3}{5}}$; (*e*) 4^{-7}, $5^{-\frac{1}{3}}$, $(67)^{-\frac{2}{3}}$.

5. The following calculations are to be made at first without the use of logarithmic tables. The results may then be checked by comparing them with the values obtained by means of the tables. (*a*) Given $\log 3 = .477121$, find $\log \{(2.7)^3 \times (81)^{\frac{4}{5}} \div (90)^{\frac{3}{4}}\}$. (*b*) Given $\log 5 = .69897$, find $\log 200$, $\log .025$, $\log \sqrt[6]{62.5}$. (*c*) Given $\log 2 = .30103$, find the logarithms of 5, $\frac{1}{125}$, $\sqrt[4]{.005}$. (*d*) Given $\log 2 = .3010$ and $\log 3 = .4771$, find $\log \frac{1}{24}$, $\log .25$, log 16.2, log $\sqrt{\frac{5}{9}}$. (*e*) Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$, find the logarithms of $\frac{21}{16}$, 175, .0054, $(12)^{\frac{2}{3}}$, $\sqrt[4]{35}$. (*f*) Given $\log 8 = .903$, $\log 9 = .954$, find the logarithms of 2, 3, 12, 500, .075.

6. Find by logarithms the values of the following quantities:

(*a*) $372.48 \times (\frac{150}{151})^{60}$, $(.006)^5 \div 125$; (*b*) $3487 \times (.00345)^{\frac{1}{2}} \div (-83)^{\frac{1}{3}}$; (*c*) $\sqrt[5]{\frac{3}{7}} \div \sqrt[7]{\frac{9}{23}}$; (*d*) $\left(\frac{345.4 \times 958.3}{23.4 \times 317.9}\right)^{\frac{1}{2}}$, $\left(\frac{417.9 \times 813.1}{964.7 \times 313.2}\right)^{\frac{1}{2}}$; (*e*) $(23\frac{3}{7})^{\frac{2}{3}} \times (41\frac{2}{9})^{\frac{3}{5}} \div (7.93)^{3.5}$.

7. Find an approximate value of x in each of the following equations: (*a*) $x^2 = 237$, (*b*) $x^3 = 17$, (*c*) $x^{-2} = 17$, (*d*) $x^{-3} = 4\frac{1}{2}$, (*e*) $2^{2x} = 9$, (*f*) $3^{5x} = 197$, (*g*) $3^x = 32$, (*h*) $(25)^{3-2x} = 2^{x+3}$, (*i*) $\log(x^2) + \log(2x) + 1 = 0$.

8. If the logarithm of 27 is $-\frac{3}{5}$, what is the base?

CHAPTER II.

1. If on a map a square inch represents 10 acres, how many yards are represented by the diagonal of a square inch?

2. Explain the English and French methods of measuring angles, and show how, when the measure of an angle according to either method is known, its measure according to the other may be found. Express 100° in grades. (See Note 2, Art. 11.)

3. If A is an acute angle, show that $\tan A$ is greater than $\sin A$.

4. By aid of an equilateral triangle find the numerical values of the six trigonometric ratios of 60° and 30°. Find the numerical values of the ratios of 45°.

5. Show that (*a*) $\sqrt{\frac{\sin 45° - \sin 30°}{\sin 45° + \sin 30°}} = \sec 45° - \tan 45°$,

(*b*) $\frac{1 + \cot 60°}{1 - \cot 60°} = \left\{\frac{1 + \cos 30°}{1 - \cos 30°}\right\}^{\frac{1}{2}}$,

(*c*) $\tan^2 60° - 2\tan^2 45° = \cot^2 30° - 2\sin^2 30° - \frac{3}{4}\operatorname{cosec}^2 45°$.

6. The sine of an angle defined as a ratio being less than unity, explain why the tabular logarithms of the sines of angles are expressed with whole numbers as characteristics. Given $\log \tan 18° = 9.51178$, show what the tabular logarithm of $\cot 18°$ must be.

7. (*a*) Given $\log 2 = .30103$, $\log 3 = .47712$, find $\log \sin 60°$ and $\log \tan 30°$. (*b*) Given $\log 5 = .69897$, find the logarithmic sine of 30°, and the logarithmic cosine of 45°.

8. Compute the trigonometric ratios of A in a right triangle ABC ($C = 90°$), when $b = \frac{4}{5}c$.

9. Construct the following right-angled triangles: (*a*) ABC, in which, $C = 90°$, $c = 5$, $\cot A = \frac{7}{3}$; (*b*) when one of the legs is 3, and the sine of the adjacent acute angle is $\frac{2}{5}$; (*c*) hypotenuse 4, and sine of one of the acute angles $\frac{2}{3}$; (*d*) $C = 90°$, $\sin A = \frac{2}{3}$, $b = 7$; (*e*) $C = 90°$, $\text{cosec } A = \frac{3}{2}$, $b = 10$; write the values of $\sin A$, $\cos A$, $\tan A$; (*f*) $C = 90°$, $\cos A = \frac{3}{5}$, $a = 9$.

10. In the triangle ABC, $C = 90°$, $\tan B = \frac{15}{8}$. If $AB = 510$ ft., find AC.

11. In ABC, $C = 90°$, $BC = 10$ ft., $\tan B = 1.05$; find the other sides.

12. The string of a kite is 250 ft. in length. How high is the kite above the ground when the string, supposed stretched quite tight, makes with the ground an angle whose tangent is $\frac{24}{7}$?

13. ABC is an isosceles triangle, right-angled at C; D is the middle point of AC. Prove that DB divides the angle B into two parts whose cotangents are as 2 : 3.

14. (*a*) Given L. cos $20° = 9.97$ and L. cot $20° = 10.44$; find *each* of the other logarithmic ratios of 20°. (*b*) Given L. sin $40° = 9.808$, L. tan $40° = 9.924$; find $\log \cot 40°$, $\log \cos 40°$, $\log \sec 40°$, $\log \text{cosec } 40°$.

15. If $\tan\theta = \dfrac{2\sqrt{ab}\sin\frac{C}{2}}{a-b}$, find θ when $a = 5$, $b = 2$, $C = 120°$.

16. Calculate $\sin^3 23° \times \sqrt{27.268} \div 2\cos^2 48°$.

17. Find x in the equations: (*a*) $x \sin 74° = 235 \tan 37° \cos 63°$,

(*b*) $x^2 \cos 39° = 47.5 \sin^2 46° \sec^2 64°$.

18. Solve $(\sin 8° + \cos 8°)^{2x} = 2 \sin 16° (\tan 32°)^x$.

CHAPTER III.

1. State what parts of a right plane triangle must be given that it may be constructed, and show how a right triangle may be solved, in each of the four possible cases.

2. Derive the formulas for computing B, a, and c of a right triangle when $C = 90°$, and A and b are given. Also find a formula that shall include only the required parts.

3. In ABC, $C = 90°$, $b = 22$ ft., and $\sin A = .42$. Find a, c, $\sin B$, and the area.

4. In ABC, $C = 90°$, $\cos A = \frac{7}{10}$, $c = 40$ ft. Find the values of $\cos B$, $\cot B$, a, b, and the area.

5. Solve the following right-angled triangles by (1) making an off-hand estimate, (2) measuring on a drawing made to scale, (3) computing without logarithms (four-place tables), (4) computing with logarithms. Check the results by computation. If a solution is impossible, explain why it is so. (Each triangle is denoted by ABC, and $C = 90°$.) (i.) $a = 45$, $b = 62$; (ii.) $a = 685$, $B = 34° 47' 25''$; (iii.) $c = 560$, $a = 310$; (iv.) $c = 327$, $b = 450$; (v.) $c = 520$, $A = 36° 40' 20''$; (vi.) $b = 720$, $B = 61° 24' 30''$; (vii.) $c = 425$, $B = 32° 45' 35''$; (viii.) $a = 11524$, $b = 35976$; (ix.) $a = 67213$, $b = 75324$; (x.) $c = 35421$, $b = 23462$.

6. Two sides of a triangle are as $5:9$, and the included angle is a right angle. Find the other angles.

7. Find the acute angles of a right-angled triangle whose hypotenuse is six times as long as the perpendicular let fall upon it from the opposite angle.

CHAPTER IV.

1. (*a*) Derive the formula for the area of a right triangle in terms of (i.) an angle and its opposite side, (ii.) an angle and its adjacent side. (*b*) One side of a triangle is seven times another, and the included angle is a right angle. Find the other angles.

2. Show how an isosceles triangle may be divided into right triangles, and how it may be solved by aid of these right triangles when the following elements are given: (*a*) Base and vertical angle, (*b*) base and side, (*c*) side and vertical angle, (*d*) base and perpendicular from vertex on the base. Discuss any other possible cases.

3. Solve the isosceles triangles (*a*) whose base is 126 ft., and vertical angle is $127°$; (*b*) whose base and perpendicular on it from the vertex are each 721.34 yd.

4. (*a*) Find the area of a regular octagon the side of which is 26 yd. (*b*) Find the side of a regular pentagon inscribed in a circle whose radius is 43 ft. (*c*) If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2:\sqrt{5}$.

5. (*a*) At 120 ft. distance, and on a level with the foot of a steeple, the angle of elevation of the top is $62° 27'$; find the height. (*b*) From a cliff 330 ft. high the angle of depression of a boat at sea is $40° 35' 25''$; how far is the boat from the foot of the cliff?

6. When the altitude of the sun is $30°$ the length of the shadow cast by Bunker Hill monument is 381 ft. What is the height of the monument?

7. The angles of depression from the top of a tower 48.6 ft. high to two points, on a level with its base and in line with the tower, are $45°$ and $30°$ respectively. Find the distances of each point from the other and from the top of the tower.

8. A pole 40 ft. high is erected at the intersection of the diagonals of a square courtyard. When the sun's altitude is $43° 40'$, the shadow just reaches a corner of the yard. Find the length of the side of the square.

9. (*a*) When the altitude of the sun was $67° 30' 45''$ the length of the shadow of a perpendicular pole was 73.4 ft. Find the length of the shadow when the sun's altitude is $35°$. (*b*) The shadow of a tower is observed to be half the known length of the tower, and some time after to be equal to the full length. How much will the sun have gone down in the interval?

10. A flagstaff which leans to the east is found to cast shadows of 198 ft. and 202 ft., when the sun is due east and west respectively, and his altitude is 7°. Find the length of the flagstaff and its inclination to the vertical.

11. What angle will a flagstaff 24 ft. high, on the top of a tower 200 ft. high, subtend to an observer on the same level with the foot of the base, and 100 yds. distant from it?

12. Looking out of a window with his eye at the height of 15 ft. above the roadway, an observer finds that the angle of elevation of the top of a telegraph post is 17° 18′ 35″, and that the angle of depression of the foot of the post is 8° 32′ 15″. Calculate the height of the telegraph post and its distance from the observer.

13. A man in a balloon, when it is one mile high, finds the angle of depression of an object on the level ground to be 35° 20′, then after ascending vertically and uniformly for 20 min., he finds the angle of depression of the same object to be 55° 40′. Find the rate of ascent of the balloon in miles per hour.

14. A man observes the elevation of a mountain top to be 15°, and after walking 3 mi. directly toward it on level ground, the elevation is 18°. Find his distance from the mountain.

15. From a boat the angle of elevation of the highest and lowest points of a flagstaff, 30 ft. high, on the edge of a cliff are observed to be 46° 12′ and 44° 13′. Determine the height of the cliff and its distance.

16. The angles of elevation of the top of a tower, observed at two points in the horizontal plane through the base of the tower, are $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}\frac{5}{12}$; the points of observation are 240 ft. apart, and lie in a direct line from the base. Find the height of the tower.

17. A person standing due south of a lighthouse observes that his shadow cast by the light at the top is 24 ft. long; on walking 100 yd. due east he finds his shadow to be 30 ft. Supposing him to be 6 ft. high, find the height of the light from the ground.

18. An observer is 384 yd. due south of a point from which a balloon ascended; he measures a horizontal base due east, and at the other extremity finds the angle of elevation to be 60° 15′. Find the height of the balloon.

19. A surveyor starts from A and runs 766 yd. due east to B, thence 622 yd. N. 20° 30′ E. to C, thence 850 yd. N. 41° 45′ W. to D, thence S. 42° 35′ W. to E. Find the distance and bearing of A from E, and determine the area of the field $ABCDE$.

20. A surveyor runs 253 yd. N.E. by E., thence N. by E. 212 yd., thence W.N.W. 156 yd., thence S.W. by S. 210 yd., thence to the starting-point. Find the bearing and distance of the starting-point from the last station, and determine the area of the field which the surveyor has gone around.

CHAPTER V.

1. Define and illustrate *angle*, *negative angle*, *complement of an angle*, *supplement of an angle*, *quadrant*, *angle in the third quadrant*.

2. Define and illustrate the six *trigonometric ratios*. Find the greatest and least values that each of them can have. Arrange in tabular form the algebraic signs of the trigonometric ratios of an angle in each quadrant.

3. Explain how the trigonometric ratios of an angle of any magnitude, positive or negative, can be found, (a) by means of tables which give these ratios for angles up to 90° only, (b) by means of tables which give these ratios for angles up to 45° only.

4. Prove that if two angles have the same sine, and also any of the other five trigonometric ratios (with one exception) the same, they will differ by a multiple of 360°.

5. State and prove the chief relations which exist between the trigonometric ratios of any angle A.

6. Express the trigonometric ratios of $90° - A$, $90° + A$, $180° - A$, $180° + A$, $270° - A$, $270° + A$, $360° - A$, $-A$, in terms of the trigonometric ratios of A.

7. Name *three* pairs of trigonometric ratios such that the product of each pair shall equal 1; *one* pair, the sum of whose squares shall equal 1; *two* pairs, the difference of whose squares shall equal 1.

8. Compare the trigonometric ratios of any angle (a) with those of its complement, (b) with those of its supplement.

9. Prove that $\sin A = \cos A \tan A$; $\sec^2 A = 1 + \tan^2 A$; $\cot A = \operatorname{cosec} A \cos A$; $\sin^2 A + \cos^2 A = 1$; $\sin\theta = \tan\theta : \sqrt{1 + \tan^2\theta}$; $\cos x = \sqrt{\operatorname{cosec}^2 x - 1} : \operatorname{cosec} x$.

10. (a) Express the following trigonometric ratios in terms of trigonometric ratios of positive angles not greater than 45°: sin 237°, cos (– 410°), tan 2000°, cot (– 137°), sec 445°, cosec (– 650°), sin 185°, tan 267°, sec 345°, cos 87°, cot (– 19°); (b) by means of the tables give the numerical values of these ratios.

11. Find, without the use of trigonometric tables, the numerical values of cos 1410°, tan (– 1260°), cosec (– 1710°), tan 225°, cot 1035°, cosec 210°, cos 1500°, sin 1665°, tan (– 1665°), all the trigonometric ratios of – 1125° and 930°.

12. Construct the angles: (a) whose secant is 3, (b) whose tangent is $\sqrt{2} + 1$, (c) whose cotangent is $\frac{3}{4}$. Find the other ratios of these angles.

13. (a) Find $\sin A$, $\cot A$, when $\cos A = -\frac{5}{13}$, and $A < 180°$. (b) Find the other ratios of A and x when $\cot A = \frac{8}{15}$ and $\cos x = -\frac{3}{5}$. (c) Find the other ratios of A when $\cos A = -\frac{1}{2}$, and A lies between 540° and 630°.

(*d*) Find the trigonometric ratios of $180° + \theta$ and $270° - \theta$, given $\tan\theta = \frac{1}{4}$. (*e*) Given $\sec x = -\frac{5}{4}$, and x in the third quadrant; find the value of $\dfrac{\sin x + \tan x}{\cos x + \cot x}$.

14. Do Ex. 9, Art. 18, A being any angle. Explain the ambiguities in the algebraic signs. If A is an angle in the third quadrant, express $\cos A$, $\tan A$, $\cot A$, $\sec A$, $\operatorname{cosec} A$ in terms of $\sin A$.

15. (*a*) If $\sec A = n \tan A$, find the other ratios of A. (*b*) If $2\sec\theta = \tan\alpha + \cot\alpha$, find $\tan\theta$ and $\operatorname{cosec}\theta$. (*c*) Solve $x^3 \cot 108° = 128° \sin 72° \cos 18°$.

16. Prove the identities: $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$; $\cos^4 A - \sin^4 A = 1 - 2\sin^2 A$; $\sin x(\cot x + 2)(2\cot x + 1) = 2\operatorname{cosec} x + 5\cos x$; $\sec^2 B - \cos^2 B = \cos^2 B \tan^2 B + \sin^2 B \sec^2 B$; $\cos^6 A + \sin^6 A = 1 - 3\cos^2 A \sin^2 A$; $\cos^6 x + 2\cos^4 x \sin^2 x + \cos^2 x \sin^4 x + \sin^2 x = 1$.

17. (*a*) Find the value of x not greater than two right angles which will satisfy the equation $4\sqrt{3}\cot x = 7\operatorname{cosec} x - 4\sin x$. (*b*) Likewise, in the case of the equation $\sin x + \cos x \cot x = 2$. (*c*) Likewise, in $\tan^4 x - 4\tan^2 x + 3 = 0$. (*d*) If $1 + \sin^2\theta = 3\sin\theta\cos\theta$, find $\tan\theta$. (*e*) Find the least positive value of A that satisfies the equation $2\sqrt{3}\cos^2 A = \sin A$. (*f*) Find all the angles between 0° and 500° which satisfy the equation $4\sin^2\theta = 3$. (*g*) If $2\cos A + \sec A = 3$, what is the value of A? (*h*) Find A when $\tan^2 A + \operatorname{cosec}^2 A = 3$.

CHAPTER VI.

1. (*a*) Write the values of $\cos(A + B)$, $\cos(A - B)$, $\sin(A + B)$, $\sin(A - B)$, $\tan(A + B)$, $\tan(A - B)$ in terms of the trigonometric ratios of A and B. (*b*) Deduce these values. (*c*) Express them in words.

2. (*a*) Express in terms of the trigonometric ratios of A *each* of the following: $\sin 2A$, $\cos 2A$ (*three* different forms), $\tan 2A$, $\cot 2A$. (*b*) Derive these expressions.

3. (*a*) Show that $\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$. (*b*) Show that $\cos 2A + \cos 2B = 2\cos(A + B)\cos(A - B)$. (*c*) State and derive an equivalent expression for the difference of two sines; (*d*) for the difference of two cosines.

4. Show how to find $\cos\frac{1}{2}A$ when $\cos A$ is known. Explain the ambiguity in the result. Determine the sign of the result when A is an angle in the third quadrant. Find the cosine of 112° 30′. [From cos 225°.]

5. Prove $2\cos\dfrac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$ if A is between 270° and 360°.

6. Derive an expression for each of the following: $\sin 3A$ in terms of $\sin A$, $\cos 3A$ in terms of $\cos A$, $\tan 3A$ in terms of $\tan A$. [SUGGESTION: $3A = 2A + A$. See Art. 93.]

7. (*a*) If $\tan A = \dfrac{\sqrt{3}}{4-\sqrt{3}}$ and $\tan B = \dfrac{\sqrt{3}}{4+\sqrt{3}}$, prove that $\tan(A-B) = .375$. (*b*) If $\sin A = \dfrac{1}{\sqrt{10}}$ and $\cos B = \dfrac{3}{5}$, find the value of $\tan(A+B)$. (*c*) Find $\tan(A+B)$, given that $\sin A = \dfrac{8}{17}$, $\sin B = \dfrac{5}{13}$.

8. (*a*) If $\tan\dfrac{A}{2} = \dfrac{a}{b}$, show that $\sin A = \dfrac{2ab}{a^2+b^2}$, $\sin 2A = \dfrac{4ab(a^2-b^2)}{(a^2+b^2)^2}$. (*b*) If $\tan A = \dfrac{b}{a}$, prove that $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}} = \dfrac{2\cos A}{\sqrt{\cos 2A}}$.

9. (*a*) Find $\sin 45°$, and thence deduce the ratios of $22° 30'$. (*b*) Prove that $\tan 67° 30' = 1 + \sqrt{2}$. (*c*) Deduce the ratios of $67° 30'$, (i.) from ratios of $45°$, $22° 30'$, (ii.) from ratios of $135°$.

10. (*a*) Given $\sin 30° = \frac{1}{2}$ and $\cos 45° = \frac{1}{2}\sqrt{2}$; find $\sin 15°$, $\cos 75°$. (*b*) Given $\sin 30° = \frac{1}{2}$; find the numerical values of the other ratios of $30°$; thence derive the ratios of $15°$, thence derive the ratios of $75°$, $105°$, $165°$, $195°$. (*c*) Prove the following: $\tan 15° + \tan 75° = 4$, $\cos 15° \cdot \cos 75° = .25$, $\sin 105° + \cos 105° = \cos 45°$, $\tan 15°(\tan 60° - \tan 30°) = \tan 60° + \tan 30° - 2$.

11. (*a*) Express $\sin 8A + \sin 2A$ as a product. (*b*) Express as a sum or difference: (i.) $2\cos A \cos B$, (ii.) $2 \sin 50° \cos 20°$. (*c*) Prove without using tables that (i.) $\sin 70° - \sin 10° = \cos 40°$, (ii.) $\cos 20° + \cos 100° + \cos 140° = 0$. Verify by the tables.

12. Show that: (1) $\cot A \cot B \cos(A+B) = \cos A \cos B(\cot A \cot B - 1)$; (2) $\cos(A+B)\cos A + \sin(A+B)\sin A = \cos B$; (3) $\cos A - \sin A = \sqrt{2}\cos(A+45°)$; (4) $2\cos^8 x - 2\sin^8 x = \cos 2x(1+\cos^2 2x)$; (5) $\cos^2 A + \sin^2 A \cos 2B = \cos^2 B + \sin^2 B \cos 2A$; (6) $\cos^2 A - \cos A \cos(60° + A) + \sin^2(30° - A) = .75$; (7) $\tan\frac{1}{2}\theta = \sin\theta : 1 + \cos\theta$; (8) $\cos(135° + A) + \sin(135° - A) = 0$; (9) $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$.

13. Prove that: (1) $\dfrac{\sin x + \sin y}{\cos x - \cos y} = -\cot\frac{1}{2}(x-y)$; (2) $\tan\dfrac{A}{2} = \sqrt{\dfrac{2\sin A - \sin 2A}{2\sin A + \sin 2A}}$; (3) $\tan(60° + A) - \tan(60° - A) = \dfrac{8\cot A}{\cot^2 A - 3}$; (4) $\dfrac{\cos(A+45°)}{\cos(A-45°)} = \sec 2A - \tan 2A$; (5) $\dfrac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \dfrac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B} = \tan(A-B)$; (6) $\sec 2A - \frac{1}{2}\tan 2A \sin 2A = \dfrac{\cot^2 A + \tan^2 A}{\cot^2 A - \tan^2 A}$.

14. (*a*) Find values of θ not greater than $180°$, which satisfy $\cot\theta = \tan\dfrac{\theta}{2}$. (*b*) Give all the positive angles less than $360°$, which satisfy the equation $\sin 2A = \sqrt{3}\cos 2A$.

15. Show that the value of $\sin(n+1)B \sin(n-1)B + \cos(n+1)B \cos(n-1)B$ is independent of n.

16. The cosines of two angles of a triangle ABC are $\frac{3}{5}$ and $\frac{12}{13}$, respectively; find all the trigonometric ratios of the third angle without using tables. Verify the results by means of the tables.

17. Two towers whose heights respectively are 180 and 80 ft., stand on a horizontal plane; from the foot of each tower the angle of elevation of the other is taken, and one angle is found to be double the other; prove that the horizontal distance between the towers is 240 ft., and show that the sine of the greater angle of elevation is .6.

CHAPTER VII.

1. In a triangle ABC, show that (1) $\sin(A+B) = \sin C$, (2) $\cos(A+B) = -\cos C$, (3) $\sin\frac{A+B}{2} = \cos\frac{C}{2}$, (4) $\cos\frac{A+B}{2} = \sin\frac{C}{2}$.

2. (*a*) State and prove the *Law of Sines* for the plane triangle. (*b*) State and prove the *Law of Cosines* for the plane triangle. (*c*) If the sines of the angles of a triangle are in the ratios of 13 : 14 : 15, prove that the cosines are in the ratios of 39 : 33 : 25.

3. (*a*) Prove that in ABC, $b + c : b - c = \tan\frac{1}{2}(B + C) : \tan\frac{1}{2}(B - C) = \cot\frac{1}{2}A : \tan\frac{1}{2}(B - C)$. (*b*) Write and derive the expressions for the cosine of an angle of a triangle, and the cosine and the sine of half that angle, in terms of the sides of the triangle. (*c*) In the triangle ABC derive the formulas expressing $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$, in terms of a, b, c. (*d*) Prove that in any triangle ABC, $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

4. (*a*) Show how to solve a triangle when the three sides are given, (i.) without logarithms, (ii.) with logarithms. Derive all the formulas necessary. (*b*) Do the same when two sides and their included angle are given. (*c*) Do the same when two angles and a side are given.

5. (*a*) Explain carefully, and illustrate by figures, the case in which the solution of a triangle is ambiguous. (*b*) Write formulas for a complete solution and check, of a triangle, when two sides and an angle opposite to one of them are given. How many solutions are there? Discuss fully all cases that may arise. (*c*) Given the angle A, and the sides a and b of a triangle ABC, determine whether there will be one solution, two solutions, or no solution, in each of the following cases: (i.) $A < 90°$, $a > b$, (ii.) $A < 90°$, $a = b$, (iii.) $A < 90°$, $a < b$, (iv.) $A > 90°$, $a > b$, (v.) $A > 90°$, $a = b$.

6. Show by the trigonometric formulas that the angles of a triangle can be found when the ratios of the three sides are given. Give the geometrical explanation.

7. Show by the trigonometric formulas that the other two angles of a triangle can be found when the third angle, and the ratio of the sides containing it, are known. Give the geometrical explanation.

8. Assuming the law of sines for a plane triangle, prove that $a + b : c = \cos \frac{1}{2}(A - B) : \sin \frac{1}{2} C$, and $a - b : c = \sin \frac{1}{2}(A - B) : \cos \frac{1}{2} C$.

9. (*a*) If $A : B : C = 2 : 3 : 4$, prove $2 \cos \frac{A}{2} = \frac{a + c}{b}$. (*b*) If $2a = b + c$, prove $\tan \frac{B}{2} \tan \frac{C}{2} = \frac{1}{3}$. (*c*) If a, b, c, the sides of a triangle, be in arithmetical progression, prove that $2 \sin \frac{A}{2} \sin \frac{C}{2} = \sin \frac{B}{2}$. [SUGGESTION: Put $2b = a + c$.]

10. [In each of the examples in Ex. 10, A, B, C, denote the angles, a, b, c, the sides of the triangle.] Solve the following triangles (1) by making an estimate, (2) by the method of construction, (3) by computation, without using the logarithms of the trigonometric ratios (four-place tables), (4) by computation, using logarithms (five-place tables), (5) by dividing some of the oblique triangles into right-angled triangles. Check the results by computation. When a solution is impossible, or ambiguous, explain why it is so. (1) $a = 753$, $b = 621$, $c = 937$; (2) $a = 9$, $b = 17$, $c = 14$; (3) $a = 1236.5$, $b = 1674.8$, $c = 2532.7$; (4) $a = 30$, $b = 42$, $c = 36$; (5) $a = 621$, $b = 237$, $c = 325$; (6) $a = 1237$, $b = 1014$, $A = 39° 42'$; (7) $a = 1114$, $b = 1345$, $A = 46° 54' 20''$; (8) $c = 832$, $b = 694$, $B = 54° 47' 30''$; (9) $a = 1020$, $b = 240$, $B = 70° 25'$; (10) $c = 794$, $b = 832$, $B = 65° 30' 20''$; (11) $c = 230$, $a = 950$, $C = 63° 47'$; (12) $a = 237$, $c = 452$, $C = 37° 49'$; (13) $a = 420$, $c = 337$, $C = 42° 46'$; (14) $a = 452$, $b = 624$, $C = 37° 23'$; (15) $a = 1237.4$, $c = 1941.6$, $B = 23° 41' 20''$; (16) $b = 237.41$, $c = 556.82$, $A = 85° 45' 35''$; (17) $A = 37° 41'$, $B = 49° 32'$, $c = 385.9$; (18) $B = 47° 21' 30''$, $C = 81° 49' 45''$, $b = 374.26$.

11. (*a*) If $b : c = 11 : 15$, and $A = 37° 40'$, find B and C. (*b*) If one side of a triangle be five times the other, and their included angle be 64°, find the remaining angles.

12. (*a*) In ABC, if $a : b : c = 8 : 7 : 5$, find the angles. (*b*) The sides of a triangle are proportional to the numbers 4, 5, 6. Find the least angle.

13. Given $a = 2b$, $C = 120°$, find A, B, and the ratio $c : a$.

14. (*a*) Two adjacent sides of a parallelogram are respectively equal to 12 and 20 in., and a diagonal is equal to 25 in. Find the angles of the parallelogram, the other diagonal, and the area. (*b*) The sides of a quadrilateral taken in order are 8, 10, 16, 18, and one diagonal is 18. Find its angles and area.

15. A ladder 52 ft. long is set 20 ft. in front of an inclined buttress, and reaches 46 ft. up its face. Find the inclination of the face of the buttress.

16. A privateer is lying 10 mi. W.S.W. of a harbour, when a merchantman leaves it, steering S.E. 8 mi. an hour. If the privateer overtakes the merchantman in 2 hr., find her course and rate of sailing.

17. A fort bore E. by N. from a beacon, and was distant from it 1500 yd. From a ship at anchor the beacon bore N.N.W. and the fort N.E. by N. How far was the ship from the beacon?

18. A and B are two points, 200 yd. apart, on the bank of a river, and C is a point on the opposite bank; the angles ABC, BAC are respectively 54° 30′ and 65° 30′. Find the breadth of the river.

19. (*a*) Two observers on the same side of a balloon, and in the same vertical plane with it, are a mile apart, and they find the angles of elevation to be 22° 18′ and 75° 30′, respectively. What is the height? (*b*) Two observers on opposite sides of a balloon, and in the same vertical plane with it, take its altitude simultaneously; one observer finds it to be 64° 15′, and the other, 48° 20′. Find the height of the balloon at the time of observation.

20. From a ship sailing along a coast a headland, C, was observed to bear N.E. by N. After the ship had sailed E. by N. 15 mi. the headland bore W.N.W. Find the distance of the headland at each observation.

21. From a certain station a fort, A, bore N., and a second fort, B, N.E. by E. Guns are fired simultaneously from the two forts, and are heard at the station in 1.5 sec. and 2 sec. respectively. Assuming that sound travels at the rate of 1142 ft. per second, find the distance of the two forts apart.

22. From a point A in the same plane as the base of a tower, the tower bears N. 62° W., and the angle of elevation of the top of the tower is 53° 37′; from B, 165 ft. due north of A, the tower bears west. What is the height of the tower?

23. From a ship steering W. by S. a beacon bore N.N.W., and after the ship had sailed 12 mi. farther, the bearing of the beacon was N.E. by E. At what distance had the ship passed the beacon?

24. From the intersection of two straight paths which are inclined to each other at an angle of 37°, two pedestrians, A and B, start at the same instant to walk along the paths, A at the rate of 5 mi. an hour, and B at a uniform rate also; after 3 hr. they are $9\frac{1}{2}$ mi. apart. Show that there are two rates at which B may walk to fulfil this condition, and find both of those rates.

25. Two straight railroads are inclined to each other at an angle of 22° 15′. At the same instant two engines, A and B, start from a station at the point of intersection, A going on one road at the rate of 20 mi. an hour, and B going uniformly on the other. After 3 hr. A and B are 25 mi. apart. Show that there are two rates at which B may go to fulfil this condition, and find those rates.

26. A tower stood at the foot of an inclined plane whose inclination to the horizon was 9°; a line was measured straight up the incline from the foot of the tower of 100 ft. in length, and at the upper extremity of this line the tower subtended an angle of 54°. Find the height of the tower.

27. The altitude of a certain rock is observed to be 47°, and after walking toward it 1000 ft. up a slope inclined at 32° to the horizon, the observer finds that this altitude is 77°. Find the vertical height of the rock above the first point of observation.

28. If, from a point at which the elevation of the observatory on Ben Nevis is $60°$, a man walks 800 ft. on a level plane toward the mountain, and then 800 ft. further up a slope of $30°$ to a point at which the elevation of the observatory is $75°$, show that the height of Ben Nevis is approximately 4478 ft., the man's path being always supposed to lie in a vertical plane passing through the observatory.

29. A man walks 40 ft. in going straight down the slope of the embankment of a railway which runs due east and west, and then walks 20 ft. along the foot of the embankment; he finds that he is exactly N.E. of the point from which he started at the top of the bank. Show that the inclination of the bank to the horizon is $60°$.

30. A man in a ship at sea sailing north observes two rocks, A and B, to bear $25°$ east of his course; he then sails in a direction northwest for 4 mi., and observes A to bear east and B northeast of his new position. Find the distance from A to B.

31. To determine the distance of two forts, C, D, at the mouth of a harbour, a boat is placed at A, with its bow toward a distant object E, and the angles CAD, DAE are observed and found to be $22° 17'$ and $48° 1'$ respectively. The boat is then rowed to B, a distance of 1000 yd., directly toward E, and the angles CBD, DBE are observed to be $53° 15'$ and $75° 43'$ respectively. Find the distance CD.

32. A fort stands on a horizontal plane; the angle of depression measured from the top of the fort to a point P on the plane is $A°$, and to a point R, a feet beyond P, is $B°$. Derive the formulas for computing h, the height of the fort, and d, the distance from P to the bottom of the fort.

33. A person standing on a level plain at the base of a hill wishes to find the height of a tower which is in full sight on the top of the hill. Describe in detail the necessary measurements and computations.

34. A surveyor wishes to find the distance and the height of a tower which is on the same level with him, but on the opposite side of an impassable chasm; illustrate the problem by a lettered figure, and describe in detail the necessary measurements and computations.

35. What measurements must be made by an observer on the shore to find the distance between two buoys? Give formulas necessary for solving.

36. Explain what measurements have to be made at two stations, A and B, in order to find the distance, CD, between two inaccessible objects (A, B, C, D being in one plane); and state clearly the steps of the calculation by which the distance is to be found therefrom.

37. (a) From the law of sines deduce that $b \cos C + c \cos B = a$. (b) Prove this geometrically. (c) Show that $a \cos B - b \cos A = \dfrac{a^2 - b^2}{c}$.

38. In any triangle ABC, prove: (a) $\tan B = \dfrac{b \sin C}{a - b \cos C}$;

(b) $\dfrac{a^2 + b^2 - ab \cos C}{a \sin A + b \sin B + c \sin C} = \dfrac{a}{2 \sin A}$; (c) $\dfrac{1 + \cos(A - B)\cos C}{1 + \cos(A - C)\cos B} = \dfrac{a^2 + b^2}{a^2 + c^2}$.

CHAPTER VIII.

[In what follows, S denotes the area of a triangle, s its semi-perimeter, R, r, r_a, r_b, r_c, the radii of its circumscribing, inscribed, and escribed circles, respectively.]

1. Prove that any side of a triangle is equal to the second side into the cosine of the angle opposite the third sine plus the third side into the cosine of the angle opposite the second side.

2. Derive expressions, in terms of the sides of a given triangle, for the radii of its circumscribing circle, and of the four circles which touch the sides.

3. Derive expressions for the radii in Ex. 2, in terms of the sides and the area of the given triangle.

4. Prove $R = \dfrac{abc}{4S}$, $r = \dfrac{S}{s}$, $r_a = \dfrac{S}{s - a}$. Write similar expressions for r_b, r_c.

5. Prove: (a) $r_a + r_b + r_c - r = 4R$; (b) $\sqrt{r \cdot r_a \cdot r_b \cdot r_c} = S$.

6. Prove: (a) $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$; (b) $Rr = \dfrac{abc}{4(a + b + c)}$;

(c) $r = 4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$.

7. Prove $r_a \cot \dfrac{A}{2} = r_b \cot \dfrac{B}{2} = r_c \cot \dfrac{C}{2} = r \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$

$$= 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

8. Prove $R = \dfrac{a}{2 \sin A}$, $r = (s - a) \tan \dfrac{A}{2}$, $r_a = s \tan \dfrac{A}{2}$. Write two other similar formulas for R and r. Write similar formulas for r_b, r_c.

9. (a) Prove $r = \dfrac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$. Write two similar formulas involving b, c.

(b) Prove $r_a = \dfrac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$. Write similar formulas for r_b, r_c.

10. Show that the length of the tangents to the inscribed circle from the angle A is $s - a$, from the angle B is $s - b$, from the angle C is $s - c$.

11. Write and derive the formula for the area of a triangle: (a) in terms of the three sides; (b) in terms of two sides and their included angle; (c) in terms of one side and the two angles adjacent to it.

12. (*a*) Prove that $S = s(s-c)$ when $C = 90°$; (*b*) if x, y, be the lengths of the two diagonals of a parallelogram, and θ the angle between them, show that area $= \frac{1}{2}xy \sin\theta$.

13. Find the areas of some of the triangles in Ex. 10, Chap. VII. Find the radii of their circumscribing, inscribed, and escribed circles.

14. (*a*) An isosceles triangle whose vertical angle is 78° contains 400 square yards; find the lengths of the sides. (*b*) Find two triangles each of which has sides 63 and 55 ft. long, and an area of 874 sq. ft. (*c*) The angles at the base of a triangle are 22° 30′ and 112° 30′ respectively; show that the area of the triangle is equal to the square of half the base.

15. (*a*) Show that the area of a regular polygon inscribed in a circle is a mean proportional between the areas of an inscribed and circumscribing polygon of half the number of sides. (*b*) The sides of a triangle are as $2:3:4$; show that the radii of the escribed circles are as $\frac{1}{5} : \frac{1}{3} : 1$.

16. Two roads form an angle of 27° 10′ 25″. At what distance from their intersection must a fence at right angles to one of them be placed so as to enclose an acre of land?

17. If the altitude of an isosceles triangle is equal to its base, the radius of the circumscribing circle is $\frac{5}{8}$ of the base.

18. An equilateral triangle and a regular hexagon have the same perimeter. Show that the areas of their inscribed circles are as $4:9$.

19. If the sides of a triangle are 51, 68, and 85 ft., show that the shortest side is divided by the point of contact of the inscribed circle into two segments, one of which is double the other.

CHAPTER IX.

1. Explain how angles are measured (1) by sexagesimal measure, (2) by radian measure. Show how to connect the radian measure of an angle with its measure in degrees. Find the number of degrees in the angle called the radian. How many degrees are there in an arc whose length is equal to the diameter? Show that the radian measure of an angle is the ratio of the lengths of two lines. What advantage is there in using radian measure?

2. (*a*) Give the number of degrees in each of the following angles: $\frac{1}{2}\pi$, $\frac{3}{4}\pi$, 2π, $\frac{3}{16}\pi$, $n\pi$, $\frac{\pi}{3}$, $-\frac{2}{3}\pi$, $\frac{2}{3}\pi$, $\frac{2}{5}\pi$, $\frac{5}{2}\pi$, $\frac{7}{2}\pi$, $\frac{7}{6}\pi$, $\frac{5}{6}\pi$, $-\frac{\pi}{7}$, $-\frac{4}{3}\pi$, $-\frac{11}{3}\pi$, $(\frac{5}{6})^{(r)}$, $(2\frac{1}{3})^{(r)}$, $(-\frac{4}{9})^{(r)}$. (*b*) Give the supplements and complements of those angles in radian measure and in degree measure. (*c*) Give the radian measures of 30°, 80°, 49°, 41° 30′ 15″, 120°, − 210°, − 175°. Give the radian measures of their supplements and complements.

3. (*a*) A central angle 1.25^r is subtended by a circular arc of 16 ft.; find the radius. (*b*) Find the number of radians and degrees in the central angle subtended by an arc 9 in. long, in a circle whose radius is 10 ft.

(c) Find the radius of a circle in which an arc 15 in. long subtends at the centre an angle containing $71° 36' 3''.6$. (d) If the radius be 8 in., find the central angle, in degrees and in radians, that is subtended by an arc 15 in. long. (e) An angle of 3^r is subtended by an arc of 5 in.; find the length of the radius; find also the number of radians, and of degrees, in an arc of 1.5 in. (f) Find the number of radians and seconds in the angle subtended at the centre of a circle whose radius is 2 mi., by an arc 11 in. long. (g) Find the length of the arc which subtends a central angle of (1) 2 radians, the radius being 10 in.; (2) 1.5 radians, radius 2 ft.; (3) 4.3 radians, radius 21 yd.; (4) 1.25 radians, radius 8 in.

4. The value of the division on the outer rim of a graduated circle is $5'$, and the distance between the two successive divisions is .1 of an inch. Find the radius of the circle.

5. Show that the distance in miles between two places on the equator, which differ in longitude by $3° 9'$, assuming the earth's equatorial radius to be 7925.6 mi., is 217.954 mi.

6. (a) The difference of two angles is $10°$, the radian measure of their sum is 2. Find the radian measure of each angle. (b) One angle of a triangle is π degrees, another is π grades. Show that the radian measure of the third angle is $\pi - \frac{19\pi^2}{1800}$. (c) If the number of degrees in an angle be equal to the number of grades in the complement of the same angle, prove that the radian measure of the angle is $\frac{5\pi}{19}$. (d) The angles of a triangle are in the ratios $1:2:3$. Express their magnitudes in each of the three systems of angular measurement. (e) One angle of a triangle is $45°$, another is 1.5 radians. Find the third, both in degrees and in radians. (f) Express in degrees and in radian measure the vertical angle of an isosceles triangle which is half of each of the angles at the base.

7. Prove the following statements, in which a denotes the length of a side of a regular polygon; P, the length of its perimeter; n, the number of its sides; r, the radius of the inscribed circle; R, the radius of the circumscribing circle:

$$a = 2R\sin\frac{\pi}{n} = 2r\tan\frac{\pi}{n};\quad P = 2nR\sin\frac{\pi}{n} = 2nr\tan\frac{\pi}{n};\quad R + r = \frac{a}{2}\cot\frac{\pi}{2n};$$

$$\text{area of polygon} = \frac{n}{2}R^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{na^2}{4}\cot\frac{\pi}{n}.$$

CHAPTER X.

1. Define and illustrate the trigonometric functions. Show in tabular form the signs of these functions in each of the four quadrants.

2. (a) Construct a table showing the values, with proper signs, of the trigonometric functions of $0°$, $30°$, $45°$, $60°$, $120°$, $180°$, $225°$, $270°$, $315°$, $360°$.

(*b*) Compare the trigonometric functions of $90° - A$, $90° + A$, $180° + A$, $180° - A$, $-A$, with those of A.

3. Show, from both the ratio and the line definitions of the trigonometric functions, that (1) the sine and cosine are never greater than unity, (2) the cosecant and secant are never less than unity, (3) the tangent and cotangent may have any values whatever from negative infinity to positive infinity, (4) the trigonometric functions change signs in passing through zero or infinity, and through no other values.

4. Given $\tan A = -\frac{8}{15}$, find the values of the other trigonometric functions of A.

5. Find geometrically an expression for the cosine of the difference of two angles in terms of the trigonometric functions of those angles.

6. Prove that:

(*a*) $\sin^2 B + \sin^2(A - B) + 2\sin B \sin(A - B)\cos A = \sin^2 A$;

(*b*) $\dfrac{\cos nA - \cos(n+2)A}{\sin(n+2)A - \sin nA} = \tan(n+1)A.$

7. Give the ratio definitions of the trigonometric functions, sine, cosine, tangent, and secant. These functions have also been defined as straight lines. Give these definitions, and show from them that $\tan 90°$, $\sec 90°$ would each be infinite. *Show that the two systems are consistent.*

8. (*a*) Trace the changes, in magnitude and sign, in the values of the trigonometric functions as the angle increases from 0° to 360°. (*b*) Trace the changes of sign of $\sin\theta$ as θ increases through 360°, and show that its equivalent $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ has always the same sign as $\sin\theta$.

9. Trace the changes, as A increases from 0° to 180°, in the sign and value of (*a*) $\cos(\pi \sin A)$, (*b*) $\sin A + \cos A$, (*c*) $\sin A - \cos A$. Draw the graphs of these functions.

10. Show that the radian measure of an acute angle is intermediate in value between the sine and the tangent of the angle.

11. (*a*) Show that the limit of $\frac{\sin\theta}{\theta}$ when θ is indefinitely diminished is 1, *i.e.* $\sin\theta = \theta$, very nearly. (*b*) Show that the limit of $\frac{\tan\theta}{\theta}$ when θ is indefinitely diminished is 1, *i.e.* $\tan\theta = \theta$, very nearly.

12. Find the area of a regular polygon of n sides inscribed in a circle, and show, by increasing the number of sides of the polygon without limit, how the expression for the area of the circle may be obtained.

13. (*a*) Find the distance at which a building 50 ft. wide will subtend an angle of 3′. (*b*) A church spire 45 ft. high subtends an angle of 9′ at the eye. Find its distance approximately. (*c*) Find approximately the distance of a tower 51 ft. high which subtends at the eye an angle of $5\frac{5}{11}'$. (*d*) How large a mark on a target 1000 yd. off will subtend an angle of 1″ at the eye?

14. Show how the functions may be represented by lines connected with a circle.

15. Explain, with illustrations, how functions may be graphically represented by means of a curve. Draw the graphs of the trigonometric functions.

NOTE.—"If a function of a variable has its magnitude unaltered when the sign of the variable is changed, that function is called *an even function*, but if the function has the same numerical value as before, but with opposite sign, then that function is called *an odd function;* for instance, x^2 is an even function of x, x^3 is an odd function of x, but $x^2 + x^3$ is neither even nor odd, since its numerical value changes when the sign of x is changed."

16. Show that the cosine, secant, and versine of an angle are even functions, and the sine, tangent, cotangent, and cosecant are odd functions, and the coversine is neither even nor odd. (See Art. 78.)

CHAPTER XI.

N.B. The problems which are purely numerical are to be solved independently of tables. The results can be verified by means of the tables.

1. (*a*) Deduce a *general expression* for all angles which have the same *sine;* (*b*) for all which have the same *cosine;* (*c*) for all which have the same *tangent.* (*d*) What are the general expressions for all angles which have the same *secant*, *cosecant*, *cotangent*, respectively.

2. Define inverse trigonometric functions; give illustrations. Define $\tan^{-1} x$, $\cos^{-1} x$.

3. (*a*) Explain fully the equations $\sin(\sin^{-1}\frac{1}{3})=\frac{1}{3}$, $\sin^{-1}(\sin\theta)=\theta$. (*b*) Construct $\sin^{-1}(\frac{2}{3})$, $\cos^{-1} 0$, $\tan^{-1}\infty$, $\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$. (*c*) Find $\tan(\cos^{-1}\frac{2}{3})$.

[Carefully state the limitations under which the following equations are true.]

4. Show that: (*a*) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$; (*b*) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$; (*c*) $\sin^{-1}x-\sin^{-1}y=\cos^{-1}\sqrt{1-x^2-y^2+x^2y^2}+xy$.

5. (*a*) From $\sin 2A=2\sin A\cos A$, show that $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$. (*b*) Show that for certain values of the angles, $2\cos^{-1}x=\cos^{-1}(2x^2-1)$; $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$; $2\cot^{-1}x=\operatorname{cosec}^{-1}\frac{1+x^2}{2x}$.

6. Show that for certain values of the angles: (*a*) $\cos^{-1}x=\sin^{-1}\sqrt{\frac{1-x}{2}}+\cos^{-1}\sqrt{\frac{1+x}{2}}$; (*b*) $\sin^{-1}\sqrt{\frac{x}{a+x}}=\tan^{-1}\sqrt{\frac{x}{a}}=\frac{1}{2}\cos^{-1}\frac{a-x}{a+x}$. (*c*) $\tan^{-1}m+\cot^{-1}m=\frac{\pi}{2}$, or $\frac{3\pi}{2}$.

7. Prove that for certain values: (1) $\sec^{-1}3=2\cot^{-1}\sqrt{2}$; (2) $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=15$; (3) $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}=90°$; (4) $\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}=\frac{\pi}{6}$; (5) $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$; (6) $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=120°$; (7) $\tan^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{3}+\sin^{-1}\frac{7\sqrt{2}}{10}=0$; (8) $\tan^{-1}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\tan^{-1}\sqrt{\frac{3}{2}}=\frac{3\pi}{4}$.

8. Prove that: (1) $\sin^{-1}\frac{2mn}{m^2+n^2}+\sin^{-1}\frac{m^2-n^2}{m^2+n^2}=\frac{\pi}{2}$; (2) $\tan^{-1}(\cot A)-\tan^{-1}(\tan A)=n\pi+\frac{\pi}{2}-2A$; (3) $\tan^{-1}t+\tan^{-1}\frac{2t}{1-t^2}=\tan^{-1}\frac{3t-t^3}{1-3t^2}$; (4) $\tan^{-1}\frac{m-1}{m}+\tan^{-1}\frac{1}{2m-1}=n\pi+\frac{\pi}{4}$; (5) $\tan^{-1}\frac{2a-b}{b\sqrt{3}}+\tan^{-1}\frac{2b-a}{a\sqrt{3}}=\frac{\pi}{3}$; (6) $\tan^{-1}m+\tan^{-1}n=\cos^{-1}\frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}$.

9. Prove that: (1) $\tan^{-1}\frac{1}{1+a}+\tan^{-1}\frac{1}{1-a}+\tan^{-1}\frac{2}{a^2}=n\pi$; (2) $\tan^{-1}1=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$; (3) $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{2}{9}=45°$; (4) $\cot^{-1}\frac{1}{3}=\cot^{-1}3+\cot^{-1}\frac{3}{4}$; (5) $\cot^{-1}\frac{1-a}{1+a}-\cot^{-1}\frac{1-b}{1+b}=\sin^{-1}\frac{a-b}{\sqrt{1+a^2}\sqrt{1+b^2}}$; (6) $4(\cot^{-1}\frac{1}{3}+\operatorname{cosec}^{-1}\sqrt{5})=\pi$; (7) $a\cos\left(\sin^{-1}\frac{b}{a}\right)=\sqrt{a^2-b^2}$; (8) $\sin^{-1}\left(\frac{x-a+b}{2b}\right)^{\frac{1}{2}}=\frac{1}{2}\cos^{-1}\frac{a-x}{b}$; (9) $\sin\cot^{-1}a=\tan\cos^{-1}\sqrt{\frac{a^2+1}{a^2+2}}$; (10) $\{\tan(\sin^{-1}a)+\cot(\cos^{-1}a)\}^2=2a\tan(2\tan^{-1}a)$.

10. Find all the angles (*i.e.* find the general values of the angles) which satisfy the following equations: (1) $\sec^2 A=\frac{4}{3}$; (2) $2\tan^2\theta=\sec^2\theta$; (3) $\tan^2\theta-\sec\theta=1$; (4) $\sqrt{3}\tan^2\theta+1=(1+\sqrt{3})\tan\theta$; (5) $\cos\theta-\sin\theta=\frac{1}{\sqrt{2}}$; (6) $2\sin x+2\operatorname{cosec}x=5$; (7) $2\sin 2y=3\tan y$; (8) $\cos B+\tan B=\sec B$; (9) $3\cos^2 x+2\sqrt{3}\cos x=5.25$; (10) $\tan z-2\sin z=0$; (11) $4\cot 2\theta=\cot^2\theta-\tan^2\theta$; (12) $\operatorname{cosec} C+\cot C=\sqrt{3}$; (13) $\cot A-\tan A=2$; (14) $\operatorname{cosec} x=\cot x+\sqrt{3}$.

11. Solve $\cos 2x=\sin x$.

[SOLUTION: $\cos 2x=\cos\left(\frac{\pi}{2}-x\right)$; or, $\sin\left(\frac{\pi}{2}-2x\right)=\sin x$. $\therefore\ \frac{\pi}{2}-x=2n\pi\pm 2x$, or $\frac{\pi}{2}-2x=n\pi+(-1)^n x$.]

12. Solve $\sin 5A=\sin 11A$.

[SOLUTION: $11A=n\pi+(-1)^n 5A$. $\therefore\ A=0,\ \frac{\pi}{16},\ \frac{\pi}{3},\ \cdots$.]

13. Find the general solutions of: (1) $\sin\theta+\cos\theta=\sqrt{2}$; (2) $\sin 4\theta=\sin\theta$; (3) $2\cos 2\theta-2\sin\theta-1=0$; (4) $\tan^2\theta+3\cot^2\theta=4$; (5) $\sin^4 x-$

$\cos^4 x=1$; (6) $\sin^2 2x-\sin^2 x=.25$; (7) $\tan B+\cot B=2$, $\cos y+\cos 2y+\cos 3y = 0$ (SUGGESTION: $\cos y + \cos 3y = 2\cos y \cos 2y$); (8) $\sin \frac{1}{2}x = \operatorname{cosec} x - \cot x$; (9) $\cos x + \cos 7x = \cos 4x$; (10) $\cos A + \cos \frac{1}{2}A = \cos \frac{3}{2}A$; (11) $\operatorname{cosec} z = 2\sin z$; (12) $2\tan^{-1}\cos A = \tan^{-1} 2 \operatorname{cosec} A$; (13) $\tan(A - 15°) = \frac{1}{3}\tan(A + 15°)$; (14) $\tan(45° + B) = 1 + \tan B$.

14. Find x in the following equations: (1) $\cos^{-1}x + \cos^{-1}(1 - x) = \cos^{-1}(-x)$; (2) $\tan^{-1}x + \tan^{-1}2x = \tan^{-1}\frac{3\sqrt{3}}{5}$; (3) $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\frac{8}{31}$; (4) $\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \tan^{-1}(-7)$; (5) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$.

15. A flagstaff a feet high is on a tower $3a$ feet high; prove that, if the observer's eye is on a level with the top of the staff, and the staff and tower subtend equal angles, the observer is at a distance $a\sqrt{2}$ from the top of the flagstaff.

16. In any triangle ABC, if $\tan\frac{A}{2} = \frac{5}{6}$, and $\tan\frac{B}{2} = \frac{20}{37}$, find $\tan C$ without tables. Verify the result by means of the tables. Show that in such a triangle, $a + c = 2b$.

CHAPTER XII.

1. Explain the advantages of measuring angles by the *sexagesimal*, *centesimal*, and *radian* methods, respectively.

2. Show that, if x be the radian measure of a positive angle less than $\frac{\pi}{2}$, then (a) $\cos x$ is less than 1 but greater than $1 - \frac{1}{2}x^2$; (b) $\sin x$ is less than x but greater than $x - \frac{1}{4}x^3$. By means of (b) show how the sine of $10''$ may be calculated approximately.

3. (a) Express in terms of functions of A, *each* of the following: $\sin 2A$, $\cos 2A$ (*three* different forms), $\tan 2A$. (b) Find $\cos 3A$ in terms of $\cos A$. (c) Find $\sin 3A$ in terms of $\sin A$. (d) Find $\tan 3A$ in terms of $\tan A$, and from the formula determine the numerical value of $\tan A$ if $3A = 90°$. (e) Investigate a formula for expressing the cosine of half an angle in terms of the sine of the whole angle; and if the angle lies between 270° and 360°, show which signs of the roots must be taken.

4. Show that $\sin(A + B - C) + \sin(A + C - B) + \sin(B + C - A) - \sin(A + B + C) = 4\sin A \sin B \sin C$.

5. If $A + B + C = 180°$ (*i.e.* if A, B, C be the three angles of a triangle), show that: (a) $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$; ($b$) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$; ($c$) $\frac{\cos A+\cos B+\cos C-1}{\sin A+\sin B+\sin C} = \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

6. If $\sin A = \frac{3}{5}$, $\sin B = \frac{12}{13}$, and $\sin C = \frac{7}{25}$, where A, B, C are positive angles and less than $90°$, find $\sin(A + B + C)$.

7. Assuming the equation $\cos 3x = 4\cos^3 x - 3\cos x$, find $\sin 18°$.

[SOLUTION: $54° + 36° = 90°$. $\therefore \cos 54° = \sin 36°$; *i.e.* $\cos 3 \cdot 18° = \sin 2 \cdot 18°$. Hence, $4\cos^3 18° - 3\cos 18° = 2\sin 18° \cos 18°$. $\therefore 4\cos^2 18° - 3 = 2\sin 18°$. On putting $1 - \sin^2 18°$ for $\cos^2 18°$, and solving for $\sin 18°$, there is obtained the result, $\sin 18° = \frac{\sqrt{5} - 1}{4}$.]

8. Assuming the result in Ex. 7, find the other trigonometric functions of $18°$ and the functions of $72°$.

9. Assuming the result in Ex. 7, show that $\cos 36° = \frac{\sqrt{5} + 1}{4} = \sin 54°$. Hence, deduce the other trigonometric functions of $36°$ and $54°$. Also, deduce the trigonometric functions of $9°$ and $81°$. (The results in Exs. 8, 9, can be verified by means of the tables.)

10. Prove the formulas:

$$\sin(36° + A) - \sin(36° - A) - \sin(72° + A) + \sin(72° - A) = \sin A,$$
$$\cos(36° + A) + \cos(36° - A) - \cos(72° + A) - \cos(72° - A) = \cos A,$$

and explain their use. (See Art. 97.)

11. (*a*) Show that $\sin^2 30° = \sin 18° \sin 54°$. (*b*) Solve $x^3 \cot 108° = 128 \sin 72° \cos 18°$, without tables. (*c*) Find the trigonometric functions of $48°$.

[HINT: $48° = 30° + 18°$.]

12. Two parallel chords of a circle lying on the same side of the centre of a circle subtend angles of $72°$ and $144°$ at the centre. Show that the distance between the chords is equal to half the radius of the circle, (*a*) using tables, (*b*) not using tables.

13. (*a*) Solve: (i.) $\cos\theta = 0$; (ii.) $\sin x + \cos x = 1$;

(iii.) $\tan y + \tan 4y + \tan 7y = \tan y \tan 4y \tan 7y$.

(*b*) If $\tan\theta \tan 3\theta = -.4$, find $\tan\theta$, $\tan 3\theta$.

14. (*a*) If in triangle ABC, $A = 3B$, show that $\sin B = \frac{1}{2}\sqrt{\frac{3b - a}{b}}$. (*b*) Given $\cos A = .28$, find $\tan\frac{A}{2}$. Explain the reason of the ambiguity that presents itself in the result. (*c*) If $\sin A = \frac{2ab'}{a^2 + b^2}$, find $\tan\frac{A}{2}$. (*d*) Given $\tan\frac{1}{2}x = 2 - \sqrt{3}$, find $\sin x$.

15. Prove the following:

(i.) $\tan A - \tan\frac{1}{2}A = \tan\frac{1}{2}A \sec A$.

(ii.) $1 + \tan^6 A = \sec^4 A(\sec^2 A - 3\sin^2 A)$.

(iii.) $\sin A + \sin 3A + \sin 5A + \sin 7A = 16 \sin A \cos^2 A \cos^2 2A$.

(iv.) $\cos 6A = 16(\cos^6 A - \sin^6 A) - 15\cos 2A$.

(v.) $\sin 3x + \sin 5x = 8 \sin x \cos^2 x \cos 2x$.

(vi.) $4\cos^3 A \sin 3A + 4\sin^3 A \cos 3A = 3\sin 4A$.

(vii.) $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$.

(viii.) $\sin^3 A + \sin^3(120^\circ + A) + \sin^3(240^\circ + A) = -\frac{3}{4}\sin 3A$.

(ix.) $1 + \cos 2(A - B)\cos 2B = \cos^2 A + \cos^2(A - 2B)$.

(x.) $2\operatorname{cosec} 4A + 2\cot 4A = \cot A - \tan A$.

16. Show that

$$\cos(36^\circ + A)\cos(36^\circ - A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A;$$
$$\sin 3A = 4\sin A \sin(60^\circ + A)\sin(60^\circ - A).$$

17. Show that

$$2\cos\frac{A}{2} = \pm\sqrt{1 + \sin A} \pm \sqrt{1 - \sin A},\quad 2\sin\frac{A}{2} = \pm\sqrt{1 + \sin A} \mp \sqrt{1 - \sin A}.$$

$\left[\text{Suggestion: } \cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1;\ 2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A.\right]$

18. Prove that the following equations are true for certain values of the angles:

(i.) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$.

(ii.) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$.

(iii.) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x + y + z - xyz}{1 - xy - yz - zx}$.

(iv.) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}\dfrac{1 - x - y - xy}{1 + x + y - xy} = \dfrac{\pi}{4}$.

(v.) Given $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{5}$, $\tan\gamma = \frac{1}{8}$, find $\tan(\alpha + \beta + \gamma)$.

(vi.) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}$.

(vii.) $\tan^{-1}\dfrac{4}{3} = \dfrac{1}{2}\tan^{-1}\left(\dfrac{-24}{7}\right) = \dfrac{1}{3}\tan^{-1}\left(\dfrac{-44}{117}\right)$.

(viii.) $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65} = \dfrac{\pi}{2}$.

(ix.) $\sin^{-1}\frac{23}{27} = 3\sin^{-1}\frac{1}{3}$.

(x.) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4} = \sin^{-1}\dfrac{1}{\sqrt{5}} + \cot^{-1}3$.

(xi.) $\sin^{-1}\dfrac{3}{\sqrt{73}} + \cos^{-1}\dfrac{11}{\sqrt{146}} + \sin^{-1}\dfrac{1}{2} = \dfrac{5\pi}{12}$.

(xii.) $\tan^{-1}\frac{2}{3} = \frac{1}{2}\tan^{-1}\frac{12}{5}$.

19. The hypotenuse and shortest side of a right-angled triangle are 5 ft. and 3 ft., respectively. Find the length of the perpendicular from the right angle upon the hypotenuse, and show that it is inclined at $\sin^{-1}\frac{7}{25}$ to the straight line drawn from the right angle to the middle point of the hypotenuse.

20. If a triangle ABC is to be solved from given parts A, a, b, show that the solution is sometimes ambiguous; and that in such a case the difference of the two values of C is $2\cos^{-1}\dfrac{b\sin A}{a}$.

21. The tangent of an angle is 2.4. Find the cosecant of the angle, the cosecant of half the angle, and the cosecant of the supplement of double the angle.

22. The angle of elevation of a tower at a distance of 20 yd. from its foot is three times as great as the angle of elevation 100 yd. from the same point; show that the height of the tower is $300 : \sqrt{7}$ ft.

23. DE is a tower on a horizontal plane. $ABCD$ is a straight line in the plane. The tower subtends an angle θ at A, $2\,\theta$ at B, and $3\,\theta$ at C. If $AB = 50$ ft., and $BC = 20$ ft., find the height of the tower and the distance CD.

24. A ship sailing at a uniform rate was observed to bear N. 30° 57′ 30″ E. After 20 minutes she bore N. 35° 32′ 15″ E., and after 10 minutes more, N. 37° 52′ 30″ E. Find the direction in which she was sailing.

[*Ans.* S. 44° 38′ E.]

25. A spectator observes the explosion of a meteor, due south of him, at an elevation of 28° 45′. To another spectator, 11 mi. S.S.W. of the former, it appears at the same instant to have an altitude of 42° 15′ 30″. Show that there are two possible heights above the earth's surface at which it may have exploded, and find these heights. [*Ans.* 4.33 mi. or 13.21 mi.]

ANSWERS TO THE EXAMPLES.

N.B. *Not all of the answers to the exercises are given. In various ways, the student should test, or check, every result that he obtains in working the problems.*

CHAPTER I.

Art 2. **1.** $\log_3 27 = 3$, $\log_4 256 = 4$, $\log_{11} 121 = 2$, $\cdots$, $\log_m p = b$. **2.** $2^3 = 8$, $5^4 = 625$, $\cdots$, $n^a = P$. **3.** 0, 1, 2, 3, 4, 5, 6, 7, 8. **6.** 1, 4, 16, 64, 256, 1024. **7.** 0, 1; 1, 2; 2, 3; 3, 4; 3, 4; 0, −1; −1, −2; −2, −3.

Art. 6. **6**(a). 1.4007. **6**(d). .09856. **7**(b). 7.2767. **8**(a). 7.937. **10.** 9.214. **12.** .6443. **13**(a). 3.236. **13**(c). 1.5563.

CHAPTER II.

Art. 8. **1.** 8, 24, 42, 54, 720, $36a$, $12b$, c. **2.** $\cdots$, a, $\frac{b}{3}$, $\frac{c}{36}$. **3.** $\cdots$, $3a$, b, $\frac{c}{12}$. **4.** $12a$, $4b$, $\frac{c}{3}$. **5.** 440, $\frac{1}{54}$.

Art. 10. **5.** 1 : 1200, 1 : 253440, 1 : 1920, $\cdots$. **7.** 2.446 mi.

Art. 11. (In these answers, h, p, b represent hypotenuse, perpendicular, and base, respectively.)

2. 35°, $h=34.86$, $p=28.56$, $\frac{b}{h}=.574$, $\frac{h}{b}=1.743$, $\frac{p}{h}=.819$, $\frac{h}{p}=1.22$, $\frac{p}{b}=1.428$, $\frac{b}{p}=.7$. **3.** 65°, $b=27.19$, $p=12.68$, $\frac{b}{h}=.906$, $\frac{h}{p}=2.37$, $\frac{p}{h}=.423$, $\frac{h}{b}=1.1$, $\frac{p}{b}=.466$, $\frac{b}{p}=2.14$. **4.** 56° 19′, 33° 41′ (nearly), $h=54.08$, $\frac{b}{p}=.67$, $\frac{p}{b}=1.5$, $\frac{h}{b}=1.8$, $\frac{b}{h}=.555$, $\frac{h}{p}=1.2$, $\frac{p}{h}=.833$. **5.** 41° 25′, 48° 35′ (nearly), $p=39.7$, $\frac{b}{h}=.75$, $\frac{h}{b}=1.33$, $\frac{p}{h}=.66$, $\frac{h}{p}=1.51$, $\frac{p}{b}=.88$, $\frac{b}{p}=1.13$. **6.** 50°, $p=59.6$, $h=77.8$, $\frac{b}{h}=.643$, $\frac{h}{b}=1.56$, $\frac{p}{h}=.766$, $\frac{h}{p}=1.31$, $\frac{p}{b}=1.19$, $\frac{b}{p}=.839$.

Art. 14. **11.** (1) tangent is $a:\sqrt{b^2-a^2}$; (2) tangent is $\sqrt{b^2-a^2}:a$; (3) sine is $a:\sqrt{a^2+b^2}$; (4) sine is $b:\sqrt{a^2+b^2}$; sine is $\sqrt{a^2-b^2}:a$; tangent is $b:\sqrt{a^2-b^2}$. **12.** 41° 24′ 35″. **13.** 19° 28 16″.

Art. 15. **1.** 2.28025. **2.** 2.3333. **3.** 5.846. **9.** 2.75. **10.** $-.708$.

Art. 18. **15.** 90°, 36° 52′ 12″. **16.** 45°. **17.** 45°, 71° 34′. **18.** 53° 7′ 48″. **19.** 30°, 48° 35′ 25″. **20.** 36° 52′ 12″, 16° 15′ 36″.

CHAPTER III.

Art. 27. **5.** $A = 65° 14'$, $b = 7.834$. **6.** $B = 50° 12' 24''$. **9.** $A = 30° 12' 12''$. **11.** $b = 215.6$. **12.** $a = 312.23$.

CHAPTER IV.

Art. 28. **1.** 24.948, 12.71. **2.** 58.78.

Art. 29. **4.** 398.19 ft. **5.** 228.4, 258 ft. **6.** 63.88 ft. **7.** 276.95 ft. **10.** 86.6, 50. **12.** 219.45 ft.

Art. 30. **1.** 26.172, 52.345 mi., second ship bears E. 19° 42′.1 N. from first. **2.** $LB = 14.197$ mi.

Art. 31. **2, 3.** 2392.18 sq. ft. **5.** 22.5 sq. ft. **6.** 435.7 sq. ft.

Art. 32. **2.** Base = 187.9 ft.; height = 350.63 ft.; area = 32,943 sq. ft. **3.** Base = 358.21 ft.; height = 161.26 ft.; area = 28,881 sq. ft.

Art. 33. **1.** 14.54 ft., 16.13 ft., 48.45 sq. ft., 105.2 sq. ft. **2.** 16.516 ft., 20.415 ft., 133.94 sq. ft., 318.4 sq. ft.

Art. 34 b. **2.** 10.954 mi. **3.** 96 ft. **4.** 14.454 mi. **5.** 67.08 mi., Dip. = 57′ 39″. **6.** 140.7 mi., Dip. = 2° 1′ 53″.

Art. 34 c. **4.** 2.852 acres. **5.** 12 acres 3 roods 6.45 poles.

CHAPTER V.

Art. 44. **18.** 45°, 135°, $-225°$, $-315°$; 45°, 135°, 405°, 495°. **19.** 60°, 240°, $-120°$, $-300°$; 60°, 240°, 420°, 600°. **20.** 135°, 225°, $-135°$, $-225°$; 135°, 225°, 495°, 585°. **21.** 150°, 330°, $-30°$, $-210°$; 150°, 330°, 510°, 690°.

CHAPTER VI.

Art. 46. **3.** $\cos(x+y) = .7874$, $\sin(x+y) = .6164$. (Verify by tables.)

Art. 47. **3.** $\sin(x-y) = -.1582$, $\cos(x-y) = .9874$. (Verify by tables.)

Art. 50. **7.** $\cos 6x = \cos^2 3x - \sin^2 3x = 1 - 2\sin^2 3x = 2\cos^2 3x - 1$, $\sin 6x = 2\sin 3x \cos 3x$. **9.** $\sin \frac{3}{4}x = 2\sin \frac{3}{8}x \cos \frac{3}{8}x$, $\cos \frac{3}{4}x = \cos^2 \frac{3}{8}x - \sin^2 \frac{3}{8}x = 1 - 2\sin^2 \frac{3}{8}x = 2\cos^2 \frac{3}{8}x - 1$. **10.** $\cos 6x = \frac{1}{2}\sqrt{1 + \cos 12x}$, $\sin 6x = \frac{1}{2}\sqrt{1 - \cos 12x}$. **12.** $\sin \frac{3}{4}x = \frac{1}{2}\sqrt{1 - \cos \frac{3}{2}x}$, $\cos \frac{3}{4}x = \sqrt{1 + \cos \frac{3}{2}x}$.

CHAPTER VII.

Art. 55. **2.** $b = 70.8$, $a = 56.1$. **4.** $b = 185$, $c = 192$. **5.** $b = 8.237$, $c = 5.464$.

Art. 56. **2.** $B = 36° 18.4'$ or $143° 41.6'$, $c = 52.71$ or 5.98. **5.** $A = 48° 25'$ or $131° 35'$.

Art. 57. **3.** $A = 80° 46.44'$, $C = 63° 48.56'$. **4.** $R = 33° 3.33'$, $S = 100° 56.67'$, $b = 39.56$.

Art. 58. **2.** $16° 47.3'$, $58° 45.07'$. **3.** $48° 11.4'$, $58° 24.7'$, $73° 23.9'$.

Art. 60. **2.** $b = 698.3$, $c = 845$. **3.** 600, 240. **6.** $b = 749.1$. **7.** $B = 46° 52' 10''$, $C = 111° 53' 25''$, $c = 1767.3$, or $B = 133° 7' 50''$, $C = 25° 37' 45''$, $c = 823.8$.

Art. 61. **2.** $c = 374.04$. **4.** $A = 109° 15' 30''$, $c = 440.46$.

Art. 62. **2.** $A = 53° 7.8'$, $B = 59° 29.4'$. **4.** $P = 44° 48.25'$, $R = 82° 15.8'$.

Art. 63. **3.** 444.72 yd. **4.** 1112.8 yd. **6.** 179.28 ft. **7.** 87.88 ft. **8.** 104.08 ft. **9.** 479.8 ft.

CHAPTER VIII.

Art. 67. 1. 11977.8 sq. ft.; $46° 13.8'$, $133° 46.2'$; 111.3 ft., 149.1 ft. **4.** $73° 30.7'$, $106° 29.3'$; area = 587637.5 sq. metres (approximately).

CHAPTER IX.

Art. 73. **2.** (*a*) $\frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{7\pi}{6}, \frac{5}{3}\pi, \frac{11}{6}\pi, \frac{3}{2}\pi, \frac{5}{4}\pi, -\frac{5}{12}\pi, \frac{7}{20}\pi, \frac{3}{20}\pi, -\frac{11}{60}\pi, -\frac{5}{6}\pi$; (*b*) .786, 1.048, 2.357, 3.667, etc. **4.** 90°, 60°, 45°, 30°, 36°, etc. **5.** $-135°$, $-900°$, $-240°$, $-165°$, $-4500°$. **7.** $28° 38' 52.4'$, $229° 10' 59.2''$, $171° 53' 14.4''$, $19° 5' 54.9''$, etc. **9.** Sine, cosine, tangent, cotangent, secant, cosecant, respectively, are: $\frac{\pi}{6}, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, \frac{2}{\sqrt{3}}, 2$; $\frac{\pi}{4}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, 1, \sqrt{2}, \sqrt{2}$; $\pi, 0, -1, 0, -\infty, -1, \infty$; $-\frac{5}{3}\pi, \frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}, \frac{1}{\sqrt{3}}, 2, \frac{2}{\sqrt{3}}$. **10** *a*. 3.75. **11.** The interior angles of the polygons are respectively, $\frac{3}{5}\pi, \frac{2}{3}\pi, \frac{5}{7}\pi, \frac{3}{4}\pi, \frac{4}{5}\pi, \frac{5}{6}\pi, \frac{13}{15}\pi$. **12.** (1) $\frac{20^r}{7}$, $163° 42' 8.3''$; (2) $\frac{1^r}{14}$, $4° 5' 33.2''$; (3) $\frac{1^r}{7920}$, $26''$. **13.** 10, 5, 2.5, $\frac{5}{3}$, 1.25, $\frac{5}{6}$, $\frac{2}{3}$, $\frac{1}{2}$, 20, 30, 25, $26\frac{2}{3}$ in. **15.** 10, 40, 70, 80, 120, 5, 3.75, 1.25 in.

CHAPTER X.

Art. 83. **4.** 238,890 mi. (approximately), 347.5. **5.** About 57′ 2″; about 1 : 13.5. **6.** About 93,757,000 mi. **7.** 206,265 times the distance of the earth from the sun, 3.26 yr. **8.** 9 ft. 2.6 in. **9.** 76 ft. 9.5 in. **10.** 4′ 35″. **11.** 15.708 yd. **12.** 13′ 1.3″.

CHAPTER XI.

Art. 85. **3.** $\theta = n\pi + (-1)^n\frac{\pi}{3}$, 60°, 120°, 420°, 480°. **4.** $n\pi + (-1)^{n+1}\frac{\pi}{6}$, 210°, 330°, 570°, 690°. **5.** $n \cdot 180° + (-1)^n 72° 30'$, 107° 30′, 432° 30′, 467° 30′.

Art. 86. **2.** $2n\pi \pm \frac{\pi}{6}$, $n \cdot 360 \pm 30°$, 30°, 330°, 390°, 690°. **3.** $n \cdot 360 \pm 7° 40'$, 7° 40′, 352° 20′, 367° 40′, 712° 20′. **5.** $n \cdot 360° \pm 136° 35'$, 136° 35′, 223° 25′, 496° 35′, 583° 25′.

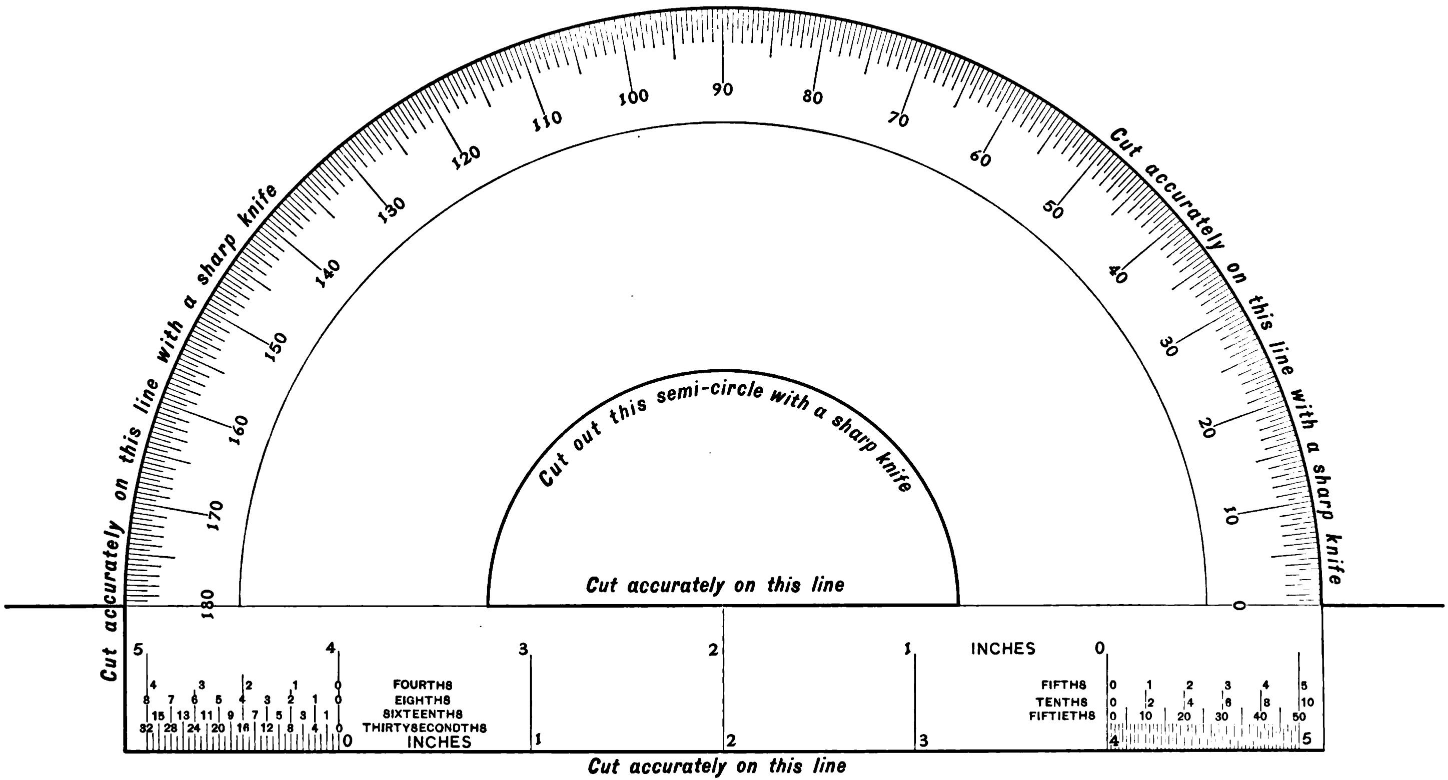
Art. 87. **2.** $n\pi + \frac{\pi}{3}$, $n \cdot 180° + 60°$, 60°, 240°, 420°, 600°. **3.** $n \cdot 180° + 20° 10'$, 20° 10′, 200° 10′, 380° 10′, 560° 10′. **6.** $n \cdot 180° + 138°$, 138°, 318°, 498°, 678°.

Art. 89. 11 (first part). (*a*) .97302; (*b*) ±.97302, ±.12117. **12.** 43° 5.5′. **13.** 45°.

Art. 90. **5.** $n\pi \pm \frac{\pi}{6}$, $n\pi \pm \frac{\pi}{3}$. **6.** $n\pi + \frac{\pi}{8}$, $n\pi + \frac{5}{8}\pi$ $\left(i.e.\ \frac{n\pi}{2} + \frac{\pi}{8}\right)$. **7.** $n \cdot 180° + 36° 52.2'$. **8.** $n \cdot 180° + 63° 26'$, $n \cdot 180° - 71° 33.9'$. **9.** $n \cdot 180° + 63° 26'$, $(4n - 1) 45°$. **10.** $\frac{n\pi}{4}$, $\frac{n\pi}{3} + (-1)^n\frac{\pi}{18}$. **11.** $n\pi + (-1)^n\frac{\pi}{2}$, $n \cdot 360° - 46° 23.85'$. **12.** $(2n+1)90°$, $\{4n + (-1)^n\}15°$. **13.** $\{6n + (-1)^n\}30°$, $\{10n + (-1)^n\}18°$, $\{10n - 3(-1)^n\}18°$. **14.** $n\pi \pm \frac{\pi}{4}$, $n\pi \pm \frac{\pi}{3}$. **15.** $n\pi \pm \frac{\pi}{6}$, $n\pi \pm \frac{\pi}{3}$. **16.** 0, $\pm\frac{1}{2}$.

CHAPTER XII.

Art. 94. **9.** $\frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$, $\frac{5 \tan A - 10 \tan^3 A + \tan^5 A}{1 - 10 \tan^2 A + 5 \tan^4 A}$, $\frac{6 \tan A - 20 \tan^3 A + 6 \tan^5 A}{1 - 15 \tan^2 A + 15 \tan^4 A - \tan^6 A}$, $\frac{7 \tan A - 35 \tan^3 A + 21 \tan^5 A - \tan^7 A}{1 - 21 \tan^2 A + 35 \tan^4 A - 7 \tan^6 A}$.


Cut accurately on this line with a sharp knife
Cut accurately on this line with a sharp knife
Cut out this semi-circle with a sharp knife
Cut accurately on this line
FOURTHS
EIGHTHS
SIXTEENTHS
THIRTYSECONDTHS
INCHES
INCHES
FIFTHS
TENTHS
FIFTIETHS
Cut accurately on this line

LOGARITHMIC AND TRIGONOMETRIC

TABLES

FIVE-PLACE AND FOUR-PLACE

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TABLES

FIVE-PLACE AND FOUR-PLACE

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CORNELL UNIVERSITY

LONGMANS, GREEN, AND CO.

91 AND 93 FIFTH AVENUE, NEW YORK

LONDON AND BOMBAY

1899

ELECTROTYPED BY J. S. CUSHING & CO., NORWOOD, MASS.

CONTENTS.

NOTE. These tables have been arranged primarily for students in elementary trigonometry, and the explanations are intended for beginners in that branch of mathematics. Tabular differences and proportional parts should be *calculated,* and not copied from tables, by those who use logarithmic and trigonometric tables for the first time. The editor may be allowed to take this opportunity of expressing his belief that the principles and use of *common logarithms* can be easily explained in the school course in arithmetic, and practical applications given which will be interesting and advantageous to young pupils.

EXPLANATION OF THE TABLES.

TABLE I.

COMMON LOGARITHMS.

N.B. *The meaning and properties of logarithms are explained in works on algebra.*

1. The first page of the table gives the characteristics and mantissas of numbers from 1 up to 100. The remainder of the table gives only mantissas. The characteristics are obtained by the following rule, which is deduced in algebra:*

When the number is greater than 1, *the characteristic is positive, and is one* less *than the number of figures to the left of the decimal point; when the number is less than* 1, *the characteristic is negative, and is one* more *than the number of zeros between the decimal point and the first significant figure.*

The first three figures of a number of four figures are found in the left-hand column marked **N**; the fourth figure of the number is found in the lines at the top and the foot of the page. The last three figures of the mantissa are found in the same line as the first three figures of the number, and in the same column as the fourth figure of the number. The first two figures of the mantissa are in the column headed **0**, and are printed only once. They are found either in the same line as the last three figures, or in the first line above which contains a whole mantissa. If, however, a * precedes the last three figures of the mantissa, the first two figures are found in the following line.

---

* This rule may be easily deduced in arithmetic.

2. To find the logarithm of a number.

RULE: *Write the characteristic, and then annex the mantissa found by means of the table.*

(*a*) *A number of four figures.*

$\log 3552 = 3.55047$; $\log 355.7 = 2.55108$; $\log 35.74 = 1.55315$;

$\log 36.34 = 1.56038$; $\log 536.2 = 2.72933$; $\log 5.371 = 0.73006$.

(*b*) *A number of less than four figures.* In this case, annex ciphers, or suppose them to be annexed, and proceed as in case (*a*).

$\log .213 = \bar{1}.32838$; $\log 47.6 = 1.67761$; $\log .0375 = \bar{2}.57403$.

(*c*) *A number of more than four figures.*

To find log 47653. The characteristic is 4. The mantissa, as shown in algebra, is the same as the mantissa of log 4765.3. Log 4765.3 lies between log 4765 and log 4766. Hence the mantissa of log 4765.3 is between the mantissas of log 4765 and log 4766. It is assumed that the change in the mantissa is proportional to the change in the number, as the latter increases from 4765 to 4766; that is,

$$\text{mantissa of } \log 4765.3 = \text{mantissa of } \log 4765 + .3 \times (\text{mantissa of } \log 4766 - \text{mantissa of } \log 4765).^*$$

$$\begin{array}{rr}
\text{mantissa of } \log 4766 = & .67815 \\
\text{mantissa of } \log 4765 = & \underline{.67806} \\
\text{difference for } 1 = & 9
\end{array}
\qquad
\begin{array}{rr}
\text{mantissa of } \log 4765 = & .67806 \\
\text{difference for } .3 = .3 \times 9 = & \underline{\quad 27} \\
\therefore \text{ mantissa of } \log 4765.3 = & .678087 \\
\text{or, } = & .67809 \\
\therefore \log 47653 = & 4.67809
\end{array}$$

NOTE 1. By general agreement, a number with six or more decimal places is reduced to a number with five in the following way:

If a number less than 5 is in the sixth decimal place, then the number in the fifth place is left unchanged; if a number greater than 5 is in the sixth place, or if there is a 5 in the sixth place and it is followed by figures other

---

* It is assumed that when a number varies from one value to another, the change in the mantissa is proportional to the change in the number if the latter change is small in comparison with the number. This is not strictly correct, but is accurate enough for practical purposes.

than zeros only, then the number in the fifth place is increased by unity; if the number in the sixth place is 5 and it is followed by zeros only, then an even number in the fifth place is left unchanged, and an odd number in the fifth place is increased by unity.

NOTE 2. The difference between the mantissas for two consecutive numbers of four figures is called their *tabular difference*, and is printed in the column marked **D.** At the lower parts of the first three pages of the table the tabular differences for the mantissas on these pages are multiplied by the nine digits expressed as tenths. The results, which are called *proportional parts*, are the amounts to be added in obtaining the logarithms of five-figure numbers. It is better for the beginner in logarithmic computation to find the tabular differences by subtraction, and make the calculations for himself. The process described above for finding the logarithms of numbers of five or more figures, is called *interpolation*.

To find log 476.532.

$$\begin{aligned} \log 476.5 &= 2.67806 \\ \text{difference for } .32 = .32 \times 9 &= \underline{288} \\ \therefore \log 476.532 &= 2.67809 \end{aligned}$$

[See Ex. above.]

NOTE 3. A five-place table of logarithms is not used, in general, with numbers of more than five figures. In numbers having more than five figures the digits beyond the fifth have little effect on logarithms that are calculated no farther than to five places of decimals.

To find log 83.946.

$$\begin{aligned} \log 83.94 &= 1.92397 \\ \text{difference} = .6 \times 5 &= \underline{3} \\ \therefore \log 83.946 &= 1.92400 \end{aligned}$$

To find log 83.9468.

$$\begin{aligned} \log 83.94 &= 1.92397 \\ \text{difference} = .68 \times 5 &= \underline{34} \\ \therefore \log 83.9468 &= 1.92400 \end{aligned}$$

To find log 1236.2.

$$\begin{aligned} \log 1236 &= 3.09202 \\ \text{difference} = .2 \times 35 &= \underline{7} \\ \therefore \log 1236.2 &= 3.09209 \end{aligned}$$

To find log 1236.24.

$$\begin{aligned} \log 1236 &= 3.09202 \\ \text{difference} = .24 \times 35 &= \underline{84} \\ \therefore \log 1236.24 &= 3.09210 \end{aligned}$$

RULE: *Find the mantissa corresponding to the first four figures of the number; multiply the tabular difference at that place in the table by the fifth and following figures treated as a decimal; and add the product to the mantissa just found.*

NOTE 4. The logarithm of the reciprocal of a number is called the *cologarithm* of the number, or the *arithmetical complement* of the logarithm of the number. For instance, $\log \frac{1}{325} = \text{colog } 325$. Now

$$\log \tfrac{1}{325} = \log 1 - \log 325 = 0 - 2.51188 = (10 - 2.51188) - 10 = 7.48812 - 10.$$

Thus the *cologarithm* of a number is equal to the *negative logarithm* of the number. The cologarithm can be written directly from the logarithm in the table. The use of cologarithms sometimes helps in computation. For example, $\log \frac{23.41 \times 375}{92.83} = \log 23.41 + \log 375 + \text{colog } 92.83.$

3. To find the number corresponding to a given logarithm. This operation is the reverse of the preceding. *The position of the decimal point in the required number is shown by the characteristic.* The number of figures before the decimal point is one more than the characteristic when the latter is positive; when the characteristic is negative the number is a decimal, and the number of ciphers between the decimal point and the first significant digit is one less than the figure in the characteristic. (See the rule for finding the characteristic.)

The sequence of figures in the number is found from the mantissa.

(*a*) *When the given mantissa is in the tables.* The first two figures of the mantissa will be found in the column headed **0**; the last three figures will be found in the same line as the first two, or in the line above (where it will be preceded by *), or in one of the lines following. The first three figures of the number are in the column headed **N**, and are in the same line as the last three figures of the mantissa; the fourth figure of the number is at the top of the page in the same column as the last three figures of the mantissa.

To find the number whose logarithm is 2.55047. On turning in the table to the mantissa 55047 it is found that the corresponding sequence of figures is 3552. The characteristic *2* shows that the required number is 355.2. The number having $\bar{2}.55047$ for its logarithm is .03552.

Given $\log N = 5.67815$, find N. The sequence of figures in the required number, as found on turning in the table to the mantissa 67815, is 4766. The characteristic *5* shows that the required number is 476600. The number having $\bar{1}.67815$ for its logarithm is .4766.

(*b*) *When the given mantissa is not in the tables.* In this case the process of interpolation is employed.

To find the number whose logarithm is 2.57072. Inspection of the table shows that the given mantissa lies between the tabulated mantissas, 57066 and 57078. Hence the required number lies between 372.1 and 372.2.

mantissa of	3722 = .57078	given mantissa	= .57072
mantissa of	3721 = .57066	mantissa of	3721 = .57066
∴ difference for	1 = 12	difference	= 6

If 12 is the difference for 1, for what is 6 the difference? Obviously for $\frac{6}{12}$ of 1, *i.e.* .5. Hence the required number is 372.15.

RULE: *Find the number corresponding to the mantissa in the table next less than the given mantissa; find the difference between these mantissas; divide this difference by the tabular difference; and annex the quotient to the four figures already found.*

TABLE II.

LOGARITHMS OF CERTAIN TRIGONOMETRIC RATIOS.

4. The numbers given in this table are sometimes called *logarithmic sines, logarithmic cosines,* etc., or the *tabular logarithms of the sines, cosines,* etc. These terms are considered necessary because these numbers, with the exception of those in one column on each page, are *not* the logarithms of the sines, cosines, etc., but are these logarithms *increased by 10.* Hence, in working examples these numbers should be diminished by 10. In the column headed **L. Cot.,** however, the logarithms are given correctly.

The degrees from 0° to 44° are given at the top of the page, and the minutes to be taken with any of these degrees are given from 0 down to 60 in the column on the left. The degrees from 45° to 89° are given at the foot of the page, and the minutes to be taken with any of these degrees are given from 0 up to 60 in the column on the right. For the degrees printed at the top of the page the contents of the columns are indicated at the top of the page; for the degrees printed at the foot of the page the contents of the columns are indicated at the foot of the page. A ratio is printed at the top of each column (excepting the columns for minutes), and the corresponding co-ratio is at the foot. This convenient arrangement of the table is possible, because, as shown in trigonometry, a trigonometric ratio of an angle is equal to the corresponding co-ratio of the complement of the angle. For instance, L. Sin. $33° 26' = 9.74113 =$ L. Cos. $56° 34'$. The column headed **L. Cot.** gives *correctly* the logarithms of the cotangents of angles from 0° to 45°, and the logarithms of the tangents of angles from 45° to 90°.

5. To find the logarithm of a trigonometric ratio of an acute angle.

(*a*) *When the angle is given in degrees and minutes. If the angle is less than* 45°, turn to where the number of degrees is given at the *top* of the page; find the number of minutes in the column on the *left* marked $'$; write the number which is in line with the number of minutes, and in the column *under* the ratio named; subtract *10* when the number found is not in the column headed L. Cot. *If the angle is* 45° *or greater than* 45°, turn to where the number of degrees is given at the *foot* of the page; find the number of minutes in the column on the *right* marked $'$; write the number which is in line with the number of minutes, and in the column *over* the ratio named; subtract 10 when the number found is not in the column headed L. Cot., or, what is the same thing, in the column that has L. Tan. at its foot.

$$\log\sin 23° 20' = 9.59778 - 10; \quad \log\tan 37° 50' = 9.89020 - 10;$$
$$\log\cos 55° 40' = 9.75128 - 10; \quad \log\cot 78° 10' = 9.32122 - 10;$$
$$\log\cot 33° 26' = 0.18032; \quad \log\tan 47° 50' = 0.04302.$$

(*b*) *When the angle is given in degrees, minutes, and seconds.* In this case the logarithms required are obtained by the process of interpolation.

To find $\log\sin 36° 42' 20''$. The required number lies between $\log\sin 36° 42'$ and $\log\sin 36° 43'$. It is assumed that the difference between the logarithms of the sines of two angles is proportional to the difference between the angles when the latter difference is small compared with either of the angles. (This is not strictly correct, but is accurate enough for practical purposes.)

$$\begin{array}{ll} \log\sin 36° 43' = 9.77660 - 10 & \log\sin 36° 42' = 9.77643 - 10 \\ \log\sin 36° 42' = \underline{9.77643 - 10} & \text{diff. for } 20'' = 17 \times \tfrac{20}{60} = \underline{\qquad\quad 56 \cdots} \\ \quad\text{diff. for } 1' = \qquad 17 & \therefore \log\sin 36° 42' 20'' = 9.77649 - 10 \end{array}$$

As the sine increases when the angle changes from 0° to 90°, log sin 36° 42′ 20″ is greater than log sin 36° 42′; and hence the difference for 20″ is added. The work indicated on the left may be omitted, since the difference for 1′ can be taken directly from the tables.

To find log cos 23° 36′ 40″.

$$
\begin{aligned}
\log\cos 23^\circ\, 36' &= 9.96207 - 10\\
\text{difference for } 40'' = 6 \times \tfrac{40}{60} &= \underline{\qquad\quad 4}\\
\therefore\ \log\cos 23^\circ\, 26'\, 40'' &= 9.96203 - 10
\end{aligned}
$$

Since the cosine decreases as the angle changes from 0° to 90°, log cos 23° 36′ 40″ is less than log cos 23° 36′; and hence, the difference for 40″ is subtracted. The differences for seconds are added in the case of the logarithm of the tangent, and subtracted in the case of the logarithm of the cotangent.

NOTE 1. Since

$$\sec A = \frac{1}{\cos A},\ \log\sec A = -\log\cos A = \operatorname{colog}\cos A;$$

since

$$\operatorname{cosec} A = \frac{1}{\sin A},\ \log\operatorname{cosec} A = -\log\sin A = \operatorname{colog}\sin A.$$

NOTE 2. It is shown in trigonometry that the trigonometric ratio of any angle can be expressed in terms of some trigonometric ratio of an angle less than 90°. Hence the logarithm of any trigonometric ratio of any angle can be found.

6. To find the acute angle that has a given logarithm of a trigonometric ratio.

This operation is the reverse of the preceding.

(*a*) *When the given logarithmic ratio is in the table.*

To find A, given that $\log\sin A = 9.77558 - 10$, and B, given that $\log\sin B = 9.88647 - 10$. Here L. Sin. $A = 9.77558$, and L. Sin. $B = 9.88647$. Look through the columns having *L. Sin.* at the top or at the foot, until the given L. Sin. is found. If this number is in the column *headed* L. Sin., write the number of degrees printed at the *top* of the page, and the number of minutes which is in the column on the *left* and in line with the given L. Sin. If the given L. Sin. is in the column having L. Sin. *at its foot*, write the number of degrees printed at the *foot* of the page, and the number of minutes which is in the column on the ***right*** and in line with the given L. Sin. The logarithms of other ratios are treated in a similar manner. In the examples given above, the acute angles that satisfy the given conditions are, $A = 36° 37'$, $B = 50° 21'$.

(*b*) *When the given logarithmic ratio is not in the table.*

To find A when $\log \sin A = 9.80218 - 10$. Examination of the columns for *L. Sin.* in the table shows that L. Sin. $39° 21' = 9.80213$, and L. Sin. $39° 22' = 9.80228$. Hence the angle required lies between $39° 21'$ and $39° 22'$.

$\log \sin 39° 22' =$	$9.80228 - 10$	$\log \sin A =$	$9.80218 - 10$
$\log \sin 39° 21' =$	$9.80213 - 10$	$\log \sin 39° 21' =$	$9.80213 - 10$
difference for $1' =$	15	difference $=$	5

If 15 is the difference for $1'$, for what is 5 the difference? Obviously for $\frac{5}{15}$ of $1'$, *i.e.* $20''$. Hence the *acute* angle that has the given logarithm of a sine is $39° 21' 20''$.

To find A when $\log \cos A = 9.58824 - 10$. Examination of the columns for *L. Cos.* in the table shows that L. Cos. $67° 12' = 9.58829$, and L. Cos. $67° 13' = 9.58799$. Hence the acute angle required lies between $67° 12'$ and $67° 13'$.

$\log \cos 66° 12' =$	$9.58829 - 10$	$\log \cos 67° 12' =$	$9.58829 - 10$
$\log \cos 67° 13' =$	$9.58799 - 10$	$\log \cos A =$	$9.58824 - 10$
difference for $1' =$	30	difference $=$	5

If 30 is the difference for $1'$, for what is 5 the difference? Obviously for $\frac{5}{30}$ of $1'$, *i.e.* $10''$. Hence the *acute* angle that has the given logarithm of a cosine is $67° 12' 10''$. The work on the left in these examples need not be written, for it can be performed mentally on inspection of the tables. The successive differences for $1'$ are called *tabular differences for one minute.*

Rule: In order to obtain the acute angle corresponding to a given logarithm of a sine or tangent, find the degrees and minutes corresponding to the logarithm next *less* than the given logarithm; divide the difference between these logarithms by the tabular difference for $1'$ at that place in the table; this gives the fraction of a minute to be added to the degrees and minutes already found. In order to obtain the acute angle corresponding to a given logarithm of a cosine or cotangent, find the degrees and minutes corresponding to the logarithm next *greater* than the given logarithm; divide the difference between these logarithms by the tabular difference for 1′ at that place in the table; this gives the fraction of a minute to be added to the degrees and minutes already found.

NOTE 1. The logarithm next *less* is taken in the case of the sine and tangent, since these ratios increase as the angle increases from 0° to 90°; the logarithm next *greater* is taken in the case of the cosine and cotangent, since these ratios decrease as the angle increases from 0° to 90°.

NOTE 2. It is shown in trigonometry that there are many angles in addition to an acute angle, which have the same trigonometric ratio, and accordingly the same logarithm of the ratio.

TABLES III.

FOUR-PLACE TABLES.

7. Four-place tables are accurate enough for many purposes. **The first two pages of Tables III.** give four-place logarithms of numbers from 1 to 999. These logarithms should not be used, in general, with numbers that contain more than four figures. The rules for using this table are similar to the rules given in connection with Table I.

$$\log 723 = 2.8591; \qquad \log 9.36 = .9713.$$

To find log 3642.

$$\begin{aligned} \log 3640 &= 3.5611 \\ \text{difference for } 2 = .2 \times 12 &= \underline{24} \\ \therefore \log 3642 &= 3.5613 \end{aligned}$$

To find the number whose logarithm is 2.6860.

$$\begin{aligned} \text{given log} &= 2.6860 \\ \log 485 &= \underline{2.6857} \end{aligned}$$

$$\text{tabular difference for } 1 = 9; \quad \text{difference} = 3$$

$$\therefore \text{addition} = \tfrac{3}{9} \text{ of } 1 = .3 \cdots \quad \therefore \text{number} = 485.3 \cdots$$

8. The second of Tables III. gives the augmented logarithms of angles at intervals of ten minutes from 0° to 90°. The angles from 0° to 45° are printed on the left, and the angles from 45° to 90° are printed on the right. This table is used in the same manner as Table II. It is necessary, however, to pay attention to the fact that the difference between the successive angles tabulated is 10′, instead of 1′ as in Table II.

To find log tan 29° 15′.

$$\begin{aligned} \log\tan 29°\,10' &= 9.7467-10 \\ \text{difference for } 5' = \tfrac{5}{10} \text{ of } 30 &= \underline{15} \\ \therefore \log\tan 29°\,15' &= 9.7482-10. \end{aligned}$$

To find A when log cot $A = .4531$.

$$\begin{aligned} \log\cot 19°\,20' &= .4549 \\ \log\cot A &= \underline{.4531} \\ \text{tabular diff. for } 10' = 40;\ \text{diff.} &= 18 \end{aligned}$$

$$\therefore \text{addition} = \tfrac{18}{40} \text{ of } 10' = 4'.5. \quad \therefore A = 19°\,24'.5.$$

9. The last of Tables III. gives the actual numerical values to four places of decimals, of the sines, cosines, tangents, and cotangents of angles, at intervals of ten minutes from 0° to 90°. These values are usually called *natural sines, natural cosines,* etc., and are denoted by *N. Sin., N. Cos.,* etc., in order to distinguish them from the so-called logarithmic sines, cosines, etc., given in the immediately preceding table and in Table II. (Logarithms were sometimes called *artificial* numbers, and ordinary numbers were regarded as *natural* numbers.) The explanations concerning this four-place table, and the rules for finding the trigonometric ratios corresponding to given angles, and for finding the angles corresponding to given ratios, are the same as the explanations and rules in the preceding table and in Table II., if all references to logarithms in the latter rules be omitted. Those who are using trigonometric tables for the first time, should test the statements made concerning the relations between the numbers in Table II. and the second of Tables III. on the one hand, and the numbers in the third of Tables III. on the other.

To find A when $\cot A = .4336$.

$$\begin{aligned} \cot 66^\circ 30' &= .4348 \\ \cot A &= .4336 \end{aligned}$$

$$\text{tabular diff. for } 10' = 34; \text{ diff.} = \quad 12$$

$\therefore$ addition $= \frac{12}{34}$ of $10' = 3'.5$. $\quad \therefore A = 66^\circ 33'.5$.

To find sin 36° 23′.

$$\begin{aligned} \sin 36^\circ 20' &= .5925 \\ \text{difference for } 3' = \tfrac{3}{10} \text{ of } 23 &= \quad 69 \\ \therefore \sin 36^\circ 23' &= .5932 \end{aligned}$$

Ex. 1. Compare the four-place mantissas of the logarithms of several numbers with the corresponding five-place mantissas. Make a similar comparison between the four-place and five-place tables in the case of the trigonometric ratios of several angles.

Ex. 2. In the four-place table of natural sines, etc., find sin 37° 25′, tan 40° 30′, cot 27° 30′, cos 31° 15′, sin 50° 20′, tan 63° 25′, cot 74° 25′, cos 51° 35′. Find the logarithms of these numbers by means of Table I. Compare the results with the values given for the logarithmic sines, etc., in Table II. and the second of Tables III.

I.

COMMON LOGARITHMS OF NUMBERS

Giving Characteristics and Mantissas of Logarithms of Numbers from 1 to 100, and Mantissas only of Numbers from 100 to 10000.

LOGARITHMS OF NUMBERS.

N	Log.	N	Log.	N	Log.	N	Log.
1	0.00000	26	1.41497	51	1.70757	76	1.88081
2	0.30103	27	1.43136	52	1.71600	77	1.88649
3	0.47712	28	1.44716	53	1.72428	78	1.89209
4	0.60206	29	1.46240	54	1.73239	79	1.89763
5	0.69897	30	1.47712	55	1.74036	80	1.90309
6	0.77815	31	1.49136	56	1.74819	81	1.90849
7	0.84510	32	1.50515	57	1.75587	82	1.91381
8	0.90309	33	1.51851	58	1.76343	83	1.91908
9	0.95424	34	1.53148	59	1.77085	84	1.92428
10	1.00000	35	1.54407	60	1.77815	85	1.92942
11	1.04139	36	1.55630	61	1.78533	86	1.93450
12	1.07918	37	1.56820	62	1.79239	87	1.93952
13	1.11394	38	1.57978	63	1.79934	88	1.94448
14	1.14613	39	1.59106	64	1.80618	89	1.94939
15	1.17609	40	1.60206	65	1.81291	90	1.95424
16	1.20412	41	1.61278	66	1.81954	91	1.95904
17	1.23045	42	1.62325	67	1.82607	92	1.96379
18	1.25527	43	1.63347	68	1.83251	93	1.96848
19	1.27875	44	1.64345	69	1.83885	94	1.97313
20	1.30103	45	1.65321	70	1.84510	95	1.97772
21	1.32222	46	1.66276	71	1.85126	96	1.98227
22	1.34242	47	1.67210	72	1.85733	97	1.98677
23	1.36173	48	1.68124	73	1.86332	98	1.99123
24	1.38021	49	1.69020	74	1.86923	99	1.99564
25	1.39794	50	1.69897	75	1.87506	100	2.00000

N	0	1	2	3	4	5	6	7	8	9	D
100	00 000	043	087	130	173	217	260	303	346	389	43
101	432	475	518	561	604	647	689	732	775	817	43
102	860	903	945	988	*030	*072	*115	*157	*199	*242	42
103	01 284	326	368	410	452	494	536	578	620	662	42
104	703	745	787	828	870	912	953	995	*036	*078	42
105	02 119	160	202	243	284	325	366	407	449	490	41
106	531	572	612	653	694	735	776	816	857	898	41
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302	40
108	03 342	383	423	463	503	543	583	623	663	703	40
109	743	782	822	862	902	941	981	*021	*060	*100	40
110	04 139	179	218	258	297	336	376	415	454	493	39
111	532	571	610	650	689	727	766	805	844	883	39
112	922	961	999	*038	*077	*115	*154	*192	*231	*269	39
113	05 308	346	385	423	461	500	538	576	614	652	38
114	690	729	767	805	843	881	918	956	994	*032	38
115	06 070	108	145	183	221	258	296	333	371	408	38
116	446	483	521	558	595	633	670	707	744	781	37
117	819	856	893	930	967	*004	*041	*078	*115	*151	37
118	07 188	225	262	298	335	372	408	445	482	518	37
119	555	591	628	664	700	737	773	809	846	882	36
120	918	954	990	*027	*063	*099	*135	*171	*207	*243	36
121	08 279	314	350	386	422	458	493	529	565	600	36
122	636	672	707	743	778	814	849	884	920	955	35
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	35
124	09 342	377	412	447	482	517	552	587	621	656	35
125	691	726	760	795	830	864	899	934	968	*003	35
126	10 037	072	106	140	175	209	243	278	312	346	34
127	380	415	449	483	517	551	585	619	653	687	34
128	721	755	789	823	857	890	924	958	992	*025	34
129	11 059	093	126	160	193	227	261	294	327	361	34
N	0	1	2	3	4	5	6	7	8	9	D

PP	44	43	42	41	40	39	38	37	36
1	4.4	4.3	4.2	4.1	4.0	3.9	3.8	3.7	3.6
2	8.8	8.6	8.4	8.2	8.0	7.8	7.6	7.4	7.2
3	13.2	12.9	12.6	12.3	12.0	11.7	11.4	11.1	10.8
4	17.6	17.2	16.8	16.4	16.0	15.6	15.2	14.8	14.4
5	22.0	21.5	21.0	20.5	20.0	19.5	19.0	18.5	18.0
6	26.4	25.8	25.2	24.6	24.0	23.4	22.8	22.2	21.6
7	30.8	30.1	29.4	28.7	28.0	27.3	26.6	25.9	25.2
8	35.2	34.4	33.6	32.8	32.0	31.2	30.4	29.6	28.8
9	39.6	38.7	37.8	36.9	36.0	35.1	34.2	33.3	32.4

N	0	1	2	3	4	5	6	7	8	9	D
130	11 394	428	461	494	528	561	594	628	661	694	33
131	727	760	793	826	860	893	926	959	992	*024	33
132	12 057	090	123	156	189	222	254	287	320	352	33
133	385	418	450	483	516	548	581	613	646	678	33
134	710	743	775	808	840	872	905	937	969	*001	32
135	13 033	066	098	130	162	194	226	258	290	322	32
136	354	386	418	450	481	513	545	577	609	640	32
137	672	704	735	767	799	830	862	893	925	956	32
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	31
139	14 301	333	364	395	426	457	489	520	551	582	31
140	613	644	675	706	737	768	799	829	860	891	31
141	922	953	983	*014	*045	*076	*106	*137	*168	*198	31
142	15 229	259	290	320	351	381	412	442	473	503	31
143	534	564	594	625	655	685	715	746	776	806	30
144	836	866	897	927	957	987	*017	*047	*077	*107	30
145	16 137	167	197	227	256	286	316	346	376	406	30
146	435	465	495	524	554	584	613	643	673	702	30
147	732	761	791	820	850	879	909	938	967	997	29
148	17 026	056	085	114	143	173	202	231	260	289	29
149	319	348	377	406	435	464	493	522	551	580	29
150	609	638	667	696	725	754	782	811	840	869	29
151	898	926	955	984	*013	*041	*070	*099	*127	*156	29
152	18 184	213	241	270	298	327	355	384	412	441	29
153	469	498	526	554	583	611	639	667	696	724	28
154	752	780	808	837	865	893	921	949	977	*005	28
155	19 033	061	089	117	145	173	201	229	257	285	28
156	312	340	368	396	424	451	479	507	535	562	28
157	590	618	645	673	700	728	756	783	811	838	28
158	866	893	921	948	976	*003	*030	*058	*085	*112	27
159	20 140	167	194	222	249	276	303	330	358	385	27
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PP	35	34	33	32	31	30	29	28	27
1	3.5	3.4	3.3	3.2	3.1	3.0	2.9	2.8	2.7
2	7.0	6.8	6.6	6.4	6.2	6.0	5.8	5.6	5.4
3	10.5	10.2	9.9	9.6	9.3	9.0	8.7	8.4	8.1
4	14.0	13.6	13.2	12.8	12.4	12.0	11.6	11.2	10.8
5	17.5	17.0	16.5	16.0	15.5	15.0	14.5	14.0	13.5
6	21.0	20.4	19.8	19.2	18.6	18.0	17.4	16.8	16.2
7	24.5	23.8	23.1	22.4	21.7	21.0	20.3	19.6	18.9
8	28.0	27.2	26.4	25.6	24.8	24.0	23.2	22.4	21.6
9	31.5	30.6	29.7	28.8	27.9	27.0	26.1	25.2	24.3

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160	20 412	439	466	493	520	548	575	602	629	656	27
161	683	710	737	763	790	817	844	871	898	925	27
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192	27
163	21 219	245	272	299	325	352	378	405	431	458	27
164	484	511	537	564	590	617	643	669	696	722	26
165	748	775	801	827	854	880	906	932	958	985	26
166	22 011	037	063	089	115	141	167	194	220	246	26
167	272	298	324	350	376	401	427	453	479	505	26
168	531	557	583	608	634	660	686	712	737	763	26
169	789	814	840	866	891	917	943	968	994	*019	26
170	23 045	070	096	121	147	172	198	223	249	274	25
171	300	325	350	376	401	426	452	477	502	528	25
172	553	578	603	629	654	679	704	729	754	779	25
173	805	830	855	880	905	930	955	980	*005	*030	25
174	24 055	080	105	130	155	180	204	229	254	279	25
175	304	329	353	378	403	428	452	477	502	527	25
176	551	576	601	625	650	674	699	724	748	773	25
177	797	822	846	871	895	920	944	969	993	*018	25
178	25 042	066	091	115	139	164	188	212	237	261	24
179	285	310	334	358	382	406	431	455	479	503	24
180	527	551	575	600	624	648	672	696	720	744	24
181	768	792	816	840	864	888	912	935	959	983	24
182	26 007	031	055	079	102	126	150	174	198	221	24
183	245	269	293	316	340	364	387	411	435	458	24
184	482	505	529	553	576	600	623	647	670	694	24
185	717	741	764	788	811	834	858	881	905	928	23
186	951	975	998	*021	*045	*068	*091	*114	*138	*161	23
187	27 184	207	231	254	277	300	323	346	370	393	23
188	416	439	462	485	508	531	554	577	600	623	23
189	646	669	692	715	738	761	784	807	830	852	23
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1	2.7	2.6	2.5	2.4	2.3	2.2
2	5.4	5.2	5.0	4.8	4.6	4.4
3	8.1	7.8	7.5	7.2	6.9	6.6
4	10.8	10.4	10.0	9.6	9.2	8.8
5	13.5	13.0	12.5	12.0	11.5	11.0
6	16.2	15.6	15.0	14.4	13.8	13.2
7	18.9	18.2	17.5	16.8	16.1	15.4
8	21.6	20.8	20.0	19.2	18.4	17.6
9	24.3	23.4	22.5	21.6	20.7	19.8

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190	875	898	921	944	967	989	*012	*035	*058	*081	23
191	28 103	126	149	171	194	217	240	262	285	307	23
192	330	353	375	398	421	443	466	488	511	533	23
193	556	578	601	623	646	668	691	713	735	758	22
194	780	803	825	847	870	892	914	937	959	981	22
195	29 003	026	048	070	092	115	137	159	181	203	22
196	226	248	270	292	314	336	358	380	403	425	22
197	447	469	491	513	535	557	579	601	623	645	22
198	667	688	710	732	754	776	798	820	842	863	22
199	885	907	929	951	973	994	*016	*038	*060	*081	22
200	30 103	125	146	168	190	211	233	255	276	298	22
201	320	341	363	384	406	428	449	471	492	514	22
202	535	557	578	600	621	643	664	685	707	728	21
203	750	771	792	814	835	856	878	899	920	942	21
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154	21
205	31 175	197	218	239	260	281	302	323	345	366	21
206	387	408	429	450	471	492	513	534	555	576	21
207	597	618	639	660	681	702	723	744	765	785	21
208	806	827	848	869	890	911	931	952	973	994	21
209	32 015	035	056	077	098	118	139	160	181	201	21
210	222	243	263	284	305	325	346	366	387	408	21
211	428	449	469	490	510	531	552	572	593	613	20
212	634	654	675	695	715	736	756	777	797	818	20
213	838	858	879	899	919	940	960	980	*001	*021	20
214	33 041	062	082	102	122	143	163	183	203	224	20
215	244	264	284	304	325	345	365	385	405	425	20
216	445	465	486	506	526	546	566	586	606	626	20
217	646	666	686	706	726	746	766	786	806	826	20
218	846	866	885	905	925	945	965	985	*005	*025	20
219	34 044	064	084	104	124	143	163	183	203	223	20
220	242	262	282	301	321	341	361	380	400	420	20
221	439	459	479	498	518	537	557	577	596	616	20
222	635	655	674	694	713	733	753	772	792	811	19
223	830	850	869	889	908	928	947	967	986	*005	19
224	35 025	044	064	083	102	122	141	160	180	199	19
225	218	238	257	276	295	315	334	353	372	392	19
226	411	430	449	468	488	507	526	545	564	583	19
227	603	622	641	660	679	698	717	736	755	774	19
228	793	813	832	851	870	889	908	927	946	965	19
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154	19
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230	36 173	192	211	229	248	267	286	305	324	342	19
231	361	380	399	418	436	455	474	493	511	530	19
232	549	568	586	605	624	642	661	680	698	717	19
233	736	754	773	791	810	829	847	866	884	903	19
234	922	940	959	977	996	*014	*033	*051	*070	*088	18
235	37 107	125	144	162	181	199	218	236	254	273	18
236	291	310	328	346	365	383	401	420	438	457	18
237	475	493	511	530	548	566	585	603	621	639	18
238	658	676	694	712	731	749	767	785	803	822	18
239	840	858	876	894	912	931	949	967	985	*003	18
240	38 021	039	057	075	093	112	130	148	166	184	18
241	202	220	238	256	274	292	310	328	346	364	18
242	382	399	417	435	453	471	489	507	525	543	18
243	561	578	596	614	632	650	668	686	703	721	18
244	739	757	775	792	810	828	846	863	881	899	18
245	917	934	952	970	987	*005	*023	*041	*058	*076	18
246	39 094	111	129	146	164	182	199	217	235	252	18
247	270	287	305	322	340	358	375	393	410	428	18
248	445	463	480	498	515	533	550	568	585	602	18
249	620	637	655	672	690	707	724	742	759	777	17
250	794	811	829	846	863	881	898	915	933	950	17
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123	17
252	40 140	157	175	192	209	226	243	261	278	295	17
253	312	329	346	364	381	398	415	432	449	466	17
254	483	500	518	535	552	569	586	603	620	637	17
255	654	671	688	705	722	739	756	773	790	807	17
256	824	841	858	875	892	909	926	943	960	976	17
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	17
258	41 162	179	196	212	229	246	263	280	296	313	17
259	330	347	363	380	397	414	430	447	464	481	17
260	497	514	531	547	564	581	597	614	631	647	17
261	664	681	697	714	731	747	764	780	797	814	17
262	830	847	863	880	896	913	929	946	963	979	16
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	16
264	42 160	177	193	210	226	243	259	275	292	308	16
265	325	341	357	374	390	406	423	439	455	472	16
266	488	504	521	537	553	570	586	602	619	635	16
267	651	667	684	700	716	732	749	765	781	797	16
268	813	830	846	862	878	894	911	927	943	959	16
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	16
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270	43 136	152	169	185	201	217	233	249	265	281	16
271	297	313	329	345	361	377	393	409	425	441	16
272	457	473	489	505	521	537	553	569	584	600	16
273	616	632	648	664	680	696	712	727	743	759	16
274	775	791	807	823	838	854	870	886	902	917	16
275	933	949	965	981	996	*012	*028	*044	*059	*075	16
276	44 091	107	122	138	154	170	185	201	217	232	16
277	248	264	279	295	311	326	342	358	373	389	16
278	404	420	436	451	467	483	498	514	529	545	16
279	560	576	592	607	623	638	654	669	685	700	16
280	716	731	747	762	778	793	809	824	840	855	15
281	871	886	902	917	932	948	963	979	994	*010	15
282	45 025	040	056	071	086	102	117	133	148	163	15
283	179	194	209	225	240	255	271	286	301	317	15
284	332	347	362	378	393	408	423	439	454	469	15
285	484	500	515	530	545	561	576	591	606	621	15
286	637	652	667	682	697	712	728	743	758	773	15
287	788	803	818	834	849	864	879	894	909	924	15
288	939	954	969	984	*000	*015	*030	*045	*060	*075	15
289	46 090	105	120	135	150	165	180	195	210	225	15
290	240	255	270	285	300	315	330	345	359	374	15
291	389	404	419	434	449	464	479	494	509	523	15
292	538	553	568	583	598	613	627	642	657	672	15
293	687	702	716	731	746	761	776	790	805	820	15
294	835	850	864	879	894	909	923	938	953	967	15
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	15
296	47 129	144	159	173	188	202	217	232	246	261	15
297	276	290	305	319	334	349	363	378	392	407	15
298	422	436	451	465	480	494	509	524	538	553	15
299	567	582	596	611	625	640	654	669	683	698	15
300	712	727	741	756	770	784	799	813	828	842	14
301	857	871	885	900	914	929	943	958	972	986	14
302	48 001	015	029	044	058	073	087	101	116	130	14
303	144	159	173	187	202	216	230	244	259	273	14
304	287	302	316	330	344	359	373	387	401	416	14
305	430	444	458	473	487	501	515	530	544	558	14
306	572	586	601	615	629	643	657	671	686	700	14
307	714	728	742	756	770	785	799	813	827	841	14
308	855	869	883	897	911	926	940	954	968	982	14
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122	14
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310	49 136	150	164	178	192	206	220	234	248	262	14
311	276	290	304	318	332	346	360	374	388	402	14
312	415	429	443	457	471	485	499	513	527	541	14
313	554	568	582	596	610	624	638	651	665	679	14
314	693	707	721	734	748	762	776	790	803	817	14
315	831	845	859	872	886	900	914	927	941	955	14
316	969	982	996	*010	*024	*037	*051	*065	*079	*092	14
317	50 106	120	133	147	161	174	188	202	215	229	14
318	243	256	270	284	297	311	325	338	352	365	14
319	379	393	406	420	433	447	461	474	488	501	14
320	515	529	542	556	569	583	596	610	623	637	14
321	651	664	678	691	705	718	732	745	759	772	14
322	786	799	813	826	840	853	866	880	893	907	13
323	920	934	947	961	974	987	*001	*014	*028	*041	13
324	51 055	068	081	095	108	121	135	148	162	175	13
325	188	202	215	228	242	255	268	282	295	308	13
326	322	335	348	362	375	388	402	415	428	441	13
327	455	468	481	495	508	521	534	548	561	574	13
328	587	601	614	627	640	654	667	680	693	706	13
329	720	733	746	759	772	786	799	812	825	838	13
330	851	865	878	891	904	917	930	943	957	970	13
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101	13
332	52 114	127	140	153	166	179	192	205	218	231	13
333	244	257	270	284	297	310	323	336	349	362	13
334	375	388	401	414	427	440	453	466	479	492	13
335	504	517	530	543	556	569	582	595	608	621	13
336	634	647	660	673	686	699	711	724	737	750	13
337	763	776	789	802	815	827	840	853	866	879	13
338	892	905	917	930	943	956	969	982	994	*007	13
339	53 020	033	046	058	071	084	097	110	122	135	13
340	148	161	173	186	199	212	224	237	250	263	13
341	275	288	301	314	326	339	352	364	377	390	13
342	403	415	428	441	453	466	479	491	504	517	13
343	529	542	555	567	580	593	605	618	631	643	13
344	656	668	681	694	706	719	732	744	757	769	13
345	782	794	807	820	832	845	857	870	882	895	13
346	908	920	933	945	958	970	983	995	*008	*020	13
347	54 033	045	058	070	083	095	108	120	133	145	13
348	158	170	183	195	208	220	233	245	258	270	12
349	283	295	307	320	332	345	357	370	382	394	12
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350	407	419	432	444	456	469	481	494	506	518	12
351	531	543	555	568	580	593	605	617	630	642	12
352	654	667	679	691	704	716	728	741	753	765	12
353	777	790	802	814	827	839	851	864	876	888	12
354	900	913	925	937	949	962	974	986	998	*011	12
355	55 023	035	047	060	072	084	096	108	121	133	12
356	145	157	169	182	194	206	218	230	242	255	12
357	267	279	291	303	315	328	340	352	364	376	12
358	388	400	413	425	437	449	461	473	485	497	12
359	509	522	534	546	558	570	582	594	606	618	12
360	630	642	654	666	678	691	703	715	727	739	12
361	751	763	775	787	799	811	823	835	847	859	12
362	871	883	895	907	919	931	943	955	967	979	12
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098	12
364	56 110	122	134	146	158	170	182	194	205	217	12
365	229	241	253	265	277	289	301	312	324	336	12
366	348	360	372	384	396	407	419	431	443	455	12
367	467	478	490	502	514	526	538	549	561	573	12
368	585	597	608	620	632	644	656	667	679	691	12
369	703	714	726	738	750	761	773	785	797	808	12
370	820	832	844	855	867	879	891	902	914	926	12
371	937	949	961	972	984	996	*008	*019	*031	*043	12
372	57 054	066	078	089	101	113	124	136	148	159	12
373	171	183	194	206	217	229	241	252	264	276	12
374	287	299	310	322	334	345	357	368	380	392	12
375	403	415	426	438	449	461	473	484	496	507	12
376	519	530	542	553	565	576	588	600	611	623	12
377	634	646	657	669	680	692	703	715	726	738	11
378	749	761	772	784	795	807	818	830	841	852	11
379	864	875	887	898	910	921	933	944	955	967	11
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081	11
381	58 092	104	115	127	138	149	161	172	184	195	11
382	206	218	229	240	252	263	274	286	297	309	11
383	320	331	343	354	365	377	388	399	410	422	11
384	433	444	456	467	478	490	501	512	524	535	11
385	546	557	569	580	591	602	614	625	636	647	11
386	659	670	681	692	704	715	726	737	749	760	11
387	771	782	794	805	816	827	838	850	861	872	11
388	883	894	906	917	928	939	950	961	973	984	11
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095	11
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390	59 106	118	129	140	151	162	173	184	195	207	11
391	218	229	240	251	262	273	284	295	306	318	11
392	329	340	351	362	373	384	395	406	417	428	11
393	439	450	461	472	483	494	506	517	528	539	11
394	550	561	572	583	594	605	616	627	638	649	11
395	660	671	682	693	704	715	726	737	748	759	11
396	770	780	791	802	813	824	835	846	857	868	11
397	879	890	901	912	923	934	945	956	966	977	11
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086	11
399	60 097	108	119	130	141	152	163	173	184	195	11
400	206	217	228	239	249	260	271	282	293	304	11
401	314	325	336	347	358	369	379	390	401	412	11
402	423	433	444	455	466	477	487	498	509	520	11
403	531	541	552	563	574	584	595	606	617	627	11
404	638	649	660	670	681	692	703	713	724	735	11
405	746	756	767	778	788	799	810	821	831	842	11
406	853	863	874	885	895	906	917	927	938	949	11
407	959	970	981	991	*002	*013	*023	*034	*045	*055	11
408	61 066	077	087	098	109	119	130	140	151	162	11
409	172	183	194	204	215	225	236	247	257	268	11
410	278	289	300	310	321	331	342	352	363	374	11
411	384	395	405	416	426	437	448	458	469	479	11
412	490	500	511	521	532	542	553	563	574	584	11
413	595	606	616	627	637	648	658	669	679	690	11
414	700	711	721	731	742	752	763	773	784	794	10
415	805	815	826	836	847	857	868	878	888	899	10
416	909	920	930	941	951	962	972	982	993	*003	10
417	62 014	024	034	045	055	066	076	086	097	107	10
418	118	128	138	149	159	170	180	190	201	211	10
419	221	232	242	252	263	273	284	294	304	315	10
420	325	335	346	356	366	377	387	397	408	418	10
421	428	439	449	459	469	480	490	500	511	521	10
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423	634	644	655	665	675	685	696	706	716	726	10
424	737	747	757	767	778	788	798	808	818	829	10
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427	63 043	053	063	073	083	094	104	114	124	134	10
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707	942	948	954	960	967	973	979	985	991	997	6
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990	564	568	572	577	581	585	590	594	599	603	4
991	607	612	616	621	625	629	634	638	642	647	4
992	651	656	660	664	669	673	677	682	686	691	4
993	695	699	704	708	712	717	721	726	730	734	4
994	739	743	747	752	756	760	765	769	774	778	4
995	782	787	791	795	800	804	808	813	817	822	4
996	826	830	835	839	843	848	852	856	861	865	4
997	870	874	878	883	887	891	896	900	904	909	4
998	913	917	922	926	930	935	939	944	948	952	4
999	957	961	965	970	974	978	983	987	991	996	4
N	0	1	2	3	4	5	6	7	8	9	D

II.

FIVE-PLACE LOGARITHMS

OF THE

SINE, COSINE, TANGENT, AND COTANGENT

FOR

EACH MINUTE FROM 0° TO 90°.

0°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	∞	∞	∞	0.00 000	**60**
1	6.46 373	6.46 373	3.53 627	0.00 000	59
2	6.76 476	6.76 476	3.23 524	0.00 000	58
3	6.94 085	6.94 085	3.05 915	0.00 000	57
4	7.06 579	7.06 579	2.93 421	0.00 000	56
5	7.16 270	7.16 270	2.83 730	0.00 000	55
6	7.24 188	7.24 188	2.75 812	0.00 000	54
7	7.30 882	7.30 882	2.69 118	0.00 000	53
8	7.36 682	7.36 682	2.63 318	0.00 000	52
9	7.41 797	7.41 797	2.58 203	0.00 000	51
10	7.46 373	7.46 373	2.53 627	0.00 000	**50**
11	7.50 512	7.50 512	2.49 488	0.00 000	49
12	7.54 291	7.54 291	2.45 709	0.00 000	48
13	7.57 767	7.57 767	2.42 233	0.00 000	47
14	7.60 985	7.60 986	2.39 014	0.00 000	46
15	7.63 982	7.63 982	2.36 018	0.00 000	45
16	7.66 784	7.66 785	2.33 215	0.00 000	44
17	7.69 417	7.69 418	2.30 582	9.99 999	43
18	7.71 900	7.71 900	2.28 100	9.99 999	42
19	7.74 248	7.74 248	2.25 752	9.99 999	41
20	7.76 475	7.76 476	2.23 524	9.99 999	**40**
21	7.78 594	7.78 595	2.21 405	9.99 999	39
22	7.80 615	7.80 615	2.19 385	9.99 999	38
23	7.82 545	7.82 546	2.17 454	9.99 999	37
24	7.84 393	7.84 394	2.15 606	9.99 999	36
25	7.86 166	7.86 167	2.13 833	9.99 999	35
26	7.87 870	7.87 871	2.12 129	9.99 999	34
27	7.89 509	7.89 510	2.10 490	9.99 999	33
28	7.91 088	7.91 089	2.08 911	9.99 999	32
29	7.92 612	7.92 613	2.07 387	9.99 998	31
30	7.94 084	7.94 086	2.05 914	9.99 998	**30**
31	7.95 508	7.95 510	2.04 490	9.99 998	29
32	7.96 887	7.96 889	2.03 111	9.99 998	28
33	7.98 223	7.98 225	2.01 775	9.99 998	27
34	7.99 520	7.99 522	2.00 478	9.99 998	26
35	8.00 779	8.00 781	1.99 219	9.99 998	25
36	8.02 002	8.02 004	1.97 996	9.99 998	24
37	8.03 192	8.03 194	1.96 806	9.99 997	23
38	8.04 350	8.04 353	1.95 647	9.99 997	22
39	8.05 478	8.05 481	1.94 519	9.99 997	21
40	8.06 578	8.06 581	1.93 419	9.99 997	**20**
41	8.07 650	8.07 653	1.92 347	9.99 997	19
42	8.08 696	8.08 700	1.91 300	9.99 997	18
43	8.09 718	8.09 722	1.90 278	9.99 997	17
44	8.10 717	8.10 720	1.89 280	9.99 996	16
45	8.11 693	8.11 696	1.88 304	9.99 996	15
46	8.12 647	8.12 651	1.87 349	9.99 996	14
47	8.13 581	8.13 585	1.86 415	9.99 996	13
48	8.14 495	8.14 500	1.85 500	9.99 996	12
49	8.15 391	8.15 395	1.84 605	9.99 996	11
50	8.16 268	8.16 273	1.83 727	9.99 995	**10**
51	8.17 128	8.17 133	1.82 867	9.99 995	9
52	8.17 971	8.17 976	1.82 024	9.99 995	8
53	8.18 798	8.18 804	1.81 196	9.99 995	7
54	8.19 610	8.19 616	1.80 384	9.99 995	6
55	8.20 407	8.20 413	1.79 587	9.99 994	5
56	8.21 189	8.21 195	1.78 805	9.99 994	4
57	8.21 958	8.21 964	1.78 036	9.99 994	3
58	8.22 713	8.22 720	1.77 280	9.99 994	2
59	8.23 456	8.23 462	1.76 538	9.99 994	1
60	8.24 186	8.24 192	1.75 808	9.99 993	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

89°

1°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.24 186	8.24 192	1.75 808	9.99 993	**60**
1	8.24 903	8.24 910	1.75 090	9.99 993	59
2	8.25 609	8.25 616	1.74 384	9.99 993	58
3	8.26 304	8.26 312	1.73 688	9.99 993	57
4	8.26 988	8.26 996	1.73 004	9.99 992	56
5	8.27 661	8.27 669	1.72 331	9.99 992	55
6	8.28 324	8.28 332	1.71 668	9.99 992	54
7	8.28 977	8.28 986	1.71 014	9.99 992	53
8	8.29 621	8.29 629	1.70 371	9.99 992	52
9	8.30 255	8.30 263	1.69 737	9.99 991	51
10	8.30 879	8.30 888	1.69 112	9.99 991	**50**
11	8.31 495	8.31 505	1.68 495	9.99 991	49
12	8.32 103	8.32 112	1.67 888	9.99 990	48
13	8.32 702	8.32 711	1.67 289	9.99 990	47
14	8.33 292	8.33 302	1.66 698	9.99 990	46
15	8.33 875	8.33 886	1.66 114	9.99 990	45
16	8.34 450	8.34 461	1.65 539	9.99 989	44
17	8.35 018	8.35 029	1.64 971	9.99 989	43
18	8.35 578	8.35 590	1.64 410	9.99 989	42
19	8.36 131	8.36 143	1.63 857	9.99 989	41
20	8.36 678	8.36 689	1.63 311	9.99 988	**40**
21	8.37 217	8.37 229	1.62 771	9.99 988	39
22	8.37 750	8.37 762	1.62 238	9.99 988	38
23	8.38 276	8.38 289	1.61 711	9.99 987	37
24	8.38 796	8.38 809	1.61 191	9.99 987	36
25	8.39 310	8.39 323	1.60 677	9.99 987	35
26	8.39 818	8.39 832	1.60 168	9.99 986	34
27	8.40 320	8.40 334	1.59 666	9.99 986	33
28	8.40 816	8.40 830	1.59 170	9.99 986	32
29	8.41 307	8.41 321	1.58 679	9.99 985	31
30	8.41 792	8.41 807	1.58 193	9.99 985	**30**
31	8.42 272	8.42 287	1.57 713	9.99 985	29
32	8.42 746	8.42 762	1.57 238	9.99 984	28
33	8.43 216	8.43 232	1.56 768	9.99 984	27
34	8.43 680	8.43 696	1.56 304	9.99 984	26
35	8.44 139	8.44 156	1.55 844	9.99 983	25
36	8.44 594	8.44 611	1.55 389	9.99 983	24
37	8.45 044	8.45 061	1.54 939	9.99 983	23
38	8.45 489	8.45 507	1.54 493	9.99 982	22
39	8.45 930	8.45 948	1.54 052	9.99 982	21
40	8.46 366	8.46 385	1.53 615	9.99 982	**20**
41	8.46 799	8.46 817	1.53 183	9.99 981	19
42	8.47 226	8.47 245	1.52 755	9.99 981	18
43	8.47 650	8.47 669	1.52 331	9.99 981	17
44	8.48 069	8.48 089	1.51 911	9.99 980	16
45	8.48 485	8.48 505	1.51 495	9.99 980	15
46	8.48 896	8.48 917	1.51 083	9.99 979	14
47	8.49 304	8.49 325	1.50 675	9.99 979	13
48	8.49 708	8.49 729	1.50 271	9.99 979	12
49	8.50 108	8.50 130	1.49 870	9.99 978	11
50	8.50 504	8.50 527	1.49 473	9.99 978	**10**
51	8.50 897	8.50 920	1.49 080	9.99 977	9
52	8.51 287	8.51 310	1.48 690	9.99 977	8
53	8.51 673	8.51 696	1.48 304	9.99 977	7
54	8.52 055	8.52 079	1.47 921	9.99 976	6
55	8.52 434	8.52 459	1.47 541	9.99 976	5
56	8.52 810	8.52 835	1.47 165	9.99 975	4
57	8.53 183	8.53 208	1.46 792	9.99 975	3
58	8.53 552	8.53 578	1.46 422	9.99 974	2
59	8.53 919	8.53 945	1.46 055	9.99 974	1
60	8.54 282	8.54 308	1.45 692	9.99 974	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

88°

2°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.54 282	8.54 308	1.45 692	9.99 974	60
1	8.54 642	8.54 669	1.45 331	9.99 973	59
2	8.54 999	8.55 027	1.44 973	9.99 973	58
3	8.55 354	8.55 382	1.44 618	9.99 972	57
4	8.55 705	8.55 734	1.44 266	9.99 972	56
5	8.56 054	8.56 083	1.43 917	9.99 971	55
6	8.56 400	8.56 429	1.43 571	9.99 971	54
7	8.56 743	8.56 773	1.43 227	9.99 970	53
8	8.57 084	8.57 114	1.42 886	9.99 970	52
9	8.57 421	8.57 452	1.42 548	9.99 969	51
10	8.57 757	8.57 788	1.42 212	9.99 969	50
11	8.58 089	8.58 121	1.41 879	9.99 968	49
12	8.58 419	8.58 451	1.41 549	9.99 968	48
13	8.58 747	8.58 779	1.41 221	9.99 967	47
14	8.59 072	8.59 105	1.40 895	9.99 967	46
15	8.59 395	8.59 428	1.40 572	9.99 967	45
16	8.59 715	8.59 749	1.40 251	9.99 966	44
17	8.60 033	8.60 068	1.39 932	9.99 966	43
18	8.60 349	8.60 384	1.39 616	9.99 965	42
19	8.60 662	8.60 698	1.39 302	9.99 964	41
20	8.60 973	8.61 009	1.38 991	9.99 964	40
21	8.61 282	8.61 319	1.38 681	9.99 963	39
22	8.61 589	8.61 626	1.38 374	9.99 963	38
23	8.61 894	8.61 931	1.38 069	9.99 962	37
24	8.62 196	8.62 234	1.37 766	9.99 962	36
25	8.62 497	8.62 535	1.37 465	9.99 961	35
26	8.62 795	8.62 834	1.37 166	9.99 961	34
27	8.63 091	8.63 131	1.36 869	9.99 960	33
28	8.63 385	8.63 426	1.36 574	9.99 960	32
29	8.63 678	8.63 718	1.36 282	9.99 959	31
30	8.63 968	8.64 009	1.35 991	9.99 959	30
31	8.64 256	8.64 298	1.35 702	9.99 958	29
32	8.64 543	8.64 585	1.35 415	9.99 958	28
33	8.64 827	8.64 870	1.35 130	9.99 957	27
34	8.65 110	8.65 154	1.34 846	9.99 956	26
35	8.65 391	8.65 435	1.34 565	9.99 956	25
36	8.65 670	8.65 715	1.34 285	9.99 955	24
37	8.65 947	8.65 993	1.34 007	9.99 955	23
38	8.66 223	8.66 269	1.33 731	9.99 954	22
39	8.66 497	8.66 543	1.33 457	9.99 954	21
40	8.66 769	8.66 816	1.33 184	9.99 953	20
41	8.67 039	8.67 087	1.32 913	9.99 952	19
42	8.67 308	8.67 356	1.32 644	9.99 952	18
43	8.67 575	8.67 624	1.32 376	9.99 951	17
44	8.67 841	8.67 890	1.32 110	9.99 951	16
45	8.68 104	8.68 154	1.31 846	9.99 950	15
46	8.68 367	8.68 417	1.31 583	9.99 949	14
47	8.68 627	8.68 678	1.31 322	9.99 949	13
48	8.68 886	8.68 938	1.31 062	9.99 948	12
49	8.69 144	8.69 196	1.30 804	9.99 948	11
50	8.69 400	8.69 453	1.30 547	9.99 947	10
51	8.69 654	8.69 708	1.30 292	9.99 946	9
52	8.69 907	8.69 962	1.30 038	9.99 946	8
53	8.70 159	8.70 214	1.29 786	9.99 945	7
54	8.70 409	8.70 465	1.29 535	9.99 944	6
55	8.70 658	8.70 714	1.29 286	9.99 944	5
56	8.70 905	8.70 962	1.29 038	9.99 943	4
57	8.71 151	8.71 208	1.28 792	9.99 942	3
58	8.71 395	8.71 453	1.28 547	9.99 942	2
59	8.71 638	8.71 697	1.28 303	9.99 941	1
60	8.71 880	8.71 940	1.28 060	9.99 940	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

87°

3°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.71 880	8.71 940	1.28 060	9.99 940	**60**
1	8.72 120	8.72 181	1.27 819	9.99 940	59
2	8.72 359	8.72 420	1.27 580	9.99 939	58
3	8.72 597	8.72 659	1.27 341	9.99 938	57
4	8.72 834	8.72 896	1.27 104	9.99 938	56
5	8.73 069	8.73 132	1.26 868	9.99 937	55
6	8.73 303	8.73 366	1.26 634	9.99 936	54
7	8.73 535	8.73 600	1.26 400	9.99 936	53
8	8.73 767	8.73 832	1.26 168	9.99 935	52
9	8.73 997	8.74 063	1.25 937	9.99 934	51
10	8.74 226	8.74 292	1.25 708	9.99 934	**50**
11	8.74 454	8.74 521	1.25 479	9.99 933	49
12	8.74 680	8.74 748	1.25 252	9.99 932	48
13	8.74 906	8.74 974	1.25 026	9.99 932	47
14	8.75 130	8.75 199	1.24 801	9.99 931	46
15	8.75 353	8.75 423	1.24 577	9.99 930	45
16	8.75 575	8.75 645	1.24 355	9.99 929	44
17	8.75 795	8.75 867	1.24 133	9.99 929	43
18	8.76 015	8.76 087	1.23 913	9.99 928	42
19	8.76 234	8.76 306	1.23 694	9.99 927	41
20	8.76 451	8.76 525	1.23 475	9.99 926	**40**
21	8.76 667	8.76 742	1.23 258	9.99 926	39
22	8.76 883	8.76 958	1.23 042	9.99 925	38
23	8.77 097	8.77 173	1.22 827	9.99 924	37
24	8.77 310	8.77 387	1.22 613	9.99 923	36
25	8.77 522	8.77 600	1.22 400	9.99 923	35
26	8.77 733	8.77 811	1.22 189	9.99 922	34
27	8.77 943	8.78 022	1.21 978	9.99 921	33
28	8.78 152	8.78 232	1.21 768	9.99 920	32
29	8.78 360	8.78 441	1.21 559	9.99 920	31
30	8.78 568	8.78 649	1.21 351	9.99 919	**30**
31	8.78 774	8.78 855	1.21 145	9.99 918	29
32	8.78 979	8.79 061	1.20 939	9.99 917	28
33	8.79 183	8.79 266	1.20 734	9.99 917	27
34	8.79 386	8.79 470	1.20 530	9.99 916	26
35	8.79 588	8.79 673	1.20 327	9.99 915	25
36	8.79 789	8.79 875	1.20 125	9.99 914	24
37	8.79 990	8.80 076	1.19 924	9.99 913	23
38	8.80 189	8.80 277	1.19 723	9.99 913	22
39	8.80 388	8.80 476	1.19 524	9.99 912	21
40	8.80 585	8.80 674	1.19 326	9.99 911	**20**
41	8.80 782	8.80 872	1.19 128	9.99 910	19
42	8.80 978	8.81 068	1.18 932	9.99 909	18
43	8.81 173	8.81 264	1.18 736	9.99 909	17
44	8.81 367	8.81 459	1.18 541	9.99 908	16
45	8.81 560	8.81 653	1.18 347	9.99 907	15
46	8.81 752	8.81 846	1.18 154	9.99 906	14
47	8.81 944	8.82 038	1.17 962	9.99 905	13
48	8.82 134	8.82 230	1.17 770	9.99 904	12
49	8.82 324	8.82 420	1.17 580	9.99 904	11
50	8.82 513	8.82 610	1.17 390	9.99 903	**10**
51	8.82 701	8.82 799	1.17 201	9.99 902	9
52	8.82 888	8.82 987	1.17 013	9.99 901	8
53	8.83 075	8.83 175	1.16 825	9.99 900	7
54	8.83 261	8.83 361	1.16 639	9.99 899	6
55	8.83 446	8.83 547	1.16 453	9.99 898	5
56	8.83 630	8.83 732	1.16 268	9.99 898	4
57	8.83 813	8.83 916	1.16 084	9.99 897	3
58	8.83 996	8.84 100	1.15 900	9.99 896	2
59	8.84 177	8.84 282	1.15 718	9.99 895	1
60	8.84 358	8.84 464	1.15 536	9.99 894	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

86°

4°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.84 358	8.84 464	1.15 536	9.99 894	60
1	8.84 539	8.84 646	1.15 354	9.99 893	59
2	8.84 718	8.84 826	1.15 174	9.99 892	58
3	8.84 897	8.85 006	1.14 994	9.99 891	57
4	8.85 075	8.85 185	1.14 815	9.99 891	56
5	8.85 252	8.85 363	1.14 637	9.99 890	55
6	8.85 429	8.85 540	1.14 460	9.99 889	54
7	8.85 605	8.85 717	1.14 283	9.99 888	53
8	8.85 780	8.85 893	1.14 107	9.99 887	52
9	8.85 955	8.86 069	1.13 931	9.99 886	51
10	8.86 128	8.86 243	1.13 757	9.99 885	50
11	8.86 301	8.86 417	1.13 583	9.99 884	49
12	8.86 474	8.86 591	1.13 409	9.99 883	48
13	8.86 645	8.86 763	1.13 237	9.99 882	47
14	8.86 816	8.86 935	1.13 065	9.99 881	46
15	8.86 987	8.87 106	1.12 894	9.99 880	45
16	8.87 156	8.87 277	1.12 723	9.99 879	44
17	8.87 325	8.87 447	1.12 553	9.99 879	43
18	8.87 494	8.87 616	1.12 384	9.99 878	42
19	8.87 661	8.87 785	1.12 215	9.99 877	41
20	8.87 829	8.87 953	1.12 047	9.99 876	40
21	8.87 995	8.88 120	1.11 880	9.99 875	39
22	8.88 161	8.88 287	1.11 713	9.99 874	38
23	8.88 326	8.88 453	1.11 547	9.99 873	37
24	8.88 490	8.88 618	1.11 382	9.99 872	36
25	8.88 654	8.88 783	1.11 217	9.99 871	35
26	8.88 817	8.88 948	1.11 052	9.99 870	34
27	8.88 980	8.89 111	1.10 889	9.99 869	33
28	8.89 142	8.89 274	1.10 726	9.99 868	32
29	8.89 304	8.89 437	1.10 563	9.99 867	31
30	8.89 464	8.89 598	1.10 402	9.99 866	30
31	8.89 625	8.89 760	1.10 240	9.99 865	29
32	8.89 784	8.89 920	1.10 080	9.99 864	28
33	8.89 943	8.90 080	1.09 920	9.99 863	27
34	8.90 102	8.90 240	1.09 760	9.99 862	26
35	8.90 260	8.90 399	1.09 601	9.99 861	25
36	8.90 417	8.90 557	1.09 443	9.99 860	24
37	8.90 574	8.90 715	1.09 285	9.99 859	23
38	8.90 730	8.90 872	1.09 128	9.99 858	22
39	8.90 885	8.91 029	1.08 971	9.99 857	21
40	8.91 040	8.91 185	1.08 815	9.99 856	20
41	8.91 195	8.91 340	1.08 660	9.99 855	19
42	8.91 349	8.91 495	1.08 505	9.99 854	18
43	8.91 502	8.91 650	1.08 350	9.99 853	17
44	8.91 655	8.91 803	1.08 197	9.99 852	16
45	8.91 807	8.91 957	1.08 043	9.99 851	15
46	8.91 959	8.92 110	1.07 890	9.99 850	14
47	8.92 110	8.92 262	1.07 738	9.99 848	13
48	8.92 261	8.92 414	1.07 586	9.99 847	12
49	8.92 411	8.92 565	1.07 435	9.99 846	11
50	8.92 561	8.92 716	1.07 284	9.99 845	10
51	8.92 710	8.92 866	1.07 134	9.99 844	9
52	8.92 859	8.93 016	1.06 984	9.99 843	8
53	8.93 007	8.93 165	1.06 835	9.99 842	7
54	8.93 154	8.93 313	1.06 687	9.99 841	6
55	8.93 301	8.93 462	1.06 538	9.99 840	5
56	8.93 448	8.93 609	1.06 391	9.99 839	4
57	8.93 594	8.93 756	1.06 244	9.99 838	3
58	8.93 740	8.93 903	1.06 097	9.99 837	2
59	8.93 885	8.94 049	1.05 951	9.99 836	1
60	8.94 030	8.94 195	1.05 805	9.99 834	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

85°

5°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	8.94 030	8.94 195	1.05 80$\bar{5}$	9.99 834	**60**
1	8.94 174	8.94 340	1.05 660	9.99 833	59
2	8.94 317	8.94 485	1.05 51$\bar{5}$	9.99 832	58
3	8.94 461	8.94 630	1.05 370	9.99 831	57
4	8.94 603	8.94 773	1.05 227	9.99 830	56
5	8.94 746	8.94 917	1.05 083	9.99 829	55
6	8.94 887	8.95 060	1.04 940	9.99 828	54
7	8.95 029	8.95 202	1.04 798	9.99 827	53
8	8.95 170	8.95 344	1.04 656	9.99 825	52
9	8.95 310	8.95 486	1.04 514	9.99 824	51
10	8.95 4$\bar{5}$0	8.95 627	1.04 373	9.99 823	**50**
11	8.95 589	8.95 767	1.04 233	9.99 822	49
12	8.95 728	8.95 908	1.04 092	9.99 821	48
13	8.95 867	8.96 047	1.03 953	9.99 820	47
14	8.96 005	8.96 187	1.03 813	9.99 819	46
15	8.96 143	8.96 325	1.03 67$\bar{5}$	9.99 817	45
16	8.96 280	8.96 464	1.03 536	9.99 816	44
17	8.96 417	8.96 602	1.03 398	9.99 815	43
18	8.96 553	8.96 739	1.03 261	9.99 814	42
19	8.96 689	8.96 877	1.03 123	9.99 813	41
20	8.96 82$\bar{5}$	8.97 013	1.02 987	9.99 812	**40**
21	8.96 960	8.97 1$\bar{5}$0	1.02 850	9.99 810	39
22	8.97 09$\bar{5}$	8.97 285	1.02 71$\bar{5}$	9.99 809	38
23	8.97 229	8.97 421	1.02 579	9.99 808	37
24	8.97 363	8.97 556	1.02 444	9.99 807	36
25	8.97 496	8.97 691	1.02 309	9.99 806	35
26	8.97 629	8.97 82$\bar{5}$	1.02 175	9.99 804	34
27	8.97 762	8.97 959	1.02 041	9.99 803	33
28	8.97 894	8.98 092	1.01 908	9.99 802	32
29	8.98 026	8.98 225	1.01 77$\bar{5}$	9.99 801	31
30	8.98 157	8.98 358	1.01 642	9.99 800	**30**
31	8.98 288	8.98 490	1.01 510	9.99 798	29
32	8.98 419	8.98 622	1.01 378	9.99 797	28
33	8.98 549	8.98 753	1.01 247	9.99 796	27
34	8.98 679	8.98 884	1.01 116	9.99 79$\bar{5}$	26
35	8.98 808	8.99 01$\bar{5}$	1.00 985	9.99 793	25
36	8.98 937	8.99 145	1.00 85$\bar{5}$	9.99 792	24
37	8.99 066	8.99 275	1.00 72$\bar{5}$	9.99 791	23
38	8.99 194	8.99 40$\bar{5}$	1.00 595	9.99 790	22
39	8.99 322	8.99 534	1.00 466	9.99 788	21
40	8.99 4$\bar{5}$0	8.99 662	1.00 338	9.99 787	**20**
41	8.99 577	8.99 791	1.00 209	9.99 786	19
42	8.99 704	8.99 919	1.00 081	9.99 78$\bar{5}$	18
43	8.99 830	9.00 046	0.99 954	9.99 783	17
44	8.99 956	9.00 174	0.99 826	9.99 782	16
45	9.00 082	9.00 301	0.99 699	9.99 781	15
46	9.00 207	9.00 427	0.99 573	9.99 780	14
47	9.00 332	9.00 553	0.99 447	9.99 778	13
48	9.00 456	9.00 679	0.99 321	9.99 777	12
49	9.00 581	9.00 80$\bar{5}$	0.99 195	9.99 776	11
50	9.00 704	9.00 930	0.99 070	9.99 77$\bar{5}$	**10**
51	9.00 828	9.01 05$\bar{5}$	0.98 945	9.99 773	9
52	9.00 951	9.01 179	0.98 821	9.99 772	8
53	9.01 074	9.01 303	0.98 697	9.99 771	7
54	9.01 196	9.01 427	0.98 573	9.99 769	6
55	9.01 318	9.01 550	0.98 4$\bar{5}$0	9.99 768	5
56	9.01 440	9.01 673	0.98 327	9.99 767	4
57	9.01 561	9.01 796	0.98 204	9.99 765	3
58	9.01 682	9.01 918	0.98 082	9.99 764	2
59	9.01 803	9.02 040	0.97 960	9.99 763	1
60	9.01 923	9.02 162	0.97 838	9.99 761	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

84°

6°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.01 923	9.02 162	0.97 838	9.99 761	**60**
1	9.02 043	9.02 283	0.97 717	9.99 760	59
2	9.02 163	9.02 404	0.97 596	9.99 759	58
3	9.02 283	9.02 525	0.97 47$\bar{5}$	9.99 757	57
4	9.02 402	9.02 645	0.97 35$\bar{5}$	9.99 756	56
5	9.02 520	9.02 766	0.97 234	9.99 75$\bar{5}$	55
6	9.02 639	9.02 885	0.97 11$\bar{5}$	9.99 753	54
7	9.02 757	9.03 00$\bar{5}$	0.96 995	9.99 752	53
8	9.02 874	9.03 124	0.96 876	9.99 751	52
9	9.02 992	9.03 242	0.96 758	9.99 749	51
10	9.03 109	9.03 361	0.96 639	9.99 748	**50**
11	9.03 226	9.03 479	0.96 521	9.99 747	49
12	9.03 342	9.03 597	0.96 403	9.99 745	48
13	9.03 458	9.03 714	0.96 286	9.99 744	47
14	9.03 574	9.03 832	0.96 168	9.99 742	46
15	9.03 690	9.03 948	0.96 052	9.99 741	45
16	9.03 80$\bar{5}$	9.04 065	0.95 93$\bar{5}$	9.99 740	44
17	9.03 920	9.04 181	0.95 819	9.99 738	43
18	9.04 034	9.04 297	0.95 703	9.99 737	42
19	9.04 149	9.04 413	0.95 587	9.99 736	41
20	9.04 262	9.04 528	0.95 472	9.99 734	**40**
21	9.04 376	9.04 643	0.95 357	9.99 733	39
22	9.04 490	9.04 758	0.95 242	9.99 731	38
23	9.04 603	9.04 873	0.95 127	9.99 730	37
24	9.04 715	9.04 987	0.95 013	9.99 728	36
25	9.04 828	9.05 101	0.94 899	9.99 727	35
26	9.04 940	9.05 214	0.94 786	9.99 726	34
27	9.05 052	9.05 328	0.94 672	9.99 724	33
28	9.05 164	9.05 441	0.94 559	9.99 723	32
29	9.05 27$\bar{5}$	9.05 553	0.94 447	9.99 721	31
30	9.05 386	9.05 666	0.94 334	9.99 720	**30**
31	9.05 497	9.05 778	0.94 222	9.99 718	29
32	9.05 607	9.05 890	0.94 110	9.99 717	28
33	9.05 717	9.06 002	0.93 998	9.99 716	27
34	9.05 827	9.06 113	0.93 887	9.99 714	26
35	9.05 937	9.06 224	0.93 776	9.99 713	25
36	9.06 046	9.06 33$\bar{5}$	0.93 665	9.99 711	24
37	9.06 155	9.06 445	0.93 55$\bar{5}$	9.99 710	23
38	9.06 264	9.06 556	0.93 444	9.99 708	22
39	9.06 372	9.06 666	0.93 334	9.99 707	21
40	9.06 481	9.06 775	0.93 22$\bar{5}$	9.99 705	**20**
41	9.06 589	9.06 88$\bar{5}$	0.93 115	9.99 704	19
42	9.06 696	9.06 994	0.93 006	9.99 702	18
43	9.06 804	9.07 103	0.92 897	9.99 701	17
44	9.06 911	9.07 211	0.92 789	9.99 699	16
45	9.07 018	9.07 320	0.92 680	9.99 698	15
46	9.07 124	9.07 428	0.92 572	9.99 696	14
47	9.07 231	9.07 536	0.92 464	9.99 69$\bar{5}$	13
48	9.07 337	9.07 643	0.92 357	9.99 693	12
49	9.07 442	9.07 751	0.92 249	9.99 692	11
50	9.07 548	9.07 858	0.92 142	9.99 690	**10**
51	9.07 653	9.07 964	0.92 036	9.99 689	9
52	9.07 758	9.08 071	0.91 929	9.99 687	8
53	9.07 863	9.08 177	0.91 823	9.99 686	7
54	9.07 968	9.08 283	0.91 717	9.99 684	6
55	9.08 072	9.08 389	0.91 611	9.99 683	5
56	9.08 176	9.08 49$\bar{5}$	0.91 505	9.99 681	4
57	9.08 280	9.08 600	0.91 400	9.99 680	3
58	9.08 383	9.08 705	0.91 29$\bar{5}$	9.99 678	2
59	9.08 486	9.08 810	0.91 190	9.99 677	1
60	9.08 589	9.08 914	0.91 086	9.99 675	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

83°

7°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.08 589	9.08 914	0.91 086	9.99 675	60
1	9.08 692	9.09 019	0.90 981	9.99 674	59
2	9.08 795	9.09 123	0.90 877	9.99 672	58
3	9.08 897	9.09 227	0.90 773	9.99 670	57
4	9.08 999	9.09 330	0.90 670	9.99 669	56
5	9.09 101	9.09 434	0.90 566	9.99 667	55
6	9.09 202	9.09 537	0.90 463	9.99 666	54
7	9.09 304	9.09 640	0.90 360	9.99 664	53
8	9.09 405	9.09 742	0.90 258	9.99 663	52
9	9.09 506	9.09 845	0.90 155	9.99 661	51
10	9.09 606	9.09 947	0.90 053	9.99 659	50
11	9.09 707	9.10 049	0.89 951	9.99 658	49
12	9.09 807	9.10 150	0.89 850	9.99 656	48
13	9.09 907	9.10 252	0.89 748	9.99 655	47
14	9.10 006	9.10 353	0.89 647	9.99 653	46
15	9.10 106	9.10 454	0.89 546	9.99 651	45
16	9.10 205	9.10 555	0.89 445	9.99 650	44
17	9.10 304	9.10 656	0.98 344	9.99 648	43
18	9.10 402	9.10 756	0.89 244	9.99 647	42
19	9.10 501	9.10 856	0.89 144	9.99 645	41
20	9.10 599	9.10 956	0.89 044	9.99 643	40
21	9.10 697	9.11 056	0.88 944	9.99 642	39
22	9.10 795	9.11 155	0.88 845	9.99 640	38
23	9.10 893	9.11 254	0.88 746	9.99 638	37
24	9.10 990	9.11 353	0.88 647	9.99 637	36
25	9.11 087	9.11 452	0.88 548	9.99 635	35
26	9.11 184	9.11 551	0.88 449	9.99 633	34
27	9.11 281	9.11 649	0.88 351	9.99 632	33
28	9.11 377	9.11 747	0.88 253	9.99 630	32
29	9.11 474	9.11 845	0.88 155	9.99 629	31
30	9.11 570	9.11 943	0.88 057	9.99 627	30
31	9.11 666	9.12 040	0.87 960	9.99 625	29
32	9.11 761	9.12 138	0.87 862	9.99 624	28
33	9.11 857	9.12 235	0.87 765	9.99 622	27
34	9.11 952	9.12 332	0.87 668	9.99 620	26
35	9.12 047	9.12 428	0.87 572	9.99 618	25
36	9.12 142	9.12 525	0.87 475	9.99 617	24
37	9.12 236	9.12 621	0.87 379	9.99 615	23
38	9.12 331	9.12 717	0.87 283	9.99 613	22
39	9.12 425	9.12 813	0.87 187	9.99 612	21
40	9.12 519	9.12 909	0.87 091	9.99 610	20
41	9.12 612	9.13 004	0.86 996	9.99 608	19
42	9.12 706	9.13 099	0.86 901	9.99 607	18
43	9.12 799	9.13 194	0.86 806	9.99 605	17
44	9.12 892	9.13 289	0.86 711	9.99 603	16
45	9.12 985	9.13 384	0.86 616	9.99 601	15
46	9.13 078	9.13 478	0.86 522	9.99 600	14
47	9.13 171	9.13 573	0.86 427	9.99 598	13
48	9.13 263	9.13 667	0.86 333	9.99 596	12
49	9.13·355	9.13 761	0.86 239	9.99 595	11
50	9.13 447	9.13 854	0.86 146	9.99 593	10
51	9.13 539	9.13 948	0.86 052	9.99 591	9
52	9.13 630	9.14 041	0.85 959	9.99 589	8
53	9.13 722	9.14 134	0.85 866	9.99 588	7
54	9.13 813	9.14 227	0.85 773	9.99 586	6
55	9.13 904	9.14 320	0.85 680	9.99 584	5
56	9.13 994	9.14 412	0.85 588	9.99 582	4
57	9.14 085	9.14 504	0.85 496	9.99 581	3
58	9.14 175	9.14 597	0.85 403	9.99 579	2
59	9.14 266	9.14 688	0.85 312	9.99 577	1
60	9.14 356	9.14 780	0.85 220	9.99 575	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

82°

8°

'	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.14 356	9.14 780	0.85 220	9.99 575	**60**
1	9.14 445	9.14 872	0.85 128	9.99 574	59
2	9.14 535	9.14 963	0.85 037	9.99 572	58
3	9.14 624	9.15 054	0.84 946	9.99 570	57
4	9.14 714	9.15 145	0.84 855	9.99 568	56
5	9.14 803	9.15 236	0.84 764	9.99 566	55
6	9.14 891	9.15 327	0.84 673	9.99 565	54
7	9.14 980	9.15 417	0.84 583	9.99 563	53
8	9.15 069	9.15 508	0.84 492	9.99 561	52
9	9.15 157	9.15 598	0.84 402	9.99 559	51
10	9.15 245	9.15 688	0.84 312	9.99 557	**50**
11	9.15 333	9.15 777	0.84 223	9.99 556	49
12	9.15 421	9.15 867	0.84 133	9.99 554	48
13	9.15 508	9.15 956	0.84 044	9.99 552	47
14	9.15 596	9.16 046	0.83 954	9.99 550	46
15	9.15 683	9.16 135	0.83 865	9.99 548	45
16	9.15 770	9.16 224	0.83 776	9.99 546	44
17	9.15 857	9.16 312	0.83 688	9.99 545	43
18	9.15 944	9.16 401	0.83 599	9.99 543	42
19	9.16 030	9.16 489	0.83 511	9.99 541	41
20	9.16 116	9.16 577	0.83 423	9.99 539	**40**
21	9.16 203	9.16 665	0.83 335	9.99 537	39
22	9.16 289	9.16 753	0.83 247	9.99 535	38
23	9.16 374	9.16 841	0.83 159	9.99 533	37
24	9.16 460	9.16 928	0.83 072	9.99 532	36
25	9.16 545	9.17 016	0.82 984	9.99 530	35
26	9.16 631	9.17 103	0.82 897	9.99 528	34
27	9.16 716	9.17 190	0.82 810	9.99 526	33
28	9.16 801	9.17 277	0.82 723	9.99 524	32
29	9.16 886	9.17 363	0.82 637	9.99 522	31
30	9.16 970	9.17 450	0.82 550	9.99 520	**30**
31	9.17 055	9.17 536	0.82 464	9.99 518	29
32	9.17 139	9.17 622	0.82 378	9.99 517	28
33	9.17 223	9.17 708	0.82 292	9.99 515	27
34	9.17 307	9.17 794	0.82 206	9.99 513	26
35	9.17 391	9.17 880	0.82 120	9.99 511	25
36	9.17 474	9.17 965	0.82 035	9.99 509	24
37	9.17 558	9.18 051	0.81 949	9.99 507	23
38	9.17 641	9.18 136	0.81 864	9.99 505	22
39	9.17 724	9.18 221	0.81 779	9.99 503	21
40	9.17 807	9.18 306	0.81 694	9.99 501	**20**
41	9.17 890	9.18 391	0.81 609	9.99 499	19
42	9.17 973	9.18 475	0.81 525	9.99 497	18
43	9.18 055	9.18 560	0.81 440	9.99 495	17
44	9.18 137	9.18 644	0.81 356	9.99 494	16
45	9.18 220	9.18 728	0.81 272	9.99 492	15
46	9.18 302	9.18 812	0.81 188	9.99 490	14
47	9.18 383	9.18 896	0.81 104	9.99 488	13
48	9.18 465	9.18 979	0.81 021	9.99 486	12
49	9.18 547	9.19 063	0.80 937	9.99 484	11
50	9.18 628	9.19 146	0.80 854	9.99 482	**10**
51	9.18 709	9.19 229	0.80 771	9.99 480	9
52	9.18 790	9.19 312	0.80 688	9.99 478	8
53	9.18 871	9.19 395	0.80 605	9.99 476	7
54	9.18 952	9.19 478	0.80 522	9.99 474	6
55	9.19 033	9.19 561	0.80 439	9.99 472	5
56	9.19 113	9.19 643	0.80 357	9.99 470	4
57	9.19 193	9.19 725	0.80 275	9.99 468	3
58	9.19 273	9.19 807	0.80 193	9.99 466	2
59	9.19 353	9.19 889	0.80 111	9.99 464	1
60	9.19 433	9.19 971	0.80 029	9.99 462	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	'

81°

9°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.19 433	9.19 971	0.80 029	9.99 462	**60**
1	9.19 513	9.20 053	0.79 947	9.99 460	59
2	9.19 592	9.20 134	0.79 866	9.99 458	58
3	9.19 672	9.20 216	0.79 784	9.99 456	57
4	9.19 751	9.20 297	0.79 703	9.99 454	56
5	9.19 830	9.20 378	0.79 622	9.99 452	55
6	9.19 909	9.20 459	0.79 541	9.99 450	54
7	9.19 988	9.20 540	0.79 460	9.99 448	53
8	9.20 067	9.20 621	0.79 379	9.99 446	52
9	9.20 145	9.20 701	0.79 299	9.99 444	51
10	9.20 223	9.20 782	0.79 218	9.99 442	**50**
11	9.20 302	9.20 862	0.79 138	9.99 440	49
12	9.20 380	9.20 942	0.79 058	9.99 438	48
13	9.20 458	9.21 022	0.78 978	9.99 436	47
14	9.20 535	9.21 102	0.78 898	9.99 434	46
15	9.20 613	9.21 182	0.78 818	9.99 432	45
16	9.20 691	9.21 261	0.78 739	9.99 429	44
17	9.20 768	9.21 341	0.78 659	9.99 427	43
18	9.20 845	9.21 420	0.78 580	9.99 425	42
19	9.20 922	9.21 499	0.78 501	9.99 423	41
20	9.20 999	9.21 578	0.78 422	9.99 421	**40**
21	9.21 076	9.21 657	0.78 343	9.99 419	39
22	9.21 153	9.21 736	0.78 264	9.99 417	38
23	9.21 229	9.21 814	0.78 186	9.99 415	37
24	9.21 306	9.21 893	0.78 107	9.99 413	36
25	9.21 382	9.21 971	0.78 029	9.99 411	35
26	9.21 458	9.22 049	0.77 951	9.99 409	34
27	9.21 534	9.22 127	0.77 873	9.99 407	33
28	9.21 610	9.22 205	0.77 795	9.99 404	32
29	9.21 685	9.22 283	0.77 717	9.99 402	31
30	9.21 761	9.22 361	0.77 639	9.99 400	**30**
31	9.21 836	9.22 438	0.77 562	9.99 398	29
32	9.21 912	9.22 516	0.77 484	9.99 396	28
33	9.21 987	9.22 593	0.77 407	9.99 394	27
34	9.22 062	9.22 670	0.77 330	9.99 392	26
35	9.22 137	9.22 747	0.77 253	9.99 390	25
36	9.22 211	9.22 824	0.77 176	9.99 388	24
37	9.22 286	9.22 901	0.77 099	9.99 385	23
38	9.22 361	9.22 977	0.77 023	9.99 383	22
39	9.22 435	9.23 054	0.76 946	9.99 381	21
40	9.22 509	9.23 130	0.76 870	9.99 379	**20**
41	9.22 583	9.23 206	0.76 794	9.99 377	19
42	9.22 657	9.23 283	0.76 717	9.99 375	18
43	9.22 731	9.23 359	0.76 641	9.99 372	17
44	9.22 805	9.23 435	0.76 565	9.99 370	16
45	9.22 878	9.23 510	0.76 490	9.99 368	15
46	9.22 952	9.23 586	0.76 414	9.99 366	14
47	9.23 025	9.23 661	0.76 339	9.99 364	13
48	9.23 098	9.23 737	0.76 263	9.99 362	12
49	9.23 171	9.23 812	0.76 188	9.99 359	11
50	9.23 244	9.23 887	0.76 113	9.99 357	**10**
51	9.23 317	9.23 962	0.76 038	9.99 355	9
52	9.23 390	9.24 037	0.75 963	9.99 353	8
53	9.23 462	9.24 112	0.75 888	9.99 351	7
54	9.23 535	9.24 186	0.75 814	9.99 348	6
55	9.23 607	9.24 261	0.75 739	9.99 346	5
56	9.23 679	9.24 335	0.75 665	9.99 344	4
57	9.23 752	9.24 410	0.75 590	9.99 342	3
58	9.23 823	9.24 484	0.75 516	9.99 340	2
59	9.23 895	9.24 558	0.75 442	9.99 337	1
60	9.23 967	9.24 632	0.75 368	9.99 335	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

80°

11°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.28 060	9.28 865	0.71 135	9.99 195	**60**
1	9.28 125	9.28 933	0.71 067	9.99 192	59
2	9.28 190	9.29 000	0.71 000	9.99 190	58
3	9.28 254	9.29 067	0.70 933	9.99 187	57
4	9.28 319	9.29 134	0.70 866	9.99 185	56
5	9.28 384	9.29 201	0.70 799	9.99 182	55
6	9.28 448	9.29 268	0.70 732	9.99 180	54
7	9.28 512	9.29 335	0.70 665	9.99 177	53
8	9.28 577	9.29 402	0.70 598	9.99 175	52
9	9.28 641	9.29 468	0.70 532	9.99 172	51
10	9.28 705	9.29 535	0.70 465	9.99 170	**50**
11	9.28 769	9.29 601	0.70 399	9.99 167	49
12	9.28 833	9.29 668	0.70 332	9.99 165	48
13	9.28 896	9.29 734	0.70 266	9.99 162	47
14	9.28 960	9.29 800	0.70 200	9.99 160	46
15	9.29 024	9.29 866	0.70 134	9.99 157	45
16	9.29 087	9.29 932	0.70 068	9.99 155	44
17	9.29 150	9.29 998	0.70 002	9.99 152	43
18	9.29 214	9.30 064	0.69 936	9.99 150	42
19	9.29 277	9.30 130	0.69 870	9.99 147	41
20	9.29 340	9.30 195	0.69 805	9.99 145	**40**
21	9.29 403	9.30 261	0.69 739	9.99 142	39
22	9.29 466	9.30 326	0.69 674	9.99 140	38
23	9.29 529	9.30 391	0.69 609	9.99 137	37
24	9.29 591	9.30 457	0.69 543	9.99 135	36
25	9.29 654	9.30 522	0.69 478	9.99 132	35
26	9.29 716	9.30 587	0.69 413	9.99 130	34
27	9.29 779	9.30 652	0.69 348	9.99 127	33
28	9.29 841	9.30 717	0.69 283	9.99 124	32
29	9.29 903	9.30 782	0.69 218	9.99 122	31
30	9.29 966	9.30 846	0.69 154	9.99 119	**30**
31	9.30 028	9.30 911	0.69 089	9.99 117	29
32	9.30 090	9.30 975	0.69 025	9.99 114	28
33	9.30 151	9.31 040	0.68 960	9.99 112	27
34	9.30 213	9.31 104	0.68 896	9.99 109	26
35	9.30 275	9.31 168	0.68 832	9.99 106	25
36	9.30 336	9.31 233	0.68 767	9.99 104	24
37	9.30 398	9.31 297	0.68 703	9.99 101	23
38	9.30 459	9.31 361	0.68 639	9.99 099	22
39	9.30 521	9.31 425	0.68 575	9.99 096	21
40	9.30 582	9.31 489	0.68 511	9.99 093	**20**
41	9.30 643	9.31 552	0.68 448	9.99 091	19
42	9.30 704	9.31 616	0.68 384	9.99 088	18
43	9.30 765	9.31 679	0.68 321	9.99 086	17
44	9.30 826	9.31 743	0.68 257	9.99 083	16
45	9.30 887	9.31 806	0.68 194	9.99 080	15
46	9.30 947	9.31 870	0.68 130	9.99 078	14
47	9.31 008	9.31 933	0.68 067	9.99 075	13
48	9.31 068	9.31 996	0.68 004	9.99 072	12
49	9.31 129	9.32 059	0.67 941	9.99 070	11
50	9.31 189	9.32 122	0.67 878	9.99 067	**10**
51	9.31 250	9.32 185	0.67 815	9.99 064	9
52	9.31 310	9.32 248	0.67 752	9.99 062	8
53	9.31 370	9.32 311	0.67 689	9.99 059	7
54	9.31 430	9.32 373	0.67 627	9.99 056	6
55	9.31 490	9.32 436	0.67 564	9.99 054	5
56	9.31 549	9.32 498	0.67 502	9.99 051	4
57	9.31 609	9.32 561	0.67 439	9.99 048	3
58	9.31 669	9.32 623	0.67 377	9.99 046	2
59	9.31 728	9.32 685	0.67 315	9.99 043	1
60	9.31 788	9.32 747	0.67 253	9.99 040	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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12°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.31 788	9.32 747	0.67 253	9.99 040	**60**
1	9.31 847	9.32 810	0.67 190	9.99 038	59
2	9.31 907	9.32 872	0.67 128	9.99 035	58
3	9.31 966	9.32 933	0.67 067	9.99 032	57
4	9.32 025	9.32 995	0.67 005	9.99 030	56
5	9.32 084	9.33 057	0.66 943	9.99 027	55
6	9.32 143	9.33 119	0.66 881	9.99 024	54
7	9.32 202	9.33 180	0.66 820	9.99 022	53
8	9.32 261	9.33 242	0.66 758	9.99 019	52
9	9.32 319	9.33 303	0.66 697	9.99 016	51
10	9.32 378	9.33 365	0.66 635	9.99 013	**50**
11	9.32 437	9.33 426	0.66 574	9.99 011	49
12	9.32 495	9.33 487	0.66 513	9.99 008	48
13	9.32 553	9.33 548	0.66 452	9.99 005	47
14	9.32 612	9.33 609	0.66 391	9.99 002	46
15	9.32 670	9.33 670	0.66 330	9.99 000	45
16	9.32 728	9.33 731	0.66 269	9.98 997	44
17	9.32 786	9.33 792	0.66 208	9.98 994	43
18	9.32 844	9.33 853	0.66 147	9.98 991	42
19	9.32 902	9.33 913	0.66 087	9.98 989	41
20	9.32 960	9.33 974	0.66 026	9.98 986	**40**
21	9.33 018	9.34 034	0.65 966	9.98 983	39
22	9.33 075	9.34 095	0.65 905	9.98 980	38
23	9.33 133	9.34 155	0.65 845	9.98 978	37
24	9.33 190	9.34 215	0.65 785	9.98 975	36
25	9.33 248	9.34 276	0.65 724	9.98 972	35
26	9.33 305	9.34 336	0.65 664	9.98 969	34
27	9.33 362	9.34 396	0.65 604	9.98 967	33
28	9.33 420	9.34 456	0.65 544	9.98 964	32
29	9.33 477	9.34 516	0.65 484	9.98 961	31
30	9.33 534	9.34 576	0.65 424	9.98 958	**30**
31	9.33 591	9.34 635	0.65 365	9.98 955	29
32	9.33 647	9.34 695	0.65 305	9.98 953	28
33	9.33 704	9.34 755	0.65 245	9.98 950	27
34	9.33 761	9.34 814	0.65 186	9.98 947	26
35	9.33 818	9.34 874	0.65 126	9.98 944	25
36	9.33 874	9.34 933	0.65 067	9.98 941	24
37	9.33 931	9.34 992	0.65 008	9.98 938	23
38	9.33 987	9.35 051	0.64 949	9.98 936	22
39	9.34 043	9.35 111	0.64 889	9.98 933	21
40	9.34 100	9.35 170	0.64 830	9.98 930	**20**
41	9.34 156	9.35 229	0.64 771	9.98 927	19
42	9.34 212	9.35 288	0.64 712	9.98 924	18
43	9.34 268	9.35 347	0.64 653	9.98 921	17
44	9.34 324	9.35 405	0.64 595	9.98 919	16
45	9.34 380	9.35 464	0.64 536	9.98 916	15
46	9.34 436	9.35 523	0.64 477	9.98 913	14
47	9.34 491	9.35 581	0.64 419	9.98 910	13
48	9.34 547	9.35 640	0.64 360	9.98 907	12
49	9.34 602	9.35 698	0.64 302	9.98 904	11
50	9.34 658	9.35 757	0.64 243	9.98 901	**10**
51	9.34 713	9.35 815	0.64 185	9.98 898	9
52	9.34 769	9.35 873	0.64 127	9.98 896	8
53	9.34 824	9.35 931	0.64 069	9.98 893	7
54	9.34 879	9.35 989	0.64 011	9.98 890	6
55	9.34 934	9.36 047	0.63 953	9.98 887	5
56	9.34 989	9.36 105	0.63 895	9.98 884	4
57	9.35 044	9.36 163	0.63 837	9.98 881	3
58	9.35 099	9.36 221	0.63 779	9.98 878	2
59	9.35 154	9.36 279	0.63 721	9.98 875	1
60	9.35 209	9.36 336	0.63 664	9.98 872	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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13°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.35 209	9.36 336	0.63 664	9.98 872	**60**
1	9.35 263	9.36 394	0.63 606	9.98 869	59
2	9.35 318	9.36 452	0.63 548	9.98 867	58
3	9.35 373	9.36 509	0.63 491	9.98 864	57
4	9.35 427	9.36 566	0.63 434	9.98 861	56
5	9.35 481	9.36 624	0.63 376	9.98 858	55
6	9.35 536	9.36 681	0.63 319	9.98 855	54
7	9.35 590	9.36 738	0.63 262	9.98 852	53
8	9.35 644	9.36 795	0.63 205	9.98 849	52
9	9.35 698	9.36 852	0.63 148	9.98 846	51
10	9.35 752	9.36 909	0.63 091	9.98 843	**50**
11	9.35 806	9.36 966	0.63 034	9.98 840	49
12	9.35 860	9.37 023	0.62 977	9.98 837	48
13	9.35 914	9.37 080	0.62 920	9.98 834	47
14	9.35 968	9.37 137	0.62 863	9.98 831	46
15	9.36 022	9.37 193	0.62 807	9.98 828	45
16	9.36 075	9.37 250	0.62 750	9.98 825	44
17	9.36 129	9.37 306	0.62 694	9.98 822	43
18	9.36 182	9.37 363	0.62 637	9.98 819	42
19	9.36 236	9.37 419	0.62 581	9.98 816	41
20	9.36 289	9.37 476	0.62 524	9.98 813	**40**
21	9.36 342	9.37 532	0.62 468	9.98 810	39
22	9.36 395	9.37 588	0.62 412	9.98 807	38
23	9.36 449	9.37 644	0.62 356	9.98 804	37
24	9.36 502	9.37 700	0.62 300	9.98 801	36
25	9.36 555	9.37 756	0.62 244	9.98 798	35
26	9.36 608	9.37 812	0.62 188	9.98 795	34
27	9.36 660	9.37 868	0.62 132	9.98 792	33
28	9.36 713	9.37 924	0.62 076	9.98 789	32
29	9.36 766	9.37 980	0.62 020	9.98 786	31
30	9.36 819	9.38 035	0.61 965	9.98 783	**30**
31	9.36 871	9.38 091	0.61 909	9.98 780	29
32	9.36 924	9.38 147	0.61 853	9.98 777	28
33	9.36 976	9.38 202	0.61 798	9.98 774	27
34	9.37 028	9.38 257	0.61 743	9.98 771	26
35	9.37 081	9.38 313	0.61 687	9.98 768	25
36	9.37 133	9.38 368	0.61 632	9.98 765	24
37	9.37 185	9.38 423	0.61 577	9.98 762	23
38	9.37 237	9.38 479	0.61 521	9.98 759	22
39	9.37 289	9.38 534	0.61 466	9.98 756	21
40	9.37 341	9.38 589	0.61 411	9.98 753	**20**
41	9.37 393	9.38 644	0.61 356	9.98 750	19
42	9.37 445	9.38 699	0.61 301	9.98 746	18
43	9.37 497	9.38 754	0.61 246	9.98 743	17
44	9.37 549	9.38 808	0.61 192	9.98 740	16
45	9.37 600	9.38 863	0.61 137	9.98 737	15
46	9.37 652	9.38 918	0.61 082	9.98 734	14
47	9.37 703	9.38 972	0.61 028	9.98 731	13
48	9.37 755	9.39 027	0.60 973	9.98 728	12
49	9.37 806	9.39 082	0.60 918	9.98 725	11
50	9.37 858	9.39 136	0.60 864	9.98 722	**10**
51	9.37 909	9.39 190	0.60 810	9.98 719	9
52	9.37 960	9.39 245	0.60 755	9.98 715	8
53	9.38 011	9.39 299	0.60 701	9.98 712	7
54	9.38 062	9.39 353	0.60 647	9.98 709	6
55	9.38 113	9.39 407	0.60 593	9.98 706	5
56	9.38 164	9.39 461	0.60 539	9.98 703	4
57	9.38 215	9.39 515	0.60 485	9.98 700	3
58	9.38 266	9.39 569	0.60 431	9.98 697	2
59	9.38 317	9.39 623	0.60 377	9.98 694	1
60	9.38 368	9.39 677	0.60 323	9.98 690	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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14°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.38 368	9.39 677	0.60 323	9.98 690	60
1	9.38 418	9.39 731	0.60 269	9.98 687	59
2	9.38 469	9.39 785	0.60 215	9.98 684	58
3	9.38 519	9.39 838	0.60 162	9.98 681	57
4	9.38 570	9.39 892	0.60 108	9.98 678	56
5	9.38 620	9.39 945	0.60 055	9.98 675	55
6	9.38 670	9.39 999	0.60 001	9.98 671	54
7	9.38 721	9.40 052	0.59 948	9.98 668	53
8	9.38 771	9.40 106	0.59 894	9.98 665	52
9	9.38 821	9.40 159	0.59 841	9.98 662	51
10	9.38 871	9.40 212	0.59 788	9.98 659	50
11	9.38 921	9.40 266	0.59 734	9.98 656	49
12	9.38 971	9.40 319	0.59 681	9.98 652	48
13	9.39 021	9.40 372	0.59 628	9.98 649	47
14	9.39 071	9.40 425	0.59 575	9.98 646	46
15	9.39 121	9.40 478	0.59 522	9.98 643	45
16	9.39 170	9.40 531	0.59 469	9.98 640	44
17	9.39 220	9.40 584	0.59 416	9.98 636	43
18	9.39 270	9.40 636	0.59 364	9.98 633	42
19	9.39 319	9.40 689	0.59 311	9.98 630	41
20	9.39 369	9.40 742	0.59 258	9.98 627	40
21	9.39 418	9.40 795	0.59 205	9.98 623	39
22	9.39 467	9.40 847	0.59 153	9.98 620	38
23	9.39 517	9.40 900	0.59 100	9.98 617	37
24	9.39 566	9.40 952	0.59 048	9.98 614	36
25	9.39 615	9.41 005	0.58 995	9.98 610	35
26	9.39 664	9.41 057	0.58 943	9.98 607	34
27	9.39 713	9.41 109	0.58 891	9.98 604	33
28	9.39 762	9.41 161	0.58 839	9.98 601	32
29	9.39 811	9.41 214	0.58 786	9.98 597	31
30	9.39 860	9.41 266	0.58 734	9.98 594	30
31	9.39 909	9.41 318	0.58 682	9.98 591	29
32	9.39 958	9.41 370	0.58 630	9.98 588	28
33	9.40 006	9.41 422	0.58 578	9.98 584	27
34	9.40 055	9.41 474	0.58 526	9.98 581	26
35	9.40 103	9.41 526	0.58 474	9.98 578	25
36	9.40 152	9.41 578	0.58 422	9.98 574	24
37	9.40 200	9.41 629	0.58 371	9.98 571	23
38	9.40 249	9.41 681	0.58 319	9.98 568	22
39	9.40 297	9.41 733	0.58 267	9.98 565	21
40	9.40 346	9.41 784	0.58 216	9.98 561	20
41	9.40 394	9.41 836	0.58 164	9.98 558	19
42	9.40 442	9.41 887	0.58 113	9.98 555	18
43	9.40 490	9.41 939	0.58 061	9.98 551	17
44	9.40 538	9.41 990	0.58 010	9.98 548	16
45	9.40 586	9.42 041	0.57 959	9.98 545	15
46	9.40 634	9.42 093	0.57 907	9.98 541	14
47	9.40 682	9.42 144	0.57 856	9.98 538	13
48	9.40 730	9.42 195	0.57 805	9.98 535	12
49	9.40 778	9.42 246	0.57 754	9.98 531	11
50	9.40 825	9.42 297	0.57 703	9.98 528	10
51	9.40 873	9.42 348	0.57 652	9.98 525	9
52	9.40 921	9.42 399	0.57 601	9.98 521	8
53	9.40 968	9.42 450	0.57 550	9.98 518	7
54	9.41 016	9.42 501	0.57 499	9.98 515	6
55	9.41 063	9.42 552	0.57 448	9.98 511	5
56	9.41 111	9.42 603	0.57 397	9.98 508	4
57	9.41 158	9.42 653	0.57 347	9.98 505	3
58	9.41 205	9.42 704	0.57 296	9.98 501	2
59	9.41 252	9.42 755	0.57 245	9.98 498	1
60	9.41 300	9.42 805	0.57 195	9.98 494	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

75°

15°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.41 300	9.42 805	0.57 195	9.98 494	**60**
1	9.41 347	9.42 856	0.57 144	9.98 491	59
2	9.41 394	9.42 906	0.57 094	9.98 488	58
3	9.41 441	9.42 957	0.57 043	9.98 484	57
4	9.41 488	9.43 007	0.56 993	9.98 481	56
5	9.41 535	9.43 057	0.56 943	9.98 477	55
6	9.41 582	9.43 108	0.56 892	9.98 474	54
7	9.41 628	9.43 158	0.56 842	9.98 471	53
8	9.41 675	9.43 208	0.56 792	9.98 467	52
9	9.41 722	9.43 258	0.56 742	9.98 464	51
10	9.41 768	9.43 308	0.56 692	9.98 460	**50**
11	9.41 815	9.43 358	0.56 642	9.98 457	49
12	9.41 861	9.43 408	0.56 592	9.98 453	48
13	9.41 908	9.43 458	0.56 542	9.98 450	47
14	9.41 954	9.43 508	0.56 492	9.98 447	46
15	9.42 001	9.43 558	0.56 442	9.98 443	45
16	9.42 047	9.43 607	0.56 393	9.98 440	44
17	9.42 093	9.43 657	0.56 343	9.98 436	43
18	9.42 140	9.43 707	0.56 293	9.98 433	42
19	9.42 186	9.43 756	0.56 244	9.98 429	41
20	9.42 232	9.43 806	0.46 194	9.98 426	**40**
21	9.42 278	9.43 855	0.56 145	9.98 422	39
22	9.42 324	9.43 905	0.56 095	9.98 419	38
23	9.42 370	9.43 954	0.56 046	9.98 415	37
24	9.42 416	9.44 004	0.55 996	9.98 412	36
25	9.42 461	9.44 053	0.55 947	9.98 409	35
26	9.42 507	9.44 102	0.55 898	9.98 405	34
27	9.42 553	9.44 151	0.55 849	9.98 402	33
28	9.42 599	9.44 201	0.55 799	9.98 398	32
29	9.42 644	9.44 250	0.55 750	9.98 395	31
30	9.42 690	9.44 299	0.55 701	9.98 391	**30**
31	9.42 735	9.44 348	0.55 652	9.98 388	29
32	9.42 781	9.44 397	0.55 603	9.98 384	28
33	9.42 826	9.44 446	0.55 554	9.98 381	27
34	9.42 872	9.44 495	0.55 505	9.98 377	26
35	9.42 917	9.44 544	0.55 456	9.98 373	25
36	9.42 962	9.44 592	0.55 408	9.98 370	24
37	9.43 008	9.44 641	0.55 359	9.98 366	23
38	9.43 053	9.44 690	0.55 310	9.98 363	22
39	9.43 098	9.44 738	0.55 262	9.98 359	21
40	9.43 143	9.44 787	0.55 213	9.98 356	**20**
41	9.43 188	9.44 836	0.55 164	9.98 352	19
42	9.43 233	9.44 884	0.55 116	9.98 349	18
43	9.43 278	9.44 933	0.55 067	9.98 345	17
44	9.43 323	9.44 981	0.55 019	9.98 342	16
45	9.43 367	9.45 029	0.54 971	9.98 338	15
46	9.43 412	9.45 078	0.54 922	9.98 334	14
47	9.43 457	9.45 126	0.54 874	9.98 331	13
48	9.43 502	9.45 174	0.54 826	9.98 327	12
49	9.43 546	9.45 222	0.54 778	9.98 324	11
50	9.43 591	9.45 271	0.54 729	9.98 320	**10**
51	9.43 635	9.45 319	0.54 681	9.98 317	9
52	9.43 680	9.45 367	0.54 633	9.98 313	8
53	9.43 724	9.45 415	0.54 585	9.98 309	7
54	9.43 769	9.45 463	0.54 537	9.98 306	6
55	9.43 813	9.45 511	0.54 489	9.98 302	5
56	9.43 857	9.45 559	0.54 441	9.98 299	4
57	9.43 901	9.45 606	0.54 394	9.98 295	3
58	9.43 946	9.45 654	0.54 346	9.98 291	2
59	9.43 990	9.45 702	0.54 298	9.98 288	1
60	9.44 034	9.45 750	0.54 250	9.98 284	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

74°

16°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.44 034	9.45 75̄0	0.54 250	9.98 284	**60**
1	9.44 078	9.45 797	0.54 203	9.98 281	59
2	9.44 122	9.45 845̄	0.54 155	9.98 277	58
3	9.44 166	9.45 892	0.54 108	9.98 273	57
4	9.44 210	9.45 940	0.54 060	9.98 270	56
5	9.44 253	9.45 987	0.54 013	9.98 266	55
6	9.44 297	9.46 035̄	0.53 965	9.98 262	54
7	9.44 341	9.46 082	0.53 918	9.98 259	53
8	9.44 385̄	9.46 130	0.53 870	9.98 255	52
9	9.44 428	9.46 177	0.53 823	9.98 251	51
10	9.44 472	9.46 224	0.53 776	9.98 248	**50**
11	9.44 516	9.46 271	0.53 729	9.98 244	49
12	9.44 559	9.46 319	0.53 681	9.98 240	48
13	9.44 602	9.46 366	0.53 634	9.98 237	47
14	9.44 646	9.46 413	0.53 587	9.98 233	46
15	9.44 689	9.46 460	0.53 540	9.98 229	45
16	9.44 733	9.46 507	0.53 493	9.98 226	44
17	9.44 776	9.46 554	0.53 446	9.98 222	43
18	9.44 819	9.46 601	0.53 399	9.98 218	42
19	9.44 862	9.46 648	0.53 352	9.98 215̄	41
20	9.44 905	9.46 694	0.53 306	9.98 211	**40**
21	9.44 948	9.46 741	0.53 259	9.98 207	39
22	9.44 992	9.46 788	0.53 212	9.98 204	38
23	9.45 035̄	9.46 835̄	0.53 165	9.98 200	37
24	9.45 077	9.46 881	0.53 119	9.98 196	36
25	9.45 120	9.46 928	0.53 072	9.98 192	35
26	9.45 163	9.46 975̄	0.53 025	9.98 189	34
27	9.45 206	9.47 021	0.52 979	9.98 185̄	33
28	9.45 249	9.47 068	0.52 932	9.98 181	32
29	9.45 292	9.47 114	0.52 886	9.98 177	31
30	9.45 334	9.47 160	0.52 840	9.98 174	**30**
31	9.45 377	9.47 207	0.52 793	9.98 170	29
32	9.45 419	9.47 253	0.52 747	9.98 166	28
33	9.45 462	9.47 299	0.52 701	9.98 162	27
34	9.45 504	9.47 346	0.52 654	9.98 159	26
35	9.45 547	9.47 392	0.52 608	9.98 155̄	25
36	9.45 589	9.47 438	0.52 562	9.98 151	24
37	9.45 632	9.47 484	0.52 516	9.98 147	23
38	9.45 674	9.47 530	0.52 470	9.98 144	22
39	9.45 716	9.47 576	0.52 424	9.98 140	21
40	9.45 758	9.47 622	0.52 378	9.98 136	**20**
41	9.45 801	9.47 668	0.52 332	9.98 132	19
42	9.45 843	9.47 714	0.52 286	9.98 129	18
43	9.45 885̄	9.47 760	0.52 240	9.98 125̄	17
44	9.45 927	9.47 806	0.52 194	9.98 121	16
45	9.45 969	9.47 852	0.52 148	9.98 117	15
46	9.46 011	9.47 897	0.52 103	9.98 113	14
47	9.46 053	9.47 943	0.52 057	9.98 110	13
48	9.46 095̄	9.47 989	0.52 011	9.98 106	12
49	9.46 136	9.48 035̄	0.51 965	9.98 102	11
50	9.46 178	9.48 080	0.51 920	9.98 098	**10**
51	9.46 220	9.48 126	0.51 874	9.98 094	9
52	9.46 262	9.48 171	0.51 829	9.98 090	8
53	9.46 303	9.48 217	0.51 783	9.98 087	7
54	9.46 345̄	9.48 262	0.51 738	9.98 083	6
55	9.46 386	9.48 307	0.51 693	9.98 079	5
56	9.46 428	9.48 353	0.51 647	9.98 075	4
57	9.46 469	9.48 398	0.51 602	9.98 071	3
58	9.46 511	9.48 443	0.51 557	9.98 067	2
59	9.46 552	9.48 489	0.51 511	9.98 063	1
60	9.46 594	9.48 534	0.51 466	9.98 060	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

73°

17°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.46 594	9.48 534	0.51 466	9.98 060	**60**
1	9.46 635	9.48 579	0.51 421	9.98 056	59
2	9.46 676	9.48 624	0.51 376	9.98 052	58
3	9.46 717	9.48 669	0.51 331	9.98 048	57
4	9.46 758	9.48 714	0.51 286	9.98 044	56
5	9.46 800	9.48 759	0.51 241	9.98 040	55
6	9.46 841	9.48 804	0.51 196	9.98 036	54
7	9.46 882	9.48 849	0.51 151	9.98 032	53
8	9.46 923	9.48 894	0.51 106	9.98 029	52
9	9.46 964	9.48 939	0.51 061	9.98 025	51
10	9.47 005	9.48 984	0.51 016	9.98 021	**50**
11	9.47 045	9.49 029	0.50 971	9.98 017	49
12	9.47 086	9.49 073	0.50 927	9.98 013	48
13	9.47 127	9.49 118	0.50 882	9.98 009	47
14	9.47 168	9.49 163	0.50 837	9.98 005	46
15	9.47 209	9.49 207	0.50 793	9.98 001	45
16	9.47 249	9.49 252	0.50 748	9.97 997	44
17	9.47 290	9.49 296	0.50 704	9.97 993	43
18	9.47 330	9.49 341	0.50 659	9.97 989	42
19	9.47 371	9.49 385	0.50 615	9.97 986	41
20	9.47 411	9.49 430	0.50 570	9.97 982	**40**
21	9.47 452	9.49 474	0.50 526	9.97 978	39
22	9.47 492	9.49 519	0.50 481	9.97 974	38
23	9.47 533	9.49 563	0.50 437	9.97 970	37
24	9.47 573	9.49 607	0.50 393	9.97 966	36
25	9.47 613	9.49 652	0.50 348	9.97 962	35
26	9.47 654	9.49 696	0.50 304	9.97 958	34
27	9.47 694	9.49 740	0.50 260	9.97 954	33
28	9.47 734	9.49 784	0.50 216	9.97 950	32
29	9.47 774	9.49 828	0.50 172	9.97 946	31
30	9.47 814	9.49 872	0.50 128	9.97 942	**30**
31	9.47 854	9.49 916	0.50 084	9.97 938	29
32	9.47 894	9.49 960	0.50 040	9.97 934	28
33	9.47 934	9.50 004	0.49 996	9.97 930	27
34	9.47 974	9.50 048	0.49 952	9.97 926	26
35	9.48 014	9.50 092	0.49 908	9.97 922	25
36	9.48 054	9.50 136	0.49 864	9.97 918	24
37	9.48 094	9.50 180	0.49 820	9.97 914	23
38	9.48 133	9.50 223	0.49 777	9.97 910	22
39	9.48 173	9.50 267	0.49 733	9.97 906	21
40	9.48 213	9.50 311	0.49 689	9.97 902	**20**
41	9.48 252	9.50 355	0.49 645	9.97 898	19
42	9.48 292	9.50 398	0.49 602	9.97 894	18
43	9.48 332	9.50 442	0.49 558	9.97 890	17
44	9.48 371	9.50 485	0.49 515	9.97 886	16
45	9.48 411	9.50 529	0.49 471	9.97 882	15
46	9.48 450	9.50 572	0.49 428	9.97 878	14
47	9.48 490	9.50 616	0.49 384	9.97 874	13
48	9.48 529	9.50 659	0.49 341	9.97 870	12
49	9.48 568	9.50 703	0.49 297	9.97 866	11
50	9.48 607	9.50 746	0.49 254	9.97 861	**10**
51	9.48 647	9.50 789	0.49 211	9.97 857	9
52	9.48 686	9.50 833	0.49 167	9.97 853	8
53	9.48 725	9.50 876	0.49 124	9.97 849	7
54	9.48 764	9.50 919	0.49 081	9.97 845	6
55	9.48 803	9.50 962	0.49 038	9.97 841	5
56	9.48 842	9.51 005	0.48 995	9.97 837	4
57	9.48 881	9.51 048	0.48 952	9.97 833	3
58	9.48 920	9.51 092	0.48 908	9.97 829	2
59	9.48 959	9.51 135	0.48 865	9.97 825	1
60	9.48 998	9.51 178	0.48 822	9.97 821	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

72°

18°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.48 998	9.51 178	0.48 822	9.97 821	**60**
1	9.49 037	9.51 221	0.48 779	9.97 817	59
2	9.49 076	9.51 264	0.48 736	9.97 812	58
3	9.49 115	9.51 306	0.48 694	9.97 808	57
4	9.49 153	9.51 349	0.48 651	9.97 804	56
5	9.49 192	9.51 392	0.48 608	9.97 800	55
6	9.49 231	9.51 435	0.48 565	9.97 796	54
7	9.49 269	9.51 478	0.48 522	9.97 792	53
8	9.49 308	9.51 520	0.48 480	9.97 788	52
9	9.49 347	9.51 563	0.48 437	9.97 784	51
10	9.49 385	9.51 606	0.48 394	9.97 779	**50**
11	9.49 424	9.51 648	0.48 352	9.97 775	49
12	9.49 462	9.51 691	0.48 309	9.97 771	48
13	9.49 500	9.51 734	0.48 266	9.97 767	47
14	9.49 539	9.51 776	0.48 224	9.97 763	46
15	9.49 577	9.51 819	0.48 181	9.97 759	45
16	9.49 615	9.51 861	0.48 139	9.97 754	44
17	9.49 654	9.51 903	0.48 097	9.97 750	43
18	9.49 692	9.51 946	0.48 054	9.97 746	42
19	9.49 730	9.51 988	0.48 012	9.97 742	41
20	9.49 768	9.52 031	0.47 969	9.97 738	**40**
21	9.49 806	9.52 073	0.47 927	9.97 734	39
22	9.49 844	9.52 115	0.47 885	9.97 729	38
23	9.49 882	9.52 157	0.47 843	9.97 725	37
24	9.49 920	9.52 200	0.47 800	9.97 721	36
25	9.49 958	9.52 242	0.47 758	9.97 717	35
26	9.49 996	9.52 284	0.47 716	9.97 713	34
27	9.50 034	9.52 326	0.47 674	9.97 708	33
28	9.50 072	9.52 368	0.47 632	9.97 704	32
29	9.50 110	9.52 410	0.47 590	9.97 700	31
30	9.50 148	9.52 452	0.47 548	9.97 696	**30**
31	9.50 185	9.52 494	0.47 506	9.97 691	29
32	9.50 223	9.52 536	0.47 464	9.97 687	28
33	9.50 261	9.52 578	0.47 422	9.97 683	27
34	9.50 298	9.52 620	0.47 380	9.97 679	26
35	9.50 336	9.52 661	0.47 339	9.97 674	25
36	9.50 374	9.52 703	0.47 297	9.97 670	24
37	9.50 411	9.52 745	0.47 255	9.97 666	23
38	9.50 449	9.52 787	0.47 213	9.97 662	22
39	9.50 486	9.52 829	0.47 171	9.97 657	21
40	9.50 523	9.52 870	0.47 130	9.97 653	**20**
41	9.50 561	9.52 912	0.47 088	9.97 649	19
42	9.50 598	9.52 953	0.47 047	9.97 645	18
43	9.50 635	9.52 995	0.47 005	9.97 640	17
44	9.50 673	9.53 037	0.46 963	9.97 636	16
45	9.50 710	9.53 078	0.46 922	9.97 632	15
46	9.50 747	9.53 120	0.46 880	9.97 628	14
47	9.50 784	9.53 161	0.46 839	9.97 623	13
48	9.50 821	9.53 202	0.46 798	9.97 619	12
49	9.50 858	9.53 244	0.46 756	9.97 615	11
50	9.50 896	9.53 285	0.46 715	9.97 610	**10**
51	9.50 933	9.53 327	0.46 673	9.97 606	9
52	9.50 970	9.53 368	0.46 632	9.97 602	8
53	9.51 007	9.53 409	0.46 591	9.97 597	7
54	9.51 043	9.53 450	0.46 550	9.97 593	6
55	9.51 080	9.53 492	0.46 508	9.97 589	5
56	9.51 117	9.53 533	0.46 467	9.97 584	4
57	9.51 154	9.53 574	0.46 426	9.97 580	3
58	9.51 191	9.53 615	0.46 385	9.97 576	2
59	9.51 227	9.53 656	0.46 344	9.97 571	1
60	9.51 264	9.53 697	0.46 303	9.97 567	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

71°

19°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.51 264	9.53 697	0.46 303	9.97 567	**60**
1	9.51 301	9.53 738	0.46 262	9.97 563	59
2	9.51 338	9.53 779	0.46 221	9.97 558	58
3	9.51 374	9.53 820	0.46 180	9.97 554	57
4	9.51 411	9.53 861	0.46 139	9.97 550	56
5	9.51 447	9.53 902	0.46 098	9.97 545	55
6	9.51 484	9.53 943	0.46 057	9.97 541	54
7	9.51 520	9.53 984	0.46 016	9.97 536	53
8	9.51 557	9.54 025	0.45 975	9.97 532	52
9	9.51 593	9.54 065	0.45 935	9.97 528	51
10	9.51 629	9.54 106	0.45 894	9.97 523	**50**
11	9.51 666	9.54 147	0.45 853	9.97 519	49
12	9.51 702	9.54 187	0.45 813	9.97 515	48
13	9.51 738	9.54 228	0.45 772	9.97 510	47
14	9.51 774	9.54 269	0.45 731	9.97 506	46
15	9.51 811	9.54 309	0.45 691	9.97 501	45
16	9.51 847	9.54 350	0.45 650	9.97 497	44
17	9.51 883	9.54 390	0.45 610	9.97 492	43
18	9.51 919	9.54 431	0.45 569	9.97 488	42
19	9.51 955	9.54 471	0.45 529	9.97 484	41
20	9.51 991	9.54 512	0.45 488	9.97 479	**40**
21	9.52 027	9.54 552	0.45 448	9.97 475	39
22	9.52 063	9.54 593	0.45 407	9.97 470	38
23	9.52 099	9.54 633	0.45 367	9.97 466	37
24	9.52 135	9.54 673	0.45 327	9.97 461	36
25	9.52 171	9.54 714	0.45 286	9.97 457	35
26	9.52 207	9.54 754	0.45 246	9.97 453	34
27	9.52 242	9.54 794	0.45 206	9.97 448	33
28	9.52 278	9.54 835	0.45 165	9.97 444	32
29	9.52 314	9.54 875	0.45 125	9.97 439	31
30	9.52 350	9.54 915	0.45 085	9.97 435	**30**
31	9.52 385	9.54 955	0.45 045	9.97 430	29
32	9.52 421	9.54 995	0.45 005	9.97 426	28
33	9.52 456	9.55 035	0.44 965	9.97 421	27
34	9.52 492	9.55 075	0.44 925	9.97 417	26
35	9.52 527	9.55 115	0.44 885	9.97 412	25
36	9.52 563	9.55 155	0.44 845	9.97 408	24
37	9.52 598	9.55 195	0.44 805	9.97 403	23
38	9.52 634	9.55 235	0.44 765	9.97 399	22
39	9.52 669	9.55 275	0.44 725	9.97 394	21
40	9.52 705	9.55 315	0.44 685	9.97 390	**20**
41	9.52 740	9.55 355	0.44 645	9.97 385	19
42	9.52 775	9.55 395	0.44 605	9.97 381	18
43	9.52 811	9.55 434	0.44 566	9.97 376	17
44	9.52 846	9.55 474	0.44 526	9.97 372	16
45	9.52 881	9.55 514	0.44 486	9.97 367	15
46	9.52 916	9.55 554	0.44 446	9.97 363	14
47	9.52 951	9.55 593	0.44 407	9.97 358	13
48	9.52 986	9.55 633	0.44 367	9.97 353	12
49	9.53 021	9.55 673	0.44 327	9.97 349	11
50	9.53 056	9.55 712	0.44 288	9.97 344	**10**
51	9.53 092	9.55 752	0.44 248	9.97 340	9
52	9.53 126	9.55 791	0.44 209	9.97 335	8
53	9.53 161	9.55 831	0.44 169	9.97 331	7
54	9.53 196	9.55 870	0.44 130	9.97 326	6
55	9.53 231	9.55 910	0.44 090	9.97 322	5
56	9.53 266	9.55 949	0.44 051	9.97 317	4
57	9.53 301	9.55 989	0.44 011	9.97 312	3
58	9.53 336	9.56 028	0.43 972	9.97 308	2
59	9.53 370	9.56 067	0.43 933	9.97 303	1
60	9.53 405	9.56 107	0.43 893	9.97 299	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

70°

20°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.53 405	9.56 107	0.43 893	9.97 299	**60**
1	9.53 440	9.56 146	0.43 854	9.97 294	59
2	9.53 475	9.56 185	0.43 815	9.97 289	58
3	9.53 509	9.56 224	0.43 776	9.97 285	57
4	9.53 544	9.56 264	0.43 736	9.97 280	56
5	9.53 578	9.56 303	0.43 697	9.97 276	55
6	9.53 613	9.56 342	0.43 658	9.97 271	54
7	9.53 647	9.56 381	0.43 619	9.97 266	53
8	9.53 682	9.56 420	0.43 580	9.97 262	52
9	9.53 716	9.56 459	0.43 541	9.97 257	51
10	9.53 751	9.56 498	0.43 502	9.97 252	**50**
11	9.53 785	9.56 537	0.43 463	9.97 248	49
12	9.53 819	9.56 576	0.43 424	9.97 243	48
13	9.53 854	9.56 615	0.43 385	9.97 238	47
14	9.53 888	9.56 654	0.43 346	9.97 234	46
15	9.53 922	9.56 693	0.43 307	9.97 229	45
16	9.53 957	9.56 732	0.43 268	9.97 224	44
17	9.53 991	9.56 771	0.43 229	9.97 220	43
18	9.54 025	9.56 810	0.43 190	9.97 215	42
19	9.54 059	9.56 849	0.43 151	9.97 210	41
20	9.54 093	9.56 887	0.43 113	9.97 206	**40**
21	9.54 127	9.56 926	0.43 074	9.97 201	39
22	9.54 161	9.56 965	0.43 035	9.97 196	38
23	9.54 195	9.57 004	0.42 996	9.97 192	37
24	9.54 229	9.57 042	0.42 958	9.97 187	36
25	9.54 263	9.57 081	0.42 919	9.97 182	35
26	9.54 297	9.57 120	0.42 880	9.97 178	34
27	9.54 331	9.57 158	0.42 842	9.97 173	33
28	9.54 365	9.57 197	0.42 803	9.97 168	32
29	9.54 399	9.57 235	0.42 765	9.97 163	31
30	9.54 433	9.57 274	0.42 726	9.97 159	**30**
31	9.54 466	9.57 312	0.42 688	9.97 154	29
32	9.54 500	9.57 351	0.42 649	9.97 149	28
33	9.54 534	9.57 389	0.42 611	9.97 145	27
34	9.54 567	9.57 428	0.42 572	9.97 140	26
35	9.54 601	9.57 466	0.42 534	9.97 135	25
36	9.54 635	9.57 504	0.42 496	9.97 130	24
37	9.54 668	9.57 543	0.42 457	9.97 126	23
38	9.54 702	9.57 581	0.42 419	9.97 121	22
39	9.54 735	9.57 619	0.42 381	9.97 116	21
40	9.54 769	9.57 658	0.42 342	9.97 111	**20**
41	9.54 802	9.57 696	0.42 304	9.97 107	19
42	9.54 836	9.57 734	0.42 266	9.97 102	18
43	9.54 869	9.57 772	0.42 228	9.97 097	17
44	9.54 903	9.57 810	0.42 190	9.97 092	16
45	9.54 936	9.57 849	0.42 151	9.97 087	15
46	9.54 969	9.57 887	0.42 113	9.97 083	14
47	9.55 003	9.57 925	0.42 075	9.97 078	13
48	9.55 036	9.57 963	0.42 037	9.97 073	12
49	9.55 069	9.58 001	0.41 999	9.97 068	11
50	9.55 102	9.58 039	0.41 961	9.97 063	**10**
51	9.55 136	9.58 077	0.41 923	9.97 059	9
52	9.55 169	9.58 115	0.41 885	9.97 054	8
53	9.55 202	9.58 153	0.41 847	9.97 049	7
54	9.55 235	9.58 191	0.41 809	9.97 044	6
55	9.55 268	9.58 229	0.41 771	9.97 039	5
56	9.55 301	9.58 267	0.41 733	9.97 035	4
57	9.55 334	9.58 304	0.41 696	9.97 030	3
58	9.55 367	9.58 342	0.41 658	9.97 025	2
59	9.55 400	9.58 380	0.41 620	9.97 020	1
60	9.55 433	9.58 418	0.41 582	9.97 015	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

69°

21°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.55 433	9.58 418	0.41 582	9.97 015	**60**
1	9.55 466	9.58 455	0.41 545	9.97 010	59
2	9.55 499	9.58 493	0.41 507	9.97 005	58
3	9.55 532	9.58 531	0.41 469	9.97 001	57
4	9.55 564	9.58 569	0.41 431	9.96 996	56
5	9.55 597	9.58 606	0.41 394	9.96 991	55
6	9.55 630	9.58 644	0.41 356	9.96 986	54
7	9.55 663	9.58 681	0.41 319	9.96 981	53
8	9.55 695	9.58 719	0.41 281	9.96 976	52
9	9.55 728	9.58 757	0.41 243	9.96 971	51
10	9.55 761	9.58 794	0.41 206	9.96 966	**50**
11	9.53 793	9.58 832	0.41 168	9.96 962	49
12	9.55 826	9.58 869	0.41 131	9.96 957	48
13	9.55 858	9.58 907	0.41 093	9.96 952	47
14	9.55 891	9.58 944	0.41 056	9.96 947	46
15	9.55 923	9.58 981	0.41 019	9.96 942	45
16	9.55 956	9.59 019	0.40 981	9.96 937	44
17	9.55 988	9.59 056	0.40 944	9.96 932	43
18	9.56 021	9.59 094	0.40 906	9.96 927	42
19	9.56 053	9.59 131	0.40 869	9.96 922	41
20	9.56 085	9.59 168	0.40 832	9.96 917	**40**
21	9.56 118	9.59 205	0.40 795	9.96 912	39
22	9.56 150	9.59 243	0.40 757	9.96 907	38
23	9.56 182	9.59 280	0.40 720	9.96 903	37
24	9.56 215	9.59 317	0.40 683	9.96 898	36
25	9.56 247	9.59 354	0.40 646	9.96 893	35
26	9.56 279	9.59 391	0.40 609	9.96 888	34
27	9.56 311	9.59 429	0.40 571	9.96 883	33
28	9.56 343	9.59 466	0.40 534	9.96 878	32
29	9.56 375	9.59 503	0.40 497	9.96 873	31
30	9.56 408	9.59 540	0.40 460	9.96 868	**30**
31	9.56 440	9.59 577	0.40 423	9.96 863	29
32	9.56 472	9.59 614	0.40 386	9.96 858	28
33	9.56 504	9.59 651	0.40 349	9.96 853	27
34	9.56 536	9.59 688	0.40 312	9.96 848	26
35	9.56 568	9.59 725	0.40 275	9.96 843	25
36	9.56 599	9.59 762	0.40 238	9.96 838	24
37	9.56 631	9.59 799	0.40 201	9.96 833	23
38	9.56 663	9.59 835	0.40 165	9.96 828	22
39	9.56 695	9.59 872	0.40 128	9.96 823	21
40	9.56 727	9.59 909	0.40 091	9.96 818	**20**
41	9.56 759	9.59 946	0.40 054	9.96 813	19
42	9.56 790	9.59 983	0.40 017	9.96 808	18
43	9.56 822	9.60 019	0.39 981	9.96 803	17
44	9.56 854	9.60 056	0.39 944	9.96 798	16
45	9.56 886	9.60 093	0.39 907	9.96 793	15
46	9.56 917	9.60 130	0.39 870	9.96 788	14
47	9.56 949	9.60 166	0.39 834	9.96 783	13
48	9.56 980	9.60 203	0.39 797	9.96 778	12
49	9.57 012	9.60 240	0.39 760	9.96 772	11
50	9.57 044	9.60 276	0.39 724	9.96 767	**10**
51	9.57 075	9.60 313	0.39 687	9.96 762	9
52	9.57 107	9.60 349	0.39 651	9.96 757	8
53	9.57 138	9.60 386	0.39 614	9.96 752	7
54	9.57 169	9.60 422	0.39 578	9.96 747	6
55	9.57 201	9.60 459	0.39 541	9.96 742	5
56	9.57 232	9.60 495	0.39 505	9.96 737	4
57	9.57 264	9.60 532	0.39 468	9.96 732	3
58	9.57 295	9.60 568	0.39 432	9.96 727	2
59	9.57 326	9.60 605	0.39 395	9.96 722	1
60	9.57 358	9.60 641	0.39 359	9.96 717	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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22°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.57 358	9.60 641	0.39 359	9.96 717	**60**
1	9.57 389	9.60 677	0.39 323	9.96 711	59
2	9.57 420	9.60 714	0.39 286	9.96 706	58
3	9.57 451	9.60 750	0.39 250	9.96 701	57
4	9.57 482	9.60 786	0.39 214	9.96 696	56
5	9.57 514	9.60 823	0.39 177	9.96 691	55
6	9.57 545	9.60 859	0.39 141	9.96 686	54
7	9.57 576	9.60 895	0.39 105	9.96 681	53
8	9.57 607	9.60 931	0.39 069	9.96 676	52
9	9.57 638	9.60 967	0.39 033	9.96 670	51
10	9.57 669	9.61 004	0.38 996	9.96 665	**50**
11	9.57 700	9.61 040	0.38 960	9.96 660	49
12	9.57 731	9.61 076	0.38 924	9.96 655	48
13	9.57 762	9.61 112	0.38 888	9.96 650	47
14	9.57 793	9.61 148	0.38 852	9.96 645	46
15	9.57 824	9.61 184	0.38 816	9.96 640	45
16	9.57 855	9.61 220	0.38 780	9.96 634	44
17	9.57 885	9.61 256	0.38 744	9.96 629	43
18	9.57 916	9.61 292	0.38 708	9.96 624	42
19	9.57 947	9.61 328	0.38 672	9.96 619	41
20	9.57 978	9.61 364	0.38 636	9.96 614	**40**
21	9.58 008	9.61 400	0.38 600	9.96 608	39
22	9.58 039	9.61 436	0.38 564	9.96 603	38
23	9.58 070	9.61 472	0.38 528	9.96 598	37
24	9.58 101	9.61 508	0.38 492	9.96 593	36
25	9.58 131	9.61 544	0.38 456	9.96 588	35
26	9.58 162	9.61 579	0.38 421	9.96 582	34
27	9.58 192	9.61 615	0.38 385	9.96 577	33
28	9.58 223	9.61 651	0.38 349	9.96 572	32
29	9.58 253	9.61 687	0.38 313	9.96 567	31
30	9.58 284	9.61 722	0.38 278	9.96 562	**30**
31	9.58 314	9.61 758	0.38 242	9.96 556	29
32	9.58 345	9.61 794	0.38 206	9.96 551	28
33	9.58 375	9.61 830	0.38 170	9.96 546	27
34	9.58 406	9.61 865	0.38 135	9.96 541	26
35	9.58 436	9.61 901	0.38 099	9.96 535	25
36	9.58 467	9.61 936	0.38 064	9.96 530	24
37	9.58 497	9.61 972	0.38 028	9.96 525	23
38	9.58 527	9.62 008	0.37 992	9.96 520	22
39	9.58 557	9.62 043	0.37 957	9.96 514	21
40	9.58 588	9.62 079	0.37 921	9.96 509	**20**
41	9.58 618	9.62 114	0.37 886	9.96 504	19
42	9.58 648	9.62 150	0.37 850	9.96 498	18
43	9.58 678	9.62 185	0.37 815	9.96 493	17
44	9.58 709	9.62 221	0.37 779	9.96 488	16
45	9.58 739	9.62 256	0.37 744	9.96 483	15
46	9.58 769	9.62 292	0.37 708	9.96 477	14
47	9.58 799	9.62 327	0.37 673	9.96 472	13
48	9.58 829	9.62 362	0.37 638	9.96 467	12
49	9.58 859	9.62 398	0.37 602	9.96 461	11
50	9.58 889	9.62 433	0.37 567	9.96 456	**10**
51	9.58 919	9.62 468	0.37 532	9.96 451	9
52	9.58 949	9.62 504	0.37 496	9.96 445	8
53	9.58 979	9.62 539	0.37 461	9.96 440	7
54	9.59 009	9.62 574	0.37 426	9.96 435	6
55	9.59 039	9.62 609	0.37 391	9.96 429	5
56	9.59 069	9.62 645	0.37 355	9.96 424	4
57	9.59 098	9.62 680	0.37 320	9.96 419	3
58	9.59 128	9.62 715	0.37 285	9.96 413	2
59	9.59 158	9.62 750	0.37 250	9.96 408	1
60	9.59 188	9.62 785	0.37 215	9.96 403	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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23°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.59 188	9.62 785	0.37 215	9.96 403	**60**
1	9.59 218	9.62 820	0.37 180	9.96 397	59
2	9.59 247	9.62 855	0.37 145	9.96 392	58
3	9.59 277	9.62 890	0.37 110	9.96 387	57
4	9.59 307	9.62 926	0.37 074	9.96 381	56
5	9.59 336	9.62 961	0.37 039	9.96 376	55
6	9.59 366	9.62 996	0.37 004	9.96 370	54
7	9.59 396	9.63 031	0.36 969	9.96 365	53
8	9.59 425	9.63 066	0.36 934	9.96 360	52
9	9.59 455	9.63 101	0.36 899	9.96 354	51
10	9.59 484	9.63 135	0.36 865	9.96 349	**50**
11	9.59 514	9.63 170	0.36 830	9.96 343	49
12	9.59 543	9.63 205	0.36 795	9.96 338	48
13	9.59 573	9.63 240	0.36 760	9.96 333	47
14	9.59 602	9.63 275	0.36 725	9.96 327	46
15	9.59 632	9.63 310	0.36 690	9.96 322	45
16	9.59 661	9.63 345	0.36 655	9.96 316	44
17	9.59 690	9.63 379	0.36 621	9.96 311	43
18	9.59 720	9.63 414	0.36 586	9.96 305	42
19	9.59 749	9.63 449	0.36 551	9.96 300	41
20	9.59 778	9.63 484	0.36 516	9.96 294	**40**
21	9.59 808	9.63 519	0.36 481	9.96 289	39
22	9.59 837	9.63 553	0.36 447	9.96 284	38
23	9.59 866	9.63 588	0.36 412	9.96 278	37
24	9.59 895	9.63 623	0.36 377	9.96 273	36
25	9.59 924	9.63 657	0.36 343	9.96 267	35
26	9.59 954	9.63 692	0.36 308	9.96 262	34
27	9.59 983	9.63 726	0.36 274	9.96 256	33
28	9.60 012	9.63 761	0.36 239	9.96 251	32
29	9.60 041	9.63 796	0.36 204	9.96 245	31
30	9.60 070	9.63 830	0.36 170	9.96 240	**30**
31	9.60 099	9.63 865	0.36 135	9.96 234	29
32	9.60 128	9.63 899	0.36 101	9.96 229	28
33	9.60 157	9.63 934	0.36 066	9.96 223	27
34	9.60 186	9.63 968	0.36 032	9.96 218	26
35	9.60 215	9.64 003	0.35 997	9.96 212	25
36	9.60 244	9.64 037	0.35 963	9.96 207	24
37	9.60 273	9.64 072	0.35 928	9.96 201	23
38	9.60 302	9.64 106	0.35 894	9.96 196	22
39	9.60 331	9.64 140	0.35 860	9.96 190	21
40	9.60 359	9.64 175	0.35 825	9.96 185	**20**
41	9.60 388	9.64 209	0.35 791	9.96 179	19
42	9.60 417	9.64 243	0.35 757	9.96 174	18
43	9.60 446	9.64 278	0.35 722	9.96 168	17
44	9.60 474	9.64 312	0.35 688	9.96 162	16
45	9.60 503	9.64 346	0.35 654	9.96 157	15
46	9.60 532	9.64 381	0.35 619	9.96 151	14
47	9.60 561	9.64 415	0.35 585	9.96 146	13
48	9.60 589	9.64 449	0.35 551	9.96 140	12
49	9.60 618	9.64 483	0.35 517	9.96 135	11
50	9.60 646	9.64 517	0.35 483	9.96 129	**10**
51	9.60 675	9.64 552	0.35 448	9.96 123	9
52	9.60 704	9.64 586	0.35 414	9.96 118	8
53	9.60 732	9.64 620	0.35 380	9.96 112	7
54	9.60 761	9.64 654	0.35 346	9.96 107	6
55	9.60 789	9.64 688	0.35 312	9.96 101	5
56	9.60 818	9.64 722	0.35 278	9.96 095	4
57	9.60 846	9.64 756	0.35 244	9.96 090	3
58	9.60 875	9.64 790	0.35 210	9.96 084	2
59	9.60 903	9.64 824	0.35 176	9.96 079	1
60	9.60 931	9.64 858	0.35 142	9.96 073	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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24°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.60 931	9.64 858	0.35 142	9.96 073	60
1	9.60 960	9.64 892	0.35 108	9.96 067	59
2	9.60 988	9.64 926	0.35 074	9.96 062	58
3	9.61 016	9.64 960	0.35 040	9.96 056	57
4	9.61 045	9.64 994	0.35 006	9.96 050	56
5	9.61 073	9.65 028	0.34 972	9.96 045	55
6	9.61 101	9.65 062	0.34 938	9.96 039	54
7	9.61 129	9.65 096	0.34 904	9.96 034	53
8	9.61 158	9.65 130	0.34 870	9.96 028	52
9	9.61 186	9.65 164	0.34 836	9.96 022	51
10	9.61 214	9.65 197	0.34 803	9.96 017	50
11	9.61 242	9.65 231	0.34 769	9.96 011	49
12	9.61 270	9.65 265	0.34 735	9.96 005	48
13	9.61 298	9.65 299	0.34 701	9.96 000	47
14	9.61 326	9.65 333	0.34 667	9.95 994	46
15	9.61 354	9.65 366	0.34 634	9.95 988	45
16	9.61 382	9.65 400	0.34 600	9.95 982	44
17	9.61 411	9.65 434	0.34 566	9.95 977	43
18	9.61 438	9.65 467	0.34 533	9.95 971	42
19	9.61 466	9.65 501	0.34 499	9.95 965	41
20	9.61 494	9.65 535	0.34 465	9.95 960	40
21	9.61 522	9.65 568	0.34 432	9.95 954	39
22	9.61 550	9.65 602	0.34 398	9.95 948	38
23	9.61 578	9.65 636	0.34 364	9.95 942	37
24	9.61 606	9.65 669	0.34 331	9.95 937	36
25	9.61 634	9.65 703	0.34 297	9.95 931	35
26	9.61 662	9.65 736	0.34 264	9.95 925	34
27	9.61 689	9.65 770	0.34 230	9.95 920	33
28	9.61 717	9.65 803	0.34 197	9.95 914	32
29	9.61 745	9.65 837	0.34 163	9.95 908	31
30	9.61 773	9.65 870	0.34 130	9.95 902	30
31	9.61 800	9.65 904	0.34 096	9.95 897	29
32	9.61 828	9.65 937	0.34 063	9.95 891	28
33	9.61 856	9.65 971	0.34 029	9.95 885	27
34	9.61 883	9.66 004	0.33 996	9.95 879	26
35	9.61 911	9.66 038	0.33 962	9.95 873	25
36	9.61 939	9.66 071	0.33 929	9.95 868	24
37	9.61 966	9.66 104	0.33 896	9.95 862	23
38	9.61 994	9.66 138	0.33 862	9.95 856	22
39	9.62 021	9.66 171	0.33 829	9.95 850	21
40	9.62 049	9.66 204	0.33 796	9.95 844	20
41	9.62 076	9.66 238	0.33 762	9.95 839	19
42	9.62 104	9.66 271	0.33 729	9.95 833	18
43	9.62 131	9.66 304	0.33 696	9.95 827	17
44	9.62 159	9.66 337	0.33 663	9.95 821	16
45	9.62 186	9.66 371	0.33 629	9.95 815	15
46	9.62 214	9.66 404	0.33 596	9.95 810	14
47	9.62 241	9.66 437	0.33 563	9.95 804	13
48	9.62 268	9.66 470	0.33 530	9.95 798	12
49	9.62 296	9.66 503	0.33 497	9.95 792	11
50	9.62 323	9.66 537	0.33 463	9.95 786	10
51	9.62 350	9.66 570	0.33 430	9.95 780	9
52	9.62 377	9.66 603	0.33 397	9.95 775	8
53	9.62 405	9.66 636	0.33 364	9.95 769	7
54	9.62 432	9.66 669	0.33 331	9.95 763	6
55	9.62 459	9.66 702	0.33 298	9.95 757	5
56	9.62 486	9.66 735	0.33 265	9.95 751	4
57	9.62 513	9.66 768	0.33 232	9.95 745	3
58	9.62 541	9.66 801	0.33 199	9.95 739	2
59	9.62 568	9.66 834	0.33 166	9.95 733	1
60	9.62 595	9.66 867	0.33 133	9.95 728	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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25°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.62 595	9.66 867	0.33 133	9.95 728	**60**
1	9.62 622	9.66 900	0.33 100	9.95 722	59
2	9.62 649	9.66 933	0.33 067	9.95 716	58
3	9.62 676	9.66 966	0.33 034	9.95 710	57
4	9.62 703	9.66 999	0.33 001	9.95 704	56
5	9.62 730	9.67 032	0.32 968	9.95 698	55
6	9.62 757	9.67 065	0.32 935	9.95 692	54
7	9.62 784	9.67 098	0.32 902	9.95 686	53
8	9.62 811	9.67 131	0.32 869	9.95 680	52
9	9.62 838	9.67 163	0.32 837	9.95 674	51
10	9.62 865	9.67 196	0.32 804	9.95 668	**50**
11	9.62 892	9.67 229	0.32 771	9.95 663	49
12	9.62 918	9.67 262	0.32 738	9.95 657	48
13	9.62 945	9.67 295	0.32 705	9.95 651	47
14	9.62 972	9.67 327	0.32 673	9.95 645	46
15	9.62 999	9.67 360	0.32 640	9.95 639	45
16	9.63 026	9.67 393	0.32 607	9.95 633	44
17	9.63 052	9.67 426	0.32 574	9.95 627	43
18	9.63 079	9.67 458	0.32 542	9.95 621	42
19	9.63 106	9.67 491	0.32 509	9.95 615	41
20	9.63 133	9.67 524	0.32 476	9.95 609	**40**
21	9.63 159	9.67 556	0.32 444	9.95 603	39
22	9.63 186	9.67 589	0.32 411	9.95 597	38
23	9.63 213	9.67 622	0.32 378	9.95 591	37
24	9.63 239	9.67 654	0.32 346	9.95 585	36
25	9.63 266	9.67 687	0.32 313	9.95 579	35
26	9.63 292	9.67 719	0.32 281	9.95 573	34
27	9.63 319	9.67 752	0.32 248	9.95 567	33
28	9.63 345	9.67 785	0.32 215	9.95 561	32
29	9.63 372	9.67 817	0.32 183	9.95 555	31
30	9.63 398	9.67 850	0.32 150	9.95 549	**30**
31	9.63 425	9.67 882	0.32 118	9.95 543	29
32	9.63 451	9.67 915	0.32 085	9.95 537	28
33	9.63 478	9.67 947	0.32 053	9.95 531	27
34	9.63 504	9.67 980	0.32 020	9.95 525	26
35	9.63 531	9.68 012	0.31 988	9.95 519	25
36	9.63 557	9.68 044	0.31 956	9.95 513	24
37	9.63 583	9.68 077	0.31 923	9.95 507	23
38	9.63 610	9.68 109	0.31 891	9.95 500	22
39	9.63 636	9.68 142	0.31 858	9.95 494	21
40	9.63 662	9.68 174	0.31 826	9.95 488	**20**
41	9.63 689	9.68 206	0.31 794	9.95 482	19
42	9.63 715	9.68 239	0.31 761	9.95 476	18
43	9.63 741	9.68 271	0.31 729	9.95 470	17
44	9.63 767	9.68 303	0.31 697	9.95 464	16
45	9.63 794	9.68 336	0.31 664	9.95 458	15
46	9.63 820	9.68 368	0.31 632	9.95 452	14
47	9.63 846	9.68 400	0.31 600	9.95 446	13
48	9.63 872	9.68 432	0.31 568	9.95 440	12
49	9.63 898	9.68 465	0.31 535	9.95 434	11
50	9.63 924	9.68 497	0.31 503	9.95 427	**10**
51	9.63 950	9.68 529	0.31 471	9.95 421	9
52	9.63 976	9.68 561	0.31 439	9.95 415	8
53	9.64 002	9.68 593	0.31 407	9.95 409	7
54	9.64 028	9.68 626	0.31 374	9.95 403	6
55	9.64 054	9.68 658	0.31 342	9.95 397	5
56	9.64 080	9.68 690	0.31 310	9.95 391	4
57	9.64 106	9.68 722	0.31 278	9.95 384	3
58	9.64 132	9.68 754	0.31 246	9.95 378	2
59	9.64 158	9.68 786	0.31 214	9.95 372	1
60	9.64 184	9.68 818	0.31 182	9.95 366	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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26°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.64 184	9.68 818	0.31 182	9.95 366	**60**
1	9.64 210	9.68 850	0.31 150	9.95 360	59
2	9.64 236	9.68 882	0.31 118	9.95 354	58
3	9.64 262	9.68 914	0.31 086	9.95 348	57
4	9.64 288	9.68 946	0.31 054	9.95 341	56
5	9.64 313	9.68 978	0.31 022	9.95 335	55
6	9.64 339	9.69 010	0.30 990	9.95 329	54
7	9.64 365	9.69 042	0.30 958	9.95 323	53
8	9.64 391	9.69 074	0.30 926	9.95 317	52
9	9.64 417	9.69 106	0.30 894	9.95 310	51
10	9.64 442	9.69 138	0.30 862	9.95 304	**50**
11	9.64 468	9.69 170	0.30 830	9.95 298	49
12	9.64 494	9.69 202	0.30 798	9.95 292	48
13	9.64 519	9.69 234	0.30 766	9.95 286	47
14	9.64 545	9.69 266	0.30 734	9.95 279	46
15	9.64 571	9.69 298	0.30 702	9.95 273	45
16	9.64 596	9.69 329	0.30 671	9.95 267	44
17	9.64 622	9.69 361	0.30 639	9.95 261	43
18	9.64 647	9.69 393	0.30 607	9.95 254	42
19	9.64 673	9.69 425	0.30 575	9.95 248	41
20	9.64 698	9.69 457	0.30 543	9.95 242	**40**
21	9.64 724	9.69 488	0.30 512	9.95 236	39
22	9.64 749	9.69 520	0.30 480	9.95 229	38
23	9.64 775	9.69 552	0.30 448	9.95 223	37
24	9.64 800	9.69 584	0.30 416	9.95 217	36
25	9.64 826	9.69 615	0.30 385	9.95 211	35
26	9.64 851	9.69 647	0.30 353	9.95 204	34
27	9.64 877	9.69 679	0.30 321	9.95 198	33
28	9.64 902	9.69 710	0.30 290	9.95 192	32
29	9.64 927	9.69 742	0.30 258	9.95 185	31
30	9.64 953	9.69 774	0.30 226	9.95 179	**30**
31	9.64 978	9.69 805	0.30 195	9.95 173	29
32	9.65 003	9.69 837	0.30 163	9.95 167	28
33	9.65 029	9.69 868	0.30 132	9.95 160	27
34	9.65 054	9.69 900	0.30 100	9.95 154	26
35	9.65 079	9.69 932	0.30 068	9.95 148	25
36	9.65 104	9.69 963	0.30 037	9.95 141	24
37	9.65 130	9.69 995	0.30 005	9.95 135	23
38	9.65 155	9.70 026	0.29 974	9.95 129	22
39	9.65 180	9.70 058	0.29 942	9.95 122	21
40	9.65 205	9.70 089	0.29 911	9.95 116	**20**
41	9.65 230	9.70 121	0.29 879	9.95 110	19
42	9.65 255	9.70 152	0.29 848	9.95 103	18
43	9.65 281	9.70 184	0.29 816	9.95 097	17
44	9.65 306	9.70 215	0.29 785	9.95 090	16
45	9.65 331	9.70 247	0.29 753	9.95 084	15
46	9.65 356	9.70 278	0.29 722	9.95 078	14
47	9.65 381	9.70 309	0.29 691	9.95 071	13
48	9.65 406	9.70 341	0.29 659	9.95 065	12
49	9.65 431	9.70 372	0.29 628	9.95 059	11
50	9.65 456	9.70 404	0.29 596	9.95 052	**10**
51	9.65 481	9.70 435	0.29 565	9.95 046	9
52	9.65 506	9.70 466	0.29 534	9.95 039	8
53	9.65 531	9.70 498	0.29 502	9.95 033	7
54	9.65 556	9.70 529	0.29 471	9.95 027	6
55	9.65 580	9.70 560	0.29 440	9.95 020	5
56	9.65 605	9.70 592	0.29 408	9.95 014	4
57	9.65 630	9.70 623	0.29 377	9.95 007	3
58	9.65 655	9.70 654	0.29 346	9.95 001	2
59	9.65 680	9.70 685	0.29 315	9.94 995	1
60	9.65 705	9.70 717	0.29 283	9.94 988	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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27°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.65 705	9.70 717	0.29 283	9.94 988	**60**
1	9.65 729	9.70 748	0.29 252	9.94 982	59
2	9.65 754	9.70 779	0.29 221	9.94 975	58
3	9.65 779	9.70 810	0.29 190	9.94 969	57
4	9.65 804	9.70 841	0.29 159	9.94 962	56
5	9.65 828	9.70 873	0.29 127	9.94 956	55
6	9.65 853	9.70 904	0.29 096	9.94 949	54
7	9.65 878	9.70 935	0.29 065	9.94 943	53
8	9.65 902	9.70 966	0.29 034	9.94 936	52
9	9.65 927	9.70 997	0.29 003	9.94 930	51
10	9.65 952	9.71 028	0.28 972	9.94 923	**50**
11	9.65 976	9.71 059	0.28 941	9.94 917	49
12	9.66 001	9.71 090	0.28 910	9.94 911	48
13	9.66 025	9.71 121	0.28 879	9.94 904	47
14	9.66 050	9.71 153	0.28 847	9.94 898	46
15	9.66 075	9.71 184	0.28 816	9.94 891	45
16	9.66 099	9.71 215	0.28 785	9.94 885	44
17	9.66 124	9.71 246	0.28 754	9.94 878	43
18	9.66 148	9.71 277	0.28 723	9.94 871	42
19	9.66 173	9.71 308	0.28 692	9.94 865	41
20	9.66 197	9.71 339	0.28 661	9.94 858	**40**
21	9.66 221	9.71 370	0.28 630	9.94 852	39
22	9.66 246	9.71 401	0.28 599	9.94 845	38
23	9.66 270	9.71 431	0.28 569	9.94 839	37
24	9.66 295	9.71 462	0.28 538	9.94 832	36
25	9.66 319	9.71 493	0.28 507	9.94 826	35
26	9.66 343	9.71 524	0.28 476	9.94 819	34
27	9.66 368	9.71 555	0.28 445	9.94 813	33
28	9.66 392	9.71 586	0.28 414	9.94 806	32
29	9.66 416	9.71 617	0.28 383	9.94 799	31
30	9.66 441	9.71 648	0.28 352	9.94 793	**30**
31	9.66 465	9.71 679	0.28 321	9.94 786	29
32	9.66 489	9.71 709	0.28 291	9.94 780	28
33	9.66 513	9.71 740	0.28 260	9.94 773	27
34	9.66 537	9.71 771	0.28 229	9.94 767	26
35	9.66 562	9.71 802	0.28 198	9.94 760	25
36	9.66 586	9.71 833	0.28 167	9.94 753	24
37	9.66 610	9.71 863	0.28 137	9.94 747	23
38	9.66 634	9.71 894	0.28 106	9.94 740	22
39	9.66 658	9.71 925	0.28 075	9.94 734	21
40	9.66 682	9.71 955	0.28 045	9.94 727	**20**
41	9.66 706	9.71 986	0.28 014	9.94 720	19
42	9.66 731	9.72 017	0.27 983	9.94 714	18
43	9.66 755	9.72 048	0.27 952	9.94 707	17
44	9.66 779	9.72 078	0.27 922	9.94 700	16
45	9.66 803	9.72 109	0.27 891	9.94 694	15
46	9.66 827	9.72 140	0.27 860	9.94 687	14
47	9.66 851	9.72 170	0.27 830	9.94 680	13
48	9.66 875	9.72 201	0.27 799	9.94 674	12
49	9.66 899	9.72 231	0.27 769	9.94 667	11
50	9.66 922	9.72 262	0.27 738	9.94 660	**10**
51	9.66 946	9.72 293	0.27 707	9.94 654	9
52	9.66 970	9.72 323	0.27 677	9.94 647	8
53	9.66 994	9.72 354	0.27 646	9.94 640	7
54	9.67 018	9.72 384	0.27 616	9.94 634	6
55	9.67 042	9.72 415	0.27 585	9.94 627	5
56	9.67 066	9.72 445	0.27 555	9.94 620	4
57	9.67 090	9.72 476	0.27 524	9.94 614	3
58	9.67 113	9.72 506	0.27 494	9.94 607	2
59	9.67 137	9.72 537	0.27 463	9.94 600	1
60	9.67 161	9.72 567	0.27 433	9.94 593	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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28°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.67 161	9.72 567	0.27 433	9.94 593	60
1	9.67 185	9.72 598	0.27 402	9.94 587	59
2	9.67 208	9.72 628	0.27 372	9.94 580	58
3	9.67 232	9.72 659	0.27 341	9.94 573	57
4	9.67 256	9.72 689	0.27 311	9.94 567	56
5	9.67 280	9.72 720	0.27 280	9.94 560	55
6	9.67 303	9.72 750	0.27 250	9.94 553	54
7	9.67 327	9.72 780	0.27 220	9.94 546	53
8	9.67 350	9.72 811	0.27 189	9.94 540	52
9	9.67 374	9.72 841	0.27 159	9.94 533	51
10	9.67 398	9.72 872	0.27 128	9.94 526	50
11	9.67 421	9.72 902	0.27 098	9.94 519	49
12	9.67 445	9.72 932	0.27 068	9.94 513	48
13	9.67 468	9.72 963	0.27 037	9.94 506	47
14	9.67 492	9.72 993	0.27 007	9.94 499	46
15	9.67 515	9.73 023	0.26 977	9.94 492	45
16	9.67 539	9.73 054	0.26 946	9.94 485	44
17	9.67 562	9.73 084	0.26 916	9.94 479	43
18	9.67 586	9.73 114	0.26 886	9.94 472	42
19	9.67 609	9.73 144	0.26 856	9.94 465	41
20	9.67 633	9.73 175	0.26 825	9.94 458	40
21	9.67 656	9.73 205	0.26 795	9.94 451	39
22	9.67 680	9.73 235	0.26 765	9.94 445	38
23	9.67 703	9.73 265	0.26 735	9.94 438	37
24	9.67 726	9.73 295	0.26 705	9.94 431	36
25	9.67 750	9.73 326	0.26 674	9.94 424	35
26	9.67 773	9.73 356	0.26 644	9.94 417	34
27	9.67 796	9.73 386	0.26 614	9.94 410	33
28	9.67 820	9.73 416	0.26 584	9.94 404	32
29	9.67 843	9.73 446	0.26 554	9.94 397	31
30	9.67 866	9.73 476	0.26 524	9.94 390	30
31	9.67 890	9.73 507	0.26 493	9.94 383	29
32	9.67 913	9.73 537	0.26 463	9.94 376	28
33	9.67 936	9.73 567	0.26 433	9.94 369	27
34	9.67 959	9.73 597	0.26 403	9.94 362	26
35	9.67 982	9.73 627	0.26 373	9.94 355	25
36	9.68 006	9.73 657	0.26 343	9.94 349	24
37	9.68 029	9.73 687	0.26 313	9.94 342	23
38	9.68 052	9.73 717	0.26 283	9.94 335	22
39	9.68 075	9.73 747	0.26 253	9.94 328	21
40	9.68 098	9.73 777	0.26 223	9.94 321	20
41	9.68 121	9.73 807	0.26 193	9.94 314	19
42	9.68 144	9.73 837	0.26 163	9.94 307	18
43	9.68 167	9.73 867	0.26 133	9.94 300	17
44	9.68 190	9.73 897	0.26 103	9.94 293	16
45	9.68 213	9.73 927	0.26 073	9.94 286	15
46	9.68 237	9.73 957	0.26 043	9.94 279	14
47	9.68 260	9.73 987	0.26 013	9.94 273	13
48	9.68 283	9.74 017	0.25 983	9.94 266	12
49	9.68 305	9.74 047	0.25 953	9.94 259	11
50	9.68 328	9.74 077	0.25 923	9.94 252	10
51	9.68 351	9.74 107	0.25 893	9.94 245	9
52	9.68 374	9.74 137	0.25 863	9.94 238	8
53	9.68 397	9.74 166	0.25 834	9.94 231	7
54	9.68 420	9.74 196	0.25 804	9.94 224	6
55	9.68 443	9.74 226	0.25 774	9.94 217	5
56	9.68 466	9.74 256	0.25 744	9.94 210	4
57	9.68 489	9.74 286	0.25 714	9.94 203	3
58	9.68 512	9.74 316	0.25 684	9.94 196	2
59	9.68 534	9.74 345	0.25 655	9.94 189	1
60	9.68 557	9.74 375	0.25 625	9.94 182	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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29°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.68 557	9.74 375	0.25 625̄	9.94 182	**60**
1	9.68 580	9.74 405̄	0.25 595	9.94 175̄	59
2	9.68 603	9.74 435̄	0.25 565	9.94 168	58
3	9.68 625	9.74 465̄	0.25 535	9.94 161	57
4	9.68 648	9.74 494	0.25 506	9.94 154	56
5	9.68 671	9.74 524	0.25 476	9.94 147	55
6	9.68 694	9.74 554	0.25 446	9.94 140	54
7	9.68 716	9.74 583	0.25 417	9.94 133	53
8	9.68 739	9.74 613	0.25 387	9.94 126	52
9	9.68 762	9.74 643	0.25 357	9.94 119	51
10	9.68 784	9.74 673	0.25 327	9.94 112	**50**
11	9.68 807	9.74 702	0.25 298	9.94 105̄	49
12	9.68 829	9.74 732	0.25 268	9.94 098	48
13	9.68 852	9.74 762	0.25 238	9.94 090	47
14	9.68 875̄	9.74 791	0.25 209	9.94 083	46
15	9.68 897	9.74 821	0.25 179	9.94 076	45
16	9.68 920	9.74 851	0.25 149	9.94 069	44
17	9.68 942	9.74 880	0.25 120	9.94 062	43
18	9.68 965̄	9.74 910	0.25 090	9.94 055	42
19	9.68 987	9.74 939	0.25 061	9.94 048	41
20	9.69 010	9.74 969	0.25 031	9.94 041	**40**
21	9.69 032	9.74 998	0.25 002	9.94 034	39
22	9.69 055̄	9.75 028	0.24 972	9.94 027	38
23	9.69 077	9.75 058	0.24 942	9.94 020	37
24	9.69 100	9.75 087	0.24 913	9.94 012	36
25	9.69 122	9.75 117	0.24 883	9.94 005	35
26	9.69 144	9.75 146	0.24 854	9.93 998	34
27	9.69 167	9.75 176	0.24 824	9.93 991	33
28	9.69 189	9.75 205	0.24 795̄	9.93 984	32
29	9.69 212	9.75 235̄	0.24 765	9.93 977	31
30	9.69 234	9.75 264	0.24 736	9.93 970	**30**
31	9.69 256	9.75 294	0.24 706	9.93 963	29
32	9.69 279	9.75 323	0.24 677	9.93 955	28
33	9.69 301	9.75 353	0.24 647	9.93 948	27
34	9.69 323	9.75 382	0.24 618	9.93 941	26
35	9.69 345	9.75 411	0.24 589	9.93 934	25
36	9.69 368	9.75 441	0.24 559	9.93 927	24
37	9.69 390	9.75 470	0.24 530	9.93 920	23
38	9.69 412	9.75 500̄	0.24 500	9.93 912	22
39	9.69 434	9.75 529	0.24 471	9.93 905	21
40	9.69 456	9.75 558	0.24 442	9.93 898	**20**
41	9.69 479	9.75 588	0.24 412	9.93 891	19
42	9.69 501	9.75 617	0.24 383	9.93 884	18
43	9.69 523	9.75 647	0.24 353	9.93 876	17
44	9.69 545	9.75 676	0.24 324	9.93 869	16
45	9.69 567	9.75 705	0.24 295̄	9.93 862	15
46	9.69 589	9.75 735̄	0.24 265	9.93 855	14
47	9.69 611	9.75 764	0.24 236	9.93 847	13
48	9.69 633	9.75 793	0.24 207	9.93 840	12
49	9.69 655	9.75 822	0.24 178	9.93 833	11
50	9.69 677	9.75 852	0.24 148	9.93 826	**10**
51	9.69 699	9.75 881	0.24 119	9.93 819	9
52	9.69 721	9.75 910	0.24 090	9.93 811	8
53	9.69 743	9.75 939	0.24 061	9.93 804	7
54	9.69 765	9.75 969	0.24 031	9.93 797	6
55	9.69 787	9.75 998	0.24 002	9.93 789	5
56	9.69 809	9.76 027	0.23 973	9.93 782	4
57	9.69 831	9.76 056	0.23 944	9.93 775̄	3
58	9.69 853	9.76 086	0.23 914	9.93 768	2
59	9.69 875	9.76 115̄	0.23 885	9.93 760	1
60	9.69 897	9.76 144	0.23 856	9.93 753	**0**
	L. Cos.	**L. Cot.**	**L. Tan.**	**L. Sin.**	**′**

60°

30°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.69 897	9.76 144	0.23 856	9.93 753	**60**
1	9.69 919	9.76 173	0.23 827	9.93 746	59
2	9.69 941	9.76 202	0.23 798	9.93 738	58
3	9.69 963	9.76 231	0.23 769	9.93 731	57
4	9.69 984	9.76 261	0.23 739	9.93 724	56
5	9.70 006	9.76 290	0.23 710	9.93 717	55
6	9.70 028	9.76 319	0.23 681	9.93 709	54
7	9.70 050	9.76 348	0.23 652	9.93 702	53
8	9.70 072	9.76 377	0.23 623	9.93 695	52
9	9.70 093	9.76 406	0.23 594	9.93 687	51
10	9.70 115	9.76 435	0.23 565	9.93 680	**50**
11	9.70 137	9.76 464	0.23 536	9.93 673	49
12	9.70 159	9.76 493	0.23 507	9.93 665	48
13	9.70 180	9.76 522	0.23 478	9.93 658	47
14	9.70 202	9.76 551	0.23 449	9.93 650	46
15	9.70 224	9.76 580	0.23 420	9.93 643	45
16	9.70 245	9.76 609	0.23 391	9.93 636	44
17	9.70 267	9.76 639	0.23 361	9.93 628	43
18	9.70 288	9.76 668	0.23 332	9.93 621	42
19	9.70 310	9.76 697	0.23 303	9.93 614	41
20	9.70 332	9.76 725	0.23 275	9.93 606	**40**
21	9.70 353	9.76 754	0.23 246	9.93 599	39
22	9.70 375	9.76 783	0.23 217	9.93 591	38
23	9.70 396	9.76 812	0.23 188	9.93 584	37
24	9.70 418	9.76 841	0.23 159	9.93 577	36
25	9.70 439	9.76 870	0.23 130	9.93 569	35
26	9.70 461	9.76 899	0.23 101	9.93 562	34
27	9.70 482	9.76 928	0.23 072	9.93 554	33
28	9.70 504	9.76 957	0.23 043	9.93 547	32
29	9.70 525	9.76 986	0.23 014	9.93 539	31
30	9.70 547	9.77 015	0.22 985	9.93 532	**30**
31	9.70 568	9.77 044	0.22 956	9.93 525	29
32	9.70 590	9.77 073	0.22 927	9.93 517	28
33	9.70 611	9.77 101	0.22 899	9.93 510	27
34	9.70 633	9.77 130	0.22 870	9.93 502	26
35	9.70 654	9.77 159	0.22 841	9.93 495	25
36	9.70 675	9.77 188	0.22 812	9.93 487	24
37	9.70 697	9.77 217	0.22 783	9.93 480	23
38	9.70 718	9.77 246	0.22 754	9.93 472	22
39	9.70 739	9.77 274	0.22 726	9.93 465	21
40	9.70 761	9.77 303	0.22 697	9.93 457	**20**
41	9.70 782	9.77 332	0.22 668	9.93 450	19
42	9.70 803	9.77 361	0.22 639	9.93 442	18
43	9.70 824	9.77 390	0.22 610	9.93 435	17
44	9.70 846	9.77 418	0.22 582	9.93 427	16
45	9.70 867	9.77 447	0.22 553	9.93 420	15
46	9.70 888	9.77 476	0.22 524	9.93 412	14
47	9.70 909	9.77 505	0.22 495	9.93 405	13
48	9.70 931	9.77 533	0.22 467	9.93 397	12
49	9.70 952	9.77 562	0.22 438	9.93 390	11
50	9.70 973	9.77 591	0.22 409	9.93 382	**10**
51	9.70 994	9.77 619	0.22 381	9.93 375	9
52	9.71 015	9.77 648	0.22 352	9.93 367	8
53	9.71 036	9.77 677	0.22 323	9.93 360	7
54	9.71 058	9.77 706	0.22 294	9.93 352	6
55	9.71 079	9.77 734	0.22 266	9.93 344	5
56	9.71 100	9.77 763	0.22 237	9.93 337	4
57	9.71 121	9.77 791	0.22 209	9.93 329	3
58	9.71 142	9.77 820	0.22 180	9.93 322	2
59	9.71 163	9.77 849	0.22 151	9.93 314	1
60	9.71 184	9.77 877	0.22 123	9.93 307	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

59°

31°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.71 184	9.77 877	0.22 123	9.93 307	**60**
1	9.71 205	9.77 906	0.22 094	9.93 299	59
2	9.71 226	9.77 935	0.22 065	9.93 291	58
3	9.71 247	9.77 963	0.22 037	9.93 284	57
4	9.71 268	9.77 992	0.22 008	9.93 276	56
5	9.71 289	9.78 020	0.21 980	9.93 269	55
6	9.71 310	9.78 049	0.21 951	9.93 261	54
7	9.71 331	9.78 077	0.21 923	9.93 253	53
8	9.71 352	9.78 106	0.21 894	9.93 246	52
9	9.71 373	9.78 135	0.21 865	9.93 238	51
10	9.71 393	9.78 163	0.21 837	9.93 230	**50**
11	9.71 414	9.78 192	0.21 808	9.93 223	49
12	9.71 435	9.78 220	0.21 780	9.93 215	48
13	9.71 456	9.78 249	0.21 751	9.93 207	47
14	9.71 477	9.78 277	0.21 723	9.93 200	46
15	9.71 498	9.78 306	0.21 694	9.93 192	45
16	9.71 519	9.78 334	0.21 666	9.93 184	44
17	9.71 539	9.78 363	0.21 637	9.93 177	43
18	9.71 560	9.78 391	0.21 609	9.93 169	42
19	9.71 581	9.78 419	0.21 581	9.93 161	41
20	9.71 602	9.78 448	0.21 552	9.93 154	**40**
21	9.71 622	9.78 476	0.21 524	9.93 146	39
22	9.71 643	9.78 505	0.21 495	9.93 138	38
23	9.71 664	9.78 533	0.21 467	9.93 131	37
24	9.71 685	9.78 562	0.21 438	9.93 123	36
25	9.71 705	9.78 590	0.21 410	9.93 115	35
26	9.71 726	9.78 618	0.21 382	9.93 108	34
27	9.71 747	9.78 647	0.21 353	9.93 100	33
28	9.71 767	9.78 675	0.21 325	9.93 092	32
29	9.71 788	9.78 704	0.21 296	9.93 084	31
30	9.71 809	9.78 732	0.21 268	9.93 077	**30**
31	9.71 829	9.78 760	0.21 240	9.93 069	29
32	9.71 850	9.78 789	0.21 211	9.93 061	28
33	9.71 870	9.78 817	0.21 183	9.93 053	27
34	9.71 891	9.78 845	0.21 155	9.93 046	26
35	9.71 911	9.78 874	0.21 126	9.93 038	25
36	9.71 932	9.78 902	0.21 098	9.93 030	24
37	9.71 952	9.78 930	0.21 070	9.93 022	23
38	9.71 973	9.78 959	0.21 041	9.93 014	22
39	9.71 994	9.78 987	0.21 013	9.93 007	21
40	9.72 014	9.79 015	0.20 985	9.92 999	**20**
41	9.72 034	9.79 043	0.20 957	9.92 991	19
42	9.72 055	9.79 072	0.20 928	9.92 983	18
43	9.72 075	9.79 100	0.20 900	9.92 976	17
44	9.72 096	9.79 128	0.20 872	9.92 968	16
45	9.72 116	9.79 156	0.20 844	9.92 960	15
46	9.72 137	9.79 185	0.20 815	9.92 952	14
47	9.72 157	9.79 213	0.20 787	9.92 944	13
48	9.72 177	9.79 241	0.20 759	9.92 936	12
49	9.72 198	9.79 269	0.20 731	9.92 929	11
50	9.72 218	9.79 297	0.20 703	9.92 921	**10**
51	9.72 238	9.79 326	0.20 674	9.92 913	9
52	9.72 259	9.79 354	0.20 646	9.92 905	8
53	9.72 279	9.79 382	0.20 618	9.92 897	7
54	9.72 299	9.79 410	0.20 590	9.92 889	6
55	9.72 320	9.79 438	0.20 562	9.92 881	5
56	9.72 340	9.79 466	0.20 534	9.92 874	4
57	9.72 360	9.79 495	0.20 505	9.92 866	3
58	9.72 381	9.79 523	0.20 477	9.92 858	2
59	9.72 401	9.79 551	0.20 449	9.92 850	1
60	9.72 421	9.79 579	0.20 421	9.92 842	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

58°

32°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.72 421	9.79 579	0.20 421	9.92 842	60
1	9.72 441	9.79 607	0.20 393	9.92 834	59
2	9.72 461	9.79 635	0.20 365	9.92 826	58
3	9.72 482	9.79 663	0.20 337	9.92 818	57
4	9.72 502	9.79 691	0.20 309	9.92 810	56
5	9.72 522	9.79 719	0.20 281	9.92 803	55
6	9.72 542	9.79 747	0.20 253	9.92 795	54
7	9.72 562	9.79 776	0.20 224	9.92 787	53
8	9.72 582	9.79 804	0.20 196	9.92 779	52
9	9.72 602	9.79 832	0.20 168	9.92 771	51
10	9.72 622	9.79 860	0.20 140	9.92 763	50
11	9.72 643	9.79 888	0.20 112	9.92 755	49
12	9.72 663	9.79 916	0.20 084	9.92 747	48
13	9.72 683	9.79 944	0.20 056	9.92 739	47
14	9.72 703	9.79 972	0.20 028	9.92 731	46
15	9.72 723	9.80 000	0.20 000	9.92 723	45
16	9.72 743	9.80 028	0.19 972	9.92 715	44
17	9.72 763	9.80 056	0.19 944	9.92 707	43
18	9.72 783	9.80 084	0.19 916	9.92 699	42
19	9.72 803	9.80 112	0.19 888	9.92 691	41
20	9.72 823	9.80 140	0.19 860	9.92 683	40
21	9.72 843	9.80 168	0.19 832	9.92 675	39
22	9.72 863	9.80 195	0.19 805	9.92 667	38
23	9.72 883	9.80 223	0.19 777	9.92 659	37
24	9.72 902	9.80 251	0.19 749	9.92 651	36
25	9.72 922	9.80 279	0.19 721	9.92 643	35
26	9.72 942	9.80 307	0.19 693	9.92 635	34
27	9.72 962	9.80 335	0.19 665	9.92 627	33
28	9.72 982	9.80 363	0.19 637	9.92 619	32
29	9.73 002	9.80 391	0.19 609	9.92 611	31
30	9.73 022	9.80 419	0.19 581	9.92 603	30
31	9.73 041	9.80 447	0.19 553	9.92 595	29
32	9.73 061	9.80 474	0.19 526	9.92 587	28
33	9.73 081	9.80 502	0.19 498	9.92 579	27
34	9.73 101	9.80 530	0.19 470	9.92 571	26
35	9.73 121	9.80 558	0.19 442	9.92 563	25
36	9.73 140	9.80 586	0.19 414	9.92 555	24
37	9.73 160	9.80 614	0.19 386	9.92 546	23
38	9.73 180	9.80 642	0.19 358	9.92 538	22
39	9.73 200	9.80 669	0.19 331	9.92 530	21
40	9.73 219	9.80 697	0.19 303	9.92 522	20
41	9.73 239	9.80 725	0.19 275	9.92 514	19
42	9.73 259	9.80 753	0.19 247	9.92 506	18
43	9.73 278	9.80 781	0.19 219	9.92 498	17
44	9.73 298	9.80 808	0.19 192	9.92 490	16
45	9.73 318	9.80 836	0.19 164	9.92 482	15
46	9.73 337	9.80 864	0.19 136	9.92 473	14
47	9.73 357	9.80 892	0.19 108	9.92 465	13
48	9.73 377	9.80 919	0.19 081	9.92 457	12
49	9.73 396	9.80 947	0.19 053	9.92 449	11
50	9.73 416	9.80 975	0.19 025	9.92 441	10
51	9.73 435	9.81 003	0.18 997	9.92 433	9
52	9.73 455	9.81 030	0.18 970	9.92 425	8
53	9.73 474	9.81 058	0.18 942	9.92 416	7
54	9.73 494	9.81 086	0.18 914	9.92 408	6
55	9.73 513	9.81 113	0.18 887	9.92 400	5
56	9.73 533	9.81 141	0.18 859	9.92 392	4
57	9.73 552	9.81 169	0.18 831	9.92 384	3
58	9.73 572	9.81 196	0.18 804	9.92 376	2
59	9.73 591	9.81 224	0.18 776	9.92 367	1
60	9.73 611	9.81 252	0.18 748	9.92 359	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

57°

33°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.73 611	9.81 252	0.18 748	9.92 359	**60**
1	9.73 630	9.81 279	0.18 721	9.92 351	59
2	9.73 650	9.81 307	0.18 693	9.92 343	58
3	9.73 669	9.81 335	0.18 665	9.92 335	57
4	9.73 689	9.81 362	0.18 638	9.92 326	56
5	9.73 708	9.81 390	0.18 610	9.92 318	55
6	9.73 727	9.81 418	0.18 582	9.92 310	54
7	9.73 747	9.81 445	0.18 555	9.92 302	53
8	9.73 766	9.81 473	0.18 527	9.92 293	52
9	9.73 785	9.81 500	0.18 500	9.92 285	51
10	9.73 805	9.81 528	0.18 472	9.92 277	**50**
11	9.73 824	9.81 556	0.18 444	9.92 269	49
12	9.73 843	9.81 583	0.18 417	9.92 260	48
13	9.73 863	9.81 611	0.18 389	9.92 252	47
14	9.73 882	9.81 638	0.18 362	9.92 244	46
15	9.73 901	9.81 666	0.18 334	9.92 235	45
16	9.73 921	9.81 693	0.18 307	9.92 227	44
17	9.73 940	9.81 721	0.18 279	9.92 219	43
18	9.73 959	9.81 748	0.18 252	9.92 211	42
19	9.73 978	9.81 776	0.18 224	9.92 202	41
20	9.73 997	9.81 803	0.18 197	9.92 194	**40**
21	9.74 017	9.81 831	0.18 169	9.92 186	39
22	9.74 036	9.81 858	0.18 142	9.92 177	38
23	9.74 055	9.81 886	0.18 114	9.92 169	37
24	9.74 074	9.81 913	0.18 087	9.92 161	36
25	9.74 093	9.81 941	0.18 059	9.92 152	35
26	9.74 113	9.81 968	0.18 032	9.92 144	34
27	9.74 132	9.81 996	0.18 004	9.92 136	33
28	9.74 151	9.82 023	0.17 977	9.92 127	32
29	9.74 170	9.82 051	0.17 949	9.92 119	31
30	9.74 189	9.82 078	0.17 922	9.92 111	**30**
31	9.74 208	9.82 106	0.17 894	9.92 102	29
32	9.74 227	9.82 133	0.17 867	9.92 094	28
33	9.74 246	9.82 161	0.17 839	9.92 086	27
34	9.74 265	9.82 188	0:17 812	9.92 077	26
35	9.74 284	9.82 215	0.17 785	9.92 069	25
36	9.74 303	9.82 243	0.17 757	9.92 060	24
37	9.74 322	9.82 270	0.17 730	9.92 052	23
38	9.74 341	9.82 298	0.17 702	9.92 044	22
39	9.74 360	9.82 325	0.17 675	9.92 035	21
40	9.74 379	9.82 352	0.17 648	9.92 027	**20**
41	9.74 398	9.82 380	0.17 620	9.92 018	19
42	9.74 417	9.82 407	0.17 593	9.92 010	18
43	9.74 436	9.82 435	0.17 565	9.92 002	17
44	9.74 455	9.82 462	0.17 538	9.91 993	16
45	9.74 474	9.82 489	0.17 511	9.91 985	15
46	9.74 493	9.82 517	0.17 483	9.91 976	14
47	9.74 512	9.82 544	0.17 456	9.91 968	13
48	9.74 531	9.82 571	0.17 429	9.91 959	12
49	9.74 549	9.82 599	0.17 401	9.91 951	11
50	9.74 568	9.82 626	0.17 374	9.91 942	**10**
51	9.74 587	9.82 653	0.17 347	9.91 934	9
52	9.74 606	9.82 681	0.17 319	9.91 925	8
53	9.74 625	9.82 708	0.17 292	9.91 917	7
54	9.74 644	9.82 735	0.17 265	9.91 908	6
55	9.74 662	9.82 762	0.17 238	9.81 900	5
56	9.74 681	9.82 790	0.17 210	9.91 891	4
57	9.74 700	9.82 817	0.18 183	9.91 883	3
58	9.74 719	9.82 844	0.17 156	9.91 874	2
59	9.74 737	9.82 871	0.17 129	9.91 866	1
60	9.74 756	9.82 899	0.17 101	9.91 857	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

56°

34°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.74 756	9.82 899	0.17 101	9.91 857	60
1	9.74 775	9.82 926	0.17 074	9.91 849	59
2	9.74 794	9.82 953	0.17 047	9.91 840	58
3	9.74 812	9.82 980	0.17 020	9.91 832	57
4	9.74 831	9.83 008	0.16 992	9.91 823	56
5	9.74 850	9.83 035	0.16 965	9.91 815	55
6	9.74 868	9.83 062	0.16 938	9.91 806	54
7	9.74 887	9.83 089	0.16 911	9.91 798	53
8	9.74 906	9.83 117	0.16 883	9.91 789	52
9	9.74 924	9.83 144	0.16 856	9.91 781	51
10	9.74 943	9.83 171	0.16 829	9.91 772	50
11	9.74 961	9.83 198	0.16 802	9.91 763	49
12	9.74 980	9.83 225	0.16 775	9.91 755	48
13	9.74 999	9.83 252	0.16 748	9.91 746	47
14	9.75 017	9.83 280	0.16 720	9.91 738	46
15	9.75 036	9.83 307	0.16 693	9.91 729	45
16	9.75 054	9.83 334	0.16 666	9.91 720	44
17	9.75 073	9.83 361	0.16 639	9.91 712	43
18	9.75 091	9.83 388	0.16 612	9.91 703	42
19	9.75 110	9.83 415	0.16 585	9.91 695	41
20	9.75 128	9.83 442	0.16 558	9.91 686	40
21	9.75 147	9.83 470	0.16 530	9.91 677	39
22	9.75 165	9.83 497	0.16 503	9.91 669	38
23	9.75 184	9.83 524	0.16 476	9.91 660	37
24	9.75 202	9.83 551	0.16 449	9.91 651	36
25	9.75 221	9.83 578	0.16 422	9.91 643	35
26	9.75 239	9.83 605	0.16 395	9.91 634	34
27	9.75 258	9.83 632	0.16 368	9.91 625	33
28	9.75 276	9.83 659	0.16 341	9.91 617	32
29	9.75 294	9.83 686	0.16 314	9.91 608	31
30	9.75 313	9.83 713	0.16 287	9.91 599	30
31	9.75 331	9.83 740	0.16 260	9.91 591	29
32	9.75 350	9.83 768	0.16 232	9.91 582	28
33	9.75 368	9.83 795	0.16 205	9.91 573	27
34	9.75 386	9.83 822	0.16 178	9.91 565	26
35	9.75 405	9.83 849	0.16 151	9.91 556	25
36	9.75 423	9.83 876	0.16 124	9.91 547	24
37	9.75 441	9.83 903	0.16 097	9.91 538	23
38	9.75 459	9.83 930	0.16 070	9.91 530	22
39	9.75 478	9.83 957	0.16 043	9.91 521	21
40	9.75 496	9.83 984	0.16 016	9.91 512	20
41	9.75 514	9.84 011	0.15 989	9.91 504	19
42	9.75 533	9.84 038	0.15 962	9.91 495	18
43	9.75 551	9.84 065	0.15 935	9.91 486	17
44	9.75 569	9.84 092	0.15 908	9.91 477	16
45	9.75 587	9.84 119	0.15 881	9.91 469	15
46	9.75 605	9.84 146	0.15 854	9.91 460	14
47	9.75 624	9.84 173	0.15 827	9.91 451	13
48	9.75 642	9.84 200	0.15 800	9.91 442	12
49	9.75 660	9.84 227	0.15 773	9.91 433	11
50	9.75 678	9.84 254	0.15 746	9.91 425	10
51	9.75 696	9.84 280	0.15 720	9.91 416	9
52	9.75 714	9.84 307	0.15 693	9.91 407	8
53	9.75 733	9.84 334	0.15 666	9.91 398	7
54	9.75 751	9.84 361	0.15 639	9.91 389	6
55	9.75 769	9.84 388	0.15 612	9.91 381	5
56	9.75 787	9.84 415	0.15 585	9.91 372	4
57	9.75 805	9.84 442	0.15 558	9.91 363	3
58	9.75 823	9.84 469	0.15 531	9.91 354	2
59	9.75 841	9.84 496	0.15 504	9.91 345	1
60	9.75 859	9.84 523	0.15 477	9.91 336	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

55°

35°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.75 859	9.84 523	0.15 477	9.91 336	**60**
1	9.75 877	9.84 550	0.15 450	9.91 328	59
2	9.75 895	9.84 576	0.15 424	9.91 319	58
3	9.75 913	9.84 603	0.15 397	9.91 310	57
4	9.75 931	9.84 630	0.15 370	9.91 301	56
5	9.75 949	9.84 657	0.15 343	9.91 292	55
6	9.75 967	9.84 684	0.15 316	9.91 283	54
7	9.75 985	9.84 711	0.15 289	9.91 274	53
8	9.76 003	9.84 738	0.15 262	9.91 266	52
9	9.76 021	9.84 764	0.15 236	9.91 257	51
10	9.76 039	9.84 791	0.15 209	9.91 248	**50**
11	9.76 057	9.84 818	0.15 182	9.91 239	49
12	9.76 075	9.84 845	0.15 155	9.91 230	48
13	9.76 093	9.84 872	0.15 128	9.91 221	47
14	9.76 111	9.84 899	0.15 101	9.91 212	46
15	9.76 129	9.84 925	0.15 075	9.91 203	45
16	9.76 146	9.84 952	0.15 048	9.91 194	44
17	9.76 164	9.84 979	0.15 021	9.91 185	43
18	9.76 182	9.85 006	0.14 994	9.91 176	42
19	9.76 200	9.85 033	0.14 967	9.91 167	41
20	9.76 218	9.85 059	0.14 941	9.91 158	**40**
21	9.76 236	9.85 086	0.14 914	9.91 149	39
22	9.76 253	9.85 113	0.14 887	9.91 141	38
23	9.76 271	9.85 140	0.14 860	9.91 132	37
24	9.76 289	9.85 166	0.14 834	9.91 123	36
25	9.76 307	9.85 193	0.14 807	9.91 114	35
26	9.76 324	9.85 220	0.14 780	9.91 105	34
27	9.76 342	9.85 247	0.14 753	9.91 096	33
28	9.76 360	9.85 273	0.14 727	9.91 087	32
29	9.76 378	9.85 300	0.14 700	9.91 078	31
30	9.76 395	9.85 327	0.14 673	9.91 069	**30**
31	9.76 413	9.85 354	0.14 646	9.91 060	29
32	9.76 431	9.85 380	0.14 620	9.91 051	28
33	9.76 448	9.85 407	0.14 593	9.91 042	27
34	9.76 466	9.85 434	0.14 566	9.91 033	26
35	9.76 484	9.85 460	0.14 540	9.91 023	25
36	9.76 501	9.85 487	0.14 513	9.91 014	24
37	9.76 519	9.85 514	0.14 486	9.91 005	23
38	9.76 537	9.85 540	0.14 460	9.90 996	22
39	9.76 554	9.85 567	0.14 433	9.90 987	21
40	9.76 572	9.85 594	0.14 406	9.90 978	**20**
41	9.76 590	9.85 620	0.14 380	9.90 969	19
42	9.76 607	9.85 647	0.14 353	9.90 960	18
43	9.76 625	9.85 674	0.14 326	9.90 951	17
44	9.76 642	9.85 700	0.14 300	9.90 942	16
45	9.76 660	9.85 727	0.14 273	9.90 933	15
46	9.76 677	9.85 754	0.14 246	9.90 924	14
47	9.76 695	9.85 780	0.14 220	9.90 915	13
48	9.76 712	9.85 807	0.14 193	9.90 906	12
49	9.76 730	9.85 834	0.14 166	9.90 896	11
50	9.76 747	9.85 860	0.14 140	9.90 887	**10**
51	9.76 765	9.85 887	0.14 113	9.90 878	9
52	9.76 782	9.85 913	0.14 087	9.90 869	8
53	9.76 800	9.85 940	0.14 060	9.90 860	7
54	9.76 817	9.85 967	0.14 033	9.90 851	6
55	9.76 835	9.85 993	0.14 007	9.90 842	5
56	9.76 852	9.86 020	0.13 980	9.90 832	4
57	9.76 870	9.86 046	0.13 954	9.90 823	3
58	9.76 887	9.86 073	0.13 927	9.90 814	2
59	9.76 904	9.86 100	0.13 900	9.90 805	1
60	9.76 922	9.86 126	0.13 874	9.90 796	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

54°

36°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.76 922	9.86 126	0.13 874	9.90 796	**60**
1	9.76 939	9.86 153	0.13 847	9.90 787	59
2	9.76 957	9.86 179	0.13 821	9.90 777	58
3	9.76 974	9.86 206	0.13 794	9.90 768	57
4	9.76 991	9.86 232	0.13 768	9.90 759	56
5	9.77 009	9.86 259	0.13 741	9.90 750	55
6	9.77 026	9.86 285	0.13 715	9.90 741	54
7	9.77 043	9.86 312	0.13 688	9.90 731	53
8	9.77 061	9.86 338	0.13 662	9.90 722	52
9	9.77 078	9.86 365	0.13 635	9.90 713	51
10	9.77 095	9.86 392	0.13 608	9.90 704	**50**
11	9.77 112	9.86 418	0.13 582	9.90 694	49
12	9.77 130	9.86 445	0.13 555	9.90 685	48
13	9.77 147	9.86 471	0.13 529	9.90 676	47
14	9.77 164	9.86 498	0.13 502	9.90 667	46
15	9.77 181	9.86 524	0.13 476	9.90 657	45
16	9.77 199	9.86 551	0.13 449	9.90 648	44
17	9.77 216	9.86 577	0.13 423	9.90 639	43
18	9.77 233	9.86 603	0.13 397	9.90 630	42
19	9.77 250	9.86 630	0.13 370	9.90 620	41
20	9.77 268	9.86 656	0.13 344	9.90 611	**40**
21	9.77 285	9.86 683	0.13 317	9.90 602	39
22	9.77 302	9.86 709	0.13 291	9.90 592	38
23	9.77 319	9.86 736	0.13 264	9.90 583	37
24	9.77 336	9.86 762	0.13 238	9.90 574	36
25	9.77 353	9.86 789	0.13 211	9.90 565	35
26	9.77 370	9.86 815	0.13 185	9.90 555	34
27	9.77 387	9.86 842	0.13 158	9.90 546	33
28	9.77 405	9.86 868	0.13 132	9.90 537	32
29	9.77 422	9.86 894	0.13 106	9.90 527	31
30	9.77 439	9.86 921	0.13 079	9.90 518	**30**
31	9.77 456	9.86 947	0.13 053	9.90 509	29
32	9.77 473	9.86 974	0.13 026	9.90 499	28
33	9.77 490	9.87 000	0.13 000	9.90 490	27
34	9.77 507	9.87 027	0.12 973	9.90 480	26
35	9.77 524	9.87 053	0.12 947	9.90 471	25
36	9.77 541	9.87 079	0.12 921	9.90 462	24
37	9.77 558	9.87 106	0.12 894	9.90 452	23
38	9.77 575	9.87 132	0.12 868	9.90 443	22
39	9.77 592	9.87 158	0.12 842	9.90 434	21
40	9.77 609	9.87 185	0.12 815	9.90 424	**20**
41	9.77 626	9.87 211	0.12 789	9.90 415	19
42	9.77 643	9.87 238	0.12 762	9.90 405	18
43	9.77 660	9.87 264	0.12 736	9.90 396	17
44	9.77 677	9.87 290	0.12 710	9.90 386	16
45	9.77 694	9.87 317	0.12 683	9.90 377	15
46	9.77 711	9.87 343	0.12 657	9.90 368	14
47.	9.77 728	9.87 369	0.12 631	9.90 358	13
48	9.77 744	9.87 396	0.12 604	9.90 349	12
49	9.77 761	9.87 422	0.12 578	9.90 339	11
50	9.77 778	9.87 448	0.12 552	9.90 330	**10**
51	9.77 795	9.87 475	0.12 525	9.90 320	9
52	9.77 812	9.87 501	0.12 499	9.90 311	8
53	9.77 829	9.87 527	0.12 473	9.90 301	7
54	9.77 846	9.87 554	0.12 446	9.90 292	6
55	9.77 862	9.87 580	0.12 420	9.90 282	5
56	9.77 879	9.87 606	0.12 394	9.90 273	4
57	9.77 896	9.87 633	0.12 367	9.90 263	3
58	9.77 913	9.87 659	0.12 341	9.90 254	2
59	9.77 930	9.87 685	0.12 315	9.90 244	1
60	9.77 946	9.87 711	0.12 289	9.90 235	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

53°

37°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.77 946	9.87 711	0.12 289	9.90 235	60
1	9.77 963	9.87 738	0.12 262	9.90 225	59
2	9.77 980	9.87 764	0.12 236	9.90 216	58
3	9.77 997	9.87 790	0.12 210	9.90 206	57
4	9.78 013	9.87 817	0.12 183	9.90 197	56
5	9.78 030	9.87 843	0.12 157	9.90 187	55
6	9.78 047	9.87 869	0.12 131	9.90 178	54
7	9.78 063	9.87 895	0.12 105	9.90 168	53
8	9.78 080	9.87 922	0.12 078	9.90 159	52
9	9.78 097	9.87 948	0.12 052	9.90 149	51
10	9.78 113	9.87 974	0.12 026	9.90 139	50
11	9.78 130	9.88 000	0.12 000	9.90 130	49
12	9.78 147	9.88 027	0.11 973	9.90 120	48
13	9.78 163	9.88 053	0.11 947	9.90 111	47
14	9.78 180	9.88 079	0.11 921	9.90 101	46
15	9.78 197	9.88 105	0.11 895	9.90 091	45
16	9.78 213	9.88 131	0.11 869	9.90 082	44
17	9.78 230	9.88 158	0.11 842	9.90 072	43
18	9.78 246	9.88 184	0.11 816	9.90 063	42
19	9.78 263	9.88 210	0.11 790	9.90 053	41
20	9.78 280	9.88 236	0.11 764	9.90 043	40
21	9.78 296	9.88 262	0.11 738	9.90 034	39
22	9.78 313	9.88 289	0.11 711	9.90 024	38
23	9.78 329	9.88 315	0.11 685	9.90 014	37
24	9.78 346	9.88 341	0.11 659	9.90 005	36
25	9.78 362	9.88 367	0.11 633	9.89 995	35
26	9.78 379	9.88 393	0.11 607	9.89 985	34
27	9.78 395	9.88 420	0.11 580	9.89 976	33
28	9.78 412	9.88 446	0.11 554	9.89 966	32
29	9.78 428	9.88 472	0.11 528	9.89 956	31
30	9.78 445	9.88 498	0.11 502	9.89 947	30
31	9.78 461	9.88 524	0.11 476	9.89 937	29
32	9.78 478	9.88 550	0.11 450	9.89 927	28
33	9.78 494	9.88 577	0.11 423	9.89 918	27
34	9.78 510	9.88 603	0.11 397	9.89 908	26
35	9.78 527	9.88 629	0.11 371	9.89 898	25
36	9.78 543	9.88 655	0.11 345	9.89 888	24
37	9.78 560	9.88 681	0.11 319	9.89 879	23
38	9.78 576	9.88 707	0.11 293	9.89 869	22
39	9.78 592	9.88 733	0.11 267	9.89 859	21
40	9.78 609	9.88 759	0.11 241	9.89 849	20
41	9.78 625	9.88 786	0.11 214	9.89 840	19
42	9.78 642	9.88 812	0.11 188	9.89 830	18
43	9.78 658	9.88 838	0.11 162	9.89 820	17
44	9.78 674	9.88 864	0.11 136	9.89 810	16
45	9.78 691	9.88 890	0.11 110	9.89 801	15
46	9.78 707	9.88 916	0.11 084	9.89 791	14
47	9.78 723	9.88 942	0.11 058	9.89 781	13
48	9.78 739	9.88 968	0.11 032	9.89 771	12
49	9.78 756	9.88 994	0.11 006	9.89 761	11
50	9.78 772	9.89 020	0.10 980	9.89 752	10
51	9.78 788	9.89 046	0.10 954	9.89 742	9
52	9.78 805	9.89 073	0.10 927	9.89 732	8
53	9.78 821	9.89 099	0.10 901	9.89 722	7
54	9.78 837	9.89 125	0.10 875	9.89 712	6
55	9.78 853	9.89 151	0.10 849	9.89 702	5
56	9.78 869	9.89 177	0.10 823	9.89 693	4
57	9.78 886	9.89 203	0.10 797	9.89 683	3
58	9.78 902	9.89 229	0.10 771	9.89 673	2
59	9.78 918	9.89 255	0.10 745	9.89 663	1
60	9.78 934	9.89 281	0.10 719	9.89 653	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

52°

38°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.78 934	9.89 281	0.10 719	9.89 653	**60**
1	9.78 950	9.89 307	0.10 693	9.89 643	59
2	9.78 967	9.89 333	0.10 667	9.89 633	58
3	9.78 983	9.89 359	0.10 641	9.89 624	57
4	9.78 999	9.89 385	0.10 615	9.89 614	56
5	9.79 015	9.89 411	0.10 589	9.89 604	55
6	9.79 031	9.89 437	0.10 563	9.89 594	54
7	9.79 047	9.89 463	0.10 537	9.89 584	53
8	9.79 063	9.89 489	0.10 511	9.89 574	52
9	9.79 079	9.89 515	0.10 485	9.89 564	51
10	9.79 095	9.89 541	0.10 459	9.89 554	**50**
11	9.79 111	9.89 567	0.10 433	9.89 544	49
12	9.79 128	9.89 593	0.10 407	9.89 534	48
13	9.79 144	9.89 619	0.10 381	9.89 524	47
14	9.79 160	9.89 645	0.10 355	9.89 514	46
15	9.79 176	9.89 671	0.10 329	9.89 504	45
16	9.79 192	9.89 697	0.10 303	9.89 495	44
17	9.79 208	9.89 723	0.10 277	9.89 485	43
18	9.79 224	9.89 749	0.10 251	9.89 475	42
19	9.79 240	9.89 775	0.10 225	9.89 465	41
20	9.79 256	9.89 801	0.10 199	9.89 455	**40**
21	9.79 272	9.89 827	0.10 173	9.89 445	39
22	9.79 288	9.89 853	0.10 147	9.89 435	38
23	9.79 304	9.89 879	0.10 121	9.89 425	37
24	9.79 319	9.89 905	0.10 095	9.89 415	36
25	9.79 335	9.89 931	0.10 069	9.89 405	35
26	9.79 351	9.89 957	0.10 043	9.89 395	34
27	9.79 367	9.89 983	0.10 017	9.89 385	33
28	9.79 383	9.90 009	0.09 991	9.89 375	32
29	9.79 399	9.90 035	0.09 965	9.89 364	31
30	9.79 415	9.90 061	0.09 939	9.89 354	**30**
31	9.79 431	9.90 086	0.09 914	9.89 344	29
32	9.79 447	9.90 112	0.09 888	9.89 334	28
33	9.79 463	9.90 138	0.09 862	9.89 324	27
34	9.79 478	9.90 164	0.09 836	9.89 314	26
35	9.79 494	9.90 190	0.09 810	9.89 304	25
36	9.79 510	9.90 216	0.09 784	9.89 294	24
37	9.79 526	9.90 242	0.09 758	9.89 284	23
38	9.79 542	9.90 268	0.09 732	9.89 274	22
39	9.79 558	9.90 294	0.09 706	9.89 264	21
40	9.79 573	9.90 320	0.09 680	9.89 254	**20**
41	9.79 589	9.90 346	0.09 654	9.89 244	19
42	9.79 605	9.90 371	0.09 629	9.89 233	18
43	9.79 621	9.90 397	0.09 603	9.89 223	17
44	9.79 636	9.90 423	0.09 577	9.89 213	16
45	9.79 652	9.90 449	0.09 551	9.89 203	15
46	9.79 668	9.90 475	0.09 525	9.89 193	14
47	9.79 684	9.90 501	0.09 499	9.89 183	13
48	9.79 699	9.90 527	0.09 473	9.89 173	12
49	9.79 715	9.90 553	0.09 447	9.89 162	11
50	9.79 731	9.90 578	0.09 422	9.89 152	**10**
51	9.79 746	9.90 604	0.09 396	9.89 142	9
52	9.79 762	9.90 630	0.09 370	9.89 132	8
53	9.79 778	9.90 656	0.09 344	9.89 122	7
54	9.79 793	9.90 682	0.09 318	9.89 112	6
55	9.79 809	9.90 708	0.09 292	9.89 101	5
56	9.79 825	9.90 734	0.09 266	9.89 091	4
57	9.79 840	9.90 759	0.09 241	9.89 081	3
58	9.79 856	9.90 785	0.09 215	9.89 071	2
59	9.79 872	9.90 811	0.09 189	9.89 060	1
60	9.79 887	9.90 837	0.09 163	9.89 050	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

51°

39°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.79 887	9.90 837	0.09 163	9.89 050	**60**
1	9.79 903	9.90 863	0.09 137	9.89 040	59
2	9.79 918	9.90 889	0.09 111	9.89 030	58
3	9.79 934	9.90 914	0.09 086	9.89 020	57
4	9.79 950	9.90 940	0.09 060	9.89 009	56
5	9.79 965	9.90 966	0.09 034	9.88 999	55
6	9.79 981	9.90 992	0.09 008	9.88 989	54
7	9.79 996	9.91 018	0.08 982	9.88 978	53
8	9.80 012	9.91 043	0.08 957	9.88 968	52
9	9.80 027	9.91 069	0.08 931	9.88 958	51
10	9.80 043	9.91 095	0.08 905	9.88 948	**50**
11	9.80 058	9.91 121	0.08 879	9.88 937	49
12	9.80 074	9.91 147	0.08 853	9.88 927	48
13	9.80 089	9.91 172	0.08 828	9.88 917	47
14	9.80 105	9.91 198	0.08 802	9.88 906	46
15	9.80 120	9.91 224	0.08 776	9.88 896	45
16	9.80 136	9.91 250	0.08 750	9.88 886	44
17	9.80 151	9.91 276	0.08 724	9.88 875	43
18	9.80 166	9.91 301	0.08 699	9.88 865	42
19	9.80 182	9.91 327	0.08 673	9.88 855	41
20	9.80 197	9.91 353	0.08 647	9.88 844	**40**
21	9.80 213	9.91 379	0.08 621	9.88 834	39
22	9.80 228	9.91 404	0.08 596	9.88 824	38
23	9.80 244	9.91 430	0.08 570	9.88 813	37
24	9.80 259	9.91 456	0.08 544	9.88 803	36
25	9.80 274	9.91 482	0.08 518	9.88 793	35
26	9.80 290	9.91 507	0.08 493	9.88 782	34
27	9.80 305	9.91 533	0.08 467	9.88 772	33
28	9.80 320	9.91 559	0.08 441	9.88 761	32
29	9.80 336	9.91 585	0.08 415	9.88 751	31
30	9.80 351	9.91 610	0.08 390	9.88 741	**30**
31	9.80 366	9.91 636	0.08 364	9.88 730	29
32	9.80 382	9.91 662	0.08 338	9.88 720	28
33	9.80 397	9.91 688	0.08 312	9.88 709	27
34	9.80 412	9.91 713	0.08 287	9.88 699	26
35	9.80 428	9.91 739	0.08 261	9.88 688	25
36	9.80 443	9.91 765	0.08 235	9.88 678	24
37	9.80 458	9.91 791	0.08 209	9.88 668	23
38	9.80 473	9.91 816	0.08 184	9.88 657	22
39	9.80 489	9.91 842	0.08 158	9.88 647	21
40	9.80 504	9.91 868	0.08 132	9.88 636	**20**
41	9.80 519	9.91 893	0.08 107	9.88 626	19
42	9.80 534	9.91 919	0.08 081	9.88 615	18
43	9.80 550	9.91 945	0.08 055	9.88 605	17
44	9.80 565	9.91 971	0.08 029	9.88 594	16
45	9.80 580	9.91 996	0.08 004	9.88 584	15
46	9.80 595	9.92 022	0.07 978	9.88 573	14
47	9.80 610	9.92 048	0.07 952	9.88 563	13
48	9.80 625	9.92 073	0.07 927	9.88 552	12
49	9.80 641	9.92 099	0.07 901	9.88 542	11
50	9.80 656	9.92 125	0.07 875	9.88 531	**10**
51	9.80 671	9.92 150	0.07 850	9.88 521	9
52	9.80 686	9.92 176	0.07 824	9.88 510	8
53	9.80 701	9.92 202	0.07 798	9.88 499	7
54	9.80 716	9.92 227	0.07 773	9.88 489	6
55	9.80 731	9.92 253	0.07 747	9.88 478	5
56	9.80 746	9.92 279	0.07 721	9.88 468	4
57	9.80 762	9.92 304	0.07 696	9.88 457	3
58	9.80 777	9.92 330	0.07 670	9.88 447	2
59	9.80 792	9.92 356	0.07 644	9.88 436	1
60	9.80 807	9.92 381	0.07 619	9.88 425	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

50°

40°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.80 807	9.92 381	0.07 619	9.88 425	60
1	9.80 822	9.92 407	0.07 593	9.88 415	59
2	9.80 837	9.92 433	0.07 567	9.88 404	58
3	9.80 852	9.92 458	0.07 542	9.88 394	57
4	9.80 867	9.92 484	0.07 516	9.88 383	56
5	9.80 882	9.92 510	0.07 490	9.88 372	55
6	9.80 897	9.92 535	0.07 465	9.88 362	54
7	9.80 912	9.92 561	0.07 439	9.88 351	53
8	9.80 927	9.92 587	0.07 413	9.88 340	52
9	9.80 942	9.92 612	0.07 388	9.88 330	51
10	9.80 957	9.92 638	0.07 362	9.88 319	50
11	9.80 972	9.92 663	0.07 337	9.88 308	49
12	9.80 987	9.92 689	0.07 311	9.88 298	48
13	9.81 002	9.92 715	0.07 285	9.88 287	47
14	9.81 017	9.92 740	0.07 260	9.88 276	46
15	9.81 032	9.92 766	0.07 234	9.88 266	45
16	9.81 047	9.92 792	0.07 208	9.88 255	44
17	9.81 061	9.92 817	0.07 183	9.88 244	43
18	9.81 076	9.92 843	0.07 157	9.88 234	42
19	9.81 091	9.92 868	0.07 132	9.88 223	41
20	9.81 106	9.92 894	0.07 106	9.88 212	40
21	9.81 121	9.92 920	0.07 080	9.88 201	39
22	9.81 136	9.92 945	0.07 055	9.88 191	38
23	9.81 151	9.92 971	0.07 029	9.88 180	37
24	9.81 166	9.92 996	0.07 004	9.88 169	36
25	9.81 180	9.93 022	0.06 978	9.88 158	35
26	9.81 195	9.93 048	0.06 952	9.88 148	34
27	9.81 210	9.93 073	0.06 927	9.88 137	33
28	9.81 225	9.93 099	0.06 901	9.88 126	32
29	9.81 240	9.93 124	0.06 876	9.88 115	31
30	9.81 254	9.93 150	0.06 850	9.88 105	30
31	9.81 269	9.93 175	0.06 825	9.88 094	29
32	9.81 284	9.93 201	0.06 799	9.88 083	28
33	9.81 299	9.93 227	0.06 773	9.88 072	27
34	9.81 314	9.93 252	0.06 748	9.88 061	26
35	9.81 328	9.93 278	0.06 722	9.88 051	25
36	9.81 343	9.93 303	0.06 697	9.88 040	24
37	9.81 358	9.93 329	0.06 671	9.88 029	23
38	9.81 372	9.93 354	0.06 646	9.88 018	22
39	9.81 387	9.93 380	0.06 620	9.88 007	21
40	9.81 402	9.93 406	0.06 594	9.87 996	20
41	9.81 417	9.93 431	0.06 569	9.87 985	19
42	9.81 431	9.93 457	0.06 543	9.87 975	18
43	9.81 446	9.93 482	0.06 518	9.87 964	17
44	9.81 461	9.93 508	0.06 492	9.87 953	16
45	9.81 475	9.93 533	0.06 467	9.87 942	15
46	9.81 490	9.93 559	0.06 441	9.87 931	14
47	9.81 505	9.93 584	0.06 416	9.87 920	13
48	9.81 519	9.93 610	0.06 390	9.87 909	12
49	9.81 534	9.93 636	0.06 364	9.87 898	11
50	9.81 549	9.93 661	0.06 339	9.87 887	10
51	9.81 563	9.93 687	0.06 313	9.87 877	9
52	9.81 578	9.93 712	0.06 288	9.87 866	8
53	9.81 592	9.93 738	0.06 262	9.87 855	7
54	9.81 607	9.93 763	0.06 237	9.87 844	6
55	9.81 622	9.93 789	0.06 211	9.87 833	5
56	9.81 636	9.93 814	0.06 186	9.87 822	4
57	9.81 651	9.93 840	0.06 160	9.87 811	3
58	9.81 665	9.93 865	0.06 135	9.87 800	2
59	9.81 680	9.93 891	0.06 109	9.87 789	1
60	9.81 694	9.93 916	0.06 084	9.87 778	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

49°

41°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.81 694	9.93 916	0.06 084	9.87 778	**60**
1	9.81 709	9.93 942	0.06 058	9.87 767	59
2	9.81 723	9.93 967	0.06 033	9.87 756	58
3	9.81 738	9.93 993	0.06 007	9.87 745	57
4	9.81 752	9.94 018	0.05 982	9.87 734	56
5	9.81 767	9.94 044	0.05 956	9.87 723	55
6	9.81 781	9.94 069	0.05 931	9.87 712	54
7	9.81 796	9.94 095	0.05 905	9.87 701	53
8	9.81 810	9.94 120	0.05 880	9.87 690	52
9	9.81 825	9.94 146	0.05 854	9.87 679	51
10	9.81 839	9.94 171	0.05 829	9.87 668	**50**
11	9.81 854	9.94 197	0.05 803	9.87 657	49
12	9.81 868	9.94 222	0.05 778	9.87 646	48
13	9.81 882	9.94 248	0.05 752	9.87 635	47
14	9.81 897	9.94 273	0.05 727	9.87 624	46
15	9.81 911	9.94 299	0.05 701	9.87 613	45
16	9.81 926	9.94 324	0.05 676	9.87 601	44
17	9.81 940	9.94 350	0.05 650	9.87 590	43
18	9.81 955	9.94 375	0.05 625	9.87 579	42
19	9.81 969	9.94 401	0.05 599	9.87 568	41
20	9.81 983	9.94 426	0.05 574	9.87 557	**40**
21	9.81 998	9.94 452	0.05 548	9.87 546	39
22	9.82 012	9.94 477	0.05 523	9.87 535	38
23	9.82 026	9.94 503	0.05 497	9.87 524	37
24	9.82 041	9.94 528	0.05 472	9.87 513	36
25	9.82 055	9.94 554	0.05 446	9.87 501	35
26	9.82 069	9.94 579	0.05 421	9.87 490	34
27	9.82 084	9.94 604	0.05 396	9.87 479	33
28	9.82 098	9.94 630	0.05 370	9.87 468	32
29	9.82 112	9.94 655	0.05 345	9.87 457	31
30	9.82 126	9.94 681	0.05 319	9.87 446	**30**
31	9.82 141	9.94 706	0.05 294	9.87 434	29
32	9.82 155	9.94 732	0.05 268	9.87 423	28
33	9.82 169	9.94 757	0.05 243	9.87 412	27
34	9.82 184	9.94 783	0.05 217	9.87 401	26
35	9.82 198	9.94 808	0.05 192	9.87 390	25
36	9.82 212	9.94 834	0.05 166	9.87 378	24
37	9.82 226	9.94 859	0.05 141	9.87 367	23
38	9.82 240	9.94 884	0.05 116	9.87 356	22
39	9.82 255	9.94 910	0.05 090	9.87 345	21
40	9.82 269	9.94 935	0.05 065	9.87 334	**20**
41	9.82 283	9.94 961	0.05 039	9.87 322	19
42	9.82 297	9.94 986	0.05 014	9.87 311	18
43	9.82 311	9.95 012	0.04 988	9.87 300	17
44	9.82 326	9.95 037	0.04 963	9.87 288	16
45	9.82 340	9.95 062	0.04 938	9.87 277	15
46	9.82 354	9.95 088	0.04 912	9.87 266	14
47	9.82 368	9.95 113	0.04 887	9.87 255	13
48	9.82 382	9.95 139	0.04 861	9.87 243	12
49	9.82 396	9.95 164	0.04 836	9.87 232	11
50	9.82 410	9.95 190	0.04 810	9.87 221	**10**
51	9.82 424	9.95 215	0.04 785	9.87 209	9
52	9.82 439	9.95 240	0.04 760	9.87 198	8
53	9.82 453	9.95 266	0.04 734	9.87 187	7
54	9.82 467	9.95 291	0.04 709	9.87 175	6
55	9.82 481	9.95 317	0.04 683	9.87 164	5
56	9.82 495	9.95 342	0.04 658	9.87 153	4
57	9.82 509	9.95 368	0.04 632	9.87 141	3
58	9.82 523	9.95 393	0.04 607	9.87 130	2
59	9.82 537	9.95 418	0.04 582	9.87 119	1
60	9.82 551	9.95 444	0.04 556	9.87 107	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

48°

42°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.82 551	9.95 444	0.04 556	9.87 107	60
1	9.82 565	9.95 469	0.04 531	9.87 096	59
2	9.82 579	9.95 495	0.04 505	9.87 085	58
3	9.82 593	9.95 520	0.04 480	9.87 073	57
4	9.82 607	9.95 545	0.04 455	9.87 062	56
5	9.82 621	9.95 571	0.04 429	9.87 050	55
6	9.82 635	9.95 596	0.04 404	9.87 039	54
7	9.82 649	9.95 622	0.04 378	9.87 028	53
8	9.82 663	9.95 647	0.04 353	9.87 016	52
9	9.82 677	9.95 672	0.04 328	9.87 005	51
10	9.82 691	9.95 698	0.04 302	9.86 993	50
11	9.82 705	9.95 723	0.04 277	9.86 982	49
12	9.82 719	9.95 748	0.04 252	9.86 970	48
13	9.82 733	9.95 774	0.04 226	9.86 959	47
14	9.82 747	9.95 799	0.04 201	9.86 947	46
15	9.82 761	9.95 825	0.04 175	9.86 936	45
16	9.82 775	9.95 850	0.04 150	9.86 924	44
17	9.82 788	9.95 875	0.04 125	9.86 913	43
18	9.82 802	9.95 901	0.04 099	9.86 902	42
19	9.82 816	9.95 926	0.04 074	9.86 890	41
20	9.82 830	9.95 952	0.04 048	9.86 879	40
21	9.82 844	9.95 977	0.04 023	9.86 867	39
22	9.82 858	9.96 002	0.03 998	9.86 855	38
23	9.82 872	9.96 028	0.03 972	9.86 844	37
24	9.82 885	9.96 053	0.03 947	9.86 832	36
25	9.82 899	9.96 078	0.03 922	9.86 821	35
26	9.82 913	9.96 104	0.03 896	9.86 809	34
27	9.82 927	9.96 129	0.03 871	9.86 798	33
28	9.82 941	9.96 155	0.03 845	9.86 786	32
29	9.82 955	9.96 180	0.03 820	9.86 775	31
30	9.82 968	9.96 205	0.03 795	9.86 763	30
31	9.82 982	9.96 231	0.03 769	9.86 752	29
32	9.82 996	9.96 256	0.03 744	9.86 740	28
33	9.83 010	9.96 281	0.03 719	9.86 728	27
34	9.83 023	9.96 307	0.03 693	9.86 717	26
35	9.83 037	9.96 332	0.03 668	9.86 705	25
36	9.83 051	9.96 357	0.03 643	9.86 694	24
37	9.83 065	9.96 383	0.03 617	9.86 682	23
38	9.83 078	9.96 408	0.03 592	9.86 670	22
39	9.83 092	9.96 433	0.03 567	9.86 659	21
40	9.83 106	9.96 459	0.03 541	9.86 647	20
41	9.83 120	9.96 484	0.03 516	9.86 635	19
42	9.83 133	9.96 510	0.03 490	9.86 624	18
43	9.83 147	9.96 535	0.03 465	9.86 612	17
44	9.83 161	9.96 560	0.03 440	9.86 600	16
45	9.83 174	9.96 586	0.03 414	9.86 589	15
46	9.83 188	9.96 611	0.03 389	9.86 577	14
47	9.83 202	9.96 636	0.03 364	9.86 565	13
48	9.83 215	9.96 662	0.03 338	9.86 554	12
49	9.83 229	9.96 687	0.03 313	9.86 542	11
50	9.83 242	9.96 712	0.03 288	9.86 530	10
51	9.83 256	9.96 738	0.03 262	9.86 518	9
52	9.83 270	9.96 763	0.03 237	9.86 507	8
53	9.83 283	9.96 788	0.03 212	9.86 495	7
54	9.83 297	9.96 814	0.03 186	9.86 483	6
55	9.83 310	9.96 839	0.03 161	9.86 472	5
56	9.83 324	9.96 864	0.03 136	9.86 460	4
57	9.83 338	9.96 890	0.03 110	9.86 448	3
58	9.83 351	9.96 915	0.03 085	9.86 436	2
59	9.83 365	9.96 940	0.03 060	9.86 425	1
60	9.83 378	9.96 966	0.03 034	9.86 413	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

47°

43°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.83 378	9.96 966	0.03 034	9.86 413	60
1	9.83 392	9.96 991	0.03 009	9.86 401	59
2	9.83 405	9.97 016	0.02 984	9.86 389	58
3	9.83 419	9.97 042	0.02 958	9.86 377	57
4	9.83 432	9.97 067	0.02 933	9.86 366	56
5	9.83 446	9.97 092	0.02 908	9.86 354	55
6	9.83 459	9.97 118	0.02 882	9.86 342	54
7	9.83 473	9.97 143	0.02 857	9.86 330	53
8	9.83 486	9.97 168	0.02 832	9.86 318	52
9	9.83 500	9.97 193	0.02 807	9.86 306	51
10	9.83 513	9.97 219	0.02 781	9.86 295	50
11	9.83 527	9.97 244	0.02 756	9.86 283	49
12	9.83 540	9.97 269	0.02 731	9.86 271	48
13	9.83 554	9.97 295	0.02 705	9.86 259	47
14	9.83 567	9.97 320	0.02 680	9.86 247	46
15	9.83 581	9.97 345	0.02 655	9.86 235	45
16	9.83 594	9.97 371	0.02 629	9.86 223	44
17	9.83 608	9.97 396	0.02 604	9.86 211	43
18	9.83 621	9.97 421	0.02 579	9.86 200	42
19	9.83 634	9.97 447	0.02 553	9.86 188	41
20	9.83 648	9.97 472	0.02 528	9.86 176	40
21	9.83 661	9.97 497	0.02 503	9.86 164	39
22	9.83 674	9.97 523	0.02 477	9.86 152	38
23	9.83 688	9.97 548	0.02 452	9.86 140	37
24	9.83 701	9.97 573	0.02 427	9.86 128	36
25	9.83 715	9.97 598	0.02 402	9.86 116	35
26	9.83 728	9.97 624	0.02 376	9.86 104	34
27	9.83 741	9.97 649	0.02 351	9.86 092	33
28	9.83 755	9.97 674	0.02 326	9.86 080	32
29	9.83 768	9.97 700	0.02 300	9.86 068	31
30	9.83 781	9.97 725	0.02 275	9.86 056	30
31	9.83 795	9.97 750	0.02 250	9.86 044	29
32	9.83 808	9.97 776	0.02 224	9.86 032	28
33	9.83 821	9.97 801	0.02 199	9.86 020	27
34	9.83 834	9.97 826	0.02 174	9.86 008	26
35	9.83 848	9.97 851	0.02 149	9.85 996	25
36	9.83 861	9.97 877	0.02 123	9.85 984	24
37	9.83 874	9.97 902	0.02 098	9.85 972	23
38	9.83 887	9.97 927	0.02 073	9.85 960	22
39	9.83 901	9.97 953	0.02 047	9.85 948	21
40	9.83 914	9.97 978	0.02 022	9.85 936	20
41	9.83 927	9.98 003	0.01 997	9.85 924	19
42	9.83 940	9.98 029	0.01 971	9.85 912	18
43	9.83 954	9.98 054	0.01 946	9.85 900	17
44	9.83 967	9.98 079	0.01 921	9.85 888	16
45	9.83 980	9.98 104	0.01 896	9.85 876	15
46	9.83 993	9.98 130	0.01 870	9.85 864	14
47	9.84 006	9.98 155	0.01 845	9.85 851	13
48	9.84 020	9.98 180	0.01 820	9.85 839	12
49	9.84 033	9.98 206	0.01 794	9.85 827	11
50	9.84 046	9.98 231	0.01 769	9.85 815	10
51	9.84 059	9.98 256	0.01 744	9.85 803	9
52	9.84 072	9.98 281	0.01 719	9.85 791	8
53	9.84 085	9.98 307	0.01 693	9.85 779	7
54	9.84 098	9.98 332	0.01 668	9.85 766	6
55	9.84 112	9.98 357	0.01 643	9.85 754	5
56	9.84 125	9.98 383	0.01 617	9.85 742	4
57	9.84 138	9.98 408	0.01 592	9.85 730	3
58	9.84 151	9.98 433	0.01 567	9.85 718	2
59	9.84 164	9.98 458	0.01 542	9.85 706	1
60	9.84 177	9.98 484	0.01 516	9.85 693	0
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

46°

44°

′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	
0	9.84 177	9.98 484	0.01 516	9.85 693	**60**
1	9.84 190	9.98 509	0.01 491	9.85 681	59
2	9.84 203	9.98 534	0.01 466	9.85 669	58
3	9.84 216	9.98 560	0.01 440	9.85 657	57
4	9.84 229	9.98 585	0.01 415	9.85 645	56
5	9.84 242	9.98 610	0.01 390	9.85 632	55
6	9.84 255	9.98 635	0.01 365	9.85 620	54
7	9.84 269	9.98 661	0.01 339	9.85 608	53
8	9.84 282	9.98 686	0.01 314	9.85 596	52
9	9.84 295	9.98 711	0.01 289	9.85 583	51
10	9.84 308	9.98 737	0.01 263	9.85 571	**50**
11	9.84 321	9.98 762	0.01 238	9.85 559	49
12	9.84 334	9.98 787	0.01 213	9.85 547	48
13	9.84 347	9.98 812	0.01 188	9.85 534	47
14	9.84 360	9.98 838	0.01 162	9.85 522	46
15	9.84 373	9.98 863	0.01 137	9.85 510	45
16	9.84 385	9.98 888	0.01 112	9.85 497	44
17	9.84 398	9.98 913	0.01 087	9.85 485	43
18	9.84 411	9.98 939	0.01 061	9.85 473	42
19	9.84 424	9.98 964	0.01 036	9.85 460	41
20	9.84 437	9.98 989	0.01 011	9.85 448	**40**
21	9.84 450	9.99 015	0.00 985	9.85 436	39
22	9.84 463	9.99 040	0.00 960	9.85 423	38
23	9.84 476	9.99 065	0.00 935	9.85 411	37
24	9.84 489	9.99 090	0.00 910	9.85 399	36
25	9.84 502	9.99 116	0.00 884	9.85 386	35
26	9.84 515	9.99 141	0.00 859	9.85 374	34
27	9.84 528	9.99 166	0.00 834	9.85 361	33
28	9.84 540	9.99 191	0.00 809	9.85 349	32
29	9.84 553	9.99 217	0.00 783	9.85 337	31
30	9.84 566	9.99 242	0.00 758	9.85 324	**30**
31	9.84 579	9.99 267	0.00 733	9.85 312	29
32	9.84 592	9.99 293	0.00 707	9.85 299	28
33	9.84 605	9.99 318	0.00 682	9.85 287	27
34	9.84 618	9.99 343	0.00 657	9.85 274	26
35	9.84 630	9.99 368	0.00 632	9.85 262	25
36	9.84 643	9.99 394	0.00 606	9.85 250	24
37	9.84 656	9.99 419	0.00 581	9.85 237	23
38	9.84 669	9.99 444	0.00 556	9.85 225	22
39	9.84 682	9.99 469	0.00 531	9.85 212	21
40	9.84 694	9.99 495	0.00 505	9.85 200	**20**
41	9.84 707	9.99 520	0.00 480	9.85 187	19
42	9.84 720	9.99 545	0.00 455	9.85 175	18
43	9.84 733	9.99 570	0.00 430	9.85 162	17
44	9.84 745	9.99 596	0.00 404	9.85 150	16
45	9.84 758	9.99 621	0.00 379	9.85 137	15
46	9.84 771	9.99 646	0.00 354	9.85 125	14
47	9.84 784	9.99 672	0.00 328	9.85 112	13
48	9.84 796	9.99 697	0.00 303	9.85 100	12
49	9.84 809	9.99 722	0.00 278	9.85 087	11
50	9.84 822	9.99 747	0.00 253	9.85 074	**10**
51	9.84 835	9.99 773	0.00 227	9.85 062	9
52	9.84 847	9.99 798	0.00 202	9.85 049	8
53	9.84 860	9.99 823	0.00 177	9.85 037	7
54	9.84 873	9.99 848	0.00 152	9.85 024	6
55	9.84 885	9.99 874	0.00 126	9.85 012	5
56	9.84 898	9.99 899	0.00 101	9.84 999	4
57	9.84 911	9.99 924	0.00 076	9.84 986	3
58	9.84 923	9.99 949	0.00 051	9.84 974	2
59	9.84 936	9.99 975	0.00 025	9.84 961	1
60	9.84 949	10.00 000	0.00 000	9.84 949	**0**
	L. Cos.	L. Cot.	L. Tan.	L. Sin.	′

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III.

FOUR-PLACE TABLES.

(1) Logarithms of Numbers.

(2) Logarithms of the Sine, Cosine, Tangent, and Cotangent, at Intervals of Ten Minutes from 0° to 90°.

(3) Values of the Sine, Cosine, Tangent, and Cotangent, at Intervals of Ten Minutes from 0° to 90°.

N	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

°	′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	′	°
0	00	∞	∞	∞	10.0000	00	90
	10	7.4637	7.4637	2.5363	0000	50	
	20	7648	7648	2352	0000	40	
	30	9408	9409	0591	0000	30	
	40	8.0658	8.0658	1.9342	0000	20	
	50	1627	1627	8373	0000	10	
1	00	8.2419	8.2419	1.7581	9.9999	00	89
	10	3088	3089	6911	9999	50	
	20	3668	3669	6331	9999	40	
	30	4179	4181	5819	9999	30	
	40	4637	4638	5362	9998	20	
	50	5050	5053	4947	9998	10	
2	00	8.5428	8.5431	1.4569	9.9997	00	88
	10	5776	5779	4221	9997	50	
	20	6097	6101	3899	9996	40	
	30	6397	6401	3599	9996	30	
	40	6677	6682	3318	9995	20	
	50	6940	6945	3055	9995	10	
3	00	8.7188	8.7194	1.2806	9.9994	00	87
	10	7423	7429	2571	9993	50	
	20	7645	7652	2348	9993	40	
	30	7857	7865	2135	9992	30	
	40	8059	8067	1933	9991	20	
	50	8251	8261	1739	9990	10	
4	00	8.8436	8.8446	1.1554	9.9989	00	86
	10	8613	8624	1376	9989	50	
	20	8783	8795	1205	9988	40	
	30	8946	8960	1040	9987	30	
	40	9104	9118	0882	9986	20	
	50	9256	9272	0728	9985	10	
5	00	8.9403	8.9420	1.0580	9.9983	00	85
	10	9545	9563	0437	9982	50	
	20	9682	9701	0299	9981	40	
	30	9816	9836	0164	9980	30	
	40	9945	9966	0034	9979	20	
	50	9.0070	9.0093	0.9907	9977	10	
6	00	9.0192	9.0216	0.9784	9.9976	00	84
	10	0311	0336	9664	9975	50	
	20	0426	0453	9547	9973	40	
	30	0539	0567	9433	9972	30	
	40	0648	0678	9322	9971	20	
	50	0755	0786	9214	9969	10	
7	00	9.0859	9.0891	0.9109	9.9968	00	83
	10	0961	0995	9005	9966	50	
	20	1060	1096	8904	9964	40	
	30	1157	1194	8806	9963	30	
	40	1252	1291	8709	9961	20	
	50	1345	1385	8615	9959	10	
8	00	9.1436	9.1478	0.8522	9.9958	00	82
	10	1525	1569	8431	9956	50	
	20	1612	1658	8342	9954	40	
	30	1697	1745	8255	9952	30	
	40	1781	1831	8169	9950	20	
	50	1863	1915	8085	9948	10	
9	00	9.1943	9.1997	0.8003	9.9946	00	81
		L. Cos.	L. Cot.	L. Tan.	L. Sin.	′	°

°	′	L. Sin.	L. Tan.	L. Cot.	L. Cos.		
9	00	9.1943	9.1997	0.8003	9.9946	00	**81**
	10	2022	2078	7922	9944	50	
	20	2100	2158	7842	9942	40	
	30	2176	2236	7764	9940	30	
	40	2251	2313	7687	9938	20	
	50	2324	2389	7611	9936	10	
10	00	9.2397	9.2463	0.7537	9.9934	00	**80**
	10	2468	2536	7464	9931	50	
	20	2538	2609	7391	9929	40	
	30	2606	2680	7320	9927	30	
	40	2674	2750	7250	9924	20	
	50	2740	2819	7181	9922	10	
11	00	9.2806	9.2887	0.7113	9.9919	00	**79**
	10	2870	2953	7047	9917	50	
	20	2934	3020	6980	9914	40	
	30	2997	3085	6915	9912	30	
	40	3058	3149	6851	9909	20	
	50	3119	3212	6788	9907	10	
12	00	9.3179	9.3275	0.6725	9.9904	00	**78**
	10	3238	3336	6664	9901	50	
	20	3296	3397	6603	9899	40	
	30	3353	3458	6542	9896	30	
	40	3410	3517	6483	9893	20	
	50	3466	3576	6424	9890	10	
13	00	9.3521	9.3634	0.6366	9.9887	00	**77**
	10	3575	3691	6309	9884	50	
	20	3629	3748	6252	9881	40	
	30	3682	3804	6196	9878	30	
	40	3734	3859	6141	9875	20	
	50	3786	3914	6086	9872	10	
14	00	9.3837	9.3968	0.6032	9.9869	00	**76**
	10	3887	4021	5979	9866	50	
	20	3937	4074	5926	9863	40	
	30	3986	4127	5873	9859	30	
	40	4035	4178	5822	9856	20	
	50	4083	4230	5770	9853	10	
15	00	9.4130	9.4281	0.5719	9.9849	00	**75**
	10	4177	4331	5669	9846	50	
	20	4223	4381	5619	9843	40	
	30	4269	4430	5570	9839	30	
	40	4314	4479	5521	9836	20	
	50	4359	4527	5473	9832	10	
16	00	9.4403	9.4575	0.5425	9.9828	00	**74**
	10	4447	4622	5378	9825	50	
	20	4491	4669	5331	9821	40	
	30	4533	4716	5284	9817	30	
	40	4576	4762	5238	9814	20	
	50	4618	4808	5192	9810	10	
17	00	9.4659	9.4853	0.5147	9.9806	00	**73**
	10	4700	4898	5102	9802	50	
	20	4741	4943	5057	9798	40	
	30	4781	4987	5013	9794	30	
	40	4821	5031	4969	9790	20	
	50	4861	5075	4925	9786	10	
18	00	9.4900	9.5118	0.4882	9.9782	00	**72**
		L. Cos.	L. Cot.	L. Tan.	L. Sin.	′	°

°	′	L. Sin.	L. Tan.	L. Cot.	L. Cos.	′	°
18	00	9.4900	9.5118	0.4882	9.9782	00	72
	10	4939	5161	4839	9778	50	
	20	4977	5203	4797	9774	40	
	30	5015	5245	4755	9770	30	
	40	5052	5287	4713	9765	20	
	50	5090	5329	4671	9761	10	
19	00	9.5126	9.5370	0.4630	9.9757	00	71
	10	5163	5411	4589	9752	50	
	20	5199	5451	4549	9748	40	
	30	5235	5491	4509	9743	30	
	40	5270	5531	4469	9739	20	
	50	5306	5571	4429	9734	10	
20	00	9.5341	9.5611	0.4389	9.9730	00	70
	10	5375	5650	4350	9725	50	
	20	5409	5689	4311	9721	40	
	30	5443	5727	4273	9716	30	
	40	5477	5766	4234	9711	20	
	50	5510	5804	4196	9706	10	
21	00	9.5543	9.5842	0.4158	9.9702	00	69
	10	5576	5879	4121	9697	50	
	20	5609	5917	4083	9692	40	
	30	5641	5954	4046	9687	30	
	40	5673	5991	4009	9682	20	
	50	5704	6028	3972	9677	10	
22	00	9.5736	9.6064	0.3936	9.9672	00	68
	10	5767	6100	3900	9667	50	
	20	5798	6136	3864	9661	40	
	30	5828	6172	3828	9656	30	
	40	5859	6208	3792	9651	20	
	50	5889	6243	3757	9646	10	
23	00	9.5919	9.6279	0.3721	9.9640	00	67
	10	5948	6314	3686	9635	50	
	20	5978	6348	3652	9629	40	
	30	6007	6383	3617	9624	30	
	40	6036	6417	3583	9618	20	
	50	6065	6452	3548	9613	10	
24	00	9.6093	9.6486	0.3514	9.9607	00	66
	10	6121	6520	3480	9602	50	
	20	6149	6553	3447	9596	40	
	30	6177	6587	3413	9590	30	
	40	6205	6620	3380	9584	20	
	50	6232	6654	3346	9579	10	
25	00	9.6259	9.6687	0.3313	9.9573	00	65
	10	6286	6720	3280	9567	50	
	20	6313	6752	3248	9561	40	
	30	6340	6785	3215	9555	30	
	40	6366	6817	3183	9549	20	
	50	6392	6850	3150	9543	10	
26	00	9.6418	9.6882	0.3118	9.9537	00	64
	10	6444	6914	3086	9530	50	
	20	6470	6946	3054	9524	40	
	30	6495	6977	3023	9518	30	
	40	6521	7009	2991	9512	20	
	50	6546	7040	2960	9505	10	
27	00	9.6570	9.7072	0.2928	9.9499	00	63
		L. Cos.	L. Cot.	L. Tan.	L. Sin.	′	°

°	'	L. Sin.	L. Tan.	L. Cot.	L. Cos.	'	°
27	00	9.6570	9.7072	0.2928	9.9499	00	**63**
	10	6595	7103	2897	9492	50	
	20	6620	7134	2866	9486	40	
	30	6644	7165	2835	9479	30	
	40	6668	7196	2804	9473	20	
	50	6692	7226	2774	9466	10	
28	00	9.6716	9.7257	0.2743	9.9459	00	**62**
	10	6740	7287	2713	9453	50	
	20	6763	7317	2683	9446	40	
	30	6787	7348	2652	9439	30	
	40	6810	7378	2622	9432	20	
	50	6833	7408	2592	9425	10	
29	00	9.6856	9.7438	0.2562	9.9418	00	**61**
	10	6878	7467	2533	9411	50	
	20	6901	7497	2503	9404	40	
	30	6923	7526	2474	9397	30	
	40	6946	7556	2444	9390	20	
	50	6968	7585	2415	9383	10	
30	00	9.6990	9.7614	0.2386	9.9875	00	**60**
	10	7012	7644	2356	9368	50	
	20	7033	7673	2327	9361	40	
	30	7055	7701	2299	9353	30	
	40	7076	7730	2270	9346	20	
	50	7097	7759	2241	9338	10	
31	00	9.7118	9.7788	0.2212	9.9331	00	**59**
	10	7139	7816	2184	9323	50	
	20	7160	7845	2155	9315	40	
	30	7181	7873	2127	9308	30	
	40	7201	7902	2098	9300	20	
	50	7222	7930	2070	9292	10	
32	00	9.7242	9.7958	0.2042	9.9284	00	**58**
	10	7262	7986	2014	9276	50	
	20	7282	8014	1986	9268	40	
	30	7302	8042	1958	9260	30	
	40	7322	8070	1930	9252	20	
	50	7342	8097	1903	9244	10	
33	00	9.7361	9.8125	0.1875	9.9236	00	**57**
	10	7380	8153	1847	9228	50	
	20	7400	8180	1820	9219	40	
	30	7419	8208	1792	9211	30	
	40	7438	8235	1765	9203	20	
	50	7457	8263	1737	9194	10	
34	00	9.7476	9.8290	0.1710	9.9186	00	**56**
	10	7494	8317	1683	9177	50	
	20	7513	8344	1656	9169	40	
	30	7531	8371	1629	9160	30	
	40	7550	8398	1602	9151	20	
	50	7568	8425	1575	9142	10	
35	00	9.7586	9.8452	0.1548	9.9134	00	**55**
	10	7604	8479	1521	9125	50	
	20	7622	8506	1494	9116	40	
	30	7640	8533	1467	9107	30	
	40	7657	8559	1441	9098	20	
	50	7675	8586	1414	9089	10	
36	00	9.7692	9.8613	0.1387	9.9080	00	**54**
		L. Cos.	L. Cot.	L. Tan.	L. Sin.	'	°

°	′	L. Sin.	L. Tan.	L. Cot.	L. Cos.		
36	00	9.7692	9.8613	0.1387	9.9080	00	**54**
	10	7710	8639	1361	9070	50	
	20	7727	8666	1334	9061	40	
	30	7744	8692	1308	9052	30	
	40	7761	8718	1282	9042	20	
	50	7778	8745	1255	9033	10	
37	00	9.7795	9.8771	0.1229	9.9023	00	**53**
	10	7811	8797	1203	9014	50	
	20	7828	8824	1176	9004	40	
	30	7844	8850	1150	8995	30	
	40	7861	8876	1124	8985	20	
	50	7877	8902	1098	8975	10	
38	00	9.7893	9.8928	0.1072	9.8965	00	**52**
	10	7910	8954	1046	8955	50	
	20	7926	8980	1020	8945	40	
	30	7941	9006	0994	8935	30	
	40	7957	9032	0968	8925	20	
	50	7973	9058	0942	8915	10	
39	00	9.7989	9.9084	0.0916	9.8905	00	**51**
	10	8004	9110	0890	8895	50	
	20	8020	9135	0865	8884	40	
	30	8035	9161	0839	8874	30	
	40	8050	9187	0813	8864	20	
	50	8066	9212	0788	8853	10	
40	00	9.8081	9.9238	0.0762	9.8843	00	**50**
	10	8096	9264	0736	8832	50	
	20	8111	9289	0711	8821	40	
	30	8125	9315	0685	8810	30	
	40	8140	9341	0659	8800	20	
	50	8155	9366	0634	8789	10	
41	00	9.8169	9.9392	0.0608	9.8778	00	**49**
	10	8184	9417	0583	8767	50	
	20	8198	9443	0557	8756	40	
	30	8213	9468	0532	8745	30	
	40	8227	9494	0506	8733	20	
	50	8241	9519	0481	8722	10	
42	00	9.8255	9.9544	0.0456	9.8711	00	**48**
	10	8269	9570	0430	8699	50	
	20	8283	9595	0405	8688	40	
	30	8297	9621	0379	8676	30	
	40	8311	9646	0354	8665	20	
	50	8324	9671	0329	8653	10	
43	00	9.8338	9.9697	0.0303	9.8641	00	**47**
	10	8351	9722	0278	8699	50	
	20	8365	9747	0253	8618	40	
	30	8378	9772	0228	8606	30	
	40	8391	9798	0202	8594	20	
	50	8405	9823	0177	8582	10	
44	00	9.8418	9.9848	0.0152	9.8569	00	**46**
	10	8431	9874	0126	8557	50	
	20	8444	9899	0101	8545	40	
	30	8457	9924	0076	8532	30	
	40	8469	9949	0051	8520	20	
	50	8482	9975	0025	8507	10	
45	00	9.8495	10.0000	0.0000	9.8495	00	**45**
		L. Cos.	L. Cot.	L. Tan.	L. Sin.	′	°

°	′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	′	°
0	00	.0000	.0000	∞	1.0000	00	**90**
	10	.0029	.0029	343.77	1.0000	50	
	20	.0058	.0058	171.89	1.0000	40	
	30	.0087	.0087	114.59	1.0000	30	
	40	.0116	.0116	85.940	.9999	20	
	50	.0145	.0145	68.750	.9999	10	
1	00	.0175	.0175	57.290	.9998	00	**89**
	10	.0204	.0204	49.104	.9998	50	
	20	.0233	.0233	42.964	.9997	40	
	30	.0262	.0262	38.188	.9997	30	
	40	.0291	.0291	34.368	.9996	20	
	50	.0320	.0320	31.242	.9995	10	
2	00	.0349	.0349	28.636	.9994	00	**88**
	10	.0378	.0378	26.432	.9993	50	
	20	.0407	.0407	24.542	.9992	40	
	30	.0436	.0437	22.904	.9990	30	
	40	.0465	.0466	21.470	.9989	20	
	50	.0494	.0495	20.206	.9988	10	
3	00	.0523	.0524	19.081	.9986	00	**87**
	10	.0552	.0553	18.075	.9985	50	
	20	.0581	.0582	17.169	.9983	40	
	30	.0610	.0612	16.350	.9981	30	
	40	.0640	.0641	15.605	.9980	20	
	50	.0669	.0670	14.924	.9978	10	
4	00	.0698	.0699	14.301	.9976	00	**86**
	10	.0727	.0729	13.727	.9974	50	
	20	.0756	.0758	13.197	.9971	40	
	30	.0785	.0787	12.706	.9969	30	
	40	.0814	.0816	12.251	.9967	20	
	50	.0843	.0846	11.826	.9964	10	
5	00	.0872	.0875	11.430	.9962	00	**85**
	10	.0901	.0904	11.059	.9959	50	
	20	.0929	.0934	10.712	.9957	40	
	30	.0958	.0963	10.385	.9954	30	
	40	.0987	.0992	10.078	.9951	20	
	50	.1016	.1022	9.7882	.9948	10	
6	00	.1045	.1051	9.5144	.9945	00	**84**
	10	.1074	.1080	9.2553	.9942	50	
	20	.1103	.1110	9.0098	.9939	40	
	30	.1132	.1139	8.7769	.9936	30	
	40	.1161	.1169	8.5555	.9932	20	
	50	.1190	.1198	8.3450	.9929	10	
7	00	.1219	.1228	8.1443	.9925	00	**83**
	10	.1248	.1257	7.9530	.9922	50	
	20	.1276	.1287	7.7704	.9918	40	
	30	.1305	.1317	7.5958	.9914	30	
	40	.1334	.1346	7.4287	.9911	20	
	50	.1363	.1376	7.2687	.9907	10	
8	00	.1392	.1405	7.1154	.9903	00	**82**
	10	.1421	.1435	6.9682	.9899	50	
	20	.1449	.1465	6.8269	.9894	40	
	30	.1478	.1495	6.6912	.9890	30	
	40	.1507	.1524	6.5606	.9886	20	
	50	.1536	.1554	6.4348	.9881	10	
9	00	.1564	.1584	6.3138	.9877	00	**81**
		N. Cos.	N. Cot.	N. Tan.	N. Sin.	′	°

°	′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	′	°
9	00	.1564	.1584	6.3138	.9877	00	81
	10	.1593	.1614	6.1970	.9872	50	
	20	.1622	.1644	6.0844	.9868	40	
	30	.1650	.1673	5.9758	.9863	30	
	40	.1679	.1703	5.8708	.9858	20	
	50	.1708	.1733	5.7694	.9853	10	
10	00	.1736	.1763	5.6713	.9848	00	80
	10	.1765	.1793	5.5764	.9843	50	
	20	.1794	.1823	5.4845	.9838	40	
	30	.1822	.1853	5.3955	.9833	30	
	40	.1851	.1883	5.3093	.9827	20	
	50	.1880	.1914	5.2257	.9822	10	
11	00	.1908	.1944	5.1446	.9816	00	79
	10	.1937	.1974	5.0658	.9811	50	
	20	.1965	.2004	4.9894	.9805	40	
	30	.1994	.2035	4.9152	.9799	30	
	40	.2022	.2065	4.8430	.9793	20	
	50	.2051	.2095	4.7729	.9787	10	
12	00	.2079	.2126	4.7046	.9781	00	78
	10	.2108	.2156	4.6382	.9775	50	
	20	.2136	.2186	4.5736	.9769	40	
	30	.2164	.2217	4.5107	.9763	30	
	40	.2193	.2247	4.4494	.9757	20	
	50	.2221	.2278	4.3897	.9750	10	
13	00	.2250	.2309	4.3315	9744	00	77
	10	.2278	.2339	4.2747	.9737	50	
	20	.2306	.2370	4.2193	.9730	40	
	30	.2334	.2401	4.1653	.9724	30	
	40	.2363	.2432	4.1126	.9717	20	
	50	.2391	.2462	4.0611	.9710	10	
14	00	.2419	.2493	4.0108	.9703	00	76
	10	.2447	.2524	3.9617	.9696	50	
	20	.2476	.2555	3.9136	.9689	40	
	30	.2504	.2586	3.8667	.9681	30	
	40	.2532	.2617	3.8208	.9674	20	
	50	.2560	.2648	3.7760	.9667	10	
15	00	.2588	.2679	3.7321	.9659	00	75
	10	.2616	.2711	3.6891	.9652	50	
	20	.2644	.2742	3.6470	.9644	40	
	30	.2672	.2773	3.6059	.9636	30	
	40	.2700	.2805	3.5656	.9628	20	
	50	.2728	.2836	3.5261	.9621	10	
16	00	.2756	.2867	3.4874	.9613	00	74
	10	.2784	.2899	3.4495	.9605	50	
	20	.2812	.2931	3.4124	.9596	40	
	30	.2840	.2962	3.3759	.9588	30	
	40	.2868	.2994	3.3402	.9580	20	
	50	.2896	.3026	3.3052	.9572	10	
17	00	.2924	.3057	3.2709	.9563	00	73
	10	.2952	.3089	3.2371	.9555	50	
	20	.2979	.3121	3.2041	.9546	40	
	30	.3007	.3153	3.1716	.9537	30	
	40	.3035	.3185	3.1397	.9528	20	
	50	.3062	.3217	3.1084	·9520	10	
18	00	.3090	.3249	3.0777	.9511	00	72
		N. Cos.	N. Cot.	N. Tan.	N. Sin.	′	°

°	′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	′	°
18	00	.3090	.3249	3.0777	.9511	00	**72**
	10	.3118	.3281	3.0475	.9502	50	
	20	.3145	.3314	3.0178	.9492	40	
	30	.3173	.3346	2.9887	.9483	30	
	40	.3201	.3378	2.9600	.9474	20	
	50	.3228	.3411	2.9319	.9465	10	
19	00	.3256	.3443	2.9042	.9455	00	**71**
	10	.3283	.3476	2.8770	.9446	50	
	20	.3311	.3508	2.8502	.9436	40	
	30	.3338	.3541	2.8239	.9426	30	
	40	.3365	.3574	2.7980	.9417	20	
	50	.3393	.3607	2.7725	.9407	10	
20	00	.3420	.3640	2.7475	.9397	00	**70**
	10	.3448	.3673	2.7228	.9387	50	
	20	.3475	.3706	2.6985	.9377	40	
	30	.3502	.3739	2.6746	.9367	30	
	40	.3529	.3772	2.6511	.9356	20	
	50	.3557	.3805	2.6279	.9346	10	
21	00	.3584	.3839	2.6051	.9336	00	**69**
	10	.3611	.3872	2.5826	.9325	50	
	20	.3638	.3906	2.5605	.9315	40	
	30	.3665	.3939	2.5386	.9304	30	
	40	.3692	.3973	2.5172	.9293	20	
	50	.3719	.4006	2.4960	.9283	10	
22	00	.3746	.4040	2.4751	9272	00	**68**
	10	.3773	.4074	2.4545	.9261	50	
	20	.3800	.4108	2.4342	.9250	40	
	30	.3827	.4142	2.4142	.9239	30	
	40	.3854	.4176	2.3945	.9228	20	
	50	.3881	.4210	2.3750	.9216	10	
23	00	.3907	.4245	2.3559	.9205	00	**67**
	10	.3934	.4279	2.3369	.9194	50	
	20	.3961	.4314	2.3183	.9182	40	
	30	.3987	.4348	2.2998	.9171	30	
	40	.4014	.4383	2.2817	.9159	20	
	50	.4041	.4417	2.2637	.9147	10	
24	00	.4067	.4452	2.2460	.9135	00	**66**
	10	.4094	.4487	2.2286	.9124	50	
	20	.4120	.4522	2.2113	.9112	40	
	30	.4147	.4557	2.1943	.9100	30	
	40	.4173	.4592	2.1775	.9088	20	
	50	.4200	.4628	2.1609	.9075	10	
25	00	.4226	.4663	2.1445	.9063	00	**65**
	10	.4253	.4699	2.1283	.9051	50	
	20	.4279	.4734	2.1123	.9038	40	
	30	.4305	.4770	2.0965	.9026	30	
	40	.4331	.4806	2.0809	.9013	20	
	50	.4358	.4841	2.0655	.9001	10	
26	00	.4384	.4877	2.0503	.8988	00	**64**
	10	.4410	.4913	2.0353	.8975	50	
	20	.4436	.4950	2.0204	.8962	40	
	30	.4462	.4986	2.0057	.8949	30	
	40	.4488	.5022	1.9912	.8936	20	
	50	.4514	.5059	1.9768	·8923	10	
27	00	.4540	.5095	1.9626	.8910	00	**63**
		N. Cos.	N. Cot.	N. Tan.	N. Sin.	′	°

FOUR-PLACE VALUES OF TRIGONOMETRIC FUNCTIONS.

°	′	N. Sin.	N. Tan.	N. Cot.	N. Cos.		
27	00	.4540	.5095	1.9626	.8910	00	**63**
	10	.4566	.5132	1.9486	.8897	50	
	20	.4592	.5169	1.9347	.8884	40	
	30	.4617	.5206	1.9210	.8870	30	
	40	.4643	.5243	1.9074	.8857	20	
	50	.4669	.5280	1.8940	.8843	10	
28	00	.4695	.5317	1.8807	.8829	00	**62**
	10	.4720	.5354	1.8676	.8816	50	
	20	.4746	.5392	1.8546	.8802	40	
	30	.4772	.5430	1.8418	.8788	30	
	40	.4797	.5467	1.8291	.8774	20	
	50	.4823	.5505	1.8165	.8760	10	
29	00	.4848	.5543	1.8040	.8746	00	**61**
	10	.4874	.5581	1.7917	.8732	50	
	20	.4899	.5619	1.7796	.8718	40	
	30	.4924	.5658	1.7675	.8704	30	
	40	.4950	.5696	1.7556	.8689	20	
	50	.4975	.5735	1.7437	.8675	10	
30	00	.5000	.5774	1.7321	.8660	00	**60**
	10	.5025	.5812	1.7205	.8646	50	
	20	.5050	.5851	1.7090	.8631	40	
	30	.5075	.5890	1.6977	.8616	30	
	40	.5100	.5930	1.6864	.8601	20	
	50	.5125	.5969	1.6753	.8587	10	
31	00	.5150	.6009	1.6643	8572	00	**59**
	10	.5175	.6048	1.6534	.8557	50	
	20	.5200	.6088	1.6426	.8542	40	
	30	.5225	.6128	1.6319	.8526	30	
	40	.5250	.6168	1.6212	.8511	20	
	50	.5275	.6208	1.6107	.8496	10	
32	00	.5299	.6249	1.6003	.8480	00	**58**
	10	.5324	.6289	1.5900	.8465	50	
	20	.5348	.6330	1.5798	.8450	40	
	30	.5373	.6371	1.5697	.8434	30	
	40	.5398	.6412	1.5597	.8418	20	
	50	.5422	.6453	1.5497	.8403	10	
33	00	.5446	.6494	1.5399	.8387	00	**57**
	10	.5471	.6536	1.5301	.8371	50	
	20	.5495	.6577	1.5204	.8355	40	
	30	.5519	.6619	1.5108	.8339	30	
	40	.5544	.6661	1.5013	.8323	20	
	50	.5568	.6703	1.4919	.8307	10	
34	00	.5592	.6745	1.4826	.8290	00	**56**
	10	.5616	.6787	1.4733	.8274	50	
	20	.5640	.6830	1.4641	.8258	40	
	30	.5664	.6873	1.4550	.8241	30	
	40	.5688	.6916	1.4460	.8225	20	
	50	.5712	.6959	1.4370	.8208	10	
35	00	.5736	.7002	1.4281	.8192	00	**55**
	10	.5760	.7046	1.4193	.8175	50	
	20	.5783	.7089	1.4106	.8158	40	
	30	.5807	.7133	1.4019	.8141	30	
	40	.5831	.7177	1.3934	.8124	20	
	50	.5854	.7221	1.3848	·8107	10	
36	00	.5878	.7265	1.3764	.8090	00	**54**
		N. Cos.	N. Cot.	N. Tan.	N. Sin.	′	°

°	′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	′	°
36	00	.5878	.7265	1.3764	.8090	00	54
	10	.5901	.7310	1.3680	.8073	50	
	20	.5925	.7355	1.3597	.8056	40	
	30	.5948	.7400	1.3514	.8039	30	
	40	.5972	.7445	1.3432	.8021	20	
	50	.5995	.7490	1.3351	.8004	10	
37	00	.6018	.7536	1.3270	.7986	00	53
	10	.6041	.7581	1.3190	.7969	50	
	20	.6065	.7627	1.3111	.7951	40	
	30	.6088	.7673	1.3032	.7934	30	
	40	.6111	.7720	1.2954	.7916	20	
	50	.6134	.7766	1.2876	.7898	10	
38	00	.6157	.7813	1.2799	.7880	00	52
	10	.6180	.7860	1.2723	.7862	50	
	20	.6202	.7907	1.2647	.7844	40	
	30	.6225	.7954	1.2572	.7826	30	
	40	.6248	.8002	1.2497	.7808	20	
	50	.6271	.8050	1.2423	.7790	10	
39	00	.6293	.8098	1.2349	.7771	00	51
	10	.6316	.8146	1.2276	.7753	50	
	20	.6338	.8195	1.2203	.7735	40	
	30	.6361	.8243	1.2131	.7716	30	
	40	.6383	.8292	1.2059	.7698	20	
	50	.6406	.8342	1.1988	.7679	10	
40	00	.6428	.8391	1.1918	.7660	00	50
	10	.6450	.8441	1.1847	.7642	50	
	20	.6472	.8491	1.1778	.7623	40	
	30	.6494	.8541	1.1708	.7604	30	
	40	.6517	.8591	1.1640	.7585	20	
	50	.6539	.8642	1.1571	.7566	10	
41	00	.6561	.8693	1.1504	.7547	00	49
	10	.6583	.8744	1.1436	.7528	50	
	20	.6604	.8796	1.1369	.7509	40	
	30	.6626	.8847	1.1303	.7490	30	
	40	.6648	.8899	1.1237	.7470	20	
	50	.6670	.8952	1.1171	.7451	10	
42	00	.6691	.9004	1.1106	.7431	00	48
	10	.6713	.9057	1.1041	.7412	50	
	20	.6734	.9110	1.0977	.7392	40	
	30	.6756	.9163	1.0913	.7373	30	
	40	.6777	.9217	1.0850	.7353	20	
	50	.6799	.9271	1.0786	.7333	10	
43	00	.6820	.9325	1.0724	.7314	00	47
	10	.6841	.9380	1.0661	.7294	50	
	20	.6862	.9435	1.0599	.7274	40	
	30	.6884	.9490	1.0538	.7254	30	
	40	.6905	.9545	1.0477	.7234	20	
	50	.6926	.9601	1.0416	.7214	10	
44	00	.6947	.9657	1.0355	.7193	00	46
	10	.6967	.9713	1.0295	.7173	50	
	20	.6988	.9770	1.0235	.7153	40	
	30	.7009	.9827	1.0176	.7133	30	
	40	.7030	.9884	1.0117	.7112	20	
	50	.7050	.9942	1.0058	.7092	10	
45	00	.7071	1.0000	1.0000	.7071	00	45
		N. Cos.	N. Cot.	N. Tan.	N. Sin.	′	°

Gore—Plane and Solid Geometry.

By James Howard Gore, Ph.D., Professor of Mathematics, Columbian University. Author of "Elements of Geodesy," "History of Geodesy," "Bibliography of Geodesy," etc., etc. Crown 8vo. 220 pages. $1.10*

"The study of Geometry is pursued with a threefold purpose.

1. To aid in the development of logical reasoning.
2. To stimulate the use of accurate and precise forms of expression.
3. To acquire facts and principles that may be of practical value in subsequent life.

The first two purposes are advocated because of their disciplinary importance; and when mathematics, because of its exactness, was the only science which furthered to a high degree these purposes, it was necessary for the student to devote a large part of his time to their study. But now other sciences, and even the languages and philosophy, claim disciplinary merit equal to that possessed by mathematics, although differing somewhat in the character of the training.

Hence it appears that the time has come when we can afford to hearken to the demands of the utilitarians, and give up those refinements in mathematics which have been retained for the mental discipline they bring about, but which are wholly lacking in practical application.

I have, therefore, out of an experience as a computer and worker in applied mathematics, as well as a teacher, eliminated from this treatise all propositions that are not of practical value or needed in the demonstration of such propositions. This exclusion leaves out about one-half of the matter usually included in our text-books on geometry. However, instructors will not entirely miss those familiar and interesting theorems which helped to swell the books they studied,—such theorems as fall below the practical standard as here given as exercises or as corollaries.

I have sought to use symbols and equations only in those cases where I could see no gain in spelling out their meaning.

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Attention is called to the solution of problems. Ordinarily the problem is presumed to be solved, and then a demonstration is given to show that the solution was correct. This does not appear to me to be in the line of discovery. I have in all cases started with a statement of those known facts which plainly suggest the first step in the solution, then introduced the next step, giving the construction in connection with each stated fact, so that with the completed construction goes its own demonstration and the student sees the road along which he travelled, and understands from the beginning why he started upon it."—Author's Note.